UNBOUNDED BANDED MATRICES, SHIFTED POSITIVE BIDIAGONAL FACTORIZATIONS, AND MIXED-TYPE MULTIPLE ORTHOGONALITY

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ABSTRACT. This work extends Favard-type spectral representations for banded matrices T beyond the bounded setting. It assumes that, for every $N \in \mathbb{N}_0$, there exists a shift $s_N \geq 0$ such that the shifted truncation $A_N := T^{[N]} + s_N I_{N+1}$ admits a positive bidiagonal factorization (PBF). Allowing s_N to depend on N leads to a natural recentering step: the discrete Gauss-type quadrature measures associated with A_N are translated by $x \mapsto x - s_N$, producing a uniformly bounded family of distribution functions. Combining moment stabilization for banded truncations with Helly-type compactness theorems yields a limiting matrix-valued measure, together with a Favard-type spectral representation and the corresponding mixed-type multiple biorthogonality relations. As a consequence, the classical Favard theorem for (possibly unbounded) Jacobi matrices is recovered as a special case. Indeed, for a tridiagonal J with positive sub- and superdiagonals, each truncation $J^{[N]}$ admits a shift $s_N \geq 0$ such that $J^{[N]} + s_N I_{N+1}$ is oscillatory and therefore admits a PBF. The preceding construction then produces the usual spectral measure for J.

1. INTRODUCTION

The spectral Favard theorem for banded matrices admitting a positive bidiagonal factorization was established in the bounded case in [**BFM1**]. In that work, boundedness makes it possible to construct a matrix-valued spectral measure and to derive a Gaussian quadrature formula, which becomes a central tool in the proof. Further structural properties, including total positivity and the normality (maximal degree pattern) of the associated mixed multiple orthogonal polynomials, were analyzed in [**BFM4**], still under the boundedness hypothesis.

The purpose of the present paper is to remove the boundedness assumption. In the unbounded setting, the positive bidiagonal factorization cannot, in general, be imposed directly at the level of the semi-infinite matrix. Accordingly, we work with finite truncations and assume that for every $N \in \mathbb{N}_0$ there exists a shift $s_N \geq 0$ such that the shifted truncation $T^{[N]} + s_N I_{N+1}$ admits a positive bidiagonal factorization. Under this hypothesis, and keeping the banded structure of T, we prove that the spectral Favard theorem remains valid in the unbounded case. The argument builds on the constructive framework introduced in [**BFM1**] and follows the same discrete-to-limit philosophy: one constructs discrete matrix-valued measures at the truncation level, and then passes to a limiting measure as the truncation size tends to infinity.

More specifically, for each finite truncation one relies on the positive bidiagonal factorization, on the spectral properties of oscillatory matrices, and on the sign-variation structure of their eigenvectors [**Gantmacher**, **Fallat**]. In the banded setting this yields positivity of the associated Christoffel numbers and Gauss-type quadrature formulas at the discrete level [**BFM1**]. The main difficulty in the unbounded regime is that the shifts s_N may depend on N, so the discrete measures must be recentered before taking limits. Following the strategy developed by Chihara for orthogonal polynomials [**Chihara**] and using a Helly-type selection principle, we combine the discrete quadrature formulas with uniform boundedness of the translated distribution functions to extract a convergent subsequence of measures and identify its limit. This provides the

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matrix-valued spectral measure and completes the proof of the spectral Favard theorem without boundedness.

A further point concerns the degree pattern of the mixed multiple orthogonal polynomials. As shown in [**BFM4**], normality (maximal degrees along the step-line) is established before taking the large-N limit. In both the bounded and unbounded settings, the required initial conditions must be constructed from the positive bidiagonal factorization. There is a natural freedom in this choice, which can be parametrized by families of positive triangular matrices, or more restrictively by totally positive triangular matrices (all nontrivial minors are positive). The latter choice guarantees that maximal degrees are attained, exactly as in the normality result proved in [**BFM4**].

Totally nonnegative and totally positive matrices constitute a fundamental class of operators with deep connections to oscillation theory, approximation theory, and spectral analysis. Their basic properties were established in the seminal work of Gantmacher and Krein [**Gantmacher**], and have since been developed extensively, see for example the monographs of Karlin [**Karlin**] and Pinkus [**Pinkus**], as well as the modern reference by Fallat and Johnson [**Fallat**]. Of particular relevance in the present context are oscillatory matrices, whose spectra consist of simple positive eigenvalues and whose eigenvectors exhibit a precise sign variation structure. These features make totally nonnegative matrices especially well adapted to the spectral analysis of banded operators admitting positive bidiagonal factorizations, and provide the key mechanism ensuring the positivity of Christoffel-type coefficients and the normality of the associated families of multiple orthogonal polynomials.

Multiple orthogonal polynomials arise as natural generalizations of classical orthogonal polynomials when orthogonality conditions are imposed with respect to several measures. They play a central role in rational approximation theory, number theory, random matrix models, and integrable systems. A systematic account of the theory can be found in the monograph of Nikishin and Sorokin [**nikishin_sorokin**], as well as in the classical treatment of Ismail [**Ismail**]. From a modern perspective, multiple orthogonal polynomials are most naturally characterized through recurrence relations associated with banded operators, leading to matrix-valued and vector-valued spectral problems. In this framework, polynomials of mixed type emerge as the appropriate objects encoding the joint left–right spectral structure of banded matrices, see for instance [**andrei_walter**, **afm**]. The mixed-type setting is particularly well suited for the study of non-symmetric banded operators, where biorthogonality and matrix-valued measures replace the scalar orthogonality of the classical theory.

1.1. **Banded matrices and mixed-type multiple orthogonal polynomials**. Let us define the recursion polynomials associated with the banded matrix

$$T=\begin{bmatrix}
T_{0,0} & \cdots & T_{0,q} & 0 & \cdots & \\
\vdots & & & \ddots & \ddots & \\
T_{p,0} & & & & \ddots & \ddots \\
0 & \ddots & & & & \ddots\\
\vdots & \ddots & \ddots & & & \\
\vdots & & \ddots & \ddots & &
\end{bmatrix}$$

as the entries of semi-infinite left and right eigenvectors,

$$A^{(a)}=\begin{bmatrix}A^{(a)}_0 & A^{(a)}_1 & \cdots\end{bmatrix},\quad a\in\{1,\dots,p\},\quad B^{(b)}=\begin{bmatrix}B^{(b)}_0\\ B^{(b)}_1\\ \vdots\end{bmatrix},\quad b\in\{1,\dots,q\},$$

which satisfy the eigenvalue relations

$$A^{(a)}T = xA^{(a)}, \quad a \in \{1,\dots,p\}, \quad TB^{(b)} = xB^{(b)}, \quad b \in \{1,\dots,q\}.$$

Throughout we assume that the semi-infinite matrix T admits a normalized positive bidiagonal factorization (PBF),

$$T = L_1 \cdots L_p \Delta U_q \cdots U_1,$$

where $\Delta = \operatorname{diag}(\Delta_0, \Delta_1, \dots)$ and the bidiagonal factors are normalized (their diagonal entries equal 1) and given by

$$L_k := \begin{bmatrix} 1 & 0 & \cdots & \cdots \\ L_{k|1,0} & 1 & \ddots & \\ 0 & L_{k|2,1} & 1 & \ddots \\ \vdots & \ddots & \ddots & \ddots \end{bmatrix}, \quad U_k := \begin{bmatrix} 1 & U_{k|0,1} & 0 & \cdots \\ 0 & 1 & U_{k|1,2} & \ddots \\ \vdots & \ddots & 1 & \ddots \\ \vdots & & \ddots & \ddots \end{bmatrix},$$

so that the positivity constraints $L_{k|i,j}, U_{k|i,j}, \Delta_i > 0$, for $i \in \mathbb{N}_0$, hold.

The components of the eigenvectors are polynomials in the spectral parameter x; we refer to them as the left and right recursion polynomials. They are fixed by the initial conditions

$$\begin{cases} A^{(1)}_0 = 1, \\ A^{(1)}_1 = \nu^{(1)}_1, \\ \quad\vdots \\ A^{(1)}_{p-1} = \nu^{(1)}_{p-1}, \end{cases} \qquad \begin{cases} A^{(2)}_0 = 0, \\ A^{(2)}_1 = 1, \\ A^{(2)}_2 = \nu^{(2)}_2, \\ \quad\vdots \\ A^{(2)}_{p-1} = \nu^{(2)}_{p-1}, \end{cases} \qquad \cdots \qquad \begin{cases} A^{(p)}_0 = 0, \\ \quad\vdots \\ A^{(p)}_{p-2} = 0, \\ A^{(p)}_{p-1} = 1, \end{cases} \tag{1}$$

with $\nu^{(i)}_j$ arbitrary constants, and

$$\begin{cases} B^{(1)}_0 = 1, \\ B^{(1)}_1 = \xi^{(1)}_1, \\ \quad\vdots \\ B^{(1)}_{q-1} = \xi^{(1)}_{q-1}, \end{cases} \qquad \begin{cases} B^{(2)}_0 = 0, \\ B^{(2)}_1 = 1, \\ B^{(2)}_2 = \xi^{(2)}_2, \\ \quad\vdots \\ B^{(2)}_{q-1} = \xi^{(2)}_{q-1}, \end{cases} \qquad \cdots \qquad \begin{cases} B^{(q)}_0 = 0, \\ \quad\vdots \\ B^{(q)}_{q-2} = 0, \\ B^{(q)}_{q-1} = 1, \end{cases} \tag{2}$$

where $\xi^{(i)}_j$ are also arbitrary. We collect these parameters into the initial condition matrices

$$\nu := \begin{bmatrix} 1 & 0 & \cdots & \cdots & 0 \\ \nu^{(1)}_1 & 1 & \ddots & & \vdots \\ \vdots & \ddots & \ddots & \ddots & \vdots \\ \vdots & & \ddots & \ddots & 0 \\ \nu^{(1)}_{p-1} & \cdots & \cdots & \nu^{(p-1)}_{p-1} & 1 \end{bmatrix}, \quad \xi := \begin{bmatrix} 1 & 0 & \cdots & \cdots & 0 \\ \xi^{(1)}_1 & 1 & \ddots & & \vdots \\ \vdots & \ddots & \ddots & \ddots & \vdots \\ \vdots & & \ddots & \ddots & 0 \\ \xi^{(1)}_{q-1} & \cdots & \cdots & \xi^{(q-1)}_{q-1} & 1 \end{bmatrix}.$$

The recursion polynomials are uniquely determined by (1.1)–(1.1) together with the banded recursions

$$A^{(a)}_{n-q}T_{n-q,n} + \cdots + A^{(a)}_{n+p}T_{n+p,n} = xA^{(a)}_n, \quad n \in \{0,1,\dots\}, \quad a \in \{1,\dots,p\}, \quad A^{(a)}_{-q} = \cdots = A^{(a)}_{-1} = 0,$$
$$T_{n,n-p}B^{(b)}_{n-p} + \cdots + T_{n,n+q}B^{(b)}_{n+q} = xB^{(b)}_n, \quad n \in \{0,1,\dots\}, \quad b \in \{1,\dots,q\}, \quad B^{(b)}_{-p} = \cdots = B^{(b)}_{-1} = 0.$$

For the semi-infinite matrix T we denote by $P_N(x)$ the characteristic polynomial of the truncation $T^{[N-1]}$, namely

$$P_N(x) := \begin{cases} 1, & N=0, \\ \det\left(xI_N - T^{[N-1]}\right), & N\in\mathbb{N}. \end{cases}$$

Introduce the blocks of left and right recursion polynomials

$$A_N := \begin{bmatrix} A^{(1)}_N & \cdots\cdots & A^{(1)}_{N+p-1} \\ \vdots & & \vdots \\ A^{(p)}_N & \cdots\cdots & A^{(p)}_{N+p-1} \end{bmatrix}, \quad B_N := \begin{bmatrix} B^{(1)}_N & \cdots\cdots\cdots & B^{(q)}_N \\ \vdots & & \vdots \\ B^{(1)}_{N+q-1} & \cdots\cdots & B^{(q)}_{N+q-1} \end{bmatrix}, \quad N\in\mathbb{N}_0,$$

and the products

$$\alpha_N := (-1)^{(p-1)N} T_{p,0}\cdots T_{N+p-1,N-1}, \quad \beta_N := (-1)^{(q-1)N} T_{0,q}\cdots T_{N-1,N+q-1}, \quad N\in\mathbb{N},$$

with $\alpha_0=\beta_0=1$. Since the entries on the extreme diagonals are nonzero, we have $\alpha_N,\beta_N\neq 0$. For $N\in\mathbb{N}_0$ these quantities relate the characteristic polynomials and the recursion blocks through

$$P_N(x) = \alpha_N \det A_N(x) = \beta_N \det B_N(x).$$

Next we introduce determinantal polynomials, built from the left and right recursion polynomials, which produce left and right eigenvectors of $T^{[N]}$. Define

$$Q_{n,N} := \begin{vmatrix} A^{(1)}_n & \cdots\cdots\cdots\cdots & A^{(p)}_n \\ A^{(1)}_{N+1} & \cdots\cdots\cdots & A^{(p)}_{N+1} \\ \vdots & & \vdots \\ A^{(1)}_{N+p-1} & \cdots\cdots & A^{(p)}_{N+p-1} \end{vmatrix}, \quad R_{n,N} := \begin{vmatrix} B^{(1)}_n & \cdots\cdots\cdots & B^{(q)}_n \\ B^{(1)}_{N+1} & \cdots\cdots\cdots & B^{(q)}_{N+1} \\ \vdots & & \vdots \\ B^{(1)}_{N+q-1} & \cdots\cdots & B^{(q)}_{N+q-1} \end{vmatrix},$$

together with the semi-infinite row and column vectors

$$Q_N := \begin{bmatrix} Q_{0,N} & Q_{1,N} \cdots\cdots \end{bmatrix}, \quad R_N := \begin{bmatrix} R_{0,N} \\ R_{1,N} \\ \vdots \end{bmatrix},$$

and their truncations

$$Q^{\langle N\rangle} := \begin{bmatrix} Q_{0,N} & Q_{1,N} \cdots\cdots Q_{N,N} \end{bmatrix}, \quad R^{\langle N\rangle} := \begin{bmatrix} R_{0,N} \\ R_{1,N} \\ \vdots \\ R_{N,N} \end{bmatrix}.$$

A generalized Christoffel–Darboux formalism for banded matrices yields the following identities.

i) The biorthogonal families of left and right eigenvectors $\{w^{\langle N\rangle}_k\}_{k=1}^{N+1}$ and $\{u^{\langle N\rangle}_k\}_{k=1}^{N+1}$ can be expressed as

$$w^{\langle N\rangle}_k = \frac{Q^{\langle N\rangle}(\lambda^{[N]}_k)}{\beta_N \sum_{l=0}^N Q_{l,N}(\lambda^{[N]}_k) R_{l,N}(\lambda^{[N]}_k)}, \quad u^{\langle N\rangle}_k = \beta_N R^{\langle N\rangle}(\lambda^{[N]}_k).$$

In particular, $w^{\langle N\rangle}_k u^{\langle N\rangle}_l = \delta_{k,l}$.

ii) In terms of the characteristic polynomial, the entries of the left eigenvectors admit the representation

$$w^{\langle N\rangle}_{k,n} = \frac{\alpha_N Q_{n,N}(\lambda^{[N]}_k)}{P_N(\lambda^{[N]}_k) P'_{N+1}(\lambda^{[N]}_k)}.$$

iii) Let $\mathscr{U}$ be the matrix whose columns are the right eigenvectors u_k in the standard order, and let $\mathscr{W}$ be the matrix whose rows are the left eigenvectors w_k in the standard order. Then $\mathscr{U}\mathscr{W} = \mathscr{W}\mathscr{U} = I_{N+1}$.
iv) Writing $D = \operatorname{diag}\big(\lambda_1^{[N]}, \lambda_2^{[N]}, \dots, \lambda_{N+1}^{[N]}\big)$, one has the spectral decomposition

$$\mathscr{U} D^n \mathscr{W} = \big(T^{[N]}\big)^n, \quad n \in \mathbb{N}_0.$$

The so-called Christoffel numbers $\mu_{k,a}^{[N]}$ and $\rho_{k,b}^{[N]}$, $a \in \{1, \dots, p\}$ and $b \in \{1, \dots, q\}$, see [**BFM1**], provide the expansions

$$\begin{aligned} w_{k,n}^{\langle N \rangle} &= A_{n-1}^{(1)}\big(\lambda_k^{[N]}\big)\mu_{k,1}^{[N]} + \dots + A_{n-1}^{(p)}\big(\lambda_k^{[N]}\big)\mu_{k,p}^{[N]}, \\ u_{k,n}^{\langle N \rangle} &= B_{n-1}^{(1)}\big(\lambda_k^{[N]}\big)\rho_{k,1}^{[N]} + \dots + B_{n-1}^{(q)}\big(\lambda_k^{[N]}\big)\rho_{k,q}^{[N]}. \end{aligned}$$

Moreover, these coefficients are related to the eigenvector entries and the initial conditions by

$$\begin{bmatrix} \mu_{k,1}^{[N]} \\ \mu_{k,2}^{[N]} \\ \vdots \\ \mu_{k,p}^{[N]} \end{bmatrix} = \nu^{-1} \begin{bmatrix} w_{k,1}^{\langle N \rangle} \\ w_{k,2}^{\langle N \rangle} \\ \vdots \\ w_{k,p}^{\langle N \rangle} \end{bmatrix}, \quad \begin{bmatrix} \rho_{k,1}^{[N]} \\ \rho_{k,2}^{[N]} \\ \vdots \\ \rho_{k,q}^{[N]} \end{bmatrix} = \xi^{-1} \begin{bmatrix} u_{k,1}^{\langle N \rangle} \\ u_{k,2}^{\langle N \rangle} \\ \vdots \\ u_{k,q}^{\langle N \rangle} \end{bmatrix}.$$

As shown in [**BFM1**], for a suitable choice of initial conditions in terms of the bidiagonal factorization the Christoffel numbers are strictly positive, namely

$$\rho_{k,b}^{[N]} > 0, \quad \mu_{k,a}^{[N]} > 0, \quad k \in \{1, \dots, N+1\}, \quad a \in \{1, \dots, p\}, \quad b \in \{1, \dots, q\}.$$

Consider the associated step functions

$$\psi_{b,a}^{[N]} := \begin{cases} 0, & x < \lambda_{N+1}^{[N]}, \\ \rho_{1,b}^{[N]}\mu_{1,a}^{[N]} + \dots + \rho_{k,b}^{[N]}\mu_{k,a}^{[N]}, & \lambda_{k+1}^{[N]} \leqslant x < \lambda_k^{[N]}, \quad k \in \{1, \dots, N\}, \\ \rho_{1,b}^{[N]}\mu_{1,a}^{[N]} + \dots + \rho_{N+1,b}^{[N]}\mu_{N+1,a}^{[N]}, & x \geqslant \lambda_1^{[N]}. \end{cases}$$

The final step is uniformly bounded: for $a \in \{1, \dots, p\}$ and $b \in \{1, \dots, q\}$ one has

$$\rho_{1,b}^{[N]}\mu_{1,a}^{[N]} + \dots + \rho_{N+1,b}^{[N]}\mu_{N+1,a}^{[N]} = (\xi^{-1} I_{q,p} \nu^{-\top})_{b,a}.$$

Here $I_{q,p} \in \mathbb{R}^{q \times p}$ denotes the rectangular identity matrix, with $(I_{q,p})_{k,l} = \delta_{k,l}$.

Define the $q \times p$ matrix

$$\Psi^{[N]} := \begin{bmatrix} \psi_{1,1}^{[N]} & \cdots & \psi_{1,p}^{[N]} \\ \vdots & & \vdots \\ \psi_{q,1}^{[N]} & \cdots & \psi_{q,p}^{[N]} \end{bmatrix}$$

and the corresponding matrix of discrete Lebesgue–Stieltjes measures supported at the zeros of P_{N+1},

$$\mathrm{d}\,\Psi^{[N]} = \begin{bmatrix} \mathrm{d}\,\psi_{1,1}^{[N]} & \cdots\cdots & \mathrm{d}\,\psi_{1,p}^{[N]} \\ \vdots & & \vdots \\ \mathrm{d}\,\psi_{q,1}^{[N]} & \cdots\cdots & \mathrm{d}\,\psi_{q,p}^{[N]} \end{bmatrix} = \sum_{k=1}^{N+1} \begin{bmatrix} \rho_{k,1}^{[N]} \\ \vdots \\ \rho_{k,q}^{[N]} \end{bmatrix} \begin{bmatrix} \mu_{k,1}^{[N]} & \cdots\cdots & \mu_{k,p}^{[N]} \end{bmatrix} \delta\big(x - \lambda_k^{[N]}\big).$$

The following biorthogonality relations hold:

$$\sum_{a=1}^{p} \sum_{b=1}^{q} \int B_n^{(b)}(x)\, \mathrm{d}\,\psi_{b,a}^{[N]}(x) A_m^{(a)}(x) = \delta_{n,m}, \quad n, m \in \{0, \dots, N\}.$$

Moreover, for $m \in \{1,\dots,N\}$ the orthogonality relations

$$\sum_{a=1}^{p}\int x^n \,\mathrm{d}\psi^{[N]}_{b,a}(x)A^{(a)}_m(x)=0,\quad n\in\left\{0,\dots,\left\lceil\frac{m+1-b}{q}\right\rceil-1\right\},\quad b\in\{1,\dots,q\},$$
$$\sum_{b=1}^{q}\int B^{(b)}_m(x)\,\mathrm{d}\psi^{[N]}_{b,a}(x)x^n=0,\quad n\in\left\{0,\dots,\left\lceil\frac{m+1-a}{p}\right\rceil-1\right\},\quad a\in\{1,\dots,p\}.$$

are satisfied. Therefore, the recursion polynomials form discrete multiple orthogonal polynomials of mixed type on the step-line. Finally, one obtains the spectral representation

$$(T^k)_{m,n}=\sum_{a=1}^{p}\sum_{b=1}^{q}\int B^{(b)}_n(x)x^k\,\mathrm{d}\psi^{[N]}_{b,a}(x)A^{(a)}_m(x),\quad m,n\in\{0,\dots,N\}.$$

In the study of the quadrature formulas in [**BFM1**], we consider

$$\int x^n\,\mathrm{d}\psi^{[N]}_{b,a}(x)=\sum_{k=1}^{N+1}\rho^{[N]}_{k,b}\mu^{[N]}_{k,a}\big(\lambda^{[N]}_k\big)^n=(e^{\xi}_b)^\top\big(T^{[N]}\big)^n e^{\nu}_a,\quad a\in\{1,\dots,p\},\quad b\in\{1,\dots,q\}. \tag{3}$$

and the following degrees of precision

$$d_{b,a}(N)=\left\lceil\frac{N+2-a}{p}\right\rceil+\left\lceil\frac{N+2-b}{q}\right\rceil-1,\quad a\in\{1,\dots,p\},\quad b\in\{1,\dots,q\}.$$

These degrees of precision or $d_{b,a}(N)$, $a\in\{1,\dots,p\}$, $b\in\{1,\dots,q\}$, are the largest natural numbers such that

$$(e^{\xi}_b)^\top\big(T^{[N]}\big)^n e^{\nu}_a=\big(u^{\xi}_b\big)^\top T^n u^{\nu}_a,\quad 0\leqslant n\leqslant d_{b,a}(N),\quad a\in\{1,\dots,p\},\quad b\in\{1,\dots,q\}.$$

2. The Favard spectral theorem for unbounded banded matrices

In [**BFM1**, **BFM4**] we proved the following result. A triangular matrix is said to be *totally positive* if all its nontrivial minors are strictly positive; see [**BFM4**].

Theorem 2.1 (Favard spectral representation for bounded banded matrices with PBF). *Let us assume that the matrix T is bounded and admits a positive bidiagonal factorization. Then the following versions of the Favard spectral theorem hold:*

i) *Let the initial condition matrices A_0 and B_0 be determined by the positive bidiagonal factorization up to left multiplication by an upper unitriangular positive matrix and a lower unitriangular positive matrix, respectively, as described in [**BFM1**]. Then there exist $p\times q$ nondecreasing positive functions $\psi_{b,a}$, $a\in\{1,\dots,p\}$ and $b\in\{1,\dots,q\}$, and corresponding positive Lebesgue–Stieltjes measures $\mathrm{d}\psi_{b,a}$ such that the following mixed-type multiple biorthogonality relations hold:*

$$\sum_{a=1}^{p}\sum_{b=1}^{q}\int_\Delta B^{(b)}_l(x)\;\mathrm{d}\psi_{b,a}(x)\;A^{(a)}_k(x)=\delta_{k,l},\quad k,l\in\mathbb{N}_0.$$

Moreover, the degrees of these mixed-type multiple orthogonal polynomials satisfy

$$\deg A^{(a)}_n\le\left\lceil\frac{n+2-a}{p}\right\rceil-1,\quad \deg B^{(b)}_n\le\left\lceil\frac{n+2-b}{q}\right\rceil-1.$$

Equality is attained at least when $n=p+a-1$ for $a\in\{1,\dots,p\}$ (left polynomials), and when $n=Nq+b-1$ for $b\in\{1,\dots,q\}$ (right polynomials).

ii) *Let the initial condition matrices A_0 and B_0 be determined by the positive bidiagonal factorization up to left multiplication by an upper unitriangular totally positive matrix and a lower unitriangular totally positive matrix, respectively, as described in [**BFM4**]. Then there exist $p\times q$ nondecreasing positive functions $\psi_{b,a}$, $a\in\{1,\dots,p\}$ and $b\in\{1,\dots,q\}$, and corresponding positive Lebesgue–Stieltjes measures $\mathrm{d}\,\psi_{b,a}$ such that the mixed-type multiple biorthogonality relations hold:*

$$\sum_{a=1}^{p}\sum_{b=1}^{q}\int_{\Delta} B^{(b)}_l(x)\ \mathrm{d}\,\psi_{b,a}(x)\,A^{(a)}_k(x)=\delta_{k,l},\quad k,l\in\mathbb{N}_0.$$

In this case, the degrees are maximal:

$$\deg A^{(a)}_n=\left\lceil\frac{n+2-a}{p}\right\rceil-1,\quad \deg B^{(b)}_n=\left\lceil\frac{n+2-b}{q}\right\rceil-1.$$

Our aim here is to extend this result to unbounded matrices. More precisely, we prove the following theorem.

Theorem 2.2 (Favard spectral representation for unbounded banded matrices with PBF). *Assume that the matrix T is unbounded and that there exists a sequence $(s_N)_{N\geq 0}$ with $s_N\geq 0$ such that, for every N, the truncation*

$$T^{[N]}+s_N I_{N+1}$$

*admits a positive bidiagonal factorization. Let the initial condition matrices A_0 and B_0 be determined by this factorization up to left multiplication by an upper unitriangular positive matrix and a lower unitriangular positive matrix, respectively, as described in [**BFM1**].*

Then there exist $p\times q$ nondecreasing positive functions $\psi_{b,a}$, $a\in\{1,\dots,p\}$ and $b\in\{1,\dots,q\}$, and the corresponding positive Lebesgue–Stieltjes measures $\mathrm{d}\,\psi_{b,a}$, such that

$$\sum_{a=1}^{p}\sum_{b=1}^{q}\int_{\mathbb{R}} B^{(b)}_l(x)\,x^n\ \mathrm{d}\,\psi_{b,a}(x)\,A^{(a)}_k(x)=(T^n)_{k,l},\quad k,l\in\mathbb{N}_0,\ n\in\mathbb{N}_0.$$

In particular, the mixed-type multiple biorthogonality relations hold:

$$\sum_{a=1}^{p}\sum_{b=1}^{q}\int_{\mathbb{R}} B^{(b)}_l(x)\ \mathrm{d}\,\psi_{b,a}(x)\,A^{(a)}_k(x)=\delta_{k,l},\quad k,l\in\mathbb{N}_0,$$

as well as the mixed-type orthogonality relations:

$$\sum_{a=1}^{p}\int_{\mathbb{R}} x^n\ \mathrm{d}\,\psi_{b,a}(x)\,A^{(a)}_m(x)=0,\quad n\in\left\{0,\dots,\left\lceil\frac{m+1-b}{q}\right\rceil-1\right\},\quad b\in\{1,\dots,q\},$$

$$\sum_{b=1}^{q}\int_{\mathbb{R}} B^{(b)}_m(x)\ \mathrm{d}\,\psi_{b,a}(x)\,x^n=0,\quad n\in\left\{0,\dots,\left\lceil\frac{m+1-a}{p}\right\rceil-1\right\},\quad a\in\{1,\dots,p\}.$$

The degrees satisfy

$$\deg A^{(a)}_n\leq\left\lceil\frac{n+2-a}{p}\right\rceil-1,\quad \deg B^{(b)}_n\leq\left\lceil\frac{n+2-b}{q}\right\rceil-1,$$

with equality attained at least when $n=\left\lceil\frac{q}{p}\right\rceil p+a-1$ for $a\in\{1,\dots,p\}$, and when $n=\left\lceil\frac{p}{q}\right\rceil q+b-1$ for $b\in\{1,\dots,q\}$. Moreover, if the initial condition matrices A_0 and B_0 are determined by the positive bidiagonal factorization up to left multiplication by an upper unitriangular totally positive matrix and a lower unitriangular totally positive matrix, respectively, then equality holds in the above degree bounds for all $n\in\mathbb{N}_0$ and for all $a\in\{1,\dots,p\}$ and $b\in\{1,\dots,q\}$.

For the proof we base our analysis on [**Chihara**]. In particular, we recall the following results due to Helly.

i) **Selection principle.**
Let $\{\psi_n\}$ be a uniformly bounded sequence of nondecreasing functions defined on $(-\infty,\infty)$. Then $\{\psi_n\}$ admits a subsequence which converges on $(-\infty,\infty)$ to a bounded nondecreasing function.

ii) **Helly's second theorem.**
Let $\{\psi_n\}$ be a uniformly bounded sequence of nondecreasing functions defined on a compact interval $[\alpha,\beta]$, and suppose that it converges on $[\alpha,\beta]$ to a limit function ψ. Then, for every real-valued function f continuous on $[\alpha,\beta]$,

$$\lim_{n\to\infty}\int_\alpha^\beta f\,\mathrm{d}\psi_n=\int_\alpha^\beta f\,\mathrm{d}\psi.$$

We also need the following lemmas:

Lemma 2.3. *Let T be a semi-infinite matrix with band structure*

$$T_{i,j}=0,\quad j<i-p \text{ or } j>i+q,$$

for some $p,q\in\mathbb{N}$. Let $T^{[N]}$ be the principal truncation of size $N+1$. Let $\nu\in\mathbb{C}^{p\times p}$ and $\xi\in\mathbb{C}^{q\times q}$ be invertible lower triangular matrices. Fix $a\in\{1,\dots,p\}$ and $b\in\{1,\dots,q\}$, and consider the vectors (Definition 9.3)

$$u_a^\nu:=E_{[p]}^\top\nu^{-\top}e_a^{[p]},\quad u_b^\xi:=(e_b^{[q]})^\top\xi^{-1}E_{[q]},\quad E_{[r]}:=\left[\begin{array}{cccc|ccc}1 & 0 & \cdots & 0 & 0 & \cdots & \cdots\\ 0 & \ddots & r\ \text{times} & \vdots & \vdots & & \\ \vdots & \ddots & \ddots & 0 & r\ \text{times} & & \\ 0 & \cdots & 0 & 1 & 0 & \cdots & \cdots\end{array}\right].\tag{4}$$

Then, for every $n\in\mathbb{N}_0$ and every $N\in\mathbb{N}_0$ such that

$$n\le d_{b,a}(N):=\left\lceil\frac{N+2-a}{p}\right\rceil+\left\lceil\frac{N+2-b}{q}\right\rceil-1,\tag{5}$$

we have the exact stabilization identity

$$u_b^\xi\,T^n\,u_a^\nu=(e_b^\xi)^\top(T^{[N]})^n e_a^\nu.\tag{6}$$

Proof. Write the path expansion

$$(T^n)_{i,j}=\sum_{i_1,\dots,i_{n-1}\in\mathbb{N}_0}T_{i,i_1}T_{i_1,i_2}\cdots T_{i_{n-1},j}.\tag{7}$$

A term in (2) can be nonzero only if each factor respects the band, namely

$$i_{k+1}\in\{i_k-p,\dots,i_k+q\},\quad k\in\{0,\dots,n-1\},$$

with $i_0=i$ and $i_n=j$. Hence, along any contributing path, the index can increase by at most q per step and decrease by at most p per step.

Assume that a contributing path starting at some $i\in\{0,\dots,b-1\}$ and ending at some $j\in\{0,\dots,a-1\}$ visits an index $>N$. Let $N+1$ be the first index strictly larger than N visited by the path, and let s be the first time such that $i_s=N+1$. Since each step increases the index by at most q, reaching $N+1$ from $i_0\le b-1$ requires

$$s\ge\left\lceil\frac{N+1-(b-1)}{q}\right\rceil=\left\lceil\frac{N+2-b}{q}\right\rceil.$$

Similarly, since each step decreases the index by at most p, going from $N+1$ down to $j\le a-1$ requires

$$n-s\ge\left\lceil\frac{N+1-(a-1)}{p}\right\rceil=\left\lceil\frac{N+2-a}{p}\right\rceil.$$

Therefore any contributing path that leaves $\{0,1,\dots,N\}$ must have length

$$n \geq \left\lceil \frac{N+2-b}{q} \right\rceil + \left\lceil \frac{N+2-a}{p} \right\rceil,$$

which contradicts (2.3). Hence, if (2.3) holds, every nonzero contributing path remains within $\{0,1,\dots,N\}$, and consequently

$$(T^n)_{i,j} = \big((T^{[N]})^n\big)_{i,j},$$

for all $i \in \{0,\dots,b-1\}$ and $j \in \{0,\dots,a-1\}$.

Since ν and ξ are lower triangular, $\nu^{-\top}$ is upper triangular and ξ^{-1} is lower triangular. Thus, by (2.3), the vector u^{ν}_a is supported in $\{0,\dots,a-1\}$ and the row vector u^{ξ}_b is supported in $\{0,\dots,b-1\}$. Multiplying the above entrywise identity by the corresponding coefficients yields (2.3). □

Lemma 2.4. i) *Let us use the notation*

$$\mathcal{N}_{p,q,a,b}(n) := \frac{n+1-\big(\frac{2-a}{p}+\frac{2-b}{q}\big)}{\frac{1}{p}+\frac{1}{q}}.$$

Then, for every $N \in \mathbb{N}_0$ such that $N \geq \mathcal{N}_{p,q,a,b}(n)$, we have

(8) $$n \leq d_{b,a}(N).$$

ii) *In particular, defining the moment $m_{n,a,b} := u^{\xi}_b\, T^n\, u^{\nu}_a$, Lemma 2.3 yields*

$$(e^{\xi}_b)^{\top}\big(T^{[N]}\big)^n e^{\nu}_a = m_{n,a,b}, \quad \text{for all } N \geq \mathcal{N}_{p,q,a,b}(n).$$

Consequently, the quantity $(e^{\xi}_b)^{\top}\big(T^{[N]}\big)^n e^{\nu}_a$ stabilizes for $N \geq \mathcal{N}_{p,q,a,b}(n)$.

Proof. Using the inequality $\lceil x \rceil \geq x$, we obtain

$$d_{b,a}(N) = \left\lceil \frac{N+2-a}{p} \right\rceil + \left\lceil \frac{N+2-b}{q} \right\rceil - 1 \geq \frac{N+2-a}{p} + \frac{N+2-b}{q} - 1.$$

Thus, the condition $n \leq d_{b,a}(N)$ is ensured as soon as

$$n \leq \frac{N+2-a}{p} + \frac{N+2-b}{q} - 1 = N\Big(\frac{1}{p}+\frac{1}{q}\Big) + \Big(\frac{2-a}{p}+\frac{2-b}{q}-1\Big).$$

Rearranging this inequality gives $N \geq \mathcal{N}_{p,q,a,b}(n)$, which proves (1). The stabilization statement in (ii) then follows from Lemma 2.3. □

Proof of the Favard spectral theorem. **Step 0 (assumption and shifted step functions).** Let us assume that for every $N \in \mathbb{N}_0$ there exists a shift $s_N \geq 0$ such that the shifted truncation

$$A_N := T^{[N]} + s_N I_{N+1}$$

admits a positive bidiagonal factorization (PBF). Fix (a,b). Let $\mathrm{d}\psi^{[N],s_N}_{b,a}$ be the corresponding discrete Gauss-type quadrature measure attached to A_N, and let $\psi^{[N],s_N}_{b,a}$ be its right-continuous distribution (step) function, i.e.

$$\psi^{[N],s_N}_{b,a}(x) := \mathrm{d}\psi^{[N],s_N}_{b,a}\big((-\infty,x]\big).$$

Since the shift depends on N, we recenter the distributions by translation and define

(9) $$\psi^{[N]}_{b,a}(x) := \psi^{[N],s_N}_{b,a}(x+s_N), \quad x \in \mathbb{R}.$$

Equivalently, in terms of measures,

(10) $$\mathrm{d}\psi^{[N]}_{b,a}(E) = \mathrm{d}\psi^{[N],s_N}_{b,a}(E+s_N) \quad \text{for every Borel set } E \subset \mathbb{R},$$

i.e. $\mathrm{d}\psi^{[N]}_{b,a}$ is the push-forward of $\mathrm{d}\psi^{[N],s_N}_{b,a}$ under the map $x \mapsto x - s_N$.

More explicitly, if $0 \leq \lambda^{[N]}_{N+1} < \lambda^{[N]}_{N} < \cdots < \lambda^{[N]}_{1}$ are the ordered simple eigenvalues of A_N, and $\rho^{[N]}_{n,b}$, $\mu^{[N]}_{n,a}$ are the Christoffel numbers (which are positive due to the PBF assumption on A_N), we have

$$\psi^{[N]}_{b,a}(x) = \begin{cases} 0, & x < \lambda^{[N]}_{N+1} - s_N, \\ \rho^{[N]}_{1,b}\mu^{[N]}_{1,a} + \cdots + \rho^{[N]}_{k,b}\mu^{[N]}_{k,a}, & \lambda^{[N]}_{k+1} - s_N \leqslant x < \lambda^{[N]}_{k} - s_N, \quad k \in \{0, \dots, N-1\}, \\ \rho^{[N]}_{1,b}\mu^{[N]}_{1,a} + \cdots + \rho^{[N]}_{N+1,b}\mu^{[N]}_{N+1,a}, & x \geqslant \lambda^{[N]}_{1} - s_N, \end{cases}$$

and the corresponding translated measures are

$$\mathrm{d}\psi^{[N]}_{b,a} = \sum_{k=1}^{N+1} \rho^{[N]}_{k,b}\mu^{[N]}_{k,a}\,\delta\Big(x - (\lambda^{[N]}_{k} - s_N)\Big).$$

Step 1 (Helly compactness on $[\alpha,\beta]$). Since translations preserve monotonicity and total mass, for every $a \in \{1, \dots, p\}$ and $b \in \{1, \dots, q\}$ the functions $\psi^{[N]}_{b,a}$ are nonnegative, nondecreasing, right-continuous, and uniformly bounded by the total mass $(\xi^{-1} I_{q,p} \nu^{-\top})_{b,a}$.

Hence $\left\{\psi^{[N]}_{b,a}\right\}_{N \in \mathbb{N}_0}$ is a uniformly bounded family of nondecreasing, right-continuous functions. By Helly's selection principle there exists a subsequence $\left\{\psi^{[N_i]}_{b,a}\right\}_{i \geq 1}$ converging pointwise at all continuity points to a bounded nondecreasing function $\psi_{b,a}$. Moreover, by Helly's second theorem, for every compact interval $[\alpha,\beta]$ and every continuous function f on $[\alpha,\beta]$,

$$\lim_{i \to \infty} \int_{\alpha}^{\beta} f(x)\,\mathrm{d}\psi^{[N_i]}_{b,a}(x) = \int_{\alpha}^{\beta} f(x)\,\mathrm{d}\psi_{b,a}(x). \tag{11}$$

Step 2 (quadrature for $(x - s_N)^n$ and recentered moments). Let us recall that for $N, k \in \mathbb{N}_0$ we have

$$\int_0^{\infty} x^k\,\mathrm{d}\psi^{[N],s_N}_{b,a}(x) = (e^{\xi}_{b})^{\top} (A_N)^k e^{\nu}_{a}. \tag{12}$$

We choose $N \in \mathbb{N}_0$ such that

$$N \geq \left\lceil \mathscr{N}_{p,q,a,b}(n) \right\rceil,$$

so that, by Lemma 2.4, we have $n \leq d_{b,a}(N)$, and therefore (2) holds for all $k \in \{0, 1, \dots, n\}$.

Since

$$(x - s_N)^n = \sum_{k=0}^{n} \binom{n}{k} (-s_N)^{n-k} x^k,$$

linearity of the integral and (2) yield

$$\begin{aligned} \int_0^{\infty} (x - s_N)^n\,\mathrm{d}\psi^{[N],s_N}_{b,a}(x) &= \sum_{k=0}^{n} \binom{n}{k} (-s_N)^{n-k} (e^{\xi}_{b})^{\top} (A_N)^k e^{\nu}_{a} \\ &= (e^{\xi}_{b})^{\top} (A_N - s_N I_{N+1})^n e^{\nu}_{a}. \end{aligned}$$

Moreover,

$$(A_N - s_N I_{N+1})^n = \Big(T^{[N]} + s_N I_{N+1} - s_N I_{N+1}\Big)^n = \big(T^{[N]}\big)^n,$$

so that

$$\int_0^{\infty} (x - s_N)^n\,\mathrm{d}\psi^{[N],s_N}_{b,a}(x) = (e^{\xi}_{b})^{\top} \big(T^{[N]}\big)^n e^{\nu}_{a}. \tag{13}$$

Now set

$$m_{n,a,b} := (u^{\xi}_{b})\, T^n\, u^{\nu}_{a}.$$

Since $n \le d_{b,a}(N)$, Lemma 2.3 gives

$$(e_b^{\xi})^{\top}(T^{[N]})^n e_a^{\nu} = m_{n,a,b}. \tag{14}$$

Combining (2) and (2), we conclude that

$$\int_0^{\infty} (x - s_N)^n \, \mathrm{d}\psi_{b,a}^{[N],s_N}(x) = m_{n,a,b}, \quad N \ge \lceil \mathcal{N}_{p,q,a,b}(n) \rceil.$$

Finally, by the push-forward relation (2), for any polynomial p we have

$$\int_{\mathbb{R}} p(u) \, \mathrm{d}\psi_{b,a}^{[N]}(u) = \int_0^{\infty} p(x - s_N) \, \mathrm{d}\psi_{b,a}^{[N],s_N}(x),$$

and in particular, taking $p(u) = u^n$,

$$\int_{\mathbb{R}} u^n \, \mathrm{d}\psi_{b,a}^{[N]}(u) = \int_0^{\infty} (x - s_N)^n \, \mathrm{d}\psi_{b,a}^{[N],s_N}(x) = m_{n,a,b}, \quad N \ge \lceil \mathcal{N}_{p,q,a,b}(n) \rceil.$$

Step 3 (tails and passage to the limit). Fix $n \in \mathbb{N}_0$ and $\alpha < \beta$, and set $R := \max\{|\alpha|, |\beta|\}$. Let $\{N_i\}_{i\ge1}$ be the subsequence from Step 1, and take i large enough so that $2n + 2 \le d_{b,a}(N_i)$ (equivalently, $N_i \ge \lceil \mathcal{N}_{p,q,a,b}(2n+2) \rceil$). Then Step 2 yields

$$\int_{\mathbb{R}} x^n \, \mathrm{d}\psi_{b,a}^{[N_i]}(x) = m_{n,a,b}, \quad \int_{\mathbb{R}} x^{2n+2} \, \mathrm{d}\psi_{b,a}^{[N_i]}(x) = m_{2n+2,a,b}. \tag{15}$$

We write

$$m_{n,a,b} - \int_{\alpha}^{\beta} x^n \, \mathrm{d}\psi_{b,a}(x) = \Big(\int_{\mathbb{R}} x^n \, \mathrm{d}\psi_{b,a}^{[N_i]}(x) - \int_{\alpha}^{\beta} x^n \, \mathrm{d}\psi_{b,a}^{[N_i]}(x) \Big) + \Big(\int_{\alpha}^{\beta} x^n \, \mathrm{d}\psi_{b,a}^{[N_i]}(x) - \int_{\alpha}^{\beta} x^n \, \mathrm{d}\psi_{b,a}(x) \Big). \tag{16}$$

Convergence on $[\alpha,\beta]$. By Helly's second theorem (2), applied to the continuous function $f(x) = x^n$ on $[\alpha,\beta]$, the second term in the RHS (2) tends to 0 as $i \to \infty$.

Tail estimate. For the first term in (2) we note that $\mathbb{R} \setminus [\alpha,\beta] \subset \{|x| > R\}$. Since $\mathrm{d}\psi_{b,a}^{[N_i]}$ is a nonnegative measure,

$$\Big| \int_{\mathbb{R}\setminus[\alpha,\beta]} x^n \, \mathrm{d}\psi_{b,a}^{[N_i]}(x) \Big| \le \int_{\mathbb{R}\setminus[\alpha,\beta]} |x|^n \, \mathrm{d}\psi_{b,a}^{[N_i]}(x) \le \int_{|x|>R} |x|^n \, \mathrm{d}\psi_{b,a}^{[N_i]}(x). \tag{17}$$

Moreover, on $\{|x| > R\}$ we have $|x| \ge R$, hence $|x|^{-(n+2)} \le R^{-(n+2)}$, and therefore

$$|x|^n = |x|^{2n+2} |x|^{-(n+2)} \le R^{-(n+2)} |x|^{2n+2}, \qquad |x| > R. \tag{18}$$

Integrating the pointwise bound (2) against the nonnegative measure $\mathrm{d}\psi_{b,a}^{[N_i]}$ over $\{|x| > R\}$ yields

$$\int_{|x|>R} |x|^n \, \mathrm{d}\psi_{b,a}^{[N_i]}(x) \le R^{-(n+2)} \int_{|x|>R} |x|^{2n+2} \, \mathrm{d}\psi_{b,a}^{[N_i]}(x).$$

Since $|x|^{2n+2} \ge 0$ and $\{|x| > R\} \subset \mathbb{R}$, we also have

$$\int_{|x|>R} |x|^{2n+2} \, \mathrm{d}\psi_{b,a}^{[N_i]}(x) \le \int_{\mathbb{R}} |x|^{2n+2} \, \mathrm{d}\psi_{b,a}^{[N_i]}(x). \tag{19}$$

Combining (2)–(2) we obtain

$$\Big| \int_{\mathbb{R}\setminus[\alpha,\beta]} x^n \, \mathrm{d}\psi_{b,a}^{[N_i]}(x) \Big| \le R^{-(n+2)} \int_{\mathbb{R}} |x|^{2n+2} \, \mathrm{d}\psi_{b,a}^{[N_i]}(x).$$

Since $2n + 2$ is even, $|x|^{2n+2} = x^{2n+2}$, and by (2),

$$\int_{\mathbb{R}} |x|^{2n+2} \, \mathrm{d}\psi_{b,a}^{[N_i]}(x) = \int_{\mathbb{R}} x^{2n+2} \, \mathrm{d}\psi_{b,a}^{[N_i]}(x) = m_{2n+2,a,b}.$$

Therefore,

$$\Big|\int_{\mathbb{R}\setminus[\alpha,\beta]} x^n \,\mathrm{d}\psi_{b,a}^{[N_i]}(x)\Big| \leq R^{-(n+2)}\, m_{2n+2,a,b}. \tag{20}$$

Conclusion. Letting $i \to \infty$ in (2) and using the convergence on $[\alpha,\beta]$ and the tail bound (2), we obtain

$$\Big|m_{n,a,b} - \int_\alpha^\beta x^n \,\mathrm{d}\psi_{b,a}(x)\Big| \leq R^{-(n+2)}\, m_{2n+2,a,b}.$$

Finally, letting $\alpha \to -\infty$ and $\beta \to +\infty$ (so that $R \to +\infty$) yields

$$\int_{\mathbb{R}} x^n \,\mathrm{d}\psi_{b,a}(x) = m_{n,a,b} = (u_b^\xi)\, T^n\, u_a^\nu, \qquad n \in \mathbb{N}_0.$$

□

In both cases, allowing positive or totally positive triangular freedom in the initial conditions of the mixed-type orthogonal polynomials, the Gauss quadrature formulas derived from (1.1) and the above proof read

$$\int_0^\infty x^n \,\mathrm{d}\psi_{b,a}(x) = \sum_{k=1}^{N+1} \rho_{k,b}^{[N]} \mu_{k,a}^{[N]} \big(\lambda_k^{[N]}\big)^n, \qquad 0 \leqslant n \leqslant d_{b,a}(N),$$

for $a \in \{1,\dots,p\}$ and $b \in \{1,\dots,q\}$. Here the degrees of precision $d_{b,a}(N)$ are optimal: for any larger power, a positive remainder appears and exactness is lost.

2.1. **Case study: unbounded Jacobi matrices and shifted PBF for truncations**.

A particularly transparent instance of the "shifted–PBF truncation" hypothesis in Theorem 2.2 arises in the context of unbounded Jacobi matrices. Let

$$J = \begin{bmatrix} b_0 & c_0 & 0 & \cdots & \\ a_1 & b_1 & c_1 & \ddots & \\ 0 & a_2 & b_2 & c_2 & \ddots \\ \vdots & \ddots & \ddots & \ddots & \ddots \end{bmatrix}, \quad a_n > 0,\ c_n > 0,\ n \geq 0,$$

where the diagonal $(b_n)_{n\geq 0}$ is real and may be unbounded (consequently, J may be unbounded as an operator). For each $N \in \mathbb{N}_0$, denote by $J^{[N]}$ the principal truncation of size $N+1$.

The goal of this case study is twofold: (i) to provide an explicit, entrywise choice of shifts $s_N \geq 0$ such that each shifted truncation $J^{[N]} + s_N I_{N+1}$ admits a PBF, and (ii) to show how the Favard spectral representation for the unbounded case follows immediately from the general apparatus developed above.

Lemma 2.5 (A convenient lower spectral bound for a truncation). *Let $M \in \mathbb{R}^{(N+1)\times(N+1)}$ be any real matrix and let $\|\cdot\|_\infty$ denote the induced ℓ^∞ operator norm, i.e.,*

$$\|M\|_\infty = \max_{0\leq i\leq N} \sum_{j=0}^N |M_{ij}|.$$

Then every eigenvalue λ of M satisfies $|\lambda| \leq \|M\|_\infty$. In particular,

$$\sigma(M) \subset \{z \in \mathbb{C} :\ |z| \leq \|M\|_\infty\}.$$

Proof. If λ is an eigenvalue of M, then λ is an eigenvalue of the bounded linear operator $x \mapsto Mx$ on the Banach space $(\mathbb{C}^{N+1}, \|\cdot\|_\infty)$, which implies $|\lambda| \leq \|M\|_\infty$. □

Theorem 2.6 (Shifted PBF for Jacobi truncations). *Let J be a (possibly unbounded) Jacobi matrix with $a_n > 0$ and $c_n > 0$. For the truncation $J^{[N]}$, we adopt the boundary convention $a_0 = 0$ and $c_N = 0$ (since these entries do not appear in the truncated matrix). Define the shift*

$$s_N := \|J^{[N]}\|_\infty + 1 = \max_{0\le i\le N}\big(|a_i| + |b_i| + |c_i|\big) + 1. \tag{21}$$

Then the shifted truncation

$$A_N := J^{[N]} + s_N I_{N+1}$$

has all leading principal minors strictly positive and is an oscillatory matrix. Consequently, A_N admits a positive bidiagonal factorization (PBF) for every $N \in \mathbb{N}_0$.

Proof. Fix $N \in \mathbb{N}_0$ and set $A_N = J^{[N]} + s_N I_{N+1}$ with s_N defined as in (2.6). For each $k \in \{0, 1, \dots, N\}$, let A_k denote the leading principal submatrix of A_N of size $k+1$, given by

$$A_k = J^{[k]} + s_N I_{k+1}.$$

Step 1: $-s_N$ lies strictly to the left of the spectrum of every $J^{[k]}$. By Lemma 2.5, every eigenvalue λ of the truncation $J^{[k]}$ satisfies $|\lambda| \le \|J^{[k]}\|_\infty$. Since $\|J^{[k]}\|_\infty \le \|J^{[N]}\|_\infty$ for all $k \le N$, we obtain the lower bound

$$\lambda \ge -|\lambda| \ge -\|J^{[k]}\|_\infty \ge -\|J^{[N]}\|_\infty > -s_N.$$

Thus, $-s_N$ is a strict lower bound for the spectrum of $J^{[k]}$ for every $k \le N$.

Step 2: Positivity of leading principal minors of A_N. For each $k \le N$, let $P_{k+1}(x) := \det(xI_{k+1} - J^{[k]})$ be the characteristic polynomial of $J^{[k]}$. Then

$$\det(A_k) = \det(J^{[k]} + s_N I_{k+1}) = \det\big((-s_N)I_{k+1} - J^{[k]}\big) = (-1)^{k+1} P_{k+1}(-s_N).$$

Since $J^{[k]}$ is a symmetric tridiagonal matrix (modulo similarity) with non-vanishing off-diagonals, all its eigenvalues $\lambda_1, \dots, \lambda_{k+1}$ are real and simple. Therefore,

$$P_{k+1}(-s_N) = \prod_{j=1}^{k+1}(-s_N - \lambda_j).$$

By Step 1, $-s_N < \lambda_j$ for all j, so each factor $(-s_N - \lambda_j)$ is strictly negative. The product of $k+1$ negative numbers has sign $(-1)^{k+1}$. Consequently,

$$\det(A_k) = (-1)^{k+1} \underbrace{P_{k+1}(-s_N)}_{(-1)^{k+1}|P(\dots)|} > 0,$$

which proves that all leading principal minors of A_N are strictly positive.

Step 3: Oscillatory implies PBF. By construction, the subdiagonal and superdiagonal entries of A_N coincide with those of $J^{[N]}$ and are strictly positive. Since A_N is a Jacobi matrix with strictly positive leading principal minors (Step 2) and positive off-diagonals, is a classical result that implies that A_N is oscillatory, [**Gantmacher0**] and [**Gantmacher**]. Finally, we recall the classical result that every oscillatory matrix admits an *LU* factorization where the factors have strictly positive diagonals and, in the tridiagonal case, strictly positive off-diagonals. Thus, A_N admits a PBF. □

Connection with the unbounded Favard spectral theorem. Theorem 2.6 provides an explicit sequence $(s_N)_{N\ge0}$ satisfying the hypothesis of Theorem 2.2: for each N, the shifted truncation $J^{[N]} + s_N I_{N+1}$ admits a PBF. Therefore, the construction of the translated distribution functions and measures (2)–(2), combined with the moment identities from Step 2 and the Helly compactness argument in Step 3 of the main proof, yields a matrix-valued measure $\mathrm{d}\,\Psi$ providing the Favard-type spectral representation for the unbounded Jacobi matrix J.

3. Conclusions and outlook

In this work we extend to the unbounded setting the spectral Favard theorem for banded matrices admitting a positive bidiagonal factorization (PBF). The bounded case in [**BFM1**] already relies on a compactness step for the finite-level spectral data, implemented via Helly-type selection theorems (see [**Chihara**]), to pass from truncations to a limiting matrix of measures. The genuinely new feature in the unbounded regime is that one must allow an N-dependent shift: for each truncation one considers $T^{[N]} + s_N I_{N+1}$, with $s_N > 0$ chosen so that a PBF is available, and one then proves that the associated measures remain uniformly controlled as $N \to \infty$. This requires a finer analysis than in the bounded case, aimed in particular at controlling the tails and preventing loss of mass at infinity.

More precisely, we work with the shifted truncations $T^{[N]} + s_N I_{N+1}$ admitting a strict PBF and with the associated mixed-type multiple biorthogonal polynomials. At the finite level, Gaussian quadrature expresses the relevant moments through positive Christoffel numbers. In the unbounded case, these identities are combined with tail estimates to obtain uniform tightness and variation bounds for the corresponding distribution functions. This, in turn, makes it possible to apply Helly-type compactness principles (as presented in [**Chihara**]) to extract convergent subsequences. Identifying the limits and verifying that they reproduce the prescribed moments yields a matrix of positive Lebesgue–Stieltjes measures realizing the mixed-type multiple biorthogonality relations and the spectral representation of the powers T^n in the unbounded case.

The argument preserves the constructive spirit of [**BFM1**] while addressing the analytic difficulties specific to the unbounded setting: controlling the limiting procedure and ensuring that positivity is not lost. As in the bounded case, the choice of initial conditions is decisive. Positivity of the triangular normalizations guarantees admissible degree bounds, whereas total positivity enforces normality and ensures that the degree upper bounds are attained for all colors and all indices.

Several directions remain open. From a structural viewpoint, it is natural to ask to what extent the method extends to broader classes of banded operators admitting factorizations beyond the bidiagonal case, or to settings where positivity holds only partially. Another problem concerns the fine spectral features of the resulting measures in the unbounded regime, including growth, regularity, and determinacy of the associated moment problems. Finally, the interplay between total positivity, factorization theory, and mixed-type multiple orthogonality suggests further connections with classical and matrix-valued extensions of orthogonal polynomial theory. A particularly promising direction is the application to continuous-time Markov processes with unbounded generator matrices, where the spectral representation may provide new tools to study recurrence, ergodicity, and stationary behavior; for related results in the discrete-time setting, see [**JP**, **BFM6**].

Declarations

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