

ON A MEASURE-THEORETIC READING OF β -GRÜSS TYPE INEQUALITIES

K. CASTILLO AND Â. MACEDO

ABSTRACT. We show that the principal β -Grüss inequalities for the positive integral can be obtained naturally from elementary measure theory. Once the positive β -integral is recognised as integration with respect to a finite positive purely atomic measure, and this measure is normalised to a probability measure, the associated Chebyshev functional becomes simply a covariance. The corresponding inequalities then follow from standard facts valid on arbitrary probability spaces: Korkine's identity, Hölder's inequality on the product space, Cauchy's inequality for covariance, and the elementary variance bound for bounded functions. The Riemann–Stieltjes β -estimates follow, in the signed case, by domination with respect to the total variation measure. Thus, rather than adding another member to this family of β -Grüss inequalities, this note identifies the elementary measure-theoretic mechanism that accounts for the family itself.

1. INTRODUCTION

Grüss type inequalities originate in the classical estimate of Grüss for the Chebyshev functional associated with the ordinary integral [5]. Since then, Grüss type estimates have been formulated in many different settings, including inner product spaces, positive linear functionals and Riemann–Stieltjes integrals; see, for instance, [6, 7]. Grüss-type inequalities for the β -integral and for a related Riemann–Stieltjes β -integral were studied in [2], which provides a recent and comprehensive account of the subject and of the relevant literature.

The aim of the present note is not to add another proof based on the defining series of the β -integral. Rather, it is to point out that, in the positive case, the β -integral is already integration with respect to a finite positive purely atomic measure, with at most countable support, concentrated on the corresponding β -interval. After normalisation by its total mass, this measure becomes a probability measure, and the Chebyshev functional associated with the β -integral becomes exactly the covariance with respect to that probability measure. This simple observation places the main β -Grüss inequalities in a more general framework. Korkine's identity, the product-space Hölder estimate, Cauchy's inequality for covariance, the variance estimate and the pre-Grüss inequality all follow from the elementary probability framework recalled below. Thus, Theorem 3.1, Proposition 3.3, Lemma 3.4, Propositions 3.5 and 3.6, Corollary 3.7, and Theorem 3.8 of [2] are recovered from general facts valid on every probability space. In this sense, the interested reader is referred to the broader literature on the subject, where the same measure-theoretic ideas apply, *mutatis mutandis*.

The same distinction is useful for the Riemann–Stieltjes β -integral. When the integrator induces a positive measure, the preceding covariance argument applies verbatim. When the integrator induces a signed measure, the corresponding estimates are no longer covariance estimates; they follow instead from domination by the total variation measure. In this way, the principal Riemann–Stieltjes β -Grüss estimates of [2], including the mechanism behind their Theorem 4.5, can also be placed within standard measure-theoretic principles. Consequently, the role of the β -integral in these inequalities is to provide a particular finite positive purely atomic measure. Once that measure is identified, the inequalities can be

Date: July 7, 2026.

2020 Mathematics Subject Classification. 28A35, 60E15.

understood as instances of general covariance and total-variation estimates, rather than as depending essentially on special properties of the β -calculus.

The note is organised as follows. Section 2 recalls the elementary measure-theoretic probability facts on which the argument rests, including Korkine's identity, Hölder's inequality on the product space, Cauchy's covariance inequality, and the standard variance estimates leading to Grüss and pre-Grüss type bounds. In Section 3 we identify the positive β -integral with integration with respect to a finite positive purely atomic measure, with at most countable support, and normalise this measure to obtain the probability measure naturally associated with the β -interval. Section 4 then shows that the β -Chebyshev functional is simply the covariance with respect to this probability measure, so that the principal β -Grüss inequalities follow directly from the general results of Section 2. Section 5 treats the corresponding Riemann–Stieltjes β -integral, separating the positive case, where the covariance argument still applies, from the signed case, where the relevant estimates are consequences of domination by the total variation measure. The concluding section places these observations in the broader context of the associated L^p_β -spaces and emphasizes the underlying atomic L^p -space structure.

2. THE ELEMENTARY MEASURE-THEORETIC FRAMEWORK

We recall a few elementary facts from measure-theoretic probability. They are included to fix notation and to make explicit the measure-theoretic framework used below. When a statement is written in a form less commonly quoted, we include a short proof for completeness. Let $(\Omega, \mathcal{A}, P)$ be a probability space. Throughout this section, all functions are real-valued. For $h \in L^1(P)$, write

$$\mathbb{E}_P h := \int_{\Omega} h dP.$$

Whenever $f, g \in L^1(P)$ and $fg \in L^1(P)$, define

$$\text{Cov}_P(f, g) := \mathbb{E}_P(fg) - \mathbb{E}_P f \mathbb{E}_P g.$$

If $f \in L^2(P)$, then $f^2 \in L^1(P)$, and we define

$$\text{Var}_P(f) := \text{Cov}_P(f, f).$$

Lemma 2.1 (Centred covariance identity). *Let $f, g \in L^1(P)$ be such that $fg \in L^1(P)$. Then, for every constant $c \in \mathbb{R}$,*

$$\text{Cov}_P(f, g) = \mathbb{E}_P[(f - c)(g - \mathbb{E}_P g)].$$

In particular, since

$$(f - \mathbb{E}_P f)(g - \mathbb{E}_P g) \in L^1(P),$$

one also has

$$\text{Cov}_P(f, g) = \mathbb{E}_P[(f - \mathbb{E}_P f)(g - \mathbb{E}_P g)].$$

Lemma 2.2 (Korkine identity). *Let $f, g \in L^1(P)$ be such that $fg \in L^1(P)$. Assume moreover that*

$$(x, y) \mapsto (f(x) - f(y))(g(x) - g(y))$$

belongs to $L^1(P \otimes P)$. Then

$$\text{Cov}_P(f, g) = \frac{1}{2} \int_{\Omega} \int_{\Omega} (f(x) - f(y))(g(x) - g(y)) dP(x) dP(y).$$

In particular, this identity holds whenever $f \in L^p(P)$ and $g \in L^q(P)$, where $p, q > 1$ and

$$\frac{1}{p} + \frac{1}{q} = 1.$$

Lemma 2.3 (Product-space Hölder estimate). *Let $p, q > 1$ satisfy*

$$\frac{1}{p} + \frac{1}{q} = 1.$$

If $f \in L^p(P)$ and $g \in L^q(P)$, then

$$\begin{aligned} |\text{Cov}_P(f, g)| &\leq \frac{1}{2} \left(\int_{\Omega} \int_{\Omega} |f(x) - f(y)|^p dP(x) dP(y) \right)^{1/p} \\ &\quad \times \left(\int_{\Omega} \int_{\Omega} |g(x) - g(y)|^q dP(x) dP(y) \right)^{1/q}. \end{aligned}$$

Proof. By Lemma 2.2,

$$|\text{Cov}_P(f, g)| \leq \frac{1}{2} \int_{\Omega} \int_{\Omega} |f(x) - f(y)| |g(x) - g(y)| dP(x) dP(y).$$

The assertion follows from Hölder's inequality on the product probability space

$$(\Omega \times \Omega, \mathcal{A} \otimes \mathcal{A}, P \otimes P).$$

□

Lemma 2.4 (Cauchy's covariance inequality). *Let $f, g \in L^2(P)$. Then*

$$|\text{Cov}_P(f, g)| \leq \sqrt{\text{Var}_P(f)} \sqrt{\text{Var}_P(g)}.$$

Proof. By Lemma 2.1,

$$\text{Cov}_P(f, g) = \mathbb{E}_P[(f - \mathbb{E}_P f)(g - \mathbb{E}_P g)].$$

The result follows from Cauchy's inequality in $L^2(P)$.

□

Lemma 2.5 (Bhatia–Davis and Popoviciu variance bounds). *Let $f \in L^\infty(P)$, and assume that*

$$m \leq f \leq M$$

P -almost everywhere. Then

$$\text{Var}_P(f) \leq (M - \mathbb{E}_P f)(\mathbb{E}_P f - m) \leq \frac{(M - m)^2}{4}.$$

Proof. The first inequality is the Bhatia–Davis variance bound [1, Theorem 1], applied to the bounded random variable f ; in the present notation this gives

$$\text{Var}_P(f) \leq (M - \mathbb{E}_P f)(\mathbb{E}_P f - m).$$

The second inequality is Popoviciu's variance bound [8]; see also [1, Corollary 1]. In the present setting it follows immediately from

$$XY \leq \frac{(X + Y)^2}{4},$$

with

$$X = M - \mathbb{E}_P f, \quad Y = \mathbb{E}_P f - m.$$

Indeed,

$$X + Y = M - m,$$

and therefore

$$(M - \mathbb{E}_P f)(\mathbb{E}_P f - m) \leq \frac{(M - m)^2}{4}.$$

□

Theorem 2.6 (A general Grüss inequality). *Let $f, g \in L^2(P)$ be real-valued functions, and assume that*

$$m \leq f \leq M, \quad n \leq g \leq N$$

P -almost everywhere. Then

$$|\operatorname{Cov}_P(f, g)| \leq \frac{1}{4}(M - m)(N - n).$$

Proof. By Lemma 2.4,

$$|\operatorname{Cov}_P(f, g)| \leq \sqrt{\operatorname{Var}_P(f)} \sqrt{\operatorname{Var}_P(g)}.$$

Since f takes values in $[m, M]$ and g takes values in $[n, N]$, Lemma 2.5 gives

$$\operatorname{Var}_P(f) \leq \frac{(M - m)^2}{4}, \quad \operatorname{Var}_P(g) \leq \frac{(N - n)^2}{4}.$$

Hence

$$|\operatorname{Cov}_P(f, g)| \leq \frac{1}{4}(M - m)(N - n).$$

□

Theorem 2.7 (A general pre-Grüss inequality). *Let $f \in L^\infty(P)$ and $g \in L^1(P)$ be real-valued functions. Assume that*

$$m \leq f \leq M$$

P -almost everywhere. Then

$$|\operatorname{Cov}_P(f, g)| \leq \frac{M - m}{2} \operatorname{E}_P |g - \operatorname{E}_P g|.$$

If, moreover, $g \in L^2(P)$, then

$$|\operatorname{Cov}_P(f, g)| \leq \frac{M - m}{2} \sqrt{\operatorname{Var}_P(g)}.$$

Proof. In Lemma 2.1, take

$$c = \frac{M + m}{2}.$$

Then

$$\operatorname{Cov}_P(f, g) = \operatorname{E}_P \left[\left(f - \frac{M + m}{2} \right) (g - \operatorname{E}_P g) \right].$$

Since

$$\left| f - \frac{M + m}{2} \right| \leq \frac{M - m}{2},$$

we obtain

$$|\operatorname{Cov}_P(f, g)| \leq \frac{M - m}{2} \operatorname{E}_P |g - \operatorname{E}_P g|.$$

If $g \in L^2(P)$, then Cauchy's inequality in the probability space $(\Omega, \mathcal{A}, P)$ gives

$$\operatorname{E}_P |g - \operatorname{E}_P g| \leq (\operatorname{E}_P |g - \operatorname{E}_P g|^2)^{1/2} = \sqrt{\operatorname{Var}_P(g)}.$$

□

3. THE β -INTEGRAL AS A MEASURE

We now specialise the preceding facts to the β -integral considered in [2]. Assume that β satisfies the standing hypotheses of [2]: it is strictly increasing and continuous, has a unique fixed point s_0 , satisfies

$$(t - s_0)(\beta(t) - t) \leq 0.$$

Consequently, for every point t for which the iterates are defined,

$$\lim_{k \rightarrow \infty} \beta^k(t) = s_0.$$

Let $a < b$, and assume that

$$a \leq s_0 \leq b.$$

Throughout the paper we use the convention $\mathbb{N} = \{0, 1, 2, \dots\}$. We write

$$[a, b]_\beta := \{\beta^k(a) : k \in \mathbb{N}\} \cup \{\beta^k(b) : k \in \mathbb{N}\} \cup \{s_0\}.$$

Only the values of the functions on this β -grid are relevant for the measure-theoretic statements below.

Then the β -integral on $[a, b]$ is integration with respect to the finite purely atomic measure

$$\mu_\beta := \sum_{k=0}^{\infty} (\beta^k(b) - \beta^{k+1}(b)) \delta_{\beta^k(b)} - \sum_{k=0}^{\infty} (\beta^k(a) - \beta^{k+1}(a)) \delta_{\beta^k(a)}.$$

Indeed, for $b \geq s_0$, the iterates decrease towards s_0 , and therefore

$$\beta^k(b) - \beta^{k+1}(b) \geq 0.$$

Similarly, for $a \leq s_0$, the iterates increase towards s_0 , and therefore

$$-(\beta^k(a) - \beta^{k+1}(a)) \geq 0.$$

Thus μ_β is a finite positive purely atomic measure, with at most countable support. Equivalently,

$$\int_a^b f(x) d_\beta x = \int_{[a, b]_\beta} f d\mu_\beta,$$

whenever the integral is defined. Moreover,

$$\mu_\beta([a, b]_\beta) = b - a,$$

because the two series telescope:

$$\sum_{k=0}^{\infty} (\beta^k(b) - \beta^{k+1}(b)) = b - s_0$$

and

$$-\sum_{k=0}^{\infty} (\beta^k(a) - \beta^{k+1}(a)) = s_0 - a.$$

We define the associated probability measure by

$$dP_\beta := \frac{1}{b - a} d\mu_\beta.$$

Definition 3.1 (The β -Chebyshev functional). For real-valued functions $f, g \in L^1(P_\beta)$ such that $fg \in L^1(P_\beta)$, define

$$T_\beta(f, g) := \mathbb{E}_{P_\beta}(fg) - \mathbb{E}_{P_\beta} f \mathbb{E}_{P_\beta} g.$$

Proposition 3.2. *For every real-valued $f, g \in L^1(P_\beta)$ such that $fg \in L^1(P_\beta)$, one has*

$$T_\beta(f, g) = \text{Cov}_{P_\beta}(f, g).$$

In particular, if $f \in L^2(P_\beta)$, then

$$T_\beta(f, f) = \text{Var}_{P_\beta}(f).$$

Proof. By definition,

$$\text{Cov}_{P_\beta}(f, g) = \mathbb{E}_{P_\beta}(fg) - \mathbb{E}_{P_\beta} f \mathbb{E}_{P_\beta} g,$$

which is exactly $T_\beta(f, g)$. The identity

$$T_\beta(f, f) = \text{Var}_{P_\beta}(f)$$

then follows immediately from the definition of variance. $\square$

4. RECOVERY OF THE β -GRÜSS TYPE INEQUALITIES

By Proposition 3.2, the β -Chebyshev functional is precisely the covariance associated with the probability measure P_β . Consequently, the inequalities obtained in Section 2 immediately give their β -integral counterparts. We record the resulting statements in order to make explicit the connection with the results of [2].

Observation 4.1 (Variance identity). *For every real-valued $f \in L^2(P_\beta)$,*

$$T_\beta(f, f) = \frac{1}{b-a} \int_a^b f^2(x) d_\beta x - \left(\frac{1}{b-a} \int_a^b f(x) d_\beta x \right)^2.$$

Moreover,

$$T_\beta(f, f) = \frac{1}{b-a} \int_a^b \left(f(x) - \frac{1}{b-a} \int_a^b f(t) d_\beta t \right)^2 d_\beta x.$$

In particular,

$$T_\beta(f, f) \geq 0.$$

Proof. This is the identity $T_\beta(f, f) = \text{Var}_{P_\beta}(f)$, written in β -integral notation. $\square$

Observation 4.2 (Korkine identity and Hölder estimate for T_β). *Let $p, q > 1$ satisfy*

$$\frac{1}{p} + \frac{1}{q} = 1.$$

If $f \in L^p(P_\beta)$ and $g \in L^q(P_\beta)$, then

$$T_\beta(f, g) = \frac{1}{2(b-a)^2} \int_a^b \int_a^b (f(x) - f(y))(g(x) - g(y)) d_\beta x d_\beta y.$$

Moreover,

$$\begin{aligned} |T_\beta(f, g)| &\leq \frac{1}{2} \left(\frac{1}{(b-a)^2} \int_a^b \int_a^b |f(x) - f(y)|^p d_\beta x d_\beta y \right)^{1/p} \\ &\quad \times \left(\frac{1}{(b-a)^2} \int_a^b \int_a^b |g(x) - g(y)|^q d_\beta x d_\beta y \right)^{1/q}. \end{aligned}$$

Proof. Both assertions are Lemma 2.2 and Lemma 2.3 applied to the probability measure P_β , using

$$dP_\beta = \frac{1}{b-a} d\mu_\beta.$$

$\square$

Observation 4.3 (The β -Grüss inequality). *Let $f, g \in L^2(P_\beta)$, and assume that there exist real constants m, M, n, N such that*

$$m \leq f(x) \leq M, \quad n \leq g(x) \leq N,$$

for every $x \in [a, b]_\beta$. Then

$$|T_\beta(f, g)| \leq \frac{1}{4}(M - m)(N - n).$$

Proof. This is Theorem 2.6 applied to the probability measure P_β . $\square$

Observation 4.4. *Let $f, g \in L^2(P_\beta)$, and assume that there exist real constants m, M such that*

$$m \leq f(x) \leq M$$

for every $x \in [a, b]_\beta$. Then

$$|T_\beta(f, g)| \leq \frac{M - m}{2} \sqrt{T_\beta(g, g)}.$$

Proof. This follows from Cauchy's covariance inequality and the Popoviciu variance bound applied to f , namely

$$T_\beta(f, f) = \text{Var}_{P_\beta}(f) \leq \frac{(M - m)^2}{4}.$$

$\square$

Observation 4.5 (The β -pre-Grüss inequality). *Let $f \in L^\infty(P_\beta)$ and $g \in L^1(P_\beta)$. Assume that there exist real constants m, M such that*

$$m \leq f(x) \leq M$$

for every $x \in [a, b]_\beta$. Then

$$|T_\beta(f, g)| \leq \frac{M - m}{2} \frac{1}{b - a} \int_a^b \left| g(x) - \frac{1}{b - a} \int_a^b g(t) d_\beta t \right| d_\beta x.$$

If, moreover, $g \in L^2(P_\beta)$, then

$$|T_\beta(f, g)| \leq \frac{M - m}{2} \sqrt{T_\beta(g, g)}.$$

Proof. This is Theorem 2.7 applied to P_β . Indeed,

$$\int \left| g - \int g dP_\beta \right| dP_\beta = \frac{1}{b - a} \int_a^b \left| g(x) - \frac{1}{b - a} \int_a^b g(t) d_\beta t \right| d_\beta x.$$

The second inequality follows from

$$\text{Var}_{P_\beta}(g) = T_\beta(g, g).$$

$\square$

The preceding statements recover, in particular, Proposition 3.3, Lemma 3.4, Proposition 3.5, Proposition 3.6, Theorem 3.1, Corollary 3.7 and Theorem 3.8 of [2]. The point is that, after the identification of the positive β -integral with the finite positive purely atomic measure μ_β , and after normalisation to P_β , these results are ordinary covariance estimates on a probability space.

5. THE RIEMANN-STIELTJES β -INTEGRAL

We finally consider the Riemann–Stieltjes β -integral. The point is again measure-theoretic, but here two mechanisms have to be distinguished. When the measure induced by the integrator is positive, the preceding covariance argument applies after normalisation. When the induced measure is signed, the estimates are no longer covariance estimates; they follow instead from domination by the total variation measure.

Whenever the following series defines a finite signed measure, the regular part of the Riemann–Stieltjes β -measure associated with u is the signed purely atomic measure

$$\nu_u^{\text{reg}} := \sum_{k=0}^{\infty} (u(\beta^k(b)) - u(\beta^{k+1}(b))) \delta_{\beta^k(b)} - \sum_{k=0}^{\infty} (u(\beta^k(a)) - u(\beta^{k+1}(a))) \delta_{\beta^k(a)}.$$

The corresponding regular part of the Riemann–Stieltjes β -integral is therefore

$$\int_{[a,b]_\beta} f d\nu_u^{\text{reg}}.$$

The possible exceptional contribution at the fixed point s_0 , when present, has to be added separately. In the positive case below, we write ν_u for the finite positive measure which represents the Riemann–Stieltjes β -integral under consideration, including any contribution at s_0 when such a contribution is part of the integral.

Observation 5.1 (The positive Riemann–Stieltjes case). *Assume that ν_u is a finite positive measure and that $0 < \nu_u([a, b]_\beta) < \infty$. Let*

$$dP_u := \frac{1}{\nu_u([a, b]_\beta)} d\nu_u.$$

For real-valued functions $f, g \in L^1(P_u)$ such that $fg \in L^1(P_u)$, set

$$T_u(f, g) := \int fg dP_u - \int f dP_u \int g dP_u.$$

Equivalently,

$$T_u(f, g) = \frac{1}{\nu_u([a, b]_\beta)} \int_a^b f(x)g(x) d\beta u(x) - \frac{1}{\nu_u([a, b]_\beta)^2} \int_a^b f(x) d\beta u(x) \int_a^b g(x) d\beta u(x).$$

Then $T_u(f, g) = \text{Cov}_{P_u}(f, g)$. Consequently, all the covariance estimates of Section 2 hold with $P = P_u$.

Proof. The definition of P_u gives

$$\int h dP_u = \frac{1}{\nu_u([a, b]_\beta)} \int_a^b h(x) d\beta u(x).$$

Substitution into the definition of covariance gives

$$T_u(f, g) = \text{Cov}_{P_u}(f, g).$$

The stated consequences are therefore exactly the results of Section 2 applied to the probability measure P_u . $\square$

Observation 5.2 (Riemann–Stieltjes β -Grüss estimates). *Assume that ν_u is a finite positive measure and that*

$$0 < \nu_u([a, b]_\beta) < \infty.$$

If $f, g \in L^2(P_u)$ and

$$m \leq f \leq M, \quad n \leq g \leq N$$

P_u -almost everywhere, then

$$|T_u(f, g)| \leq \frac{1}{4}(M - m)(N - n).$$

Moreover, if $f \in L^\infty(P_u)$, $g \in L^1(P_u)$, and

$$m \leq f \leq M$$

P_u -almost everywhere, then

$$|T_u(f, g)| \leq \frac{M - m}{2} \int \left| g - \int g dP_u \right| dP_u.$$

If, in addition, $g \in L^2(P_u)$, then

$$|T_u(f, g)| \leq \frac{M - m}{2} \sqrt{T_u(g, g)}.$$

Proof. This follows from Observation 5.1, together with Theorem 2.6 and Theorem 2.7. $\square$

Observation 5.3 (Total-variation domination). *Let ν be a finite signed or complex measure on a measurable space. Then, for every $|\nu|$ -integrable function f ,*

$$\left| \int f d\nu \right| \leq \int |f| d|\nu|.$$

Proof. This is the defining domination property of the total variation measure. $\square$

Observation 5.4 (The β -Lipschitzian estimate). *Let ν_u^{reg} denote the non-exceptional part of the Riemann–Stieltjes β -measure associated with u . Assume that*

$$|\nu_u^{\text{reg}}| \leq L\mu_\beta.$$

This is the measure-theoretic form of the corresponding β -Lipschitz condition: the total variation of the increments of u along the β -orbits is dominated by L times the measure defining the positive β -integral. Then, for every absolutely β -integrable function f ,

$$\left| \int_{[a, b]_\beta} f d\nu_u^{\text{reg}} \right| \leq L \int_a^b |f(x)| d_\beta x.$$

If the Riemann–Stieltjes β -measure has an exceptional contribution at s_0 , that contribution must be treated separately.

Proof. By Observation 5.3 and by the assumed domination

$$|\nu_u^{\text{reg}}| \leq L\mu_\beta,$$

one has

$$\left| \int_{[a, b]_\beta} f d\nu_u^{\text{reg}} \right| \leq \int_{[a, b]_\beta} |f| d|\nu_u^{\text{reg}}| \leq L \int_{[a, b]_\beta} |f| d\mu_\beta.$$

This is precisely the asserted inequality. $\square$

Observation 5.5 (The centred Riemann–Stieltjes β -Grüss estimate). *Assume that the non-exceptional part ν_u^{reg} of the Riemann–Stieltjes β -measure associated with u satisfies*

$$|\nu_u^{\text{reg}}| \leq L\mu_\beta.$$

Let $f \in L^\infty(P_\beta)$, and assume that there exist real constants m, M such that

$$m \leq f(x) \leq M$$

for every $x \in [a, b]_\beta$. Then

$$\left| \int_{[a, b]_\beta} \left(f(x) - \frac{1}{b-a} \int_a^b f(t) d_\beta t \right) d\nu_u^{\text{reg}}(x) \right| \leq \frac{1}{2} L(M - m)(b - a).$$

If an exceptional contribution at s_0 is present, it has to be added separately.

Proof. By Observation 5.3 and by the assumed domination $|\nu_u^{\text{reg}}| \leq L\mu_\beta$, applied to

$$f - \frac{1}{b-a} \int_a^b f(t) d_\beta t,$$

we obtain

$$\left| \int_{[a,b]_\beta} \left(f(x) - \frac{1}{b-a} \int_a^b f(t) d_\beta t \right) d\nu_u^{\text{reg}}(x) \right| \leq L \int_a^b \left| f(x) - \frac{1}{b-a} \int_a^b f(t) d_\beta t \right| d_\beta x.$$

Since P_β is a probability measure, Cauchy's inequality gives

$$\frac{1}{b-a} \int_a^b \left| f(x) - \frac{1}{b-a} \int_a^b f(t) d_\beta t \right| d_\beta x \leq \sqrt{T_\beta(f, f)}.$$

By the Popoviciu variance bound,

$$T_\beta(f, f) = \text{Var}_{P_\beta}(f) \leq \frac{(M-m)^2}{4}.$$

Hence

$$\int_a^b \left| f(x) - \frac{1}{b-a} \int_a^b f(t) d_\beta t \right| d_\beta x \leq \frac{1}{2}(M-m)(b-a).$$

Combining the preceding estimates gives the result. $\square$

Remark 5.6. In the notation of [2], and whenever the one-sided limits involved exist, the identity

$$\int_a^b d_\beta u(x) = u(b) - u(a) - (u(s_0^+) - u(s_0^-))$$

separates the contribution coming from the regular Riemann–Stieltjes β -measure from the possible exceptional contribution at the fixed point s_0 . Thus the measure-theoretic core of Theorem 4.5 is obtained by applying Observation 5.5 to the regular part and then accounting for the exceptional term through the displayed identity. In the cases where the jump term at s_0 vanishes, the estimate reduces directly to the total-variation domination argument above. The remaining Riemann–Stieltjes consequences in that paper are obtained by specialising this same argument to the corresponding choices of u .

This section uses two distinct mechanisms. When the Riemann–Stieltjes β -measure is positive, the estimates are covariance estimates after normalisation. When the measure is signed, the estimates are not covariance estimates; they follow from total-variation domination.

6. CONCLUSIONS

The same point also explains, at a structural level, the results obtained for the spaces $L_\beta^p[a, b]$ associated with certain operators, see [3] and [4]. Once the β -integral is written as integration with respect to its associated finite positive purely atomic measure, the Hölder and Minkowski inequalities are the usual inequalities for L^p -spaces over a positive measure. Likewise, after quotienting by equality μ_β -almost everywhere, the space $L_\beta^p[a, b]$ is naturally identified with the L^p -space of a purely atomic finite measure, equivalently with the product of the two weighted sequence spaces associated with the two β -orbits issuing from a and b , with points of zero μ_β -mass playing no role. Its completeness, separability for $1 \leq p < \infty$, reflexivity for $1 < p < \infty$, and the Hilbert structure of $L_\beta^2[a, b]$ can therefore be understood as consequences of the elementary structure of atomic L^p -spaces.

It is worth stressing that this point is not specific to the β -calculus. The same reasoning applies, mutatis mutandis, to other discrete positive calculi, such as the usual q -calculus, whenever the corresponding integral can be represented as integration with respect to a

finite positive atomic measure. In such situations, inequalities whose proofs use only positivity, Hölder-type estimates, power-mean arguments, or normalisation of the underlying discrete integral are governed by the same measure-theoretic mechanism. The particular form of the nodes, weights and auxiliary identities may vary from one calculus to another, but the underlying principle remains the elementary structure of integration with respect to a positive atomic measure.

The references collected in [3, 4, 2] show that these constructions have generated a substantial body of work. The papers just cited are among its more significant contributions. From the present viewpoint, however, a substantial part of this circle of problems can be traced back to the same elementary measure-theoretic mechanism: the identification of the relevant β -integral with integration against a finite positive purely atomic measure with at most countable support. This does not diminish the usefulness of the β -formalism, nor of analogous formalisms such as the usual q -calculus. It indicates, rather, that for the material discussed in this note, and for a significant part of the surrounding work, the essential mechanism is not a special feature of any one discrete calculus but the elementary structure of integration with respect to an atomic finite measure.

ACKNOWLEDGEMENTS

This work arose from conversations between the authors and was motivated by the previous work of ÂM and coauthors on the subject. The measure-theoretic approach adopted in this note was proposed by KC during those discussions. KC acknowledges financial support from the Centre for Mathematics of the University of Coimbra (CMUC), funded by the Portuguese Foundation for Science and Technology (FCT), under the projects UID/00324/2025 (<https://doi.org/10.54499/UID/00324/2025>) and UID/PRR/00324/2025, and from FCT under the grant <https://doi.org/10.54499/2022.00143.CEECIND/CP1714/CT0002>. The research ÂM was partially financed by Portuguese funds through FCT, under the project UID/00013/2025 (<https://doi.org/10.54499/UID/00013/2025>).

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CMUC, DEPARTMENT OF MATHEMATICS, UNIVERSITY OF COIMBRA, 3000-143 COIMBRA, PORTUGAL
 Email address: kenier@mat.uc.pt

CENTRO DE MATEMÁTICA DA UNIVERSIDADE DO MINHO – POLO CMAT-UTAD, AND DEPARTAMENTO DE MATEMÁTICA, UNIVERSIDADE DE TRÁS-OS-MONTES E ALTO DOURO, QUINTA DE PRADOS, 5001–801 VILA REAL, PORTUGAL
 Email address: amacedo@utad.pt