BIDIAGONAL FACTORIZATION OF BANDED RECURSION MATRICES FOR MIXED-TYPE MULTIPLE ORTHOGONAL POLYNOMIALS

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ABSTRACT. Given a banded matrix $\mathscr{T}_N$ with p subdiagonals and q superdiagonals arising from the Gauss–Borel factorization $\mathscr{M}_N = \mathscr{L}_N^{-1}\mathscr{U}_N^{-1}$ of a moment matrix, this paper constructs explicitly its bidiagonal factorization

$$\mathscr{T}_N = L_1 \cdots L_p U_q \cdots U_1.$$

Bidiagonal factorizations of this type are central to the study of oscillatory banded matrices and to the spectral Favard theorem for multiple orthogonal polynomials [8].

The factorization is obtained via Christoffel transformations of the moment matrix. Provided that the perturbed moment matrices $\mathscr{M}_{N,(b,0)}$ and $\mathscr{M}_{N,(0,a)}$ admit a Gauss–Borel factorization, each bidiagonal factor is a quotient of the corresponding Gauss–Borel factors:

$$U_b = \mathscr{U}_{N,(b,0)}^{-1}\mathscr{U}_{N,(b-1,0)}, \qquad L_a = \mathscr{L}_{N,(0,a-1)}\mathscr{L}_{N,(0,a)}^{-1}.$$

Explicit Christoffel-type formulas for the entries of the bidiagonal factors are then derived in terms of certain tau-determinants evaluated at the origin:

$$U_{b,n} = -\frac{\tau^B_{b-1,n}\,\tau^B_{b,n+1}}{\tau^B_{b-1,n+1}\,\tau^B_{b,n}}, \qquad L_{a,n+1} = -\frac{\tau^A_{a-1,n+2}\,\tau^A_{a,n}}{\tau^A_{a-1,n+1}\,\tau^A_{a,n+1}}.$$

As an illustration, the theory is applied to the recurrence matrices of multiple Hahn orthogonal polynomials. For two weights the tetradiagonal case is handled via contiguous hypergeometric relations [13, 9]; for three weights, i.e. the pentadiagonal case, the direct hypergeometric representations of [15] are required. In both cases fully explicit bidiagonal factorizations are obtained.

1. INTRODUCTION

The bidiagonal factorization of banded matrices plays a central role in two related but distinct directions. The nonnegative bidiagonal factorization is a fundamental tool in the structural theory of total positivity of matrices, providing an explicit and computable description of totally nonnegative banded matrices in terms of elementary factors [16, 20]. The positive bidiagonal factorization, in turn, leads to a spectral Favard theorem for banded matrices in terms of mixed-type multiple orthogonal polynomials [8], and has been applied to construct explicit stochastic factorizations of finite-time Markov chains [12, 14]. Constructing bidiagonal factorizations explicitly is therefore of direct relevance both for the theory of totally positive matrices and for the spectral analysis of banded operators.

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Key words and phrases. Multiple orthogonal polynomials of mixed type,Gauss–Borel factorization, bidiagonal factorization of banded recursion matrices, Christoffel transformations, Christoffel formulas, multiple Hahn polyomials.

The setting of this paper is the following. A rectangular $q \times p$ matrix of measures $\mathrm{d}\mu$ determines a moment matrix $\mathscr{M}_N$ whose Gauss–Borel factorization

$$\mathscr{M}_N = \mathscr{L}_N^{-1}\mathscr{U}_N^{-1}$$

produces two dual families of mixed-type multiple orthogonal polynomials $B^{[N]}$ and $A^{[N]}$, together with a banded recurrence matrix $\mathscr{T}_N$ with p subdiagonals and q superdiagonals. This framework was introduced in [1, 2] and further developed in [17, 18].

The first main result (Theorem 4.8) gives the bidiagonal factorization $\mathscr{T}_N = L_1 \cdots L_p\, U_q \cdots U_1$, under the condition that the Christoffel perturbations $\mathscr{M}_{N,(b,0)}$ and $\mathscr{M}_{N,(0,a)}$ of the moment matrix admit a Gauss–Borel factorization. The bidiagonal factors are then quotients of the corresponding Gauss–Borel factors:

$$U_b = \mathscr{U}_{N,(b,0)}^{-1}\mathscr{U}_{N,(b-1,0)}, \qquad L_a = \mathscr{L}_{N,(0,a-1)}\mathscr{L}_{N,(0,a)}^{-1}.$$

This establishes a precise correspondence: the bidiagonal factorization of the recurrence matrix reflects exactly the Gauss–Borel structure of the Christoffel-transformed moment matrices.

The second main result (Theorem 5.5) makes this fully explicit. The entries of the bidiagonal factors are expressed as cross-ratios of tau-determinants:

$$U_{b,n} = -\frac{\tau^B_{b-1,n}\,\tau^B_{b,n+1}}{\tau^B_{b-1,n+1}\,\tau^B_{b,n}}, \qquad L_{a,n+1} = -\frac{\tau^A_{a-1,n+2}\,\tau^A_{a,n}}{\tau^A_{a-1,n+1}\,\tau^A_{a,n+1}}, \tag{1}$$

where $\tau^B_{b,n}$ and $\tau^A_{a,n}$ are $b \times b$ and $a \times a$ determinants of values of the mixed-type polynomials at $x = 0$. The cross-ratio structure in (1) is the matrix counterpart of the classical Christoffel formula for scalar orthogonal polynomials [6], and the condition for the factorization to exist — the nonvanishing of the tau-determinants — coincides with the condition for the transformed orthogonality to be well-defined [17].

Section 6 subjects these formulas to a substantial computational test on multiple Hahn orthogonal polynomials. For two weights the recurrence matrix is tetradiagonal, and the tau-determinants can be evaluated using contiguous hypergeometric relations from [13, 9]; the bidiagonal factorization of tetradiagonal matrices in this context was studied in [7]. For three weights this approach is no longer viable; the direct hypergeometric representations of [15] are required, and a more involved computation delivers the explicit factorization. The hypergeometric functions appearing throughout follow the standard notation of [21], and analogous representations for other classical families are available in [10, 11].

The paper is organized as follows. Section 2 introduces moment matrices, their Gauss–Borel factorization, and the mixed-type multiple orthogonal polynomials. Section 3 defines the banded recurrence matrices. Section 4 constructs the bidiagonal factorization via Christoffel transformations. Section 5 derives the explicit Christoffel formulas for the bidiagonal factors. Section 6 applies the theory to multiple Hahn polynomials with two and three weights.

1.1. **Moment matrices.**

We will study orthogonality with respect a rectangular matrix of measures, namely

Definition 1.1 (The matrix of measures). *Given two natural numbers $p, q \in \mathbb{N}$, we consider a rectangular $q \times p$ matrix of measures*

$$\mu = \left.\begin{bmatrix} \overbrace{\mu_{1,1} \cdots\cdots \mu_{1,p}}^{p} \\ \vdots \qquad\qquad \vdots \\ \vdots \qquad\qquad \vdots \\ \vdots \qquad\qquad \vdots \\ \mu_{q,1} \cdots\cdots \mu_{q,p} \end{bmatrix}\right\}q .$$

First, we need to introduce moment matrices of finite size for this matrix of measures and characterize its symmetries. We collect all the monomials in the following matrices

Definition 1.2. *We consider the following semi-infinite matrices on monomials*

$$X_{[r]} = \begin{bmatrix} \overbrace{I_r}^{r} \\ xI_r \\ \vdots \\ \vdots \end{bmatrix} \in \mathbb{R}^{\infty \times r}[x].$$

as well as its N truncations, $N \in \mathbb{N}$, $X_{[r]}^{[N]}$.

For $r \in \mathbb{N}$ we consider the Euclidean division $N = N_r r + s_r$, with $s_r \in \{0, 1, \dots, r-1\}$, $N_r = \lfloor \frac{N}{r} \rfloor$ and

$$I_{s,r} := \left.\begin{bmatrix} \overbrace{\begin{matrix} 1 & 0 & \cdots & \cdots & \cdots & \cdots & 0 \\ 0 & 1 & 0 & \cdots & \cdots & \cdots & 0 \\ \vdots & \ddots & \ddots & \ddots & & & \vdots \\ \underbrace{0 \cdots\cdots 0}_{s} & & 1 & & \underbrace{0 \cdots\cdots 0}_{r-s} & & \end{matrix}}^{r} \end{bmatrix}\right\}s.$$

Proposition 1.3. *The truncated matrix of monomials are given by*

$$X_{[r]}^{[N]} = \begin{bmatrix} \left.\begin{matrix} \overbrace{I_r}^{r} \\ xI_r \\ \vdots \\ x^{N_r - 1} I_r \end{matrix}\right\} N_r \\ \left. x^{N_r} I_{s_r,r} \right\} s_r \end{bmatrix} \in \mathbb{R}^{N \times r}[x].$$

These matrices of monomials have important properties with respect the following matrices:

Definition 1.4 (Shift matrices). *The shift matrix is*

$$\Lambda = \begin{bmatrix} 0 & 1 & 0 & \cdots & \cdots \\ 0 & 0 & 1 & \ddots & \\ \vdots & \ddots & \ddots & \ddots & \ddots \\ \vdots & & \ddots & \ddots & \ddots \end{bmatrix},$$

in terms of which we have the corresponding block shift matrix

$$\Lambda_{[r]}=\begin{bmatrix}0_r & I_r & 0_r & \cdots & \\ 0_r & 0_r & I_r & \ddots & \\ \vdots & \ddots & \ddots & \ddots & \ddots\\ \vdots & & \ddots & \ddots & \ddots\end{bmatrix}=\Lambda^r.$$

Our focus will be in truncated versions of these matrices as follows:

Proposition 1.5 (Truncated block shift matrices). *Truncations $\Lambda_{[r]}^{[N]}\in\mathbb{R}^{N\times N}$ of this shift matrix are*

$$\Lambda_{[r]}^{[N]}=\left.\left[\begin{array}{ccccc|c} \multicolumn{5}{c|}{\overbrace{\hphantom{0_r\quad I_r\quad 0_r\cdots\cdots 0_r\quad 0_{r\times s_r}}}^{N}} & \\ 0_r & I_r & 0_r & \cdots & 0_r & 0_{r\times s_r}\\ \vdots & \ddots & \ddots & \ddots & \vdots & \vdots\\ \vdots & & \ddots & \ddots & 0_r & 0_{r\times s_r}\\ \vdots & & & \ddots & I_r & 0_{r\times s_r}\\ \vdots & & & & 0_r & I_{s_r,r}^{\top}\\ 0_{s_r\times r} & \cdots & \cdots & \cdots & 0_{s_r\times r} & 0_{s_r}\\ \multicolumn{5}{c|}{\underbrace{\hphantom{0_{s_r\times r}\cdots\cdots\cdots 0_{s_r\times r}}}_{N_r r}} & \underbrace{\hphantom{0_{s_r}}}_{s_r}\end{array}\right]\right\}N$$

We will also need to border these truncated shift matrices

Definition 1.6 (Bordered shift matrices). *Bordered truncated matrices $\Lambda_{[r]}^{[N,N+r]}\in\mathbb{R}^{N\times(N+r)}$ are obtained by augmenting with r columns as follows*

$$\Lambda_{[r]}^{[N,N+r]}=\left[\begin{array}{ccccc|c|c} 0_r & I_r & 0_r & \cdots & 0_r & 0_{r\times s_r} & \\ \vdots & \ddots & \ddots & \ddots & \vdots & \vdots & 0_{(N-r)\times r}\\ \vdots & & \ddots & \ddots & 0_r & 0_{r\times s_r} & \\ \vdots & & & \ddots & I_r & 0_{r\times s_r} & \\ \cline{7-7} \vdots & & & & 0_r & I_{s_r,r}^{\top} & \\ 0_{s_r\times r} & \cdots & \cdots & \cdots & 0_{s_r\times r} & 0_{s_r} & I_r\end{array}\right].$$

We have the key spectral type relations

$$\Lambda_{[r]}^{[N,N+r]}X_{[r]}^{[N+r]}(x)=xX_{[r]}^{[N]}(x). \tag{2}$$

We first give an slightly different view of these bordered truncated shift matrices

Proposition 1.7. *The bordered truncated shift matrices $\Lambda_{[r]}^{[N,N+r]}\in\mathbb{R}^{N\times(N+r)}$ have the following expression*

$$\Lambda_{[r]}^{[N,N+r]}=\left[\begin{array}{ccccc|c} 0_r & I_r & 0_r & \cdots & 0_r & \\ \vdots & \ddots & \ddots & \ddots & \vdots & 0_{(N-r-s_r)\times(r+s_r)}\\ \vdots & & \ddots & \ddots & 0_r & \\ \vdots & & & \ddots & I_r & \\ \cline{6-6} \vdots & & & & 0_r & \\ 0_{s_r\times r} & \cdots & \cdots & \cdots & 0_{s_r\times r} & I_{r+s_r}\end{array}\right].$$

Then, we are ready to state the simple but key spectral property of these shift matrices with respect the matrix of monomials:

Proposition 1.8 (Monomial spectrality). *We have the spectral type relations:*

$$\Lambda^{[N,N+r]}_{[r]} X^{[N+r]}_{[r]}(x) = x X^{[N]}_{[r]}(x). \tag{3}$$

Now, we are prepared to consider matrices of moments of the matrix of measures $\mathrm{d}\,\mu$:

Definition 1.9 (Moment matrices). *For $N, M \in \mathbb{N}$, we have the following matrices of moments of μ:*

$$\begin{aligned}
\mathscr{M}^{[N,M]} &:= \int X^{[N]}_{[q]}(x)\,\mathrm{d}\,\mu(x)\left(X^{[M]}_{[p]}(x)\right)^{\top} \\
&= \int \begin{bmatrix} I_q \\ xI_q \\ \vdots \\ x^{N_q-1} I_q \\ x^{N_q} I_{s_q,q} \end{bmatrix} \mathrm{d}\,\mu(x) \begin{bmatrix} I_p & xI_p & \cdots\cdots & x^{M_p-1} I_p & x^{M_p} I^{\top}_{s_p,p} \end{bmatrix} \\
&= \left.\begin{bmatrix} \overbrace{\begin{matrix} \int \mathrm{d}\,\mu(x) & \int x\,\mathrm{d}\,\mu(x) \cdots\cdots & \int x^{M_p-1}\,\mathrm{d}\,\mu(x) & \int x^{M_p}\,\mathrm{d}\,\mu(x) I^{\top}_{s_p,q} \end{matrix}}^{M} \\ \begin{matrix} \int x\,\mathrm{d}\,\mu(x) & \int x^2\,\mathrm{d}\,\mu(x) \cdots\cdots & \int x^{M_p}\,\mathrm{d}\,\mu(x) & \int x^{M_p+1}\,\mathrm{d}\,\mu(x) I^{\top}_{s_p,p} \\ \vdots & \vdots & \vdots & \vdots \\ \int x^{N_q-1}\,\mathrm{d}\,\mu(x) & \int x^{N_q}\,\mathrm{d}\,\mu(x) \cdots\cdots & \int x^{N_q+M_p-2}\,\mathrm{d}\,\mu(x) & \int x^{N_q+M_p-1}\,\mathrm{d}\,\mu(x) I^{\top}_{s_p,p} \\ \int x^{N_q} I_{s_q,q}\,\mathrm{d}\,\mu(x) & \int x^{N_q+1} I_{s_q,q}\,\mathrm{d}\,\mu(x) \cdots & \int I_{s_q,q} x^{N_q+M_p-1}\,\mathrm{d}\,\mu(x) & \int x^{N_q+M_p} I_{s_q,q}\,\mathrm{d}\,\mu(x) I^{\top}_{s_p,p} \end{matrix} \end{bmatrix}\right\} N .
\end{aligned}$$

Using the spectral properties of the matrix of monomials we find the following block Hankel type symmetries of these matrices of moments:

Proposition 1.10 (Hankel type symmetries). *We have the following symmetries for the moment matrices*

$$\Lambda^{[N,N+q]}_{[q]} \mathscr{M}^{[N+q,M]} = \mathscr{M}^{[N,M+p]} \left(\Lambda^{[M,M+p]}_{[p]}\right)^{\top}.$$

Proof. Is a direct consequence of Equation (3). Indeed,

$$\begin{aligned}
\Lambda^{[N,N+q]}_{[q]} \mathscr{M}^{[N+q,M]} &= \Lambda^{[N,N+q]}_{[q]} \int X^{[N+q]}_{[q]}(x)\,\mathrm{d}\,\mu(x)\left(X^{[M]}_{[p]}(x)\right)^{\top} \\
&= \int x X^{[N]}_{[q]}(x)\,\mathrm{d}\,\mu(x)\left(X^{[M]}_{[p]}(x)\right)^{\top} = \int X^{[N]}_{[q]}(x)\,\mathrm{d}\,\mu(x)\left(x X^{[M]}_{[p]}(x)\right)^{\top} \\
&= \int X^{[N]}_{[q]}(x)\,\mathrm{d}\,\mu(x)\left(\Lambda^{[M,M+p]}_{[p]} X^{[M+p]}_{[p]}(x)\right)^{\top} = \mathscr{M}^{[N,M+p]}\left(\Lambda^{[M,M+p]}_{[p]}\right)^{\top}.
\end{aligned}$$

□

In particular,

Corollary 1.11. *We have the following Hankel symmetry type relations*

$$\Lambda^{[N,N+q]}_{[q]} \mathscr{M}^{[N+q,N]} = \mathscr{M}^{[N,N+p]} \left(\Lambda^{[N,N+p]}_{[p]}\right)^{\top}, \tag{4a}$$

$$\Lambda^{[N-q,N]}_{[q]} \mathscr{M}^{[N,N]} = \mathscr{M}^{[N-q,N+p]} \left(\Lambda^{[N,N+p]}_{[p]}\right)^{\top}, \tag{4b}$$

$$\Lambda^{[N,N+q]}_{[q]} \mathscr{M}^{[N+q,N-p]} = \mathscr{M}^{[N,N]} \left(\Lambda^{[N-p,N]}_{[p]}\right)^{\top}. \tag{4c}$$

Note that for $N\neq M$ the matrix $\mathscr{M}^{[N,M]}$ is a rectangular matrix, and it happens to be a square matrix if and only if $N=M$. In this case we will use the notation

$$\mathscr{M}^{[N,N]}=:\mathscr{M}_N.$$

2. The Gauss-Borel factorization of the moment matrices

We assume that the matrix of moments $\mathscr{M}_N$ admits a Gauss–Borel factorization

$$\mathscr{M}_N=\mathscr{L}_N^{-1}\mathscr{U}_N^{-1}$$

in terms of nonsingular lower and upper triangular matrices $\mathscr{L}_N$ and $\mathscr{U}_N$, respectively. This is equivalent to all leading principal submatrices $\mathscr{M}_n$, $n\in\{1,\dots,N\}$, of $\mathscr{M}_N$ to be non singular. This is equivalent that for all these matrices submatrices there exists a Gauss–Borel factorization

$$\mathscr{M}_n=\mathscr{L}_n^{-1}\mathscr{U}_n^{-1},$$

where $\mathscr{L}_n$ and $\mathscr{U}_n$, can be taken, for example, as the n-th leading principal submatrices of $\mathscr{L}_N$ and $\mathscr{U}_N$, respectively.

Definition 2.1 (Polynomial matrices). *We define the following polynomials vectors*

$$B^{[N]}:=\mathscr{L}_N X_{[q]}^{[N]},\quad A^{[N]}:=\left(X_{[p]}^{[N]}\right)^{\top}\mathscr{U}_N.$$

We will use the notation

$$B^{[N]}=\begin{bmatrix} B_0^{(1)} & \cdots\cdots & B_0^{(q)}\\ B_1^{(1)} & \cdots\cdots & B_1^{(q)}\\ \vdots & & \vdots\\ \vdots & & \vdots\\ B_{N-1}^{(1)} & \cdots\cdots & B_{N-1}^{(q)}\end{bmatrix},\quad A^{[N]}=\begin{bmatrix} A_0^{(1)} & A_1^{(1)} & \cdots\cdots & A_{N-1}^{(1)}\\ \vdots & \vdots & & \vdots\\ A_0^{(p)} & A_1^{(p)} & \cdots\cdots & A_{N-1}^{(p)}\end{bmatrix},$$

Proposition 2.2. *Whenever $\mathscr{M}_N$ admits a Gauss–Borel factorization the following relations hold:*

$$\begin{aligned}
\mathscr{M}^{[N,N+p]}&=\mathscr{L}_N^{-1}\left[\mathscr{U}_N^{-1}\;\middle|\;U^{[N,N+p]}\right],\\
\mathscr{M}^{[N+q,N]}&=\left[\frac{\mathscr{L}_N^{-1}}{L^{[N+q,N]}}\right]\mathscr{U}_N^{-1},\\
\mathscr{M}^{[N-q,N+p]}&=\mathscr{L}_{N-q}^{-1}\left[\mathscr{U}_{N-q}^{-1}\;\middle|\;U^{[N-q,N+p]}\right],\\
\mathscr{M}^{[N+q,N-p]}&=\left[\frac{\mathscr{L}_{N-p}^{-1}}{L^{[N+q,N-p]}}\right]\mathscr{U}_{N-p}^{-1}
\end{aligned}$$

where $U^{[N,N+p]}\in\mathbb{R}^{N\times p}$, $L^{[N+q,N]}\in\mathbb{R}^{q\times N}$, $U^{[N-q,N+p]}\in\mathbb{R}^{(N-q)\times(q+p)}$ and $L^{[N+q,N-p]}\in\mathbb{R}^{(q+p)\times(N-p)}$.

We now introduce an important polynomial matrix:

Definition 2.3. *For $s \in \{0,1,\dots,r\}$, let us consider*

$$\mathfrak{X}_{[r,s]}(x) := \left[\begin{array}{c|c} 0_{(r-s)\times s} & I_{(r-s)\times(r-s)} \\ \hline xI_s & 0_{s\times(r-s)} \end{array}\right].$$

This matrix enjoys some important properties:

Proposition 2.4. *We have*

i)

$$\mathfrak{X}_{[r,r]} = xI_r.$$

ii)

$$\mathfrak{X}_{[r,s]} = \mathfrak{X}^s_{[r,1]},$$

iii)

$$\mathfrak{X}_{[r,s]}\mathfrak{X}_{[r,s']} = \mathfrak{X}_{[r,s']}\mathfrak{X}_{[r,s]} = \mathfrak{X}_{[r,s+s']}.$$

This matrix is useful to represent the matrix of monomials

Proposition 2.5. *It holds that*

$$X^{[N+r]}_{[r]} = \begin{bmatrix} X^{[N]}_{[r]} \\ x^{N_r}\mathfrak{X}_{[r,s_r]} \end{bmatrix}$$

It also allows for explicit expressions for block related to the truncations of the inverse of the triangular factors of the Gauss–Borel factorization

Proposition 2.6. *The following relations are fulfilled*

$$L^{[N+q,N]} = \int x^{N_q}\mathfrak{X}_{[q,s_q]}(x)\,\mathrm{d}\,\mu(x)A^{[N]}(x), \tag{5}$$

$$U^{[N,N+p]} = \int B^{[N]}(x)\,\mathrm{d}\,\mu(x)\left(\mathfrak{X}_{[p,s_p]}(x)\right)^{\top} x^{N_p}. \tag{6}$$

Proof. First, we compute that

$$
\begin{aligned}
\mathscr{M}^{[N+q,N]} &= \int \left[\begin{array}{c} X^{[N]}_{[q]}(x) \\ \hline x^{N_q}\mathfrak{X}_{[q,s_q]}(x) \end{array}\right] \mathrm{d}\,\mu(x) \left(X^{[N]}_{[p]}(x)\right)^\top \\
&= \left[\begin{array}{c} \mathscr{M}_N \\ \hline x^{N_q}\left(\mathfrak{X}_{[q,s_q]}(x)\right)^\top \mathrm{d}\,\mu(x) \left(X^{[N]}_{[p]}(x)\right)^\top \end{array}\right] \\
&= \left[\begin{array}{c} \mathscr{L}_N^{-1} \\ \hline \int x^{N_q}\mathfrak{X}_{[q,s_q]}(x)\,\mathrm{d}\,\mu(x) A^{[N]}(x) \end{array}\right] \mathscr{U}_N^{-1}, \\
\mathscr{M}^{[N,N+p]} &= \int X^{[N]}_{[q]}(x)\,\mathrm{d}\,\mu(x) \left[\begin{array}{c|c} \left(X^{[N]}(x)\right)^\top & x^{N_p}\left(\mathfrak{X}_{[p,s_p]}(x)\right)^\top \end{array}\right] \\
&= \left[\begin{array}{c|c} \mathscr{M}_N & \int X^{[N]}_{[q]}(x)\,\mathrm{d}\,\mu(x) x^{N_p}\left(\mathfrak{X}_{[p,s_p]}(x)\right)^\top \end{array}\right] \\
&= \mathscr{L}_N^{-1}\left[\begin{array}{c|c} \mathscr{U}_N^{-1} & \int B^{[N]}(x)\,\mathrm{d}\,\mu(x) x^{N_p}\left(\mathfrak{X}_{[p,s_p]}(x)\right)^\top \end{array}\right].
\end{aligned}
$$

□

Being $\mathscr{L}_N$ and $\mathscr{U}_N$ lower and upper triangular non singular matrices, respectively, they have the block structure

$$
\mathscr{L}_N = \left[\begin{array}{c|c} \mathscr{L}_{N-q} & 0_{(N-q)\times q} \\ \hline \ell^{[N,N-q]} & \mathscr{L}_{[N-q,N]} \end{array}\right], \quad \mathscr{U}_N = \left[\begin{array}{c|c} \mathscr{U}_{N-p} & u^{[N-p,N]} \\ \hline 0_{p\times(N-p)} & \mathscr{U}_{[N-p,N]} \end{array}\right],
$$

where $\mathscr{L}_{[N-q,N]} \in \mathbb{R}^{q\times q}$ and $\mathscr{U}_{[N-p,N]} \in \mathbb{R}^{p\times p}$ are lower and upper triangular non singular matrices, respectively. Hence, their inverses have the form

$$
\mathscr{L}_N^{-1} = \left[\begin{array}{c|c} \mathscr{L}_{N-q}^{-1} & 0_{(N-q)\times q} \\ \hline \tilde{\ell}^{[N,N-q]} & \mathscr{L}_{[N-q,N]}^{-1} \end{array}\right], \quad \mathscr{U}_N^{-1} = \left[\begin{array}{c|c} \mathscr{U}_{N-p}^{-1} & \tilde{u}^{[N-p,N]} \\ \hline 0_{p\times(N-p)} & \mathscr{U}_{[N-p,N]}^{-1} \end{array}\right]. \tag{7}
$$

For a better block representation of these inverses we write

$$
A^{[N]} = \left[\begin{array}{c|c} A^{[N-q]} & A^{[N,q]} \end{array}\right], \quad B^{[N]} = \left[\begin{array}{c} B^{[N-p]} \\ \hline B^{[N,p]} \end{array}\right], \quad A^{[N,q]}, B^{[N,p]} \in \mathbb{R}^{p\times q}[x],
$$

with

$$
B^{[N,p]} = \begin{bmatrix} B^{(1)}_{N-p} \cdots\cdots B^{(q)}_{N-p} \\ B^{(1)}_{N-p+1} \cdots\cdots B^{(q)}_{N-p+1} \\ \vdots \qquad\qquad \vdots \\ B^{(1)}_{N-1} \cdots\cdots B^{(q)}_{N-1} \end{bmatrix}, \quad A^{[N,q]} = \begin{bmatrix} A^{(1)}_{N-q} & A^{(1)}_{N-q+1} \cdots\cdots A^{(1)}_{N-1} \\ \vdots & \vdots \qquad \vdots \\ A^{(p)}_{N-q} & A^{(p)}_{N-q+1} \cdots\cdots A^{(p)}_{N-1} \end{bmatrix}.
$$

Proposition 2.7 (Block structure of triangular inverses). *The inverse matrices of the triangular factors of the Gauss–Borel factorization have the following block structure:*

$$
\begin{aligned}
\mathscr{L}_N^{-1} &= \left[\begin{array}{c|c} \mathscr{L}_{N-q}^{-1} & 0_{(N-q)\times q} \\ \hline L^{[N,N-q]} & \int x^{N_q-1}\mathfrak{X}_{[q,s_q]}(x)\,\mathrm{d}\,\mu(x)A^{[N,q]}(x) \end{array}\right], \\
\mathscr{U}_N^{-1} &= \left[\begin{array}{c|c} \mathscr{U}_{N-p}^{-1} & U^{[N-p,N]} \\ \hline 0_{p\times(N-q)} & \int B^{[N,p]}(x)\,\mathrm{d}\,\mu(x)x^{N_p-1}\left(\mathfrak{X}_{[p,s_p]}(x)\right)^\top \end{array}\right].
\end{aligned}
$$

Proof. It follows from

$$
\begin{aligned}
\mathscr{L}_N^{-1} &= \mathscr{M}_N\mathscr{U}_N = \int X^{[N]}(x)\,\mathrm{d}\,\mu(x)\left(X^{[N]}(x)\right)^\top\mathscr{U}_N = \int \left[\begin{array}{c} X_{[q]}^{[N-q]}(x) \\ \hline x^{N_q-1}\mathfrak{X}_{[q,s_q]}(x) \end{array}\right]\mathrm{d}\,\mu(x)A^{[N]}(x) \\
&= \left[\begin{array}{c|c} \mathscr{L}_{N-q}^{-1} & 0_{(N-q)\times q} \\ \hline L^{[N,N-q]} & \int x^{N_q-1}\mathfrak{X}_{[q,s_q]}(x)\,\mathrm{d}\,\mu(x)A^{[N,q]}(x) \end{array}\right], \\
\mathscr{U}_N^{-1} &= \mathscr{L}_N\mathscr{M}_N = \mathscr{L}_N\int X^{[N]}(x)\,\mathrm{d}\,\mu(x)\left(X^{[N]}(x)\right)^\top \\
&= \int B^{[N]}(x)\,\mathrm{d}\,\mu(x)\left[\begin{array}{c|c} \left(X_{[p]}^{[N-p]}(x)\right)^\top & x^{N_p-1}\left(\mathfrak{X}_{[p,s_p]}(x)\right)^\top \end{array}\right] \\
&= \left[\begin{array}{c|c} \mathscr{U}_{N-p}^{-1} & U^{[N-p,N]} \\ \hline 0_{p\times(N-q)} & \int B^{[N,p]}(x)\,\mathrm{d}\,\mu(x)x^{N_p-1}\left(\mathfrak{X}_{[p,s_p]}(x)\right)^\top \end{array}\right].
\end{aligned}
$$

□

3. The banded recursion matrices

The aim of this section is to find matrices that models the recursion relations that the polynomial matrices $A^{[N]}$ and $B^{[N]}$ fulfill. We will do it at the end of the section. For that aim we need some preparation.

In this paper we assume that $\mathscr{M}_N$ has a Gauss-Borel factorization, but $\det\mathscr{M}_{N+1} = 0$. Hence, $\mathscr{M}_{N+1}$ do not admit a Gauss-Borel factorization, and we only have a finite number of polynomials collected in $A^{[N]}$ and $B^{[N]}$ which satisfy multiple orthogonal relations of mixed type in the step-line. This is the reason the treatment we have given to the recursion relations working only with truncations for which the orthogonal polynomials do exist.

First note that

Proposition 3.1. *The following relations hold*

$$\mathscr{L}_N\Lambda^{[N,N+q]}_{[q]}\left[\frac{\mathscr{L}_N^{-1}}{L^{[N+q,N]}}\right]=\left[\mathscr{U}_N^{-1}\;\middle|\;U^{[N,N+p]}\right]\left(\Lambda^{[N,N+p]}_{[p]}\right)^\top\mathscr{U}_N, \tag{8a}$$

$$\mathscr{L}_{N-q}\Lambda^{[N-q,N]}_{[q]}\mathscr{L}_N^{-1}=\left[\mathscr{U}_{N-q}^{-1}\;\middle|\;U^{[N-q,N+p]}\right]\left(\Lambda^{[N,N+p]}_{[p]}\right)^\top\mathscr{U}_N, \tag{8b}$$

$$\mathscr{L}_N\Lambda^{[N,N+q]}_{[q]}\left[\frac{\mathscr{L}_{N-p}^{-1}}{L^{[N+q,N-p]}}\right]=\mathscr{U}_N^{-1}\left(\Lambda^{[N-p,N]}_{[p]}\right)^\top\mathscr{U}_{N-p}. \tag{8c}$$

Proof. An immediate consequence of the Hankel symmetry given in Equations (4) and the Gauss–Borel factorization. □

We are now ready to introduce three matrices that will represent the mentioned recursion relations:

Definition 3.2 (Recursion matrices). *We define the square matrix $\mathscr{T}_N$ and the rectangular matrices $\mathscr{T}^{[N-q,N]}$ and $\mathscr{T}^{[N,N-p]}$ as follows:*

$$\mathscr{T}_N:=\mathscr{L}_N\Lambda^{[N,N+q]}_{[q]}\left[\frac{\mathscr{L}_N^{-1}}{L^{[N+q,N]}}\right]=\left[\mathscr{U}_N^{-1}\;\middle|\;U^{[N,N+p]}\right]\left(\Lambda^{[N,N+p]}_{[p]}\right)^\top\mathscr{U}_N\in\mathbb{R}^{N\times N}, \tag{9a}$$

$$\mathscr{T}^{[N-q,N]}:=\mathscr{L}_{N-q}\Lambda^{[N-q,N]}_{[q]}\mathscr{L}_N^{-1}=\left[\mathscr{U}_{N-q}^{-1}\;\middle|\;U^{[N-q,N+p]}\right]\left(\Lambda^{[N,N+p]}_{[p]}\right)^\top\mathscr{U}_N\in\mathbb{R}^{(N-q)\times N}, \tag{9b}$$

$$\mathscr{T}^{[N,N-p]}:=\mathscr{U}_N^{-1}\left(\Lambda^{[N-p,N]}_{[p]}\right)^\top\mathscr{U}_{N-p}=\mathscr{L}_N\Lambda^{[N,N+q]}_{[q]}\left[\frac{\mathscr{L}_{N-p}^{-1}}{L^{[N+q,N-p]}}\right]\in\mathbb{R}^{N\times(N-p)}, \tag{9c}$$

Proposition 3.3 (Banded structure of the recursion matrices). *Recursion matrices $\mathscr{T}_N$, $\mathscr{T}^{[N-q,N]}$ and $\mathscr{T}^{[N,N-p]}$ are banded matrices with q superdiagonals and p subdiagonals.*

Proof. It directly follows from Equations (8) or (9). □

We now discuss the relation between these three matrices. First, we need to introduce the following:

Definition 3.4. *Let us consider the following lower and upper triangular matrices, respectively:*

$$T^{[N-q,N]}:=\mathscr{L}_{[N-2q,N-q]}\mathscr{L}^{-1}_{[N-q,N]}=\int B^{[N-q,q]}(x)\,\mathrm{d}\,\mu(x)A^{[N,q]}(x)\in\mathbb{R}^{q\times q},$$

$$T^{[N,N-p]}:=\mathscr{U}^{-1}_{[N,N-p]}\mathscr{U}_{[N-p,N-2p]}=\int B^{[N,p]}(x)\,\mathrm{d}\,\mu(x)A^{[N-p,p]}(x)\in\mathbb{R}^{p\times p},$$

Proposition 3.5 (Relation between recursion matrices). *The following block structure of the recursion matrices holds:*

$$\mathscr{T}^{[N-q,N]}=\left[\begin{array}{c|c}\mathscr{T}_{N-q} & \begin{array}{c}0_{(N-2q)\times q}\\ \hline T^{[N-q,N]}\end{array}\end{array}\right],$$

$$\mathscr{T}^{[N,N-p]}=\left[\begin{array}{c}\mathscr{T}_{N-p}\\ \hline \begin{array}{c|c}0_{p\times(N-2p)} & T^{[N,N-p]}\end{array}\end{array}\right],$$

Proof. From Equation (9a) we get

$$\mathscr{T}_{N-q}=\mathscr{L}_{N-q}\Lambda^{[N-q,N]}_{[q]}\left[\begin{array}{c}\mathscr{L}^{-1}_{N-q}\\ \hline L^{[N,N-q]}\end{array}\right],\quad \mathscr{T}_{N-p}=\left[\begin{array}{c|c}\mathscr{U}^{-1}_{N-p} & U^{[N-p,N]}\end{array}\right]\left(\Lambda^{[N-p,N]}_{[p]}\right)^{\top}\mathscr{U}_{N-p}.$$

Using (7), (9b) and (9c) we obtain

$$\mathscr{T}^{[N-q,N]}:=\mathscr{L}_{N-q}\Lambda^{[N-q,N]}_{[q]}\left[\begin{array}{c|c}\mathscr{L}^{-1}_{N-q} & 0_{(N-q)\times q}\\ \hline L^{[N,N-q]} & \mathscr{L}^{-1}_{[N,N-q]}\end{array}\right]=\left[\begin{array}{c|c}\mathscr{T}_{N-q} & \mathscr{T}^{[N-q,N,q]}\end{array}\right],\tag{10}$$

$$\mathscr{T}^{[N,N-p]}:=\left[\begin{array}{c|c}\mathscr{U}^{-1}_{N-p} & U^{[N-p,N]}\\ \hline 0_{p\times(N-q)} & \mathscr{U}^{-1}_{[N-p,N]}\end{array}\right]\left(\Lambda^{[N-p,N]}_{[p]}\right)^{\top}\mathscr{U}_{N-p}=\left[\begin{array}{c}\mathscr{T}_{N-q}\\ \hline \mathscr{T}^{[N,N-p,p]}\end{array}\right],\tag{11}$$

with $\mathscr{T}^{[N-q,N,q]}\in\mathbb{R}^{(N-q)\times q}$, $\mathscr{T}^{[N,N-p,p]}\in\mathbb{R}^{p\times(N-p)}$ given by

$$\begin{aligned}\mathscr{T}^{[N-q,N,q]}&=\mathscr{L}_{N-q}\Lambda^{[N-q,N]}_{[q]}\left[\begin{array}{c}0_{(N-q)\times q}\\ \hline \int x^{N_q-1}\mathfrak{X}_{[q,s_q]}(x)\,\mathrm{d}\mu(x)A^{[N,q]}(x)\end{array}\right]\\
&=\left[\begin{array}{c|c}\mathscr{L}_{N-2q} & 0_{(N-2q)\times q}\\ \hline \ell^{[N-q,N-2q]} & \mathscr{L}_{[N-2q,N-q]}\end{array}\right]\left[\begin{array}{c}0_{(N-2q)\times q}\\ \hline \int x^{N_q-1}\mathfrak{X}_{[q,s_q]}(x)\,\mathrm{d}\mu(x)A^{[N,q]}(x)\end{array}\right]\\
&=\left[\begin{array}{c}0_{(N-2q)\times q}\\ \hline \int B^{[N-q,q]}(x)\,\mathrm{d}\mu(x)A^{[N,q]}(x)\end{array}\right],\\
\mathscr{T}^{[N,N-p,p]}&=\left[\begin{array}{c|c}0_{p\times(N-p)} & \int B^{[N,p]}(x)\,\mathrm{d}\mu(x)x^{N_p-1}\left(\mathfrak{X}_{[p,s_p]}(x)\right)^{\top}\end{array}\right]\left(\Lambda^{[N-p,N]}_{[p]}\right)^{\top}\mathscr{U}_{N-p}\end{aligned}$$

$$
\begin{aligned}
&= \left[\begin{array}{c|c} 0_{p\times(N-2p)} & \int B^{[N,p]}(x)\,\mathrm{d}\mu(x)x^{N_p-1}\left(\mathfrak{X}_{[p,s_p]}(x)\right)^\top \end{array}\right]\left[\begin{array}{c|c} \mathscr{U}_{N-2p} & \mathcal{u}^{[N-2p,N-p]} \\ \hline 0_{p\times(N-2p)} & \mathscr{U}_{[N-2p,N-p]}\end{array}\right]\\
&= \left[\begin{array}{c|c} 0_{p\times(N-2p)} & \int B^{[N,p]}(x)\,\mathrm{d}\mu(x)A^{[N-p,p]}(x) \end{array}\right].
\end{aligned}
$$

□

Proposition 3.6 (Recursion relations). *The rectangular banded matrices $\mathscr{T}^{[N-q,N]}$ and $\mathscr{T}^{[N,N-p]}$ model recursion relations for the B and A polynomials, respectively,*

$$
\mathscr{T}^{[N-q,N]}B^{[N]}(x) = xB^{[N-q]}(x), \quad A^{[N]}(x)\mathscr{T}^{[N,N-p]} = xA^{[N-p]}(x).
$$

The square banded matrix $\mathscr{T}_N$ needs a correction to describe such recursion relations. For that aim we require of:

Definition 3.7. *Let us consider $A^{[N]} = \left[A^{[N-p]}|A^{[N,p]}\right]$ and $B^{[N]} = \left[\frac{B^{[N-q]}}{B^{[N,q]}}\right]$, $A^{[N,p]} \in \mathbb{R}^{p\times p}[x]$, $B^{[N,p]} \in \mathbb{R}^{q\times q}[x]$, with*

$$
B^{[N,q]} = \begin{bmatrix} B^{(1)}_{N-q} \cdots\cdots\cdots B^{(q)}_{N-q} \\ B^{(1)}_{N-q+1} \cdots\cdots B^{(q)}_{N-q+1} \\ \vdots \qquad\qquad \vdots \\ B^{(1)}_{N-1} \cdots\cdots\cdots B^{(q)}_{N-1}\end{bmatrix}, \quad A^{[N,p]} = \begin{bmatrix} A^{(1)}_{N-p} & A^{(1)}_{N-p+1} & \cdots\cdots & A^{(1)}_{N-1} \\ \vdots & \vdots & & \vdots \\ A^{(p)}_{N-p} & A^{(p)}_{N-p+1} & \cdots\cdots & A^{(p)}_{N-1}\end{bmatrix}.
$$

Corollary 3.8 (Recursion relations). *The recursion relations can be alternatively written as*

$$
\begin{aligned}
\mathscr{T}_{N-q}B^{[N-q]}(x) + \left[\frac{0_{(N-q)\times q}}{T^{[N-q,N]}B^{[N,q]}(x)}\right] &= xB^{[N-q]}(x),\\
A^{[N-p]}(x)\mathscr{T}_{N-p} + \left[0_{(N-p)\times p} \,\middle|\, A^{[N,p]}(x)T^{[N,N-p]}\right] &= xA^{[N-p]}(x).
\end{aligned}
$$

4. Christoffel transformations and bidiagonal factorizations

In this section we discuss bidiagonal factorizations of the recursion matrices $\mathscr{T}_N$.

Definition 4.1 (Bidiagonal factorization). *Given bidiagonal matrices of the form*

$$
U_b = \begin{bmatrix} U_{b,0} & 1 & 0 & \cdots & 0 \\ 0 & U_{b,1} & 1 & 0 & \vdots \\ \vdots & \ddots & \ddots & \ddots & 0 \\ \vdots & & \ddots & \ddots & 1 \\ 0 & \cdots & \cdots & 0 & U_{b,N-1}\end{bmatrix}, \quad b\in\{1,\dots,q\},
$$

$$
L_a = \begin{bmatrix} 1 & 0 & \cdots & \cdots & 0 \\ L_{a,1} & 1 & 0 & & \vdots \\ 0 & L_{a,2} & 1 & 0 & \vdots \\ \vdots & \ddots & \ddots & \ddots & 0 \\ 0 & \cdots & 0 & L_{a,N-1} & 1\end{bmatrix}, \quad a\in\{1,\dots,p\}.
$$

we say that a monic banded matrix $\mathscr{T}_N$, with q superdiagonal and p subdiagonals and with the highest superdiagonal normalized with 1's, admits a bidiagonal factorization if it can written as

$$\mathscr{T}_N = L_1 \cdots L_p U_q \cdots U_1.$$

In this section we discuss this possibility and find conditions for the existence of such factorization, and also expressions for its factors. We will do it considering simple Christoffel transformations of the matrix of measures $\mathrm{d}\,\mu$.

For a more clear presentation, within this section we consider that $\mathscr{L}_N$ is lower unitriangular. To start with, we need truncations of the basic shift matrix Λ:

Definition 4.2 (Truncated basic shift matrices). *For $i \in \{0, 1, \dots, r-1\}$, let us define*

$$\Lambda^{[N+i,N+i+1]} := \left.\overbrace{\left[\begin{array}{ccccc|c} 0 & 1 & 0 & \cdots & 0 & 0 \\ \vdots & \ddots & \ddots & \ddots & \vdots & \vdots \\ \vdots & & \ddots & \ddots & 0 & 0 \\ \vdots & & & \ddots & 1 & 0 \\ 0 & \cdots & \cdots & \cdots & 0 & 1 \end{array}\right]}^{N+i+1}\right\} {\scriptstyle N+i}.$$

Similar to the block case we have a spectral property on monomials:

Proposition 4.3. *The action of these truncated basic shift matrices on the monomial vectors is*

$$\Lambda^{[N+r-1,N+r]} X^{[N+r]}_{[q]} = X^{[N+r-1]}_{[r]} \mathfrak{X}_{[r,1]}(x).$$

Importantly, we have that:

Proposition 4.4. *We can factor out the matrix $\Lambda^{[N,N+r]}_{[r]}$ as follows*

$$\Lambda^{[N,N+r]}_{[r]} = \Lambda^{[N,N+1]} \Lambda^{[N+1,N+2]} \cdots \Lambda^{[N+r-1,N+r]}.$$

We now give a rewriting of the equation defining the recursion matrices:

Proposition 4.5. *The recursion matrices $\mathscr{T}_N$ satisfy*

$$\mathscr{T}_N = \mathscr{L}_N \Lambda^{[N,N+1]} \Lambda^{[N+1,N+2]} \cdots \Lambda^{[N+q-1,N+q]} \mathscr{M}^{[N+q,N]} \mathscr{U}_N \tag{12}$$

$$= \mathscr{L}_N \mathscr{M}^{[N,N+p]} \left(\Lambda^{[N+p-1,N+p]}\right)^\top \cdots \left(\Lambda^{[N+1,N+2]}\right)^\top \left(\Lambda^{[N,N+1]}\right)^\top \mathscr{U}_N. \tag{13}$$

Proof. Rewrite (9a) as follows

$$\mathscr{T}_N = \mathscr{L}_N \Lambda^{[N,N+q]}_{[q]} \mathscr{M}^{[N+q,N]} \mathscr{U}_N = \mathscr{L}_N \mathscr{M}^{[N,N+p]} \left(\Lambda^{[N,N+p]}_{[p]}\right)^\top \mathscr{U}_N, \tag{14}$$

and Proposition 4.4. □

Now, we introduce some basic Christoffel type perturbations of the matrix of measures. Namely perturbations by left or right multiplication by the first degree matrix polynomials $\mathfrak{X}_{[r,1]}$:

Definition 4.6 (Christoffel perturbed matrix of measures). *Let us define*

$$\mathrm{d}\,\mu_{(n,m)} := \mathfrak{X}^n_{[q,1]} \,\mathrm{d}\,\mu \left(\mathfrak{X}^m_{[p,1]}\right)^\top,$$

and the corresponding moment matrices

$$\mathscr{M}^{[N,M]}_{(n,m)} = \int X^{[N]}(x) \,\mathrm{d}\,\mu_{(n,m)}(x) \left(X^{[M]}(x)\right)^\top.$$

Remark 4.7. *Observe that*
$$x \,\mathrm{d}\,\mu = \mathrm{d}\,\mu^{(p,0)} = \mathrm{d}\,\mu^{(0,q)},$$
and the Christoffel transformation coincides.

Theorem 4.8. *The recursion matrix $\mathscr{T}_N$ admits a bidiagonal factorization*
$$\mathscr{T}_N = L_1 \cdots L_p U_q \cdots U_1,$$
if all the moment matrices $\mathscr{M}_N$, $\mathscr{M}_{N,(b,0)}$, $b \in \{1,\dots,q\}$, and $\mathscr{M}_{N,(0,a)}$, $a \in \{1,\dots,p\}$, admit a Gauss–Borel factorization. The bidiagonal factors are given by

$$(15)\quad \begin{aligned} U_b &:= \mathscr{U}^{-1}_{N,(b,0)}\mathscr{U}_{N,(b-1,0)} = \begin{bmatrix} U_{b,0} & 1 & 0 & \cdots & \cdots & 0\\ 0 & U_{b,1} & 1 & 0 & & \vdots\\ \vdots & \ddots & \ddots & \ddots & \ddots & \vdots\\ \vdots & & & \ddots & \ddots & 0\\ \vdots & & & & \ddots & 1\\ 0 & \cdots & \cdots & \cdots & 0 & U_{b,N-1} \end{bmatrix}, \quad b \in \{1,\dots,q\},\\ L_a &:= \mathscr{L}_{N,(0,a-1)}\mathscr{L}^{-1}_{N,(0,a)} = \begin{bmatrix} 1 & 0 & \cdots & \cdots & \cdots & 0\\ L_{a,1} & 1 & 0 & & & \vdots\\ 0 & L_{a,2} & 1 & 0 & & \vdots\\ \vdots & \ddots & \ddots & \ddots & \ddots & \vdots\\ \vdots & & \ddots & \ddots & \ddots & 0\\ 0 & \cdots & \cdots & 0 & L_{a,N-1} & 1 \end{bmatrix}, \quad a \in \{1,\dots,p\}. \end{aligned}$$

Proof. We start the proof by finding the bidiagonal factors U_n. Firstly, we note that
$$\Lambda^{[N+q-1,N+q]}\mathscr{M}^{[N+q,N]} = \int X^{[N+q-1]}(x)\mathfrak{X}_{[q]}(x)\,\mathrm{d}\,\mu(x)\left(X^{[N]}_{[p]}(x)\right)^\top = \mathscr{M}^{[N+q-1,N]}_{(1,0)}$$
and
$$(16)\qquad \Lambda^{[N+q-1,N+q]}\mathscr{M}^{[N+q,N]}\mathscr{U}_N = \Lambda^{[N+q-1,N+q]}\left[\frac{\mathscr{L}_N^{-1}}{L^{[N+q,N]}}\right] = \left[\frac{\mathscr{L}^{-1}_{N,(1,0)}}{L^{[N+q-1,N]}_{(1,0)}}\right]\mathscr{U}^{-1}_{N,(1,0)}\mathscr{U}_N.$$
Now, the matrix $\Lambda^{[N+q-1,N+q]}\left[\frac{\mathscr{L}_N^{-1}}{L^{[N+q,N]}}\right]$ is a upper Hessenberg matrix, that assuming that $\mathscr{L}_N$ is monic, is also monic. Therefore,
$$U_1 := \mathscr{U}^{-1}_{N,(1,0)}\mathscr{U}_N$$
is a monic upper bidiagonal matrix (with its first superdiagonal being monic).

Now we consider
$$\begin{aligned}\Lambda^{[N+q-2,N+q-1]}\Lambda^{[N+q-1,N+q]}\mathscr{M}^{[N+q,N]}\mathscr{U}_N &= \Lambda^{[N+q-2,N+q-1]}\mathscr{M}^{[N+q-1,N]}_{(1,0)}\mathscr{U}_N\\ &= \Lambda^{[N+q-2,N+q-1]}\mathscr{M}^{[N+q-1,N]}_{(1,0)}\mathscr{U}_{N,(1,0)}U_1\end{aligned}$$
and using (16) for $\mathrm{d}\,\mu_{(1,0)}$ and $N+q \to N+q-1$ we find

(17)
$$\Lambda^{[N+q-2,N+q-1]}\mathscr{M}^{[N+q-1,N]}_{(1,0)}\mathscr{U}_{N,(1,0)} = \Lambda^{[N+q-2,N+q-1]}\left[\frac{\mathscr{L}^{-1}_{N,(1,0)}}{L^{[N+q-1,N]}_{(1,0)}}\right] = \left[\frac{\mathscr{L}^{-1}_{N,(2,0)}}{L^{[N+q-2,N]}_{(2,0)}}\right]\mathscr{U}^{-1}_{N,(2,0)}\mathscr{U}_{N,(1,0)}.$$

Hence,

$$U_2 \coloneqq \mathscr{U}^{-1}_{N,(2,0)}\mathscr{U}_{N,(1,0)}$$

is a monic upper bidiagonal matrix and

$$\Lambda^{[N+q-2,N+q-1]}\Lambda^{[N+q-1,N+q]}\mathscr{M}^{[N+q,N]}\mathscr{U}_N = \left[\begin{array}{c}\mathscr{L}^{-1}_{N,(2,0)}\\ \hline L^{[N+q-2,N]}_{(2,0)}\end{array}\right]U_2U_1. \tag{18}$$

Repeating this process q times we find

$$\Lambda^{[N,N+1]}\Lambda^{[N+1,N+2]}\cdots\Lambda^{[N+q-1,N+q]}\mathscr{M}^{[N+q,N]}\mathscr{U}_N = \mathscr{L}^{-1}_{N,(q,0)}U_q\cdots U_1$$

where

$$U_b \coloneqq \mathscr{U}^{-1}_{N,(b,0)}\mathscr{U}_{N,(b-1,0)} = \begin{bmatrix} U_{b,0} & 1 & 0 & \cdots & 0\\ 0 & U_{b,1} & 1 & 0 & \vdots\\ \vdots & \ddots & \ddots & \ddots & 0\\ \vdots & & & \ddots & 1\\ 0 & \cdots & & 0 & U_{b,N-1}\end{bmatrix}, \quad b\in\{1,\dots,q\}.$$

so that

$$\mathscr{T}_N = \mathscr{L}_N\mathscr{L}^{-1}_{N,(q,0)}U_q\cdots U_1.$$

We now find the lower bidiagonal factors L_m. Following an similar procedure as above we get

$$\mathscr{M}^{[N,N+p]}\left(\Lambda^{[N+p-1,N+p]}\right)^\top = \int X^{[N]}(x)\,\mathrm{d}\,\mu(x)\,\left(\mathfrak{X}_{[q,1]}(x)\right)^\top\left(X^{[N+p-1]}_{[p]}(x)\right)^\top = \mathscr{M}^{[N,N+p-1]}_{(0,1)}$$

and also that

$$\begin{aligned}\mathscr{L}_N\mathscr{M}^{[N,N+p]}\left(\Lambda^{[N+p-1,N+p]}\right)^\top &= \left[\begin{array}{c|c}\mathscr{U}^{-1}_N & U^{[N,N+p]}\end{array}\right]\left(\Lambda^{[N+p-1,N+p]}\right)^\top\\ &= \mathscr{L}_N\mathscr{L}^{-1}_{N,(0,1)}\left[\begin{array}{c|c}\mathscr{U}^{-1}_{N,(1,0)} & U^{[N,N+p-1]}_{(0,1)}\end{array}\right].\end{aligned} \tag{19}$$

Thus,

$$L_1 \coloneqq \mathscr{L}_N\mathscr{L}^{-1}_{N,(0,1)}$$

is a monic lower bidiagonal matrix (with is main diagonal being monic). A iterative procedure leads to

$$\mathscr{T}_N = L_1\cdots L_p\mathscr{U}^{-1}_{N,(0,p)}\mathscr{U}_N,$$

with

$$L_a \coloneqq \mathscr{L}_{N,(0,a-1)}\mathscr{L}^{-1}_{N,(0,a)} = \begin{bmatrix} 1 & 0 & \cdots & & 0\\ L_{a,1} & 1 & 0 & & \vdots\\ 0 & L_{a,2} & 1 & 0 & \vdots\\ \vdots & \ddots & \ddots & \ddots & 0\\ 0 & \cdots & 0 & L_{a,N-1} & 1\end{bmatrix}, \quad a\in\{1,\dots,p\}.$$

Given the normalization of these bidiagonal matrices on its higher diagonal, we get that the bidiagonal factorization

$$\mathscr{T}_N = L_1\cdots L_pU_q\cdots U_1$$

is satisfied. □

Remark 4.9. *We observe that given the triangular form these bidiagonal matrices for the M-th, $M\leq N$, truncations we have*
$$\mathscr{T}_M=L_1^{[M]}\cdots L_p^{[M]}U_q^{[M]}\cdots U_1^{[M]},$$
where $U_n^{[M]}, L_m^{[M]}$ are the M-th truncations of the corresponding bidiagonal matrices.

Remark 4.10. *Note that for $a\in\{1,\dots,p\}$ and $b\in\{1,\dots,q\}$ we have a the connection formulas*
$$B^{[N]}_{(a-1,0)}=L_aB^{[N]}_{(a,0)},\quad A^{[N]}_{(0,b-1)}=A^{[N]}_{(0,b)}U_b.$$

Theorem 4.11. *The entries of the bidiagonal matrices can be expressed as follows:*
$$\begin{aligned}L_{a,n}&=\big(\mathscr{L}_{N,(0,a-1)}\big)_{n,n-1}-\big(\mathscr{L}_{N,(0,a)}\big)_{n,n-1}, && a\in\{1,\dots,p\},\quad n\in\{1,\dots,N-1\},\\ U_{b,n}&=\frac{\big(\mathscr{U}_{N,(b-1,0)}\big)_{n,n}}{\big(\mathscr{U}_{N,(b,0)}\big)_{n,n}}, && b\in\{1,\dots,q\},\quad n\in\{0,1,\dots,N-1\}.\end{aligned}\tag{20}$$
Alternatively, we also find for $b\in\{1,\dots,q\}$ and $a\in\{1,\dots,p\}$
$$\begin{aligned}L_{a,n}&=\frac{\big(\mathscr{U}_{N,(0,a))}\big)_{n-1,n-1}}{\big(\mathscr{U}_{N,(0,a-1))}\big)_{n,n}}, && n\in\{1,\dots,N-1\},\\ U_{b,n}&=(1-\delta_{n,0})\big(\mathscr{L}_{N,(b,0)}\big)_{n,n-1}-\big(\mathscr{L}_{N,(b-1,0)}\big)_{n+1,n}, && n\in\{0,1,\dots,N-1\}.\end{aligned}\tag{21}$$

Proof. Equations (20) follows from Equations (15) for the bidiagonal matrices. Equations (21) follow from
$$\Lambda^{[N+q-b,N+q-b+1]}\left[\begin{array}{c}\mathscr{L}^{-1}_{N,(b-1,0)}\\\hline L^{[N+q-b+1,N]}_{(b-1,0)}\end{array}\right]=\left[\begin{array}{c}\mathscr{L}^{-1}_{N,(b,0)}\\\hline L^{[N+q-b,N]}_{(b,0)}\end{array}\right]U_b,\qquad b\in\{1,\dots,q\},$$
$$\left[\begin{array}{c|c}\mathscr{U}^{-1}_{N,(0,a-1)}&U^{[N,N+p-a+1]}\end{array}\right]\left(\Lambda^{[N+p-a,N+p-a+1]}\right)^\top=L_a\left[\begin{array}{c|c}\mathscr{U}^{-1}_{N,(a,0)}&U^{[N,N+p-a]}_{(0,a)}\end{array}\right],\quad a\in\{1,\dots,p\}.$$
□

Corollary 4.12. *In terms of the sub-leading coefficients $\beta^{(s+1)}_{n,(0,b)}$ of the polynomials $B^{(s+1)}_{n,(0,b)}(x)$ and the leading coefficients $\alpha^{(s+1)}_{n,(0,b)}$ of the polynomials $A^{(s+1)}_{n,(0,a)}(x)$, $b\in\{1,\dots,q\}$, $a\in\{1,\dots,p\}$, $n\in\{1,\dots,N-1\}$, we have*
$$\begin{aligned}L_{a,n}&=\beta^{(s_n+1)}_{n+1,(0,a-1)}-\beta^{(s_n+1)}_{n,(0,a)}=\frac{\alpha^{(s_{n-1}+1)}_{n-1,(0,a)}}{\alpha^{(s_n+1)}_{n,(0,a-1)}},\\ U_{b,n}&=\frac{\alpha^{(s_n+1)}_{n,(b-1,0)}}{\alpha^{(s_n+1)}_{n,(b,0)}}=(1-\delta_{n,0})\beta^{(s_n+1)}_{n,(b,0)}-\beta^{(s_{n+1}+1)}_{n+1,(b-1,0)}.\end{aligned}$$

Proposition 4.13. *For $b\in\{1,\dots,q\}$, $a\in\{1,\dots,p\}$, the following identities between Christoffel transformed recurrence matrices hold:*
$$U_b\mathscr{T}^{[N,N-p]}_{(b-1,0)}\left(U_b^{[N-p]}\right)^{-1}=\mathscr{T}^{[N,N-p]}_{(b,0)},\quad\left(L_a^{[N-q]}\right)^{-1}\mathscr{T}^{[N-q,N]}_{(0,a-1)}L_a=\mathscr{T}^{[N-q,N]}_{(0,a)}.$$

Proof. It is a direct consequence of
$$\mathscr{T}^{[N-q,N]}_{(0,a)}=\mathscr{L}_{N-q,(0,a)}\Lambda^{[N-q,N]}_{[q]}\mathscr{L}^{-1}_{N,(0,a)},\qquad\mathscr{T}^{[N,N-p]}_{(b,0)}=\mathscr{U}^{-1}_{N,(b,0)}\left(\Lambda^{[N-p,N]}_{[p]}\right)^\top\mathscr{U}_{N-p,(b,0)},$$

and

$$U_b := \mathscr{U}^{-1}_{N,(b,0)}\mathscr{U}_{N,(b-1,0)}, \quad L_a := \mathscr{L}_{N,(0,a-1)}\mathscr{L}^{-1}_{N,(0,a)}.$$

□

Remark 4.14. *In the truncated scenario the basic Christoffel transformations do not induce anymore a permutation of the bidiagonal factors for the corresponding transformed recurrence matrix, as it does in the infinite case. To see this let put $b=1$, and look at*

$$U_1\mathscr{T}^{[N,N-p]}\left(U_1^{[N-p]}\right)^{-1} = \mathscr{T}^{[N,N-p]}_{(1,0)}$$

and, recalling Proposition 3.5, we know that

$$\mathscr{T}^{[N,N-p]} = \left[\begin{array}{c|c} \multicolumn{2}{c}{\mathscr{T}_{N-p}} \\ \hline 0_{p\times(N-2p)} & T^{[N,N-p]} \end{array}\right],$$

and we get

$$\mathscr{T}_{N-p,(1,0)} = U_1^{[N-p]}L_1^{[N-p]}\cdots L_p^{[N-p]}U_q^{[N-p]}\cdots U_2^{[N-p]} + \left[\begin{array}{c|c} \multicolumn{2}{c}{0_{(N-p-1)\times(N-p)}} \\ \hline 0_{1\times(N-2p)} & e_1^\top T^{[N,N-p]} \end{array}\right].$$

5. Christoffel formulas for the bidiagonal factors

We now proceed to use the basic Christoffel transformations and the corresponding Christoffel formulas in order to derive closed determinantal expressions of the entries in the bidiagonal matrices in terms of determinants of the original polynomials $A^{[N]}(x)$ and $B^{[N]}(X)$ and its truncations evaluated at $x=0$. The required determinants are given by:

Definition 5.1 (τ-determinants). *Let us introduce the following determinants*

$$\tau^B_{b,n} := \begin{vmatrix} B^{(1)}_n(0) & \cdots\cdots & B^{(b)}_n(0) \\ \vdots & & \vdots \\ B^{(1)}_{n+b-1}(0) & \cdots\cdots & B^{(b)}_{n+b-1}(0)\end{vmatrix}, \quad b\in\{1,\dots,q\}, \quad n\in\{0,1,\dots,N-b-1\},$$

$$\tau^A_{a,n} := \begin{vmatrix} A^{(1)}_{n+a-1}(0) & \cdots\cdots & A^{(1)}_n(0) \\ \vdots & & \vdots \\ A^{(a)}_{n+a-1}(0) & \cdots\cdots & A^{(a)}_n(0)\end{vmatrix}, \quad a\in\{1,\dots,p\}, \quad n\in\{0,1,\dots,N-a-1\},$$

and $\tau^A_{0,n}=\tau^B_{0,n}=1$.

These determinants allow to characterize the existence of the orthogonality as was shown in [17], see also [18] for the Geronimus situation.

Theorem 5.2 (Existence of orthogonality (Mañas & Rojas)).

i) *The existence of the transformed orthogonality for* $\mathrm{d}\,\mu_{(0,a)}$, $a \in \{1,\dots,p\}$, *up to* $N-a-1$ *polynomials is equivalent to*
$$\tau^A_{a,n} \neq 0, \quad a \in \{1,\dots,p\}, \quad n \in \{0,1,\dots,N-a-1\}.$$

ii) *The existence of the transformed orthogonality for* $\mathrm{d}\,\mu_{(b,0)}$, $b \in \{1,\dots,q\}$, *up to* $N-b-1$ *polynomial is equivalent to*
$$\tau^B_{b,n} \neq 0 \quad b \in \{1,\dots,q\}, \quad n \in \{0,1,\dots,N-b-1\}.$$

Proof. Is a direct consequence of the existence results described in [17]. □

We proved in [8] that

Theorem 5.3. *For* $n \in \{0,1,\dots,\min(N-p,N-q)\}$ *we find*
$$(-1)^{(q-1)n}\begin{vmatrix} B^{(1)}_n(x) & \cdots\cdots & B^{(q)}_n(x) \\ \vdots & & \vdots \\ B^{(1)}_{n+q-1}(x) & \cdots\cdots & B^{(b)}_{n+q-1}(x)\end{vmatrix} = (-1)^{(p-1)(n+1)}\mathscr{T}_{p,0}\cdots\mathscr{T}_{n+p-1,n-1}\begin{vmatrix} A^{(1)}_{n+p-1}(x) & \cdots\cdots & A^{(1)}_n(x) \\ \vdots & & \vdots \\ A^{(p)}_{n+p-1}(x) & \cdots\cdots & A^{(p)}_n(x)\end{vmatrix}.$$

That immediately leads to:

Corollary 5.4. *The following identities hold*
$$(-1)^{(p-1)(n+1)}\mathscr{T}_{p,0}\cdots\mathscr{T}_{n+p-1,n-1}\tau^A_{p,n} = (-1)^{(q-1)n}\tau^B_{q,n}, \tag{22}$$
$$(-1)^{p-1}\frac{1}{\mathscr{T}_{n+p,n}}\frac{\tau^A_{p,n}}{\tau^A_{p,n+1}} = (-1)^{q-1}\frac{\tau^B_{q,n}}{\tau^B_{q,n+1}}. \tag{23}$$

Proof. Relation (22) follows from the evaluations of the determinants in Theorem 5.3, Equation (23) is an trivial consequence derived by taking ratios of the previous relations. □

Theorem 5.5. *The entries of the bidiagonal matrices have the following expressions*
$$U_{b,n} = -\frac{\tau^B_{b-1,n}\tau^B_{b,n+1}}{\tau^B_{b-1,n+1}\tau^B_{b,n}}, \qquad b \in \{1,\dots,q\}, \quad n \in \{0,1,\dots,N-b-1\},$$
$$L_{a,n+1} = -\frac{\tau^A_{a-1,n+2}\tau^A_{a,n}}{\tau^A_{a-1,n+1}\tau^A_{a,n+1}}, \quad a \in \{1,\dots,p\}, \quad n \in \{0,1,\dots,N-a-1\}.$$

Proof. Equation (16) imply
$$\left[\frac{\mathscr{L}^{-1}_{N,(1,0)}}{L^{[N+q-1,N]}_{(1,0)}}\right]U_1 = \Lambda^{[N+q-1,N+q]}\left[\frac{I_N}{L^{[N+q,N]}\mathscr{L}_N}\right]\mathscr{L}^{-1}_N$$
and recalling that by definition
$$\mathscr{L}_{N+q-1,(1,0)}\left[\frac{\mathscr{L}^{-1}_{N,(1,0)}}{L^{[N+q-1,N]}_{(1,0)}}\right] = \begin{bmatrix} I_N \\ 0_{(q-1)\times N}\end{bmatrix}$$

we find that

$$
\begin{aligned}
\left[\frac{I_N}{0_{(q-1)\times N}}\right]U_1B^{[N]}(x)&=\mathscr{L}_{N+q-1,(1,0)}\Lambda^{[N+q-1,N+q]}\left[\frac{I_N}{L^{[N+q,N]}\mathscr{L}_N}\right]\mathscr{L}_N^{-1}B^{[N]}(x)\\
&=\mathscr{L}_{N+q-1,(1,0)}\Lambda^{[N+q-1,N+q]}\left[\frac{I_N}{L^{[N+q,N]}\mathscr{L}_N}\right]X^{[N]}_{[q]}(x)\\
&=\mathscr{L}_{N+q-1,(1,0)}\Lambda^{[N+q-1,N+q]}\left[\frac{X^{[N]}_{[q]}(x)}{L^{[N+q,N]}B^{[N]}(x)}\right]\\
&=\begin{bmatrix}\mathscr{L}_{N-1,(1,0)} & 0_{(N-1)\times q}\\ \ell^{[N-1+q,N-1]}_{(1,0)} & \mathscr{L}_{[N-1,N-1+q],(1,0)}\end{bmatrix}\left[\frac{X^{[N-1]}_{[q]}(x)\mathfrak{X}_{[q,1]}(x)}{L^{[N+q,N]}B^{[N]}(x)}\right]
\end{aligned}
$$

Hence, if $U_1^{[N-1,N]}$ is obtained by removing the last row in U_1, we get

$$U_1^{[N-1,N]}B^{[N]}(x)=B^{[N-1]}_{(1,0)}(x)\mathfrak{X}_{[q,1]}(x).$$

The only eigenvalue of $\mathfrak{X}_{[q,1]}(x)$ is 0 with right eigenvector give by $e_1\in\mathbb{R}^q$, that zero entries but for 1 in the first entry.

$$U_1^{[N-1,N]}B^{[N]}(0)e_1=0_{(N-1)\times 1}.$$

and, consequently,

$$U_1^{[N-1,N]}\begin{bmatrix}B^{(1)}_0(0)\\ B^{(1)}_1(0)\\ \vdots\\ B^{(1)}_{N-1}(0)\end{bmatrix}=0_{(N-1)\times 1}$$

Therefore,

$$U_{1,n}=-\frac{B^{(1)}_{n+1}(0)}{B^{(1)}_n(0)},\quad n\in\{0,1,\dots,N-2\}.$$

Now, Equation (18) can be written

$$\left[\frac{\mathscr{L}^{-1}_{N,(2,0)}}{L^{[N+q-2,N]}_{(2,0)}}\right]U_2U_1=\Lambda^{[N+q-2,N+q-1]}\Lambda^{[N+q-1,N+q]}\left[\frac{I_N}{L^{[N+q-1,N]}\mathscr{L}_N}\right]\mathscr{L}_N^{-1}$$

and recalling that

$$\mathscr{L}_{N+q-1,(2,0)}\left[\frac{\mathscr{L}^{-1}_{N,(2,0)}}{L^{[N+q-2,N]}_{(2,0)}}\right]=\begin{bmatrix}I_N\\ 0_{(q-2)\times N}\end{bmatrix}$$

we express it as

$$\left[\frac{I_N}{0_{(q-2)\times N}}\right]U_2U_1=\mathscr{L}_{N+q-2,(2,0)}\Lambda^{[N+q-2,N+q-1]}\Lambda^{[N+q-1,N+q]}\left[\frac{I_N}{L^{[N+q-1,N]}\mathscr{L}_N}\right]\mathscr{L}_N^{-1}.$$

Then, proceeding as we did for U_1:

$$
\begin{aligned}
\left[\frac{I_N}{0_{(q-2)\times N}}\right]U_2U_1B^{[N]}(x)&=\mathscr{L}_{N+q-2,(2,0)}\Lambda^{[N+q-2,N+q-1]}\Lambda^{[N+q-1,N+q]}\left[\frac{I_N}{L^{[N+q-1,N]}\mathscr{L}_N}\right]\mathscr{L}_N^{-1}B^{[N]}(x)\\
&=\mathscr{L}_{N+q-2,(2,0)}\Lambda^{[N+q-2,N+q-1]}\Lambda^{[N+q-1,N+q]}\left[\frac{I_N}{L^{[N+q-1,N]}\mathscr{L}_N}\right]X^{[N]}_{[q]}(x)\\
&=\mathscr{L}_{N+q-2,(2,0)}\Lambda^{[N+q-2,N+q-1]}\Lambda^{[N+q-1,N+q]}\left[\frac{X^{[N]}_{[q]}(x)}{L^{[N+q-1,N]}B^{[N]}(x)}\right]\\
&=\begin{bmatrix}\mathscr{L}_{N-2,(2,0)} & 0_{(N-2)\times q}\\ \ell^{[N-2+q,N-2]}_{(2,0)} & \mathscr{L}_{[N-2,N-2+q],(2,0)}\end{bmatrix}\left[\frac{X^{[N-2]}_{[q]}(x)\mathfrak{X}^2_{[q,1]}(x)}{L^{[N+q,N]}B^{[N]}(x)}\right]
\end{aligned}
$$

Thus, if $(U_2U_1)^{[N-2,N]}$ is obtained by removing the last two rows in U_2U_1, we obtain

$$
(U_2U_1)^{[N-2,N]}B^{[N]}(x)=B^{[N-2]}_{(2,0)}(x)\mathfrak{X}^2_{[q,1]}(x).
$$

The only eigenvalue of $\mathfrak{X}^2_{[q,1]}(x)$ is 0, which is double, and with right eigenvector given by $e_1,e_2\in\mathbb{R}^q$, that zero entries but for 1 in the first and second entries, respectively

$$
(U_2U_1)^{[N-2,N]}B^{[N]}(0)\left[e_1\mid e_2\right]=0_{(N-2)\times 2}.
$$

so that

$$
(U_2U_1)^{[N-2,N]}\begin{bmatrix}B^{(1)}_0(0) & B^{(2)}_0(0)\\ B^{(1)}_1(0) & B^{(2)}_1(0)\\ \vdots & \vdots\\ B^{(1)}_{N-1}(0) & B^{(2)}_{N-1}(0)\end{bmatrix}=0_{(N-2)\times 2}.
$$

That is, for $n\in\{0,1,\dots N-3\}$,

$$
\begin{aligned}
(U_2U_1)_{n,n}B^{(1)}_n(0)+(U_2U_1)_{n,n+1}B^{(1)}_{n+1}(0)+B^{(1)}_{n+2}(0)&=0,\\
(U_2U_1)_{n,n}B^{(2)}_n(0)+(U_2U_1)_{n,n+1}B^{(2)}_{n+1}(0)+B^{(2)}_{n+2}(0)&=0.
\end{aligned}
$$

This system can be written as follows

$$
\begin{bmatrix}(U_2U_1)_{n,n} & (U_2U_1)_{n,n+1}\end{bmatrix}=-\begin{bmatrix}B^{(1)}_{n+2}(0) & B^{(2)}_{n+2}(0)\end{bmatrix}\begin{bmatrix}B^{(1)}_n(0) & B^{(2)}_n(0)\\ B^{(1)}_{n+1}(0) & B^{(2)}_{n+1}(0)\end{bmatrix}^{-1}
$$

and, consequently,

$$
\begin{aligned}
(U_2U_1)_{n,n}&=-\begin{bmatrix}B^{(1)}_{n+2}(0) & B^{(2)}_{n+2}(0)\end{bmatrix}\begin{bmatrix}B^{(1)}_n(0) & B^{(2)}_n(0)\\ B^{(1)}_{n+1}(0) & B^{(2)}_{n+1}(0)\end{bmatrix}^{-1}\begin{bmatrix}1\\0\end{bmatrix}=\Theta_*\begin{bmatrix}B^{(1)}_n(0) & B^{(2)}_n(0) & 1\\ B^{(1)}_{n+1}(0) & B^{(2)}_{n+1}(0) & 0\\ B^{(1)}_{n+2}(0) & B^{(2)}_{n+2}(0) & 0\end{bmatrix},\\
(U_2U_1)_{n,n+1}&=-\begin{bmatrix}B^{(1)}_{n+2}(0) & B^{(2)}_{n+2}(0)\end{bmatrix}\begin{bmatrix}B^{(1)}_n(0) & B^{(2)}_n(0)\\ B^{(1)}_{n+1}(0) & B^{(2)}_{n+1}(0)\end{bmatrix}^{-1}\begin{bmatrix}0\\1\end{bmatrix}=\Theta_*\begin{bmatrix}B^{(1)}_n(0) & B^{(2)}_n(0) & 0\\ B^{(1)}_{n+1}(0) & B^{(2)}_{n+1}(0) & 1\\ B^{(1)}_{n+2}(0) & B^{(2)}_{n+2}(0) & 0\end{bmatrix}.
\end{aligned}
$$

Where Θ_* denotes the last quasideterminant [19], that in the scalar case leads to the following expressions

$$(U_2U_1)_{n,n} = U_{2,n}U_{1,n} = \frac{\begin{vmatrix} B^{(1)}_{n+1}(0) & B^{(2)}_{n+1}(0) \\ B^{(1)}_{n+2}(0) & B^{(2)}_{n+2}(0) \end{vmatrix}}{\begin{vmatrix} B^{(1)}_{n}(0) & B^{(2)}_{n}(0) \\ B^{(1)}_{n+1}(0) & B^{(2)}_{n+1}(0) \end{vmatrix}},$$

for $n \in \{0, 1, \dots N-3\}$. Hence, we find

$$U_{2,n} = -\frac{B^{(1)}_{n}(0)}{B^{(1)}_{n+1}(0)} \frac{\begin{vmatrix} B^{(1)}_{n+1}(0) & B^{(2)}_{n+1}(0) \\ B^{(1)}_{n+2}(0) & B^{(2)}_{n+2}(0) \end{vmatrix}}{\begin{vmatrix} B^{(1)}_{n}(0) & B^{(2)}_{n}(0) \\ B^{(1)}_{n+1}(0) & B^{(2)}_{n+1}(0) \end{vmatrix}}, \quad n \in \{0, 1, \dots N-3\}.$$

In general, it holds that

$$(U_b \cdots U_1)^{[N-b,N]} B^{[N]}(x) = B^{[N-b]}_{(b,0)}(x) \mathfrak{X}^b_{[q,1]}(x),$$

so that

$$(U_b \cdots U_1)^{[N-b,N]} \begin{bmatrix} B^{(1)}_0(0) \cdots\cdots\cdots B^{(b)}_0(0) \\ B^{(1)}_1(0) \cdots\cdots\cdots B^{(b)}_1(0) \\ \vdots \qquad\qquad \vdots \\ B^{(1)}_{N-1}(0) \cdots\cdots B^{(b)}_{N-1}(0) \end{bmatrix} = 0_{(N-b)\times b}.$$

This can be rewritten as

$$\begin{aligned}(U_b \cdots U_1)_{n,n} &= -\begin{bmatrix} B^{(1)}_{n+b}(0) \cdots\cdots B^{(b)}_{n+b}(0) \end{bmatrix} \begin{bmatrix} B^{(1)}_n(0) \cdots\cdots\cdots B^{(b)}_n(0) \\ \vdots \qquad\qquad \vdots \\ B^{(1)}_{n+b-1}(0) \cdots\cdots B^{(b)}_{n+b-1}(0) \end{bmatrix}^{-1} \begin{bmatrix} 1 \\ 0 \\ \vdots \\ 0 \end{bmatrix} \\ &= (-1)^b \frac{\tau^B_{b,n+1}}{\tau^B_{b,n}},\end{aligned}$$

for $b \in \{1, \dots, q\}, n \in \{0, 1, \dots, N-b-1\}$. Therefore, we finally obtain the stated formula.

We now compute the L_m. Using (19) we deduce that

$$
\begin{aligned}
A^{[N]}(x)L_1\left[\begin{array}{c|c} I_N & 0_{N\times(p-1)}\end{array}\right] &= A^{[N]}(x)\mathscr{U}_N^{-1}\left[\begin{array}{c|c} I_N & \mathscr{U}_N U^{[N,N+p]}\end{array}\right]\left(\Lambda^{[N+p-1,N+p]}\right)^\top \mathscr{U}_{N+p-1,(0,1)}\\
&= \left(X^{[N]}_{[p]}(x)\right)^\top\left[\begin{array}{c|c} I_N & \mathscr{U}_N U^{[N,N+p]}\end{array}\right]\left(\Lambda^{[N+p-1,N+p]}\right)^\top \mathscr{U}_{N+p-1,(0,1)}\\
&= \left[\begin{array}{c|c} \left(X^{[N]}_{[p]}(x)\right)^\top & A^{[N]}(x)U^{[N,N+p]}\end{array}\right]\left(\Lambda^{[N+p-1,N+p]}\right)^\top \mathscr{U}_{N+p-1,(0,1)}\\
&= \left[\begin{array}{c|c} \left(\mathfrak{X}_{[p,1]}(x)\right)^\top\left(X^{[N-1]}_{[p]}(x)\right)^\top & A^{[N]}(x)U^{[N,N+p]}\end{array}\right]\mathscr{U}_{N+p-1,(0,1)}\\
&= \left[\begin{array}{c|c} \left(\mathfrak{X}_{[p,1]}(x)\right)^\top\left(X^{[N-1]}_{[p]}(x)\right)^\top & A^{[N]}(x)U^{[N,N+p]}\end{array}\right]\\
&\qquad\times\begin{bmatrix} \mathscr{U}_{N-1,(0,1)} & \mathcal{U}^{[N+p-1,N-1]}_{(0,1)}\\ 0_{(p-1)\times(N-1)} & \mathscr{U}_{[N+p-1,N-1],(1,1,0)}\end{bmatrix}
\end{aligned}
$$

Hence, if we denote by $L_1^{[N,N-1]}$ the truncated matrix obtained from L_1 by erasing the last column, we get

$$
A^{[N]}(x)L_1^{[N,N-1]} = \left(\mathfrak{X}_{[p,1]}(x)\right)^\top A^{[N-1]}_{(0,1)}(x).
$$

Taking into account that for the perturbation polynomial its det $\left(\mathfrak{X}_{[p]}(x)\right)^\top$ has only a root that is located at $x=0$, and its left eigenvector is $e_1^\top\in(\mathbb{R}^p)^*$, that has all its components equal to zero but for the first which equals one, we have

$$
e_1^\top A^{[N]}(0)L_1^{[N,N-1]} = e_1^\top\left(\mathfrak{X}_{[p]}(x)\right)^\top A^{[N-1]}_{(0,1)}(x) = 0_{1\times(N-1)}.
$$

and, consequently we obtain

$$
\begin{bmatrix} A_0^{(1)}(0) & A_1^{(1)}(0)\cdots\cdots A_{N-1}^{(1)}(0)\end{bmatrix}L_1^{[N,N-1]} = 0_{1\times(N-1)},
$$

that leads to

$$
L_{1,n+1} = -\frac{A_n^{(1)}(0)}{A_{n+1}^{(1)}(0)},\quad n\in\{0,\dots,N-2\}.
$$

For the next bidiagonal matrix L_2 we use

$$
\begin{aligned}
&A^{[N]}(x)L_1L_2\left[\begin{array}{c|c} I_N & 0_{N\times(p-2)}\end{array}\right]\\
&\quad= A^{[N]}(x)\mathscr{U}_N^{-1}\left[\begin{array}{c|c} I_N & \mathscr{U}_N U^{[N,N+p-1]}\end{array}\right]\left(\Lambda^{[N+p-1,N+p]}\right)^\top\left(\Lambda^{[N+p-2,N+p-1]}\right)^\top \mathscr{U}_{N+p-2,(0,2)}
\end{aligned}
$$

that implies

$$
A^{[N]}(x)(L_1L_2)^{[N,N-2]} = \left(\mathfrak{X}^2_{[p,1]}(x)\right)^\top A^{[N-2]}_{(0,2)}(x).
$$

Consequently,

$$
\begin{bmatrix} A_0^{(1)}(0) & A_1^{(1)}(0)\cdots\cdots A_{N-1}^{(1)}\\ A_0^{(2)}(0) & A_1^{(2)}(0)\cdots\cdots A_{N-1}^{(2)}\end{bmatrix}(L_1L_2)^{[N,N-2]} = 0_{2\times(N-2)}.
$$

Hence, for $n\in\{0,1,\dots,N-3\}$ we get

$$
\begin{aligned}
A_{n+2}^{(1)}(0)(L_1L_2)_{n+2,n} + A_{n+1}^{(1)}(0)(L_1L_2)_{n+1,n} + A_n^{(1)}(0) = 0,\\
A_{n+2}^{(2)}(0)(L_1L_2)_{n+2,n} + A_{n+1}^{(2)}(0)(L_1L_2)_{n+1,n} + A_n^{(2)}(0) = 0,
\end{aligned}
$$

or

$$\begin{bmatrix}(L_1L_2)_{n+2,n}\\(L_1L_2)_{n+1,n}\end{bmatrix}=-\begin{bmatrix}A^{(1)}_{n+2}(0) & A^{(1)}_{n+1}(0)\\A^{(2)}_{n+2}(0) & A^{(2)}_{n+1}(0)\end{bmatrix}^{-1}\begin{bmatrix}A^{(1)}_{n}(0)\\A^{(2)}_{n}(0)\end{bmatrix}.$$

The solution is

$$(L_1L_2)_{n+2,n}=-\begin{bmatrix}1 & 0\end{bmatrix}\begin{bmatrix}A^{(1)}_{n+2}(0) & A^{(1)}_{n+1}(0)\\A^{(2)}_{n+2}(0) & A^{(2)}_{n+1}(0)\end{bmatrix}^{-1}\begin{bmatrix}A^{(1)}_{n}(0)\\A^{(2)}_{n}(0)\end{bmatrix}=\frac{\begin{vmatrix}A^{(1)}_{n+1}(0) & A^{(1)}_{n}(0)\\A^{(2)}_{n+1}(0) & A^{(2)}_{n}(0)\end{vmatrix}}{\begin{vmatrix}A^{(1)}_{n+2}(0) & A^{(1)}_{n+1}(0)\\A^{(2)}_{n+2}(0) & A^{(2)}_{n+1}(0)\end{vmatrix}},$$

$$(L_1L_2)_{n+1,n}=-\begin{bmatrix}0 & 1\end{bmatrix}\begin{bmatrix}A^{(1)}_{n+2}(0) & A^{(1)}_{n+1}(0)\\A^{(2)}_{n+2}(0) & A^{(2)}_{n+1}(0)\end{bmatrix}^{-1}\begin{bmatrix}A^{(1)}_{n}(0)\\A^{(2)}_{n}(0)\end{bmatrix}=-\frac{\begin{vmatrix}A^{(1)}_{n+2}(0) & A^{(1)}_{n}(0)\\A^{(2)}_{n+2}(0) & A^{(2)}_{n}(0)\end{vmatrix}}{\begin{vmatrix}A^{(1)}_{n+2}(0) & A^{(1)}_{n+1}(0)\\A^{(2)}_{n+2}(0) & A^{(2)}_{n+1}(0)\end{vmatrix}}.$$

As

$$(L_1L_2)_{n+2,n}=L_{1,n+2}L_{2,n+1}$$

we obtain that

$$L_{2,n+1}=-\frac{A^{(1)}_{n+2}(0)}{A^{(1)}_{n+1}(0)}\frac{\begin{vmatrix}A^{(1)}_{n+1}(0) & A^{(1)}_{n}(0)\\A^{(2)}_{n+1}(0) & A^{(2)}_{n}(0)\end{vmatrix}}{\begin{vmatrix}A^{(1)}_{n+2}(0) & A^{(1)}_{n+1}(0)\\A^{(2)}_{n+2}(0) & A^{(2)}_{n+1}(0)\end{vmatrix}},\quad n\in\{0,\dots,N-3\}.$$

In general, it holds that

$$A^{[N]}(x)(L_1\cdots L_a)^{[N,N-a]}=\left(\mathfrak{X}^{a}_{[p,1]}(x)\right)^{\top}A^{[N-a]}_{(a,0)}(x),$$

so that

$$\begin{bmatrix}A^{(1)}_{0}(0) & A^{(1)}_{1}(0)\cdots\cdots A^{(1)}_{N-1}(0)\\ \vdots & \vdots \qquad\qquad \vdots\\ A^{(a)}_{0}(0) & A^{(a)}_{1}(0)\cdots\cdots A^{(a)}_{N-1}(0)\end{bmatrix}(L_1\cdots L_a)^{[N,N-a]}=0_{a\times(N-a)}.$$

that can be rewritten as

$$\begin{aligned}(L_1\cdots L_j)_{n+a,n}&=-\begin{bmatrix}1 & 0\cdots\cdots 0\end{bmatrix}\begin{bmatrix}A^{(1)}_{n+a-1}(0)\cdots\cdots A^{(1)}_{n}(0)\\ \vdots \qquad\qquad \vdots\\ A^{(a)}_{n+a-1}(0)\cdots\cdots A^{(a)}_{n}(0)\end{bmatrix}^{-1}\begin{bmatrix}A^{(1)}_{n}(0)\\ \vdots\\ A^{(p)}_{n}(0)\end{bmatrix}\\&=(-1)^a\frac{\tau^A_{a,n}}{\tau^A_{a,n+1}},\end{aligned}$$

for $a \in \{1,\dots,q\}, n \in \{0,1,\dots,N-a-1\}$. Note that

$$(L_1 \dots L_a)_{n+a,n} = (L_1 \dots L_{a-1})_{n+a,n+1} L_{a,n+1}$$

and we obtain the desired result. □

Remark 5.6. *An expression for $U_{1,N-1}$ is missing in this formula. However, we can get an expression for it. In the one hand, the N-th entry of $U_1 B^{[}N](0)e_1$ is $U_{1,N-1}B^{(1)}_{N-1}(0)$. In the other hand, this last entry must be the first entry of*

$$\Big(\ell^{[N-1+q,N-1]}_{(1,0)} X^{[N-1]}(x)\mathfrak{X}_{[q,1]}(0) + \mathscr{L}_{[N-1,N-1+q],(1,0)} L^{[N+q,N]} B^{[N]}(0)\Big)\, e_1,$$

and as $\mathfrak{X}_{[q,1]}(0)e_1$ is the zero-vector we get that we are only interested in the first entry

$$e_1^\top \mathscr{L}_{[N-1,N-1+q],(1,0)} L^{[N+q,N]} B^{[N]}(0)e_1.$$

Hence, as $\mathscr{L}_{[N-1,N-1+q],(1,0)}$ is a lower unitriangular, it does not intervene in the first entry , so we finally obtain

$$U_{1,N-1} = \frac{1}{B^{(1)}_{N-1}(0)} e_1^\top \int x^{N_q} \mathfrak{X}_{[q,s_q]}(x)\,\mathrm{d}\,\mu(x) A^{[N]}(x) \begin{bmatrix} B^{(1)}_0(0) \\ B^{(1)}_2(0) \\ \vdots \\ B^{(1)}_{N-1}(0) \end{bmatrix},$$

were (5) has been used.

Remark 5.7. *From*

$$(U_2U_1)_{n,n+1} = U_{2,n} + U_{1,n+1} = -\frac{\begin{vmatrix} B^{(1)}_n(0) & B^{(2)}_n(0) \\ B^{(1)}_{n+2}(0) & B^{(2)}_{n+2}(0) \end{vmatrix}}{\begin{vmatrix} B^{(1)}_n(0) & B^{(2)}_n(0) \\ B^{(1)}_{n+1}(0) & B^{(2)}_{n+1}(0) \end{vmatrix}},$$

we derive alternative expressions

$$U_{2,n} = -\frac{\begin{vmatrix} B^{(1)}_n(0) & B^{(2)}_n(0) \\ B^{(1)}_{n+2}(0) & B^{(2)}_{n+2}(0) \end{vmatrix}}{\begin{vmatrix} B^{(1)}_n(0) & B^{(2)}_n(0) \\ B^{(1)}_{n+1}(0) & B^{(2)}_{n+1}(0) \end{vmatrix}} + \frac{B^{(1)}_{n+2}(0)}{B^{(1)}_{n+1}(0)}, \quad n \in \{0,1,\dots N-3\}.$$

6. A study case: Hahn multiple orthogonal polynomials

In this section we put $q = 1$, while p remains arbitrary. That is we are dealing with multiple orthogonality, which is the standard one and not of mixed type, Now, the step-line recursion relations read

$$xB_n(x) = B_{n+1}(x) + b^0_n B_n(x) + \sum_{j=1}^{p} b^j_n B_{n-j}(x), \qquad n \in \{0,1,\dots,N-2\},$$

$$xA^{(a)}_n(x) = A^{(a)}_{n-1}(x) + b^0_n A^{(a)}_n(x) + \sum_{j=1}^{p} b^j_{n+j} A^{(a)}_{n+j}(x), \quad n \in \{0,1,\dots,N-p-1\}, \quad a \in \{0,1,\dots,p\}$$

with $B_{-1},\dots,B_{-p}=0$ and $A^{(1)}_{-1},\dots,A^{(p)}_{-1}=0$. and the recursion matrix is

$$\mathscr{T}_{N-p} := \begin{bmatrix} b_0^0 & 1 & 0 & \cdots & \cdots & \cdots & 0\\ b_1^1 & b_1^0 & 1 & \ddots & & & \vdots\\ \vdots & b_2^1 & b_2^0 & \ddots & \ddots & & \vdots\\ \vdots & & \ddots & \ddots & \ddots & \ddots & \vdots \\ b_p^p & & & \ddots & \ddots & \ddots & 0\\ 0 & \ddots & & & \ddots & \ddots & 1\\ 0 & \cdots & 0 & b_{N-p-1}^p & \cdots & b_{N-p-1}^1 & b_{N-p-1}^0 \end{bmatrix}$$

Before delving into the multiple Hahn polynomials, it's important to recall that these polynomials can be represented using generalized hypergeometric series [3, 21],

$$ {}_pF_q\left[\begin{matrix} a_1,\dots,a_p\\ \alpha_1,\dots,\alpha_q\end{matrix};x\right] := \sum_{l=0}^{\infty}\frac{(a_1)_l\cdots(a_p)_l}{(\alpha_1)_l\cdots(\alpha_q)_l}\frac{x^l}{l!}, \tag{24}$$

with the Pochhammer symbols given by

$$(x)_n := \frac{\Gamma(x+n)}{\Gamma(x)} = \begin{cases} x(x+1)\cdots(x+n-1) \text{ if } n\in\mathbb{N},\\ 1 \text{ if } n=0.\end{cases}$$

The weight functions for the Hahn family are defined as

$$w_i(x;\alpha_i,\beta,N) = \frac{\Gamma(\alpha_i+x+1)}{\Gamma(\alpha_i+1)\Gamma(x+1)}\frac{\Gamma(\beta+N-x+1)}{\Gamma(\beta+1)\Gamma(N-x+1)},\quad i\in\{1,\dots,p\},\quad \Delta=\{0,\dots,N\}, \tag{25}$$

with $\alpha_1,\dots,\alpha_p,\beta>-1$. We need a vector on nonnegative integers indexes $\vec{n}=(n_1,\dots,n_p)$, with $n_i\in\mathbb{N}_0$ In order to have an AT system, we require that $|\vec{n}|\leqslant N\in\mathbb{N}_0$ and $\alpha_i-\alpha_j\notin\mathbb{Z}$ for $i\neq j$. This ensures the existence of the orthogonal polynomial sup to $|\vec{n}|=N$. We will denote $\vec{\alpha}:=(\alpha_1,\dots,\alpha_p)$.

The corresponding type II polynomials were first found for $p=2$ in [4] and then generalized for $p\geqslant 2$ in [5]. In [15], the following alternative representation was proven

$$\begin{aligned} B_{\vec{n}}(x;\vec{\alpha},\beta,N) = (-1)^{|\vec{n}|}\frac{\Gamma(N+\beta+1)}{(N-|\vec{n}|)!}\prod_{i=1}^{p}\frac{(\alpha_i+1)_{n_i}}{(\alpha_i+\beta+|\vec{n}|+1)_{n_i}}\frac{\Gamma(N-x+1)}{\Gamma(\beta+N-x+1)}\\ \times {}_{p+2}F_{p+1}\left[\begin{matrix}-|\vec{n}|-\beta,-x,\vec{\alpha}+\vec{n}+\vec{1}_p\\ -N-\beta,\vec{\alpha}+\vec{1}_p\end{matrix};1\right]. \end{aligned}\tag{26}$$

For $p=2$, the first explicit hypergeometric representation of the Hahn polynomials of type I were given in [9]. Then, the case for $p\geqslant 2$ was presented in in [15] and reads as follows:

$$\begin{aligned} A^{(a)}_{\vec{n}}(x;\vec{\alpha},\beta,N) = \frac{(-1)^{|\vec{n}|-1}(N+1-|\vec{n}|)!}{(n_a-1)!(\beta+1)_{|\vec{n}|-1}(\alpha_a+\beta+|\vec{n}|)_{N+2-|\vec{n}|}}\frac{\prod_{k=1}^{p}(\alpha_k+\beta+|\vec{n}|)_{n_k}}{\prod_{k=1,k\neq a}^{p}(\alpha_k-\alpha_a)_{n_k}}\\ \times {}_{p+2}F_{p+1}\left[\begin{matrix}-n_a+1,\alpha_a+\beta+|\vec{n}|,(\alpha_a+1)\vec{1}_{p-1}-\vec{\alpha}^{*a}-\vec{n}^{*a},x+\alpha_a+1\\ \alpha_a+1,(\alpha_a+1)\vec{1}_{p-1}-\vec{\alpha}^{*a},\alpha_a+\beta+N+2\end{matrix};1\right]. \end{aligned}\tag{27}$$

Remark 6.1. *Notice that we used the following notation. We denoted the unit vector by* $\vec{1}_p := (1,\dots,1) \in \mathbb{R}^p$. *Given a vector* $\vec{v} \in \mathbb{R}^p$, *we wrote* $\vec{v}^{*q} \in \mathbb{R}^{p-1}$ *for the vector obtained from* $\vec{v}$ *after removing its* q*-th entry.*

The evaluations at $x = 0$ give

$$B_{\vec{n}}(0) = (-1)^{|\vec{n}|}\frac{N!}{(N-|\vec{n}|)!}\prod_{i=1}^{p}\frac{(\alpha_i+1)_{n_i}}{(\alpha_i+\beta+|\vec{n}|+1)_{n_i}} \tag{28}$$

and

$$A^{(a)}_{\vec{n}}(0) = \frac{(-1)^{|\vec{n}|-1}(N+1-|\vec{n}|)!}{(n_a-1)!(\beta+1)_{|\vec{n}|-1}(\alpha_a+\beta+|\vec{n}|)_{N+2-|\vec{n}|}}\frac{\prod_{k=1}^{p}(\alpha_k+\beta+|\vec{n}|)_{n_k}}{\prod_{k=1,k\neq a}^{p}(\alpha_k-\alpha_a)_{n_k}} \tag{29}$$
$$\times\, {}_{p+1}F_p\left[\begin{matrix}-n_a+1, \alpha_a+\beta+|\vec{n}|, (\alpha_a+1)\vec{1}_{p-1}-\vec{\alpha}^{*a}-\vec{n}^{*a}\\ (\alpha_a+1)\vec{1}_{p-1}-\vec{\alpha}^{*a}, \alpha_a+\beta+N+2\end{matrix};1\right].$$

The indices considered are general. The step-line corresponds to considering multi-index of the form

$$\vec{n} = (\underbrace{m+1,\dots,m+1}_{s\text{ times}},\underbrace{m,\dots,m}_{p-s\text{ times}}), \quad m \in \mathbb{N}_0, \quad s \in \{0,\dots,p-1\}$$

This way, these multi-index can be ordered in a sequence such that the modulus

$$|\vec{n}| = pm + s$$

increase by 1 as follows

$$\{(0,0,\dots,0),(1,0,\dots,0),(1,1,\dots,0),\dots,(1,1,\dots,1),(2,1,\dots,1),\dots\}$$

This is what is known as the step-line and allows to relabel the multiple polynomials

$$B_{pm+s} := B_{(m+1,\dots,m+1,m,\dots,m)}$$
$$A^{(a)}_{pm+s-1} := A^{(a)}_{(m+1,\dots,m+1,m,\dots,m)}, \quad a \in \{1,\dots,p\}$$

For the recurrence coefficients in the step-line we introduce the following notation. Consider $\alpha_1,\dots,\alpha_p \in \mathbb{R}$ then

$$\alpha_{i+pn} := \alpha_i + n, \quad i \in \{1,\dots,r\}, \quad n \in \mathbb{N}_0 \tag{30}$$

For example

$$\alpha_{p+1} := \alpha_1+1, \quad \alpha_{p+2} := \alpha_2+1, \quad \dots, \quad \alpha_{2p} := \alpha_p+1, \quad \alpha_{2p+1} := \alpha_1+2, \dots$$

This cyclic notation is useful to get an compact expression for the recurrence coefficients in the step-line

$$b^{j}_{pm+k} = -(\alpha_{k+1}+m+1)\delta_{j,0}+\frac{(N-pm-k+1)_j(\beta+pm+k+1-j)_j(\alpha_{k+1}+\beta+(p+1)m+k+1-j)}{\prod_{l=p+k+2-j}^{p+k+1}(\alpha_l+\beta+(p+1)m+k-j)}$$
$$\times\frac{\prod_{l=1}^{p}(\alpha_l+\beta+pm+k+1-j)_j}{\prod_{l=k+1}^{p+k}(\alpha_l+\beta+(p+1)m+k+1-j)_j}\sum_{i=k+1}^{p+k+1-j}\frac{(\alpha_i+m)(\alpha_i+\beta+N+m+1)}{(\alpha_i+\beta+(p+1)m+k-j)_{j+2}}\frac{\prod_{l=1}^{p}(\alpha_i-\alpha_l+m)}{\prod_{l=k+1,l\neq i}^{p+k+1-j}(\alpha_i-\alpha_l)}$$

for $j \in \{0,1,\dots,p\}$, $m \geqslant 0$ and $k \in \{0,\dots,p-1\}$. This appears for the first time in the PhD thesis [?]. The symbol $\prod_{l=p+k+2-j}^{p+k+1}$ for $j \geq 1$ is the standard product and for $j = 0$ is 1.

6.1. **Two weights**. Let us additionally assume that $p=2$. Therefore our matrices have one superdiagonal and two subdiagonals, and we are looking for a factorization of the form $\mathscr{T}_{N-2}=L_1L_2U_1$.

Theorem 6.2. *The Hahn multiple orthogonal polynomials with two weights we have the factorization*

$$\mathscr{T}_{N-2}=L_1L_2U_1$$

in terms of two lower bidiagonal matrices L_1 and L_2 and an upper unitriangular matrix U_1. The corresponding entries of these matrices are

$$\begin{aligned}
U_{1,2n}&=\frac{(N-2n)(\alpha_1+n+1)(\alpha_1+\beta+2n+1)(\alpha_2+\beta+2n+1)}{(\alpha_1+\beta+3n+1)_2(\alpha_2+\beta+3n+1)},\\
U_{1,2n+1}&=\frac{(N-2n-1)(\alpha_2+n+1)(\alpha_1+\beta+2n+2)(\alpha_2+\beta+2n+2)}{(\alpha_1+\beta+3n+3)(\alpha_2+\beta+3n+2)_2},\\
L_{1,2n}&=\frac{(N+1-2n)n(\beta+2n)(\alpha_2+\beta+2n)}{(\alpha_1+\beta+3n)_2(\alpha_2+\beta+3n)}\frac{{}_3F_2\left[{-n+1,\alpha_1+\beta+2n,\alpha_1-\alpha_2-n+1 \atop \alpha_1-\alpha_2+1,\alpha_1+\beta+N+2};1\right]}{{}_3F_2\left[{-n,\alpha_1+\beta+2n+1,\alpha_1-\alpha_2-n+1 \atop \alpha_1-\alpha_2+1,\alpha_1+\beta+N+2};1\right]},\\
L_{1,2n+1}&=\frac{(N-2n)(\beta+2n+1)(\alpha_2-\alpha_1+n)(\alpha_2+\beta+2n+1)}{(\alpha_1+\beta+3n+2)(\alpha_2+\beta+3n+1)_2}\frac{{}_3F_2\left[{-n,\alpha_1+\beta+2n+1,\alpha_1-\alpha_2-n+1 \atop \alpha_1-\alpha_2+1,\alpha_1+\beta+N+2};1\right]}{{}_3F_2\left[{-n,\alpha_1+\beta+2n+2,\alpha_1-\alpha_2-n \atop \alpha_1-\alpha_2+1,\alpha_1+\beta+N+2};1\right]},\\
L_{2,2n}&=\frac{n(\beta+2n)(\alpha_1+\beta+2n+1)(\alpha_2+\beta+N+n+1)}{(\alpha_1+\beta+3n+1)(\alpha_2+\beta+3n)_2}\frac{{}_3F_2\left[{-n,\alpha_1+\beta+2n+2,\alpha_1-\alpha_2-n \atop \alpha_1-\alpha_2+1,\alpha_1+\beta+N+2};1\right]}{{}_3F_2\left[{-n,\alpha_1+\beta+2n+1,\alpha_1-\alpha_2-n+1 \atop \alpha_1-\alpha_2+1,\alpha_1+\beta+N+2};1\right]},\\
L_{2,2n+1}&=\frac{(\beta+2n+1)(\alpha_1+\beta+2n+2)(\alpha_1+\beta+N+n+2)(\alpha_1-\alpha_2+n+1)}{(\alpha_1+\beta+3n+2)_2(\alpha_2+\beta+3n+2)}\frac{{}_3F_2\left[{-n-1,\alpha_1+\beta+2n+3,\alpha_1-\alpha_2-n \atop \alpha_1-\alpha_2+1,\alpha_1+\beta+N+2};1\right]}{{}_3F_2\left[{-n,\alpha_1+\beta+2n+2,\alpha_1-\alpha_2-n \atop \alpha_1-\alpha_2+1,\alpha_1+\beta+N+2};1\right]}.
\end{aligned}$$

Proof. We first note that

$$B_{2m}=B_{(m,m)},\quad B_{2m+1}=B_{(m+1,m)}\quad A^{(1)}_{2m-1}=A^{(1)}_{(m,m)},\quad A^{(1)}_{2m}=A^{(1)}_{(m+1,m)},\quad A^{(2)}_{2m-1}=A^{(a)}_{(m,m)},\quad A^{(2)}_{2m}=A^{(2)}_{(m+1,m)}.$$

From (28) we get

$$B_{2n}(0)=\frac{N!(\alpha_1+1)_n(\alpha_2+1)_n}{(N-2n)!(\alpha_1+\beta+2n+1)_n(\alpha_2+\beta+2n+1)_n},\quad B_{2n+1}(0)=-\frac{N!(\alpha_1+1)_{n+1}(\alpha_2+1)_n}{(N-2n-1)!(\alpha_1+\beta+2n+2)_{n+1}(\alpha_2+\beta+2n+2)_n}$$

and

$$\begin{aligned}
A^{(1)}_{2n-1}(0)&=\frac{-(N+1-2n)!\alpha_1+\beta+2n)_n(\alpha_2+\beta+2n)_n}{(n-1)!(\beta+1)_{2n-1}(\alpha_1+\beta+2n)_{N+2-2n}(\alpha_2-\alpha_1)_n}{}_3F_2\left[{-n+1,\alpha_1+\beta+2n,\alpha_1-\alpha_2-n+1 \atop \alpha_1-\alpha_2+1,\alpha_1+\beta+N+2};1\right],\\
A^{(2)}_{2n-1}(0)&=\frac{-(N+1-2n)!\alpha_1+\beta+2n)_n(\alpha_2+\beta+2n)_n}{(n-1)!(\beta+1)_{2n-1}(\alpha_2+\beta+2n)_{N+2-2n}(\alpha_1-\alpha_2)_n}{}_3F_2\left[{-n+1,\alpha_2+\beta+2n,\alpha_2-\alpha_1-n+1 \atop \alpha_2-\alpha_1+1,\alpha_2+\beta+N+2};1\right],\\
A^{(1)}_{2n+1}(0)&=\frac{-(N-1-2n)!(\alpha_1+\beta+2n+2)_{n+1}(\alpha_2+\beta+2n+2)_{n+1}}{(n)!(\beta+1)_{2n+1}(\alpha_1+\beta+2n+2)_{N-2n}(\alpha_2-\alpha_1)_{n+1}}{}_3F_2\left[{-n,\alpha_1+\beta+2n+2,\alpha_1-\alpha_2-n \atop \alpha_1-\alpha_2+1,\alpha_1+\beta+N+2};1\right],\\
A^{(2)}_{2n+1}(0)&=\frac{-(N-1-2n)!(\alpha_1+\beta+2n+2)_{n+1}(\alpha_2+\beta+2n+2)_{n+1}}{(n)!(\beta+1)_{2n+1}(\alpha_2+\beta+2n+2)_{N-2n}(\alpha_1-\alpha_2)_{n+1}}{}_3F_2\left[{-n,\alpha_2+\beta+2n+2,\alpha_2-\alpha_1-n \atop \alpha_2-\alpha_1+1,\alpha_2+\beta+N+2};1\right],\\
A^{(1)}_{2n}(0)&=\frac{(N-2n)!(\alpha_1+\beta+2n+1)_{n+1}(\alpha_2+\beta+2n+1)_n}{n!(\beta+1)_{2n}(\alpha_1+\beta+2n+1)_{N+1-2n}(\alpha_2-\alpha_1)_n}{}_3F_2\left[{-n,\alpha_1+\beta+2n+1,\alpha_1-\alpha_2-n+1 \atop \alpha_1-\alpha_2+1,\alpha_1+\beta+N+2};1\right],\\
A^{(2)}_{2n}(0)&=\frac{(N-2n)!(\alpha_1+\beta+2n+1)_{n+1}(\alpha_2+\beta+2n+1)_n}{(n-1)!(\beta+1)_{2n}(\alpha_2+\beta+2n+1)_{N+1-2n}(\alpha_1-\alpha_2)_{n+1}}{}_3F_2\left[{-n+1,\alpha_2+\beta+2n+1,\alpha_2-\alpha_1-n \atop \alpha_2-\alpha_1+1,\alpha_2+\beta+N+2};1\right],\\
A^{(1)}_{2n+2}(0)&=\frac{(N-2n-2)!(\alpha_1+\beta+2n+3)_{n+2}(\alpha_2+\beta+2n+3)_{n+1}}{(n+1)!(\beta+1)_{2n+2}(\alpha_1+\beta+2n+3)_{N-1-2n}(\alpha_2-\alpha_1)_{n+1}}{}_3F_2\left[{-n-1,\alpha_1+\beta+2n+3,\alpha_1-\alpha_2-n \atop \alpha_1-\alpha_2+1,\alpha_1+\beta+N+2};1\right],\\
A^{(2)}_{2n+2}(0)&=\frac{(N-2n-2)!(\alpha_1+\beta+2n+3)_{n+2}(\alpha_2+\beta+2n+3)_{n+1}}{n!(\beta+1)_{2n+2}(\alpha_2+\beta+2n+3)_{N-1-2n}(\alpha_1-\alpha_2)_{n+2}}{}_3F_2\left[{-n,\alpha_2+\beta+2n+3,\alpha_2-\alpha_1-n-1 \atop \alpha_2-\alpha_1+1,\alpha_2+\beta+N+2};1\right].
\end{aligned}$$

For for $j\in\{0,1,2\}$, the recurrence coefficients are

$$
\begin{aligned}
b^j_{2m}= &-(\alpha_1+m+1)\delta_{j,0}+\frac{(N-2m+1)_j(\beta+2m+1-j)_j(\alpha_1+\beta+3m+1-j)}{\prod_{l=4-j}^3(\alpha_l+\beta+3m-j)}\\
&\times\frac{\prod_{l=1}^2(\alpha_l+\beta+2m+1-j)_j}{\prod_{l=1}^2(\alpha_l+\beta+3m+1-j)_j}\sum_{i=1}^{3-j}\frac{(\alpha_i+m)(\alpha_i+\beta+N+m+1)}{(\alpha_i+\beta+3m-j)_{j+2}}\frac{\prod_{l=1}^2(\alpha_i-\alpha_l+m)}{\prod_{l=1,l\neq i}^{3-j}(\alpha_i-\alpha_l)}\\
b^j_{2m+1}= &-(\alpha_2+m+1)\delta_{j,0}+\frac{(N-2m)_j(\beta+2m+2-j)_j(\alpha_2+\beta+3m+2-j)}{\prod_{l=5-j}^4(\alpha_l+\beta+3m+1-j)}\\
&\times\frac{\prod_{l=1}^2(\alpha_l+\beta+2m+2-j)_j}{\prod_{l=2}^3(\alpha_l+\beta+3m+2-j)_j}\sum_{i=2}^{4-j}\frac{(\alpha_i+m)(\alpha_i+\beta+N+m+1)}{(\alpha_i+\beta+3m+1-j)_{j+2}}\frac{\prod_{l=1}^2(\alpha_i-\alpha_l+m)}{\prod_{l=2,l\neq i}^{4-j}(\alpha_i-\alpha_l)}.
\end{aligned}
$$

The corresponding τ-determinants are given by

$$\tau^B_{1,n}=B_n(0),\quad n\in\{0,1,\dots,N-2\},\quad \tau^A_{1,n}=A^{(1)}_n(0),\quad n\in\{0,1,\dots,N-3\},$$

and

$$\tau^A_{2,n}=\begin{vmatrix}A^{(1)}_{n+1}(0)&A^{(1)}_n(0)\\A^{(2)}_{n+1}(0)&A^{(2)}_n(0)\end{vmatrix},\quad n\in\{0,1,\dots,N-3\},\quad (-1)^{(n+1)}b^2_2\cdots b^2_{n+1}\tau^A_{2,n}=B_n(0).$$

Thus, following Theorem 5.5, for $n\in\{0,1,\dots,N-2\}$, we get

$$U_{1,n}=-\frac{B_{n+1}(0)}{B_n(0)},\quad L_{1,n+1}=-\frac{A^{(1)}_n(0)}{A^{(1)}_{n+1}(0)},$$

and

$$L_{2,n+1}=-\frac{A^{(1)}_{n+2}(0)}{A^{(1)}_{n+1}(0)}\frac{\begin{vmatrix}A^{(1)}_{n+1}(0)&A^{(1)}_n(0)\\A^{(2)}_{n+1}(0)&A^{(2)}_n(0)\end{vmatrix}}{\begin{vmatrix}A^{(1)}_{n+2}(0)&A^{(1)}_{n+1}(0)\\A^{(2)}_{n+2}(0)&A^{(2)}_{n+1}(0)\end{vmatrix}}=b^2_{n+2}\frac{A^{(1)}_{n+2}(0)}{A^{(1)}_{n+1}(0)}\frac{B_n(0)}{B_{n+1}(0)},\quad n\in\{0,1,\dots,N-3\}.$$

Therefore,

$$
\begin{aligned}
U_{1,2n}&=-\frac{-\frac{N!}{(N-2n-1)!}\frac{(\alpha_1+1)_{n+1}(\alpha_2+1)_n}{(\alpha_1+\beta+2n+2)_{n+1}(\alpha_2+\beta+2n+2)_n}}{\frac{N!}{(N-2n)!}\frac{(\alpha_1+1)_n(\alpha_2+1)_n}{(\alpha_1+\beta+2n+1)_n(\alpha_2+\beta+2n+1)_n}}=\frac{(N-2n)(\alpha_1+n+1)(\alpha_1+\beta+2n+1)(\alpha_2+\beta+2n+1)}{(\alpha_1+\beta+3n+1)(\alpha_1+\beta+3n+2)(\alpha_2+\beta+3n+1)},\\
U_{1,2n+1}&=-\frac{\frac{N!}{(N-2n-2)!}\frac{(\alpha_1+1)_{n+1}(\alpha_2+1)_{n+1}}{(\alpha_1+\beta+2n+3)_{n+1}(\alpha_2+\beta+2n+3)_{n+1}}}{-\frac{N!}{(N-2n-1)!}\frac{(\alpha_1+1)_{n+1}(\alpha_2+1)_n}{(\alpha_1+\beta+2n+2)_{n+1}(\alpha_2+\beta+2n+2)_n}}=\frac{(N-2n-1)(\alpha_2+n+1)(\alpha_1+\beta+2n+2)(\alpha_2+\beta+2n+2)}{(\alpha_1+\beta+3n+3)(\alpha_2+\beta+3n+3)(\alpha_2+\beta+3n+2)},
\end{aligned}
$$

and also, for the lower bidiagonal we find

$$L_{1,2n}=-\frac{\frac{-(N+1-2n)!}{(n-1)!(\beta+1)_{2n-1}(\alpha_1+\beta+2n)_{N+2-2n}}\frac{(\alpha_1+\beta+2n)_n(\alpha_2+\beta+2n)_n}{(\alpha_2-\alpha_1)_n}{}_3F_2\left[{-n+1,\alpha_1+\beta+2n,\alpha_1-\alpha_2-n+1 \atop \alpha_1-\alpha_2+1,\alpha_1+\beta+N+2};1\right]}{\frac{(N-2n)!}{n!(\beta+1)_{2n}(\alpha_1+\beta+2n+1)_{N+1-2n}}\frac{(\alpha_1+\beta+2n+1)_{n+1}(\alpha_2+\beta+2n+1)_n}{(\alpha_2-\alpha_1)_n}{}_3F_2\left[{-n,\alpha_1+\beta+2n+1,\alpha_1-\alpha_2-n+1 \atop \alpha_1-\alpha_2+1,\alpha_1+\beta+N+2};1\right]}$$

$$=\frac{(N+1-2n)n(\beta+2n)(\alpha_2+\beta+2n)}{(\alpha_1+\beta+3n)(\alpha_1+\beta+3n+1)(\alpha_2+\beta+3n)}\frac{{}_3F_2\left[{-n+1,\alpha_1+\beta+2n,\alpha_1-\alpha_2-n+1 \atop \alpha_1-\alpha_2+1,\alpha_1+\beta+N+2};1\right]}{{}_3F_2\left[{-n,\alpha_1+\beta+2n+1,\alpha_1-\alpha_2-n+1 \atop \alpha_1-\alpha_2+1,\alpha_1+\beta+N+2};1\right]},$$

$$L_{1,2n+1}=-\frac{\frac{(N-2n)!}{n!(\beta+1)_{2n}(\alpha_1+\beta+2n+1)_{N+1-2n}}\frac{(\alpha_1+\beta+2n+1)_{n+1}(\alpha_2+\beta+2n+1)_n}{(\alpha_2-\alpha_1)_n}{}_3F_2\left[{-n,\alpha_1+\beta+2n+1,\alpha_1-\alpha_2-n+1 \atop \alpha_1-\alpha_2+1,\alpha_1+\beta+N+2};1\right]}{\frac{-(N-2n-1)!}{n!(\beta+1)_{2n+1}(\alpha_1+\beta+2n+2)_{N-2n}}\frac{(\alpha_1+\beta+2n+2)_{n+1}(\alpha_2+\beta+2n+2)_{n+1}}{(\alpha_2-\alpha_1)_{n+1}}{}_3F_2\left[{-n,\alpha_1+\beta+2n+2,\alpha_1-\alpha_2-n \atop \alpha_1-\alpha_2+1,\alpha_1+\beta+N+2};1\right]}$$

$$=\frac{(N-2n)(\beta+2n+1)(\alpha_2-\alpha_1+n)(\alpha_2+\beta+2n+1)}{(\alpha_1+\beta+3n+2)(\alpha_2+\beta+3n+1)(\alpha_2+\beta+3n+2)}\frac{{}_3F_2\left[{-n,\alpha_1+\beta+2n+1,\alpha_1-\alpha_2-n+1 \atop \alpha_1-\alpha_2+1,\alpha_1+\beta+N+2};1\right]}{{}_3F_2\left[{-n,\alpha_1+\beta+2n+2,\alpha_1-\alpha_2-n \atop \alpha_1-\alpha_2+1,\alpha_1+\beta+N+2};1\right]}$$

Let us continue by considering $L_{2,2n}=\frac{b^2_{2n+1}}{L_{1,2n+1}U_{1,2n-1}}$. Substituting

$$b^2_{2n+1}=\frac{(N-2n)_2(\beta+2n)_2(\alpha_2+\beta+3n)}{\prod_{l=3}^4(\alpha_l+\beta+3n-1)}\frac{\prod_{l=1}^2(\alpha_l+\beta+2n)_2(\alpha_2-\alpha_l+n)}{\prod_{l=2}^3(\alpha_l+\beta+3n)_2}\times\frac{(\alpha_2+n)(\alpha_2+\beta+N+n+1)}{(\alpha_2+\beta+3n-1)_4}.$$

where $\alpha_3=\alpha_1+1$ and $\alpha_4=\alpha_2+1$ and

$$L_{1,2n+1}=\frac{(N-2n)(\beta+2n+1)(\alpha_2-\alpha_1+n)(\alpha_2+\beta+2n+1)}{(\alpha_1+\beta+3n+2)(\alpha_2+\beta+3n+1)(\alpha_2+\beta+3n+2)}\frac{{}_3F_2\left[{-n,\alpha_1+\beta+2n+1,\alpha_1-\alpha_2-n+1 \atop \alpha_1-\alpha_2+1,\alpha_1+\beta+N+2};1\right]}{{}_3F_2\left[{-n,\alpha_1+\beta+2n+2,\alpha_1-\alpha_2-n \atop \alpha_1-\alpha_2+1,\alpha_1+\beta+N+2};1\right]},$$

$$U_{1,2n-1}=\frac{(N-2n+1)(\alpha_2+n)(\alpha_1+\beta+2n)(\alpha_2+\beta+2n)}{(\alpha_1+\beta+3n)(\alpha_2+\beta+3n)(\alpha_2+\beta+3n-1)}$$

we arrive to the identity

$$L_{2,2n}=\frac{n(\beta+2n)(\alpha_1+\beta+2n+1)(\alpha_2+\beta+N+n+1)}{(\alpha_1+\beta+3n+1)(\alpha_2+\beta+3n)_2}\frac{{}_3F_2\left[{-n,\alpha_1+\beta+2n+2,\alpha_1-\alpha_2-n \atop \alpha_1-\alpha_2+1,\alpha_1+\beta+N+2};1\right]}{{}_3F_2\left[{-n,\alpha_1+\beta+2n+1,\alpha_1-\alpha_2-n+1 \atop \alpha_1-\alpha_2+1,\alpha_1+\beta+N+2};1\right]}.$$

Let us finally discuss $L_{2,2n+1}=\frac{b^2_{2n+2}}{L_{1,2n+2}U_{1,2n}}$. With the substitutions

$$b^2_{2n+2}=\frac{(N-2n-1)_2(\beta+2n+1)_2(\alpha_1+\beta+3n+2)}{\prod_{l=2}^3(\alpha_l+\beta+3n+1)}\frac{\prod_{l=1}^2(\alpha_l+\beta+2n+1)_2}{\prod_{l=1}^2(\alpha_l+\beta+3n+2)_2}\frac{(\alpha_1+n+1)(\alpha_1+\beta+N+n+2)}{(\alpha_1+\beta+3n+1)_4}\frac{\prod_{l=1}^2(\alpha_1-\alpha_l+n+1)}{1},$$

$$L_{1,2n+2}=\frac{(N-1-2n)(n+1)(\beta+2n+2)(\alpha_2+\beta+2n+2)}{(\alpha_1+\beta+3n+3)(\alpha_1+\beta+3n+4)(\alpha_2+\beta+3n+3)}\frac{{}_3F_2\left[{-n,\alpha_1+\beta+2n+2,\alpha_1-\alpha_2-n \atop \alpha_1-\alpha_2+1,\alpha_1+\beta+N+2};1\right]}{{}_3F_2\left[{-n-1,\alpha_1+\beta+2n+3,\alpha_1-\alpha_2-n \atop \alpha_1-\alpha_2+1,\alpha_1+\beta+N+2};1\right]},$$

$$U_{1,2n}=\frac{(N-2n)(\alpha_1+n+1)(\alpha_1+\beta+2n+1)(\alpha_2+\beta+2n+1)}{(\alpha_1+\beta+3n+1)(\alpha_1+\beta+3n+2)(\alpha_2+\beta+3n+1)},$$

We obtain that

$$L_{2,2n+1}=\frac{(\beta+2n+1)(\alpha_1+\beta+2n+2)(\alpha_1+\beta+N+n+2)(\alpha_1-\alpha_2+n+1)}{(\alpha_1+\beta+3n+2)_2(\alpha_2+\beta+3n+2)}\frac{{}_3F_2\left[{-n-1,\alpha_1+\beta+2n+3,\alpha_1-\alpha_2-n \atop \alpha_1-\alpha_2+1,\alpha_1+\beta+N+2};1\right]}{{}_3F_2\left[{-n,\alpha_1+\beta+2n+2,\alpha_1-\alpha_2-n \atop \alpha_1-\alpha_2+1,\alpha_1+\beta+N+2};1\right]}.$$

□

Lemma 6.3. *The following hypergeometrical identities hold:*

$$\frac{(-1)^n(2n-2)!(\alpha_2+\beta+n+1)_{2n-1}}{(n-1)!(\alpha_1-\alpha_2-n+1)_{2n-1}}{}_3F_2\left[{-n+1,-N,\alpha_2-\alpha_1-n+1 \atop -2n+2,\alpha_2+\beta+n+1};1\right]=\frac{(\alpha_2+\beta+2n)_n}{(\alpha_1+\beta+N+1)(\alpha_2-\alpha_1)_n}{}_3F_2\left[{-n+1,\alpha_1+\beta+2n,\alpha_1-\alpha_2-n+1 \atop \alpha_1-\alpha_2+1,\alpha_1+\beta+N+2};1\right],$$

$$\frac{(-1)^n(2n-1)!(\alpha_2+\beta+n+1)_{2n}}{(n-1)!(\alpha_1-\alpha_2-n+1)_{2n}}{}_3F_2\left[{-n,-N,\alpha_2-\alpha_1-n \atop -2n+1,\alpha_2+\beta+n+1};1\right]=\frac{(\alpha_2+\beta+2n+1)_n}{(\alpha_1+\beta+N+1)(\alpha_2-\alpha_1)_n}{}_3F_2\left[{-n,\alpha_1+\beta+2n+1,\alpha_1-\alpha_2-n+1 \atop \alpha_1-\alpha_2+1,\alpha_1+\beta+N+2};1\right].$$

Proof. Using our previous results in [9] we obtain the alternative expressions for the type I Hahn multiple orthogonal polynomials at $x = 0$

$$
\begin{aligned}
A^{(1)}_{2n-1}(0) &= \frac{(-1)^{n-1}(N-2n+1)!(2n-2)!(\alpha_2+\beta+n+1)_{2n-1}}{(n-1)!(n-1)!(\beta+1)_{2n-1}(\alpha_1+\beta+3n)_{N-2n+1}(\alpha_1-\alpha_2-n+1)_{2n-1}}\,{}_3F_2\left[\begin{matrix}-n+1 & -N & \alpha_2-\alpha_1-n+1\\ & -2n+2 & \alpha_2+\beta+n+1\end{matrix};1\right],\\
A^{(1)}_{2n}(0) &= \frac{(-1)^{n}(N-2n)!(2n-1)!(\alpha_2+\beta+n+1)_{2n}}{n!(n-1)!(\beta+1)_{2n}(\alpha_1+\beta+3n+2)_{N-2n}(\alpha_1-\alpha_2-n+1)_{2n}}\,{}_3F_2\left[\begin{matrix}-n & -N & \alpha_2-\alpha_1-n\\ -2n+1 & & \alpha_2+\beta+n+1\end{matrix};1\right]
\end{aligned}
$$

Comparing these expressions with the one used previously, deduced from [15], we get the result. □

In our [13], with the notation

$$
a_{6n+1} = U_{1,2n},\quad a_{6n+4} = U_{1,2n+1},\quad a_{6n+2} = L_{1,2n},\quad a_{6n+5} = L_{1,2n+1},\quad a_{6n+3} = L_{2,2n},\quad a_{6n+6} = L_{1,2n+1},
$$

we proved that:

Theorem 6.4. *The bidiagonal factorization of the recursion matrix for the Hahn multiple orthogonal polynomials with respect two weights is:*

$$
\begin{aligned}
a_{6n+1} &= \frac{(N-2n)(\alpha_1+1+n)(\alpha_1+\beta+2n+1)(\alpha_2+\beta+2n+1)}{(\alpha_1+\beta+3n+1)_2(\alpha_2+\beta+3n+1)}\\
a_{6n+4} &= \frac{(N-2n-1)(\alpha_2+1+n)(\alpha_1+\beta+2n+2)(\alpha_2+\beta+2n+2)}{(\alpha_1+\beta+3n+3)(\alpha_2+\beta+3n+2)_2},\\
a_{6n+2} &= \frac{(N-2n)(n)_n(\beta+2n+1)(\alpha_2-\alpha_1+n)(\alpha_2+\beta+n+1)}{(n+1)_n(\alpha_1+\beta+3n+2)(\alpha_2+\beta+3n+1)_2}\,\frac{{}_3F_2\left[\begin{matrix}-n,-N,\alpha_2-\alpha_1-n\\ -2n+1,\alpha_2+\beta+n+1\end{matrix};1\right]}{{}_3F_2\left[\begin{matrix}-n,-N,\alpha_2-\alpha_1-n\\ -2n,\alpha_2+\beta+n+2\end{matrix};1\right]},\\
a_{6n+5} &= \frac{(n+1)(N-2n-1)(\beta+2n+2)(\alpha_1-\alpha_2+n+1)(\alpha_1+\beta+2+n+N)}{(2n+1)(\alpha_1+\beta+3n+3)_2(\alpha_2+\beta+3n+3)}\,\frac{{}_3F_2\left[\begin{matrix}-n,-N,\alpha_2-\alpha_1-n\\ -2n,\alpha_2+\beta+n+2\end{matrix};1\right]}{{}_3F_2\left[\begin{matrix}-n-1,-N,\alpha_2-\alpha_1-n-1\\ -2n-1,\alpha_2+\beta+n+2\end{matrix};1\right]},\\
a_{6n+3} &= \frac{(2n+1)(\beta+2n+1)(\alpha_1+\beta+2n+2)(\alpha_2+\beta+2n+2)}{(\alpha_1+\beta+3n+2)_2(\alpha_2+\beta+3n+2)}\,\frac{{}_3F_2\left[\begin{matrix}-n-1,-N,\alpha_2-\alpha_1-n-1\\ -2n-1,\alpha_2+\beta+n+2\end{matrix};1\right]}{{}_3F_2\left[\begin{matrix}-n,-N,\alpha_2-\alpha_1-n\\ -2n,\alpha_2+\beta+n+2\end{matrix};1\right]},\\
a_{6n+6} &= \frac{2(n+1)(\beta+2n+2)(\alpha_1+\beta+2n+3)(\alpha_2+\beta+2n+3)(\alpha_2+\beta+2+n+N)}{(\alpha_1+\beta+3n+4)(\alpha_2+\beta+3n+3)_2(\alpha_2+\beta+n+2)}\,\frac{{}_3F_2\left[\begin{matrix}-n-1,-N,\alpha_2-\alpha_1-n-1\\ -2n-2,\alpha_2+\beta+n+3\end{matrix};1\right]}{{}_3F_2\left[\begin{matrix}-n-1,-N,\alpha_2-\alpha_1-n-1\\ -2n-1,\alpha_2+\beta+n+2\end{matrix};1\right]}.
\end{aligned}
$$

Theorem 6.5. *The bidiagonal factorization in Theorems 6.2 and 6.4 coincide.*

Proof. Use Lemma 6.3. □

6.2. **Three weights.** Let us now assume that $p = 3$. Therefore our matrices have one superdiagonal and two subdiagonals, and we are looking for a factorization of the form $\mathscr{T}_{N-3} = L_1L_2L_3U_1$.

We first note that

$$
\begin{aligned}
&B_{3m} = B_{(m,m,m)}, && B_{3m+1} = B_{(m+1,m,m)}, && B_{3m+2} = B_{(m+1,m+1,m)},\\
&A^{(1)}_{3m-1} = A^{(1)}_{(m,m,m)}, && A^{(1)}_{3m} = A^{(1)}_{(m+1,m,m)}, && A^{(1)}_{3m+1} = A^{(1)}_{(m+1,m+1,m)},\\
&A^{(2)}_{3m-1} = A^{(2)}_{(m,m,m)}, && A^{(2)}_{3m} = A^{(2)}_{(m+1,m,m)}, && A^{(2)}_{3m+1} = A^{(2)}_{(m+1,m+1,m)},\\
&A^{(3)}_{3m-1} = A^{(3)}_{(m,m,m)}, && A^{(3)}_{3m} = A^{(3)}_{(m+1,m,m)}, && A^{(3)}_{3m+1} = A^{(3)}_{(m+1,m+1,m)},
\end{aligned}
$$

Theorem 6.6. *The Hahn multiple orthogonal polynomials with two weights we have the factorization*

$$
\mathscr{T}_{N-3} = L_1L_2L_3U_1
$$

in terms of three lower bidiagonal matrices L_1, L_2 and L_3 and an upper unitriangular matrix U_1. The corresponding entries of these matrices are

$$
\begin{aligned}
U_{1,3n} &= \frac{(N-3n)(\alpha_1+n+1)(\alpha_1+\beta+3n+1)(\alpha_2+\beta+3n+1)(\alpha_3+\beta+3n+1)}{(\alpha_1+\beta+4n+1)_2(\alpha_2+\beta+4n+1)(\alpha_3+\beta+4n+1)},\\
U_{1,3n+1} &= -\frac{(N-3n-1)(\alpha_2+n+1)\,(\alpha_1+\beta+3n+2)(\alpha_2+\beta+3n+2)(\alpha_3+\beta+3n+2)}{(\alpha_1+\beta+4n+3)(\alpha_2+\beta+4n+2)_2(\alpha_3+\beta+4n+2)},\\
U_{1,3n+2} &= \frac{(N-3n-2)(\alpha_3+n+1)\,(\alpha_1+\beta+3n+3)(\alpha_2+\beta+3n+3)(\alpha_3+\beta+3n+3)}{(\alpha_1+\beta+4n+4)(\alpha_2+\beta+4n+4)(\alpha_3+\beta+4n+3)_2},\\
L_{1,3n+1} &= \frac{(N-3n)\,(\beta+3n+1)\,(\alpha_2-\alpha_1+n)(\alpha_2+\beta+3n+1)\,(\alpha_3+\beta+3n+1)}{(\alpha_1+\beta+4n+2)\,(\alpha_2+\beta+4n+1)_2(\alpha_3+\beta+4n+1)}\frac{{}_4F_3\left[\begin{matrix}-n,\alpha_1+\beta+3n+1,\alpha_1-\alpha_2-n+1,\alpha_1-\alpha_3-n+1\\ \alpha_1-\alpha_2+1,\alpha_1-\alpha_3+1,\alpha_1+\beta+N+2\end{matrix};1\right]}{{}_4F_3\left[\begin{matrix}-n,\alpha_1+\beta+3n+2,\alpha_1-\alpha_2-n,\alpha_1-\alpha_3-n+1\\ \alpha_1-\alpha_2+1,\alpha_1-\alpha_3+1,\alpha_1+\beta+N+2\end{matrix};1\right]},\\
L_{1,3n+2} &= \frac{(N-3n-1)\,(\beta+3n+2)\,(\alpha_3-\alpha_1+n)(\alpha_2+\beta+3n+2)(\alpha_3+\beta+3n+2)}{(\alpha_1+\beta+4n+3)(\alpha_2+\beta+4n+3)(\alpha_3+\beta+4n+2)_2}\frac{{}_4F_3\left[\begin{matrix}-n,\alpha_1+\beta+3n+2,\alpha_1-\alpha_2-n,\alpha_1-\alpha_3-n+1\\ \alpha_1-\alpha_2+1,\alpha_1-\alpha_3+1,\alpha_1+\beta+N+2\end{matrix};1\right]}{{}_4F_3\left[\begin{matrix}-n,\alpha_1+\beta+3n+3,\alpha_1-\alpha_2-n,\alpha_1-\alpha_3-n\\ \alpha_1-\alpha_2+1,\alpha_1-\alpha_3+1,\alpha_1+\beta+N+2\end{matrix};1\right]},\\
L_{1,3n+3} &= \frac{(N-3n-2)(n+1)(\beta+3n+3)(\alpha_2+\beta+3n+3)(\alpha_3+\beta+3n+3)}{(\alpha_1+\beta+4n+4)_2(\alpha_2+\beta+4n+4)(\alpha_3+\beta+4n+4)}\frac{{}_4F_3\left[\begin{matrix}-n,\alpha_1+\beta+3n+3,\alpha_1-\alpha_2-n,\alpha_1-\alpha_3-n\\ \alpha_1-\alpha_2+1,\alpha_1-\alpha_3+1,\alpha_1+\beta+N+2\end{matrix};1\right]}{{}_4F_3\left[\begin{matrix}-n-1,\alpha_1+\beta+3n+4,\alpha_1-\alpha_2-n,\alpha_1-\alpha_3-n\\ \alpha_1-\alpha_2+1,\alpha_1-\alpha_3+1,\alpha_1+\beta+N+2\end{matrix};1\right]},
\end{aligned}
$$

$$
\begin{aligned}
L_{2,3n+1} = &-\frac{n(N-3n)_2(\beta+3n)_2(\alpha_1+\beta+3n)(\alpha_1+\beta+4n+3)(\alpha_2+\beta+3n)(\alpha_2+\beta+4n+3)(\alpha_3+\beta+3n)_2(\alpha_3+\beta+4n+2)_2(\alpha_1-\alpha_2+n)}{(N-3n-1)(\beta+3n+2)(\alpha_1+\beta+4n)_3(\alpha_2+\beta+3n+2)(\alpha_2+\beta+4n)_3(\alpha_3+\beta+3n+2)(\alpha_3+\beta+4n)_2(\alpha_3-\alpha_1+n)}\\
&\times\frac{{}_4F_3\left[\begin{matrix}-n,\alpha_1+\beta+3n+3,\alpha_1-\alpha_2-n,\alpha_1-\alpha_3-n\\ \alpha_1-\alpha_2+1,\alpha_1-\alpha_3+1,\alpha_1+\beta+N+2\end{matrix};1\right]}{{}_4F_3\left[\begin{matrix}-n,\alpha_1+\beta+3n+2,\alpha_1-\alpha_2-n,\alpha_1-\alpha_3-n+1\\ \alpha_1-\alpha_2+1,\alpha_1-\alpha_3+1,\alpha_1+\beta+N+2\end{matrix};1\right]}
\frac{\begin{vmatrix}
\frac{{}_4F_3\left[\begin{matrix}-n,\alpha_1+\beta+3n+1,\alpha_1-\alpha_2-n+1,\alpha_1-\alpha_3-n+1\\ \alpha_1-\alpha_2+1,\alpha_1-\alpha_3+1,\alpha_1+\beta+N+2\end{matrix};1\right]}{n} & \frac{{}_4F_3\left[\begin{matrix}-n+1,\alpha_1+\beta+3n,\alpha_1-\alpha_2-n+1,\alpha_1-\alpha_3-n+1\\ \alpha_1-\alpha_2+1,\alpha_1-\alpha_3+1,\alpha_1+\beta+N+2\end{matrix};1\right]}{\alpha_1+\beta+3n}\\
\frac{{}_4F_3\left[\begin{matrix}-n+1,\alpha_2+\beta+3n+1,\alpha_2-\alpha_1-n,\alpha_2-\alpha_3-n+1\\ \alpha_2-\alpha_1+1,\alpha_2-\alpha_3+1,\alpha_2+\beta+N+2\end{matrix};1\right]}{\alpha_1-\alpha_2+n} & \frac{{}_4F_3\left[\begin{matrix}-n+1,\alpha_2+\beta+3n,\alpha_2-\alpha_1-n+1,\alpha_2-\alpha_3-n+1\\ \alpha_2-\alpha_1+1,\alpha_2-\alpha_3+1,\alpha_2+\beta+N+2\end{matrix};1\right]}{\alpha_2+\beta+3n}
\end{vmatrix}}{\begin{vmatrix}
\frac{{}_4F_3\left[\begin{matrix}-n,\alpha_1+\beta+3n+2,\alpha_1-\alpha_2-n,\alpha_1-\alpha_3-n+1\\ \alpha_1-\alpha_2+1,\alpha_1-\alpha_3+1,\alpha_1+\beta+N+2\end{matrix};1\right]}{\alpha_2-\alpha_1+n} & \frac{{}_4F_3\left[\begin{matrix}-n,\alpha_1+\beta+3n+1,\alpha_1-\alpha_2-n+1,\alpha_1-\alpha_3-n+1\\ \alpha_1-\alpha_2+1,\alpha_1-\alpha_3+1,\alpha_1+\beta+N+2\end{matrix};1\right]}{n(\alpha_1+\beta+3n+1)}\\
{}_4F_3\left[\begin{matrix}-n,\alpha_2+\beta+3n+2,\alpha_2-\alpha_1-n,\alpha_2-\alpha_3-n+1\\ \alpha_2-\alpha_1+1,\alpha_2-\alpha_3+1,\alpha_2+\beta+N+2\end{matrix};1\right] & \frac{{}_4F_3\left[\begin{matrix}-n+1,\alpha_2+\beta+3n+1,\alpha_2-\alpha_1-n,\alpha_2-\alpha_3-n+1\\ \alpha_2-\alpha_1+1,\alpha_2-\alpha_3+1,\alpha_2+\beta+N+2\end{matrix};1\right]}{\alpha_2+\beta+3n+1}
\end{vmatrix}},
\end{aligned}
$$

$$
\begin{aligned}
L_{2,3n+2} = &-\frac{n(N-3n-1)_2(\beta+3n+1)_2(\alpha_1+\beta+3n+1)(\alpha_1+\beta+4n+4)_2(\alpha_2+\beta+3n+1)(\alpha_2+\beta+4n+4)(\alpha_3+\beta+3n+1)_2(\alpha_3+\beta+4n+4)(\alpha_2-\alpha_1+n)}{(n+1)(N-3n-2)(\beta+3n+3)(\alpha_1+\beta+4n+2)_2(\alpha_2+\beta+3n+3)(\alpha_2+\beta+4n+1)_3(\alpha_3+\beta+3n+3)(\alpha_3+\beta+4n+1)_3}\\
&\times\frac{{}_4F_3\left[\begin{matrix}-n-1,\alpha_1+\beta+3n+4,\alpha_1-\alpha_2-n,\alpha_1-\alpha_3-n\\ \alpha_1-\alpha_2+1,\alpha_1-\alpha_3+1,\alpha_1+\beta+N+2\end{matrix};1\right]}{{}_4F_3\left[\begin{matrix}-n,\alpha_1+\beta+3n+,\alpha_1-\alpha_2-n,\alpha_1-\alpha_3-n\\ \alpha_1-\alpha_2+1,\alpha_1-\alpha_3+1,\alpha_1+\beta+N+2\end{matrix};1\right]}
\frac{\begin{vmatrix}
\frac{{}_4F_3\left[\begin{matrix}-n,\alpha_1+\beta+3n+2,\alpha_1-\alpha_2-n,\alpha_1-\alpha_3-n+1\\ \alpha_1-\alpha_2+1,\alpha_1-\alpha_3+1,\alpha_1+\beta+N+2\end{matrix};1\right]}{\alpha_2-\alpha_1+n} & \frac{{}_4F_3\left[\begin{matrix}-n,\alpha_1+\beta+3n+1,\alpha_1-\alpha_2-n+1,\alpha_1-\alpha_3-n+1\\ \alpha_1-\alpha_2+1,\alpha_1-\alpha_3+1,\alpha_1+\beta+N+2\end{matrix};1\right]}{n(\alpha_1+\beta+3n+1)}\\
{}_4F_3\left[\begin{matrix}-n,\alpha_2+\beta+3n+2,\alpha_2-\alpha_1-n,\alpha_2-\alpha_3-n+1\\ \alpha_2-\alpha_1+1,\alpha_2-\alpha_3+1,\alpha_2+\beta+N+2\end{matrix};1\right] & \frac{{}_4F_3\left[\begin{matrix}-n+1,\alpha_2+\beta+3n+1,\alpha_2-\alpha_1-n,\alpha_2-\alpha_3-n+1\\ \alpha_2-\alpha_1+1,\alpha_2-\alpha_3+1,\alpha_2+\beta+N+2\end{matrix};1\right]}{\alpha_2+\beta+3n+1}
\end{vmatrix}}{\begin{vmatrix}
\frac{{}_4F_3\left[\begin{matrix}-n,\alpha_1+\beta+3n+3,\alpha_1-\alpha_2-n,\alpha_1-\alpha_3-n\\ \alpha_1-\alpha_2+1,\alpha_1-\alpha_3+1,\alpha_1+\beta+N+2\end{matrix};1\right]}{\alpha_3-\alpha_1+n} & \frac{{}_4F_3\left[\begin{matrix}-n,\alpha_1+\beta+3n+2,\alpha_1-\alpha_2-n,\alpha_1-\alpha_3-n+1\\ \alpha_1-\alpha_2+1,\alpha_1-\alpha_3+1,\alpha_1+\beta+N+2\end{matrix};1\right]}{(\alpha_1+\beta+3n+2)}\\
\frac{{}_4F_3\left[\begin{matrix}-n,\alpha_2+\beta+3n+3,\alpha_2-\alpha_1-n,\alpha_2-\alpha_3-n\\ \alpha_2-\alpha_1+1,\alpha_2-\alpha_3+1,\alpha_2+\beta+N+2\end{matrix};1\right]}{\alpha_3-\alpha_2+n} & \frac{{}_4F_3\left[\begin{matrix}-n,\alpha_2+\beta+3n+2,\alpha_2-\alpha_1-n,\alpha_2-\alpha_3-n+1\\ \alpha_2-\alpha_1+1,\alpha_2-\alpha_3+1,\alpha_2+\beta+N+2\end{matrix};1\right]}{\alpha_2+\beta+3n+2}
\end{vmatrix}},
\end{aligned}
$$

$$
\begin{aligned}
L_{2,3n+3} = &-\frac{(N-3n-2)_2(\beta+3n+2)_2(\alpha_1+\beta+3n+2)(\alpha_1+\beta+4n+6)(\alpha_2+\beta+3n+2)(\alpha_2+\beta+4n+5)_2(\alpha_3+\beta+3n+2)_2(\alpha_3+\beta+4n+5)(\alpha_3-\alpha_1+n)(\alpha_3-\alpha_2+n)}{(N-3n-3)(\beta+3n+4)(\alpha_1+\beta+4n+3)_3(\alpha_2+\beta+3n+4)(\alpha_2+\beta+4n+3)_2(\alpha_3+\beta+3n+4)(\alpha_3+\beta+4n+2)_3(\alpha_2-\alpha_1+n+1)}\\
&\times\frac{{}_4F_3\left[\begin{matrix}-n-1,\alpha_1+\beta+3n+5,\alpha_1-\alpha_2-n-1,\alpha_1-\alpha_3-n\\ \alpha_1-\alpha_2+1,\alpha_1-\alpha_3+1,\alpha_1+\beta+N+2\end{matrix};1\right]}{{}_4F_3\left[\begin{matrix}-n-1,\alpha_1+\beta+3n+4,\alpha_1-\alpha_2-n,\alpha_1-\alpha_3-n\\ \alpha_1-\alpha_2+1,\alpha_1-\alpha_3+1,\alpha_1+\beta+N+2\end{matrix};1\right]}
\frac{\begin{vmatrix}
\frac{{}_4F_3\left[\begin{matrix}-n,\alpha_1+\beta+3n+3,\alpha_1-\alpha_2-n,\alpha_1-\alpha_3-n\\ \alpha_1-\alpha_2+1,\alpha_1-\alpha_3+1,\alpha_1+\beta+N+2\end{matrix};1\right]}{\alpha_3-\alpha_1+n} & \frac{{}_4F_3\left[\begin{matrix}-n,\alpha_1+\beta+3n+2,\alpha_1-\alpha_2-n,\alpha_1-\alpha_3-n+1\\ \alpha_1-\alpha_2+1,\alpha_1-\alpha_3+1,\alpha_1+\beta+N+2\end{matrix};1\right]}{(\alpha_1+\beta+3n+2)}\\
\frac{{}_4F_3\left[\begin{matrix}-n,\alpha_2+\beta+3n+3,\alpha_2-\alpha_1-n,\alpha_2-\alpha_3-n\\ \alpha_2-\alpha_1+1,\alpha_2-\alpha_3+1,\alpha_2+\beta+N+2\end{matrix};1\right]}{\alpha_3-\alpha_2+n} & \frac{{}_4F_3\left[\begin{matrix}-n,\alpha_2+\beta+3n+2,\alpha_2-\alpha_1-n,\alpha_2-\alpha_3-n+1\\ \alpha_2-\alpha_1+1,\alpha_2-\alpha_3+1,\alpha_2+\beta+N+2\end{matrix};1\right]}{\alpha_2+\beta+3n+2}
\end{vmatrix}}{\begin{vmatrix}
\frac{{}_4F_3\left[\begin{matrix}-n-1,\alpha_1+\beta+3n+4,\alpha_1-\alpha_2-n,\alpha_1-\alpha_3-n\\ \alpha_1-\alpha_2+1,\alpha_1-\alpha_3+1,\alpha_1+\beta+N+2\end{matrix};1\right]}{n+1} & \frac{{}_4F_3\left[\begin{matrix}-n,\alpha_1+\beta+3n+3,\alpha_1-\alpha_2-n,\alpha_1-\alpha_3-n\\ \alpha_1-\alpha_2+1,\alpha_1-\alpha_3+1,\alpha_1+\beta+N+2\end{matrix};1\right]}{\alpha_1+\beta+3n+3}\\
\frac{{}_4F_3\left[\begin{matrix}-n,\alpha_2+\beta+3n+4,\alpha_2-\alpha_1-n-1,\alpha_2-\alpha_3-n\\ \alpha_2-\alpha_1+1,\alpha_2-\alpha_3+1,\alpha_2+\beta+N+2\end{matrix};1\right]}{\alpha_1-\alpha_2+n+1} & \frac{{}_4F_3\left[\begin{matrix}-n,\alpha_2+\beta+3n+3,\alpha_2-\alpha_1-n,\alpha_2-\alpha_3-n\\ \alpha_2-\alpha_1+1,\alpha_2-\alpha_3+1,\alpha_2+\beta+N+2\end{matrix};1\right]}{\alpha_2+\beta+3n+3}
\end{vmatrix}}
\end{aligned}
$$

$$
\begin{aligned}
L_{3,3n+1} = &-\frac{(N-3n-2)(\beta+3n+3)(\alpha_1+\beta+3n+2)_2(\alpha_2+\beta+3n+2)_2(\alpha_3+\beta+3n+2)_2}{(\alpha_1+\beta+4n+2)_3(\alpha_2+\beta+4n+1)_4(\alpha_3+\beta+4n+1)_4}\\
&\times\frac{(\alpha_1+n+1)(\alpha_1+\beta+N+n+2)(n+1)(\alpha_1-\alpha_2+n+1)(\alpha_1-\alpha_3+n+1)}{(\alpha_1+\beta+4n+1)_5}\\
&\times\frac{(\alpha_1+\beta+4n)(\alpha_2+\beta+4n)_4(\alpha_3+\beta+4n-1)_5(\alpha_1+\beta+4n+2)_2}{(N-3n+1)(\alpha_3+n)(\alpha_1+\beta+3n)(\alpha_2+\beta+3n)(\alpha_3+\beta+3n)n(\alpha_2-\alpha_1+n)}\\
&\times\frac{\begin{vmatrix}
\frac{{}_4F_3\left[{-n,\alpha_1+\beta+3n+3,\alpha_1-\alpha_2-n,\alpha_1-\alpha_3-n \atop \alpha_1-\alpha_2+1,\alpha_1-\alpha_3+1,\alpha_1+\beta+N+2};1\right]}{\alpha_3-\alpha_1+n} & \frac{{}_4F_3\left[{-n,\alpha_1+\beta+3n+2,\alpha_1-\alpha_2-n,\alpha_1-\alpha_3-n+1 \atop \alpha_1-\alpha_2+1,\alpha_1-\alpha_3+1,\alpha_1+\beta+N+2};1\right]}{(\alpha_1+\beta+3n+2)}\\
\frac{{}_4F_3\left[{-n,\alpha_2+\beta+3n+3,\alpha_2-\alpha_1-n,\alpha_2-\alpha_3-n \atop \alpha_2-\alpha_1+1,\alpha_2-\alpha_3+1,\alpha_2+\beta+N+2};1\right]}{\alpha_3-\alpha_2+n} & \frac{{}_4F_3\left[{-n,\alpha_2+\beta+3n+2,\alpha_2-\alpha_1-n,\alpha_2-\alpha_3-n+1 \atop \alpha_2-\alpha_1+1,\alpha_2-\alpha_3+1,\alpha_2+\beta+N+2};1\right]}{\alpha_2+\beta+3n+2}
\end{vmatrix}}{\begin{vmatrix}
\frac{{}_4F_3\left[{-n,\alpha_1+\beta+3n+2,\alpha_1-\alpha_2-n,\alpha_1-\alpha_3-n+1 \atop \alpha_1-\alpha_2+1,\alpha_1-\alpha_3+1,\alpha_1+\beta+N+2};1\right]}{\alpha_2-\alpha_1+n} & \frac{{}_4F_3\left[{-n,\alpha_1+\beta+3n+1,\alpha_1-\alpha_2-n+1,\alpha_1-\alpha_3-n+1 \atop \alpha_1-\alpha_2+1,\alpha_1-\alpha_3+1,\alpha_1+\beta+N+2};1\right]}{n(\alpha_1+\beta+3n+1)}\\
{}_4F_3\left[{-n,\alpha_2+\beta+3n+2,\alpha_2-\alpha_1-n,\alpha_2-\alpha_3-n+1 \atop \alpha_2-\alpha_1+1,\alpha_2-\alpha_3+1,\alpha_2+\beta+N+2};1\right] & \frac{{}_4F_3\left[{-n+1,\alpha_2+\beta+3n+1,\alpha_2-\alpha_1-n,\alpha_2-\alpha_3-n+1 \atop \alpha_2-\alpha_1+1,\alpha_2-\alpha_3+1,\alpha_2+\beta+N+2};1\right]}{\alpha_2+\beta+3n+1}
\end{vmatrix}},
\end{aligned}
$$

$$
\begin{aligned}
L_{3,3n+2} = &-\frac{(N-3n-3)(\beta+3n+4)(\alpha_1+\beta+3n+3)_2(\alpha_2+\beta+3n+3)_2(\alpha_3+\beta+3n+4)(\alpha_2+\beta+N+n+2)}{(N-3n)(\alpha_1+\beta+3n+1)(\alpha_2+\beta+3n+1)(\alpha_3+\beta+3n+1)(\alpha_2+\beta+4n+2)_4}\\
&\times\frac{(n+1)(\alpha_2+n+1)(\alpha_2-\alpha_1+n+1)(\alpha_2-\alpha_3+n+1)(\alpha_1+\beta+4n+1)_2(\alpha_2+\beta+4n+1)(\alpha_3+\beta+4n+1)}{(\alpha_1+n+1)(\alpha_3-\alpha_2+n)(\alpha_3-\alpha_1+n)(\alpha_1+\beta+4n+6)(\alpha_2+\beta+4n+5)_2(\alpha_3+\beta+4n+5)}\\
&\times\frac{\begin{vmatrix}
\frac{{}_4F_3\left[{-n-1,\alpha_1+\beta+3n+4,\alpha_1-\alpha_2-n,\alpha_1-\alpha_3-n \atop \alpha_1-\alpha_2+1,\alpha_1-\alpha_3+1,\alpha_1+\beta+N+2};1\right]}{n+1} & \frac{{}_4F_3\left[{-n,\alpha_1+\beta+3n+3,\alpha_1-\alpha_2-n,\alpha_1-\alpha_3-n \atop \alpha_1-\alpha_2+1,\alpha_1-\alpha_3+1,\alpha_1+\beta+N+2};1\right]}{\alpha_1+\beta+3n+3}\\
\frac{{}_4F_3\left[{-n,\alpha_2+\beta+3n+4,\alpha_2-\alpha_1-n-1,\alpha_2-\alpha_3-n \atop \alpha_2-\alpha_1+1,\alpha_2-\alpha_3+1,\alpha_2+\beta+N+2};1\right]}{\alpha_1-\alpha_2+n+1} & \frac{{}_4F_3\left[{-n,\alpha_2+\beta+3n+3,\alpha_2-\alpha_1-n,\alpha_2-\alpha_3-n \atop \alpha_2-\alpha_1+1,\alpha_2-\alpha_3+1,\alpha_2+\beta+N+2};1\right]}{\alpha_2+\beta+3n+3}
\end{vmatrix}}{\begin{vmatrix}
\frac{{}_4F_3\left[{-n,\alpha_1+\beta+3n+3,\alpha_1-\alpha_2-n,\alpha_1-\alpha_3-n \atop \alpha_1-\alpha_2+1,\alpha_1-\alpha_3+1,\alpha_1+\beta+N+2};1\right]}{\alpha_3-\alpha_1+n} & \frac{{}_4F_3\left[{-n,\alpha_1+\beta+3n+2,\alpha_1-\alpha_2-n,\alpha_1-\alpha_3-n+1 \atop \alpha_1-\alpha_2+1,\alpha_1-\alpha_3+1,\alpha_1+\beta+N+2};1\right]}{(\alpha_1+\beta+3n+2)}\\
\frac{{}_4F_3\left[{-n,\alpha_2+\beta+3n+3,\alpha_2-\alpha_1-n,\alpha_2-\alpha_3-n \atop \alpha_2-\alpha_1+1,\alpha_2-\alpha_3+1,\alpha_2+\beta+N+2};1\right]}{\alpha_3-\alpha_2+n} & \frac{{}_4F_3\left[{-n,\alpha_2+\beta+3n+2,\alpha_2-\alpha_1-n,\alpha_2-\alpha_3-n+1 \atop \alpha_2-\alpha_1+1,\alpha_2-\alpha_3+1,\alpha_2+\beta+N+2};1\right]}{\alpha_2+\beta+3n+2}
\end{vmatrix}},
\end{aligned}
$$

$$
\begin{aligned}
L_{3,3n+3} = &-\frac{(N-3n-4)(\beta+3n+5)(\alpha_1+\beta+3n+4)_2(\alpha_2+\beta+3n+4)_2(\alpha_3+\beta+3n+5)}{(N-3n-1)(\alpha_1+\beta+3n+2)(\alpha_2+\beta+3n+2)(\alpha_3+\beta+3n+2)(\alpha_3+\beta+4n+3)_5}\\
&\times\frac{(n+1)(\alpha_3+n+1)(\alpha_3-\alpha_1+n+1)(\alpha_3-\alpha_2+n+1)(\alpha_3+\beta+N+n+2)}{(\alpha_2+n+1)(\alpha_1-\alpha_2+n+1)}\\
&\times\frac{(\alpha_1+\beta+4n+3)(\alpha_2+\beta+4n+2)_2(\alpha_3+\beta+4n+2)}{(\alpha_1+\beta+4n+7)(\alpha_2+\beta+4n+7)(\alpha_3+\beta+4n+6)}\\
&\times\frac{\begin{vmatrix}
\frac{{}_4F_3\left[{-n-1,\alpha_1+\beta+3n+5,\alpha_1-\alpha_2-n-1,\alpha_1-\alpha_3-n \atop \alpha_1-\alpha_2+1,\alpha_1-\alpha_3+1,\alpha_1+\beta+N+2};1\right]}{\alpha_2-\alpha_1+n+1} & \frac{{}_4F_3\left[{-n-1,\alpha_1+\beta+3n+4,\alpha_1-\alpha_2-n,\alpha_1-\alpha_3-n \atop \alpha_1-\alpha_2+1,\alpha_1-\alpha_3+1,\alpha_1+\beta+N+2};1\right]}{(n+1)(\alpha_1+\beta+3n+4)}\\
{}_4F_3\left[{-n-1,\alpha_2+\beta+3n+5,\alpha_2-\alpha_1-n-1,\alpha_2-\alpha_3-n \atop \alpha_2-\alpha_1+1,\alpha_2-\alpha_3+1,\alpha_2+\beta+N+2};1\right] & \frac{{}_4F_3\left[{-n,\alpha_2+\beta+3n+4,\alpha_2-\alpha_1-n-1,\alpha_2-\alpha_3-n \atop \alpha_2-\alpha_1+1,\alpha_2-\alpha_3+1,\alpha_2+\beta+N+2};1\right]}{\alpha_2+\beta+3n+4}
\end{vmatrix}}{\begin{vmatrix}
\frac{{}_4F_3\left[{-n-1,\alpha_1+\beta+3n+4,\alpha_1-\alpha_2-n,\alpha_1-\alpha_3-n \atop \alpha_1-\alpha_2+1,\alpha_1-\alpha_3+1,\alpha_1+\beta+N+2};1\right]}{n+1} & \frac{{}_4F_3\left[{-n,\alpha_1+\beta+3n+3,\alpha_1-\alpha_2-n,\alpha_1-\alpha_3-n \atop \alpha_1-\alpha_2+1,\alpha_1-\alpha_3+1,\alpha_1+\beta+N+2};1\right]}{\alpha_1+\beta+3n+3}\\
\frac{{}_4F_3\left[{-n,\alpha_2+\beta+3n+4,\alpha_2-\alpha_1-n-1,\alpha_2-\alpha_3-n \atop \alpha_2-\alpha_1+1,\alpha_2-\alpha_3+1,\alpha_2+\beta+N+2};1\right]}{\alpha_1-\alpha_2+n+1} & \frac{{}_4F_3\left[{-n,\alpha_2+\beta+3n+3,\alpha_2-\alpha_1-n,\alpha_2-\alpha_3-n \atop \alpha_2-\alpha_1+1,\alpha_2-\alpha_3+1,\alpha_2+\beta+N+2};1\right]}{\alpha_2+\beta+3n+3}
\end{vmatrix}}
\end{aligned}
$$

Proof. From (28) we get

$$
\begin{aligned}
B_{3n}(0) &= (-1)^n\frac{N!}{(N-3n)!}\frac{(\alpha_1+1)_n(\alpha_2+1)_n(\alpha_3+1)_n}{(\alpha_1+\beta+3n+1)_n(\alpha_2+\beta+3n+1)_n(\alpha_3+\beta+3n+1)_n},\\
B_{3n+1}(0) &= (-1)^{n+1}\frac{N!}{(N-3n-1)!}\frac{(\alpha_1+1)_{n+1}(\alpha_2+1)_n(\alpha_3+1)_n}{(\alpha_1+\beta+3n+2)_{n+1}(\alpha_2+\beta+3n+2)_n(\alpha_3+\beta+3n+2)_n},\\
B_{3n+2}(0) &= (-1)^n\frac{N!}{(N-3n-2)!}\frac{(\alpha_1+1)_{n+1}(\alpha_2+1)_{n+1}(\alpha_3+1)_n}{(\alpha_1+\beta+3n+3)_{n+1}(\alpha_2+\beta+3n+3)_{n+1}(\alpha_3+\beta+3n+3)_n},
\end{aligned}
$$

and

$$
\begin{aligned}
A^{(1)}_{3n-1}(0) &= \frac{(-1)^{n-1}(N+1-3n)!(\alpha_1+\beta+3n)_n(\alpha_2+\beta+3n)_n(\alpha_3+\beta+3n)_n}{(n-1)!(\beta+1)_{3n-1}(\alpha_1+\beta+3n)_{N+2-3n}(\alpha_2-\alpha_1)_n(\alpha_3-\alpha_1)_n}\,{}_4F_3\left[\begin{matrix}-n+1,\alpha_1+\beta+3n,\alpha_1-\alpha_2-n+1,\alpha_1-\alpha_3-n+1\\ \alpha_1-\alpha_2+1,\alpha_1-\alpha_3+1,\alpha_1+\beta+N+2\end{matrix};1\right],\\
A^{(2)}_{3n-1}(0) &= \frac{(-1)^{n-1}(N+1-3n)!(\alpha_1+\beta+3n)_n(\alpha_2+\beta+3n)_n(\alpha_3+\beta+3n)_n}{(n-1)!(\beta+1)_{3n-1}(\alpha_2+\beta+3n)_{N+2-3n}(\alpha_1-\alpha_2)_n(\alpha_3-\alpha_2)_n}\,{}_4F_3\left[\begin{matrix}-n+1,\alpha_2+\beta+3n,\alpha_2-\alpha_1-n+1,\alpha_2-\alpha_3-n+1\\ \alpha_2-\alpha_1+1,\alpha_2-\alpha_3+1,\alpha_2+\beta+N+2\end{matrix};1\right],\\
A^{(3)}_{3n-1}(0) &= \frac{(-1)^{n-1}(N+1-3n)!(\alpha_1+\beta+3n)_n(\alpha_2+\beta+3n)_n(\alpha_3+\beta+3n)_n}{(n-1)!(\beta+1)_{3n-1}(\alpha_3+\beta+3n)_{N+2-3n}(\alpha_1-\alpha_3)_n(\alpha_2-\alpha_3)_n}\,{}_4F_3\left[\begin{matrix}-n+1,\alpha_3+\beta+3n,\alpha_3-\alpha_1-n+1,\alpha_3-\alpha_2-n+1\\ \alpha_3-\alpha_1+1,\alpha_3-\alpha_2+1,\alpha_3+\beta+N+2\end{matrix};1\right],\\
A^{(1)}_{3n}(0) &= \frac{(-1)^{n}(N-3n)!(\alpha_1+\beta+3n+1)_{n+1}(\alpha_2+\beta+3n+1)_n(\alpha_3+\beta+3n+1)_n}{n!(\beta+1)_{3n}(\alpha_1+\beta+3n+1)_{N+1-3n}(\alpha_2-\alpha_1)_n(\alpha_3-\alpha_1)_n}\,{}_4F_3\left[\begin{matrix}-n,\alpha_1+\beta+3n+1,\alpha_1-\alpha_2-n+1,\alpha_1-\alpha_3-n+1\\ \alpha_1-\alpha_2+1,\alpha_1-\alpha_3+1,\alpha_1+\beta+N+2\end{matrix};1\right],\\
A^{(2)}_{3n}(0) &= \frac{(-1)^{n}(N-3n)!(\alpha_1+\beta+3n+1)_{n+1}(\alpha_2+\beta+3n+1)_n(\alpha_3+\beta+3n+1)_n}{(n-1)!(\beta+1)_{3n}(\alpha_2+\beta+3n+1)_{N+1-3n}(\alpha_1-\alpha_2)_{n+1}(\alpha_3-\alpha_2)_n}\,{}_4F_3\left[\begin{matrix}-n+1,\alpha_2+\beta+3n+1,\alpha_2-\alpha_1-n,\alpha_2-\alpha_3-n+1\\ \alpha_2-\alpha_1+1,\alpha_2-\alpha_3+1,\alpha_2+\beta+N+2\end{matrix};1\right],\\
A^{(3)}_{3n}(0) &= \frac{(-1)^{n}(N-3n)!(\alpha_1+\beta+3n+1)_{n+1}(\alpha_2+\beta+3n+1)_n(\alpha_3+\beta+3n+1)_n}{(n-1)!(\beta+1)_{3n}(\alpha_3+\beta+3n+1)_{N+1-3n}(\alpha_1-\alpha_3)_{n+1}(\alpha_2-\alpha_3)_n}\,{}_4F_3\left[\begin{matrix}-n+1,\alpha_3+\beta+3n+1,\alpha_3-\alpha_1-n,\alpha_3-\alpha_2-n+1\\ \alpha_3-\alpha_1+1,\alpha_3-\alpha_2+1,\alpha_3+\beta+N+2\end{matrix};1\right],\\
A^{(1)}_{3n+1}(0) &= \frac{(-1)^{n+1}(N-3n-1)!(\alpha_1+\beta+3n+2)_{n+1}(\alpha_2+\beta+3n+2)_{n+1}(\alpha_3+\beta+3n+2)_n}{n!(\beta+1)_{3n+1}(\alpha_1+\beta+3n+2)_{N-3n}(\alpha_2-\alpha_1)_{n+1}(\alpha_3-\alpha_1)_n}\,{}_4F_3\left[\begin{matrix}-n,\alpha_1+\beta+3n+2,\alpha_1-\alpha_2-n,\alpha_1-\alpha_3-n+1\\ \alpha_1-\alpha_2+1,\alpha_1-\alpha_3+1,\alpha_1+\beta+N+2\end{matrix};1\right],\\
A^{(2)}_{3n+1}(0) &= \frac{(-1)^{n+1}(N-3n-1)!(\alpha_1+\beta+3n+2)_{n+1}(\alpha_2+\beta+3n+2)_{n+1}(\alpha_3+\beta+3n+2)_n}{n!(\beta+1)_{3n+1}(\alpha_2+\beta+3n+2)_{N-3n}(\alpha_1-\alpha_2)_{n+1}(\alpha_3-\alpha_2)_n}\,{}_4F_3\left[\begin{matrix}-n,\alpha_2+\beta+3n+2,\alpha_2-\alpha_1-n,\alpha_2-\alpha_3-n+1\\ \alpha_2-\alpha_1+1,\alpha_2-\alpha_3+1,\alpha_2+\beta+N+2\end{matrix};1\right],\\
A^{(3)}_{3n+1}(0) &= \frac{(-1)^{n+1}(N-3n-1)!(\alpha_1+\beta+3n+2)_{n+1}(\alpha_2+\beta+3n+2)_{n+1}(\alpha_3+\beta+3n+2)_n}{(n-1)!(\beta+1)_{3n+1}(\alpha_3+\beta+3n+2)_{N-3n}(\alpha_1-\alpha_3)_{n+1}(\alpha_2-\alpha_3)_{n+1}}\,{}_4F_3\left[\begin{matrix}-n+1,\alpha_3+\beta+3n+2,\alpha_3-\alpha_1-n,\alpha_3-\alpha_2-n\\ \alpha_3-\alpha_1+1,\alpha_3-\alpha_2+1,\alpha_3+\beta+N+2\end{matrix};1\right].
\end{aligned}
$$

For for $j \in \{0,1,2,3\}$, the recurrence coefficients are

$$
\begin{aligned}
b^{j}_{3m} &= -(\alpha_1+m+1)\delta_{j,0} + \frac{(N-3m+1)_j(\beta+3m+1-j)_j(\alpha_1+\beta+4m+1-j)}{\prod_{l=5-j}^{4}(\alpha_l+\beta+4m-j)}\,\frac{\prod_{l=1}^{3}(\alpha_l+\beta+3m+1-j)_j}{\prod_{l=1}^{3}(\alpha_l+\beta+4m+1-j)_j}\sum_{i=1}^{4-j}\frac{(\alpha_i+m)(\alpha_i+\beta+N+m+1)}{(\alpha_i+\beta+4m-j)_{j+2}}\,\frac{\prod_{l=1}^{3}(\alpha_i-\alpha_l+m)}{\prod_{l=1,l\neq i}^{4-j}(\alpha_i-\alpha_l)}\\
b^{j}_{3m+1} &= -(\alpha_2+m+1)\delta_{j,0} + \frac{(N-3m)_j(\beta+3m+2-j)_j(\alpha_2+\beta+4m+2-j)}{\prod_{l=6-j}^{5}(\alpha_l+\beta+4m+1-j)}\,\frac{\prod_{l=1}^{3}(\alpha_l+\beta+3m+2-j)_j}{\prod_{l=2}^{4}(\alpha_l+\beta+4m+2-j)_j}\sum_{i=2}^{5-j}\frac{(\alpha_i+m)(\alpha_i+\beta+N+m+1)}{(\alpha_i+\beta+4m+1-j)_{j+2}}\,\frac{\prod_{l=1}^{3}(\alpha_i-\alpha_l+m)}{\prod_{l=2,l\neq i}^{5-j}(\alpha_i-\alpha_l)},\\
b^{j}_{3m+2} &= -(\alpha_3+m+1)\delta_{j,0} + \frac{(N-3m-1)_j(\beta+3m+3-j)_j(\alpha_3+\beta+4m+3-j)}{\prod_{l=7-j}^{6}(\alpha_l+\beta+4m+2-j)}\,\frac{\prod_{l=1}^{3}(\alpha_l+\beta+3m+3-j)_j}{\prod_{l=3}^{5}(\alpha_l+\beta+4m+3-j)_j}\sum_{i=3}^{6-j}\frac{(\alpha_i+m)(\alpha_i+\beta+N+m+1)}{(\alpha_i+\beta+4m+2-j)_{j+2}}\,\frac{\prod_{l=1}^{3}(\alpha_i-\alpha_l+m)}{\prod_{l=3,l\neq i}^{6-j}(\alpha_i-\alpha_l)}.
\end{aligned}
$$

In particular

$$
\begin{aligned}
b^{3}_{3m} &= \frac{(N-3m+1)_3(\beta+3m-2)_3(\alpha_1+\beta+4m-2)}{\prod_{l=2}^{4}(\alpha_l+\beta+4m-3)}\,\frac{\prod_{l=1}^{3}(\alpha_l+\beta+3m-2)_3}{\prod_{l=1}^{3}(\alpha_l+\beta+4m-2)_3}\,\frac{(\alpha_1+m)(\alpha_1+\beta+N+m+1)}{(\alpha_1+\beta+4m-3)_5}\prod_{l=1}^{3}(\alpha_1-\alpha_l+m)\\
b^{3}_{3m+1} &= \frac{(N-3m)_3(\beta+3m-1)_3(\alpha_2+\beta+4m-1)}{\prod_{l=3}^{5}(\alpha_l+\beta+4m-2)}\,\frac{\prod_{l=1}^{3}(\alpha_l+\beta+3m-1)_3}{\prod_{l=2}^{4}(\alpha_l+\beta+4m-1)_3}\,\frac{(\alpha_2+m)(\alpha_2+\beta+N+m+1)}{(\alpha_2+\beta+4m-2)_5}\prod_{l=1}^{3}(\alpha_2-\alpha_l+m),\\
b^{3}_{3m+2} &= \frac{(N-3m-1)_3(\beta+3m)_3(\alpha_3+\beta+4m)}{\prod_{l=4}^{6}(\alpha_l+\beta+4m-1)}\,\frac{\prod_{l=1}^{3}(\alpha_l+\beta+3m)_3}{\prod_{l=3}^{5}(\alpha_l+\beta+4m)_3}\,\frac{(\alpha_3+m)(\alpha_3+\beta+N+m+1)}{(\alpha_3+\beta+4m-1)_5}\prod_{l=1}^{3}(\alpha_3-\alpha_l+m).
\end{aligned}
$$

The corresponding τ-determinants are given by

$$
\tau^{B}_{1,n} = B_n(0), \quad n \in \{0,1,\dots,N-2\}, \quad \tau^{A}_{1,n} = A^{(1)}_n(0), \quad n \in \{0,1,\dots,N-4\},
$$

and

$$
\tau^{A}_{2,n} = \begin{vmatrix} A^{(1)}_{n+1}(0) & A^{(1)}_{n}(0)\\ A^{(2)}_{n+1}(0) & A^{(2)}_{n}(0)\end{vmatrix}, \quad n \in \{0,1,\dots,N-5\}, \quad \tau^{A}_{3,n} = \begin{vmatrix} A^{(1)}_{n+2}(0) & A^{(1)}_{n+1}(0) & A^{(1)}_{n}(0)\\ A^{(2)}_{n+2}(0) & A^{(2)}_{n+1}(0) & A^{(2)}_{n}(0)\\ A^{(3)}_{n+2}(0) & A^{(3)}_{n+1}(0) & A^{(3)}_{n}(0)\end{vmatrix}, \quad n \in \{0,1,\dots,N-6\}.
$$

Thus, following Theorem 5.5 we get

$$
U_{1,n} = -\frac{B_{n+1}(0)}{B_n(0)}, \quad n \in \{0,1,\dots,N-2\}, \quad L_{1,n+1} = -\frac{A^{(1)}_n(0)}{A^{(1)}_{n+1}(0)}, \quad n \in \{0,1,\dots,N-4\},
$$

and

$$L_{2,n+1}=-\frac{A^{(1)}_{n+2}(0)}{A^{(1)}_{n+1}(0)}\frac{\begin{vmatrix}A^{(1)}_{n+1}(0) & A^{(1)}_{n}(0)\\ A^{(2)}_{n+1}(0) & A^{(2)}_{n}(0)\end{vmatrix}}{\begin{vmatrix}A^{(1)}_{n+2}(0) & A^{(1)}_{n+1}(0)\\ A^{(2)}_{n+2}(0) & A^{(2)}_{n+1}(0)\end{vmatrix}}\quad n\in\{0,1,\dots,N-5\},\quad L_{3,n+1}=-\frac{\begin{vmatrix}A^{(1)}_{n+3}(0) & A^{(1)}_{n+2}(0)\\ A^{(2)}_{n+3}(0) & A^{(2)}_{n+2}(0)\end{vmatrix}}{\begin{vmatrix}A^{(1)}_{n+2}(0) & A^{(1)}_{n+1}(0)\\ A^{(2)}_{n+2}(0) & A^{(2)}_{n+1}(0)\end{vmatrix}}\frac{\begin{vmatrix}A^{(1)}_{n+2}(0) & A^{(1)}_{n+1}(0) & A^{(1)}_{n}(0)\\ A^{(2)}_{n+2}(0) & A^{(2)}_{n+1}(0) & A^{(2)}_{n}(0)\\ A^{(3)}_{n+2}(0) & A^{(3)}_{n+1}(0) & A^{(3)}_{n}(0)\end{vmatrix}}{\begin{vmatrix}A^{(1)}_{n+3}(0) & A^{(1)}_{n+2}(0) & A^{(1)}_{n+1}(0)\\ A^{(2)}_{n+3}(0) & A^{(2)}_{n+2}(0) & A^{(2)}_{n+1}(0)\\ A^{(3)}_{n+3}(0) & A^{(3)}_{n+2}(0) & A^{(3)}_{n+1}(0)\end{vmatrix}},\quad n\in\{0,1,\dots,N-6\}.$$

In particular

$$L_{3,n+1}=-b^3_{n+3}\frac{\begin{vmatrix}A^{(1)}_{n+3}(0) & A^{(1)}_{n+2}(0)\\ A^{(2)}_{n+3}(0) & A^{(2)}_{n+2}(0)\end{vmatrix}}{\begin{vmatrix}A^{(1)}_{n+2}(0) & A^{(1)}_{n+1}(0)\\ A^{(2)}_{n+2}(0) & A^{(2)}_{n+1}(0)\end{vmatrix}}\frac{B_n(0)}{B_{n+1}(0)},\quad n\in\{0,1,\dots,N-6\}.$$

The expressions for the U's are obtained form

$$U_{1,3n}=-\frac{B_{3n+1}(0)}{B_{3n}(0)},\quad U_{1,3n+1}=-\frac{B_{3n+2}(0)}{B_{3n+1}(0)},\quad U_{1,3n+2}=-\frac{B_{3n+3}(0)}{B_{3n+2}(0)}.$$

For the L_1's we obtain:

$$L_{1,3n+1}=-\frac{A^{(1)}_{3n}(0)}{A^{(1)}_{3n+1}(0)}=-\frac{\frac{(-1)^n(N-3n)!(\alpha_1+\beta+3n+1)_{n+1}(\alpha_2+\beta+3n+1)_n(\alpha_3+\beta+3n+1)_n}{n!(\beta+1)_{3n}(\alpha_1+\beta+3n+1)_{N+1-3n}(\alpha_2-\alpha_1)_n(\alpha_3-\alpha_1)_n}}{\frac{(-1)^{n+1}(N-3n-1)!(\alpha_1+\beta+3n+2)_{n+1}(\alpha_2+\beta+3n+2)_{n+1}(\alpha_3+\beta+3n+2)_n}{n!(\beta+1)_{3n+1}(\alpha_1+\beta+3n+2)_{N-3n}(\alpha_2-\alpha_1)_{n+1}(\alpha_3-\alpha_1)_n}}\times\frac{{}_4F_3\left[{-n,\alpha_1+\beta+3n+1,\alpha_1-\alpha_2-n+1,\alpha_1-\alpha_3-n+1 \atop \alpha_1-\alpha_2+1,\alpha_1-\alpha_3+1,\alpha_1+\beta+N+2};1\right]}{{}_4F_3\left[{-n,\alpha_1+\beta+3n+2,\alpha_1-\alpha_2-n,\alpha_1-\alpha_3-n+1 \atop \alpha_1-\alpha_2+1,\alpha_1-\alpha_3+1,\alpha_1+\beta+N+2};1\right]}$$

$$L_{1,3n+2}=-\frac{A^{(1)}_{3n+1}(0)}{A^{(1)}_{3n+2}(0)}=-\frac{\frac{(-1)^{n+1}(N-3n-1)!(\alpha_1+\beta+3n+2)_{n+1}(\alpha_2+\beta+3n+2)_{n+1}(\alpha_3+\beta+3n+2)_n}{n!(\beta+1)_{3n+1}(\alpha_1+\beta+3n+2)_{N-3n}(\alpha_2-\alpha_1)_{n+1}(\alpha_3-\alpha_1)_n}}{\frac{(-1)^n(N+1-3(n+1))!(\alpha_1+\beta+3(n+1))_{n+1}(\alpha_2+\beta+3(n+1))_{n+1}(\alpha_3+\beta+3(n+1))_{n+1}}{n!(\beta+1)_{3(n+1)-1}(\alpha_1+\beta+3(n+1))_{N+2-3(n+1)}(\alpha_2-\alpha_1)_{n+1}(\alpha_3-\alpha_1)_{n+1}}}\times\frac{{}_4F_3\left[{-n,\alpha_1+\beta+3n+2,\alpha_1-\alpha_2-n,\alpha_1-\alpha_3-n+1 \atop \alpha_1-\alpha_2+1,\alpha_1-\alpha_3+1,\alpha_1+\beta+N+2};1\right]}{{}_4F_3\left[{-(n+1)+1,\alpha_1+\beta+3(n+1),\alpha_1-\alpha_2-(n+1)+1,\alpha_1-\alpha_3-(n+1)+1 \atop \alpha_1-\alpha_2+1,\alpha_1-\alpha_3+1,\alpha_1+\beta+N+2};1\right]}$$

$$L_{1,3n+3}=-\frac{A^{(1)}_{3n+2}(0)}{A^{(1)}_{3n+3}(0)}=-\frac{\frac{(-1)^n(N+1-3(n+1))!(\alpha_1+\beta+3(n+1))_{n+1}(\alpha_2+\beta+3(n+1))_{n+1}(\alpha_3+\beta+3(n+1))_{n+1}}{n!(\beta+1)_{3(n+1)-1}(\alpha_1+\beta+3(n+1))_{N+2-3(n+1)}(\alpha_2-\alpha_1)_{n+1}(\alpha_3-\alpha_1)_{n+1}}}{\frac{(-1)^{n+1}(N-3(n+1))!(\alpha_1+\beta+3(n+1)+1)_{n+2}(\alpha_2+\beta+3(n+1)+1)_{n+1}(\alpha_3+\beta+3(n+1)+1)_{n+1}}{(n+1)!(\beta+1)_{3(n+1)}(\alpha_1+\beta+3(n+1)+1)_{N+1-3(n+1)}(\alpha_2-\alpha_1)_{n+1}(\alpha_3-\alpha_1)_{n+1}}}\times\frac{{}_4F_3\left[{-(n+1)+1,\alpha_1+\beta+3(n+1),\alpha_1-\alpha_2-(n+1)+1,\alpha_1-\alpha_3-(n+1)+1 \atop \alpha_1-\alpha_2+1,\alpha_1-\alpha_3+1,\alpha_1+\beta+N+2};1\right]}{{}_4F_3\left[{-(n+1),\alpha_1+\beta+3(n+1)+1,\alpha_1-\alpha_2-(n+1)+1,\alpha_1-\alpha_3-(n+1)+1 \atop \alpha_1-\alpha_2+1,\alpha_1-\alpha_3+1,\alpha_1+\beta+N+2};1\right]}$$

That simplifies to the exspressions for the L_1's in the theorem.

Now, we need to deal with 2×2 determinants

$$\tau^A_{2,3n+1}=\begin{vmatrix}A^{(1)}_{3n+1}(0) & A^{(1)}_{3n}(0)\\ A^{(2)}_{3n+1}(0) & A^{(2)}_{3n}(0)\end{vmatrix},\quad \tau^A_{2,3n+2}=\begin{vmatrix}A^{(1)}_{3n+2}(0) & A^{(1)}_{3n+1}(0)\\ A^{(2)}_{3n+2}(0) & A^{(2)}_{3n+1}(0)\end{vmatrix},\quad \tau^A_{2,3n+3}=\begin{vmatrix}A^{(1)}_{3n+3}(0) & A^{(1)}_{3n+2}(0)\\ A^{(2)}_{3n+3}(0) & A^{(2)}_{3n+2}(0)\end{vmatrix},$$

Hence,

$$\begin{aligned}\tau^A_{2,3n+1}&=\frac{(-1)^n(N-3n)!(\alpha_1+\beta+3n+1)_{n+1}(\alpha_2+\beta+3n+1)_n(\alpha_3+\beta+3n+1)_n}{(n-1)!(\beta+1)_{3n}}\\
&\quad\frac{(-1)^{n+1}(N-3n-1)!(\alpha_1+\beta+3n+2)_{n+1}(\alpha_2+\beta+3n+2)_{n+1}(\alpha_3+\beta+3n+2)_n}{n!(\beta+1)_{3n+1}}\\
&\quad\begin{vmatrix}\dfrac{{}_4F_3\left[{-n,\alpha_1+\beta+3n+2,\alpha_1-\alpha_2-n,\alpha_1-\alpha_3-n+1 \atop \alpha_1-\alpha_2+1,\alpha_1-\alpha_3+1,\alpha_1+\beta+N+2};1\right]}{(\alpha_1+\beta+3n+2)_{N-3n}(\alpha_2-\alpha_1)_{n+1}(\alpha_3-\alpha_1)_n} & \dfrac{{}_4F_3\left[{-n,\alpha_1+\beta+3n+1,\alpha_1-\alpha_2-n+1,\alpha_1-\alpha_3-n+1 \atop \alpha_1-\alpha_2+1,\alpha_1-\alpha_3+1,\alpha_1+\beta+N+2};1\right]}{n(\alpha_1+\beta+3n+1)_{N+1-3n}(\alpha_2-\alpha_1)_n(\alpha_3-\alpha_1)_n}\\ \dfrac{{}_4F_3\left[{-n,\alpha_2+\beta+3n+2,\alpha_2-\alpha_1-n,\alpha_2-\alpha_3-n+1 \atop \alpha_2-\alpha_1+1,\alpha_2-\alpha_3+1,\alpha_2+\beta+N+2};1\right]}{(\alpha_2+\beta+3n+2)_{N-3n}(\alpha_1-\alpha_2)_{n+1}(\alpha_3-\alpha_2)_n} & \dfrac{{}_4F_3\left[{-n+1,\alpha_2+\beta+3n+1,\alpha_2-\alpha_1-n,\alpha_2-\alpha_3-n+1 \atop \alpha_2-\alpha_1+1,\alpha_2-\alpha_3+1,\alpha_2+\beta+N+2};1\right]}{(\alpha_2+\beta+3n+1)_{N+1-3n}(\alpha_1-\alpha_2)_{n+1}(\alpha_3-\alpha_2)_n}\end{vmatrix}\\
&=-\frac{(N-3n)!(\alpha_1+\beta+3n+1)_{n+1}(\alpha_2+\beta+3n+1)_n(\alpha_3+\beta+3n+1)_n}{(n-1)!(\beta+1)_{3n}(\alpha_2-\alpha_1)_n(\alpha_3-\alpha_1)_n(\alpha_1+\beta+3n+2)_{N-3n}}\\
&\quad\frac{(N-3n-1)!(\alpha_1+\beta+3n+2)_{n+1}(\alpha_2+\beta+3n+2)_{n+1}(\alpha_3+\beta+3n+2)_n}{n!(\beta+1)_{3n+1}(\alpha_1-\alpha_2)_{n+1}(\alpha_3-\alpha_2)_n(\alpha_2+\beta+3n+2)_{N-3n}}\\
&\quad\begin{vmatrix}\dfrac{{}_4F_3\left[{-n,\alpha_1+\beta+3n+2,\alpha_1-\alpha_2-n,\alpha_1-\alpha_3-n+1 \atop \alpha_1-\alpha_2+1,\alpha_1-\alpha_3+1,\alpha_1+\beta+N+2};1\right]}{\alpha_2-\alpha_1+n} & \dfrac{{}_4F_3\left[{-n,\alpha_1+\beta+3n+1,\alpha_1-\alpha_2-n+1,\alpha_1-\alpha_3-n+1 \atop \alpha_1-\alpha_2+1,\alpha_1-\alpha_3+1,\alpha_1+\beta+N+2};1\right]}{n(\alpha_1+\beta+3n+1)}\\ {}_4F_3\left[{-n,\alpha_2+\beta+3n+2,\alpha_2-\alpha_1-n,\alpha_2-\alpha_3-n+1 \atop \alpha_2-\alpha_1+1,\alpha_2-\alpha_3+1,\alpha_2+\beta+N+2};1\right] & \dfrac{{}_4F_3\left[{-n+1,\alpha_2+\beta+3n+1,\alpha_2-\alpha_1-n,\alpha_2-\alpha_3-n+1 \atop \alpha_2-\alpha_1+1,\alpha_2-\alpha_3+1,\alpha_2+\beta+N+2};1\right]}{\alpha_2+\beta+3n+1}\end{vmatrix}\end{aligned}$$

$$\begin{aligned}\tau^A_{2,3n+2}&=-\frac{(N-3n-2)!(\alpha_1+\beta+3n+3)_{n+1}(\alpha_2+\beta+3n+3)_{n+1}(\alpha_3+\beta+3n+3)_{n+1}}{(n)!(\beta+1)_{3n+2}(\alpha_1+\beta+3n+3)_{N-3n-1}(\alpha_2-\alpha_1)_{n+1}(\alpha_3-\alpha_1)_n}\\
&\quad\frac{(N-3n-1)!(\alpha_1+\beta+3n+2)_{n+1}(\alpha_2+\beta+3n+2)_{n+1}(\alpha_3+\beta+3n+2)_n}{n!(\beta+1)_{3n+1}(\alpha_2+\beta+3n+3)_{N-3n-1}(\alpha_1-\alpha_2)_{n+1}(\alpha_3-\alpha_2)_n}\\
&\quad\begin{vmatrix}\dfrac{{}_4F_3\left[{-n,\alpha_1+\beta+3n+3,\alpha_1-\alpha_2-n,\alpha_1-\alpha_3-n \atop \alpha_1-\alpha_2+1,\alpha_1-\alpha_3+1,\alpha_1+\beta+N+2};1\right]}{\alpha_3-\alpha_1+n} & \dfrac{{}_4F_3\left[{-n,\alpha_1+\beta+3n+2,\alpha_1-\alpha_2-n,\alpha_1-\alpha_3-n+1 \atop \alpha_1-\alpha_2+1,\alpha_1-\alpha_3+1,\alpha_1+\beta+N+2};1\right]}{(\alpha_1+\beta+3n+2)}\\ \dfrac{{}_4F_3\left[{-n,\alpha_2+\beta+3n+3,\alpha_2-\alpha_1-n,\alpha_2-\alpha_3-n \atop \alpha_2-\alpha_1+1,\alpha_2-\alpha_3+1,\alpha_2+\beta+N+2};1\right]}{\alpha_3-\alpha_2+n} & \dfrac{{}_4F_3\left[{-n,\alpha_2+\beta+3n+2,\alpha_2-\alpha_1-n,\alpha_2-\alpha_3-n+1 \atop \alpha_2-\alpha_1+1,\alpha_2-\alpha_3+1,\alpha_2+\beta+N+2};1\right]}{\alpha_2+\beta+3n+2}\end{vmatrix}\end{aligned}$$

$$\begin{aligned}\tau^A_{2,3n}&=-\frac{(N-3n)!(\alpha_1+\beta+3n+1)_{n+1}(\alpha_2+\beta+3n+1)_n(\alpha_3+\beta+3n+1)_n}{(n-1)!(\beta+1)_{3n}(\alpha_1+\beta+3n+1)_{N-3n+1}(\alpha_2-\alpha_1)_n(\alpha_3-\alpha_1)_n}\\
&\quad\frac{(N-3n+1)!(\alpha_1+\beta+3n)_n(\alpha_2+\beta+3n)_n(\alpha_3+\beta+3n)_n}{(n-1)!(\beta+1)_{3n-1}(\alpha_2+\beta+3n+1)_{N-3n+1}(\alpha_1-\alpha_2)_n(\alpha_3-\alpha_2)_n}\\
&\quad\begin{vmatrix}\dfrac{{}_4F_3\left[{-n,\alpha_1+\beta+3n+1,\alpha_1-\alpha_2-n+1,\alpha_1-\alpha_3-n+1 \atop \alpha_1-\alpha_2+1,\alpha_1-\alpha_3+1,\alpha_1+\beta+N+2};1\right]}{n} & \dfrac{{}_4F_3\left[{-n+1,\alpha_1+\beta+3n,\alpha_1-\alpha_2-n+1,\alpha_1-\alpha_3-n+1 \atop \alpha_1-\alpha_2+1,\alpha_1-\alpha_3+1,\alpha_1+\beta+N+2};1\right]}{\alpha_1+\beta+3n}\\ \dfrac{{}_4F_3\left[{-n+1,\alpha_2+\beta+3n+1,\alpha_2-\alpha_1-n,\alpha_2-\alpha_3-n+1 \atop \alpha_2-\alpha_1+1,\alpha_2-\alpha_3+1,\alpha_2+\beta+N+2};1\right]}{\alpha_1-\alpha_2+n} & \dfrac{{}_4F_3\left[{-n+1,\alpha_2+\beta+3n,\alpha_2-\alpha_1-n+1,\alpha_2-\alpha_3-n+1 \atop \alpha_2-\alpha_1+1,\alpha_2-\alpha_3+1,\alpha_2+\beta+N+2};1\right]}{\alpha_2+\beta+3n}\end{vmatrix}\end{aligned}$$

$$
\begin{aligned}
\tau^A_{2,3n+3} = {}&-\frac{(N-3n-3)!(\alpha_1+\beta+3n+4)_{n+2}(\alpha_2+\beta+3n+4)_{n+1}(\alpha_3+\beta+3n+4)_{n+1}}{(n)!(\beta+1)_{3n+3}(\alpha_1+\beta+3n+4)_{N-3n-2}(\alpha_2-\alpha_1)_{n+1}(\alpha_3-\alpha_1)_{n+1}}\\
&\frac{(N-3n-2)!(\alpha_1+\beta+3n+3)_{n+1}(\alpha_2+\beta+3n+3)_{n+1}(\alpha_3+\beta+3n+3)_{n+1}}{n!(\beta+1)_{3n+2}(\alpha_2+\beta+3n+4)_{N-3n-2}(\alpha_1-\alpha_2)_{n+1}(\alpha_3-\alpha_2)_{n+1}}\\
&\begin{vmatrix}
\dfrac{{}_4F_3\left[\begin{matrix}-n-1,\alpha_1+\beta+3n+4,\alpha_1-\alpha_2-n,\alpha_1-\alpha_3-n\\ \alpha_1-\alpha_2+1,\alpha_1-\alpha_3+1,\alpha_1+\beta+N+2\end{matrix};1\right]}{n+1} & \dfrac{{}_4F_3\left[\begin{matrix}-n,\alpha_1+\beta+3n+3,\alpha_1-\alpha_2-n,\alpha_1-\alpha_3-n\\ \alpha_1-\alpha_2+1,\alpha_1-\alpha_3+1,\alpha_1+\beta+N+2\end{matrix};1\right]}{\alpha_1+\beta+3n+3}\\
\dfrac{{}_4F_3\left[\begin{matrix}-n,\alpha_2+\beta+3n+4,\alpha_2-\alpha_1-n-1,\alpha_2-\alpha_3-n\\ \alpha_2-\alpha_1+1,\alpha_2-\alpha_3+1,\alpha_2+\beta+N+2\end{matrix};1\right]}{\alpha_1-\alpha_2+n+1} & \dfrac{{}_4F_3\left[\begin{matrix}-n,\alpha_2+\beta+3n+3,\alpha_2-\alpha_1-n,\alpha_2-\alpha_3-n\\ \alpha_2-\alpha_1+1,\alpha_2-\alpha_3+1,\alpha_2+\beta+N+2\end{matrix};1\right]}{\alpha_2+\beta+3n+3}
\end{vmatrix}
\end{aligned}
$$

$$
\begin{aligned}
\frac{\tau^A_{2,3n}}{\tau^A_{2,3n+1}} = {}&\frac{n(N-3n)_2(\beta+3n)_2(\alpha_1+\beta+3n)(\alpha_2+\beta+3n)(\alpha_3+\beta+3n)_2(\alpha_1-\alpha_2+n)}{(\alpha_1+\beta+4n)_3(\alpha_2+\beta+4n)_3(\alpha_3+\beta+4n)_2}\\
&\times\frac{\begin{vmatrix}
\dfrac{{}_4F_3\left[\begin{matrix}-n,\alpha_1+\beta+3n+1,\alpha_1-\alpha_2-n+1,\alpha_1-\alpha_3-n+1\\ \alpha_1-\alpha_2+1,\alpha_1-\alpha_3+1,\alpha_1+\beta+N+2\end{matrix};1\right]}{n} & \dfrac{{}_4F_3\left[\begin{matrix}-n+1,\alpha_1+\beta+3n,\alpha_1-\alpha_2-n+1,\alpha_1-\alpha_3-n+1\\ \alpha_1-\alpha_2+1,\alpha_1-\alpha_3+1,\alpha_1+\beta+N+2\end{matrix};1\right]}{\alpha_1+\beta+3n}\\
\dfrac{{}_4F_3\left[\begin{matrix}-n+1,\alpha_2+\beta+3n+1,\alpha_2-\alpha_1-n,\alpha_2-\alpha_3-n+1\\ \alpha_2-\alpha_1+1,\alpha_2-\alpha_3+1,\alpha_2+\beta+N+2\end{matrix};1\right]}{\alpha_1-\alpha_2+n} & \dfrac{{}_4F_3\left[\begin{matrix}-n+1,\alpha_2+\beta+3n,\alpha_2-\alpha_1-n+1,\alpha_2-\alpha_3-n+1\\ \alpha_2-\alpha_1+1,\alpha_2-\alpha_3+1,\alpha_2+\beta+N+2\end{matrix};1\right]}{\alpha_2+\beta+3n}
\end{vmatrix}}{\begin{vmatrix}
\dfrac{{}_4F_3\left[\begin{matrix}-n,\alpha_1+\beta+3n+2,\alpha_1-\alpha_2-n,\alpha_1-\alpha_3-n+1\\ \alpha_1-\alpha_2+1,\alpha_1-\alpha_3+1,\alpha_1+\beta+N+2\end{matrix};1\right]}{\alpha_2-\alpha_1+n} & \dfrac{{}_4F_3\left[\begin{matrix}-n,\alpha_1+\beta+3n+1,\alpha_1-\alpha_2-n+1,\alpha_1-\alpha_3-n+1\\ \alpha_1-\alpha_2+1,\alpha_1-\alpha_3+1,\alpha_1+\beta+N+2\end{matrix};1\right]}{n(\alpha_1+\beta+3n+1)}\\
{}_4F_3\left[\begin{matrix}-n,\alpha_2+\beta+3n+2,\alpha_2-\alpha_1-n,\alpha_2-\alpha_3-n+1\\ \alpha_2-\alpha_1+1,\alpha_2-\alpha_3+1,\alpha_2+\beta+N+2\end{matrix};1\right] & \dfrac{{}_4F_3\left[\begin{matrix}-n+1,\alpha_2+\beta+3n+1,\alpha_2-\alpha_1-n,\alpha_2-\alpha_3-n+1\\ \alpha_2-\alpha_1+1,\alpha_2-\alpha_3+1,\alpha_2+\beta+N+2\end{matrix};1\right]}{\alpha_2+\beta+3n+1}
\end{vmatrix}}
\end{aligned}
$$

$$
\begin{aligned}
\frac{\tau^A_{2,3n+1}}{\tau^A_{2,3n+2}} = {}&\frac{n(N-3n-1)_2(\beta+1+3n)_2(\alpha_1+\beta+3n+1)(\alpha_2+\beta+3n+1)(\alpha_3+\beta+3n+1)_2}{(\alpha_1+\beta+4n+2)_2(\alpha_2+\beta+4n+1)_3(\alpha_3+\beta+4n+1)_3}\\
&\times(\alpha_2-\alpha_1+n)\frac{\begin{vmatrix}
\dfrac{{}_4F_3\left[\begin{matrix}-n,\alpha_1+\beta+3n+2,\alpha_1-\alpha_2-n,\alpha_1-\alpha_3-n+1\\ \alpha_1-\alpha_2+1,\alpha_1-\alpha_3+1,\alpha_1+\beta+N+2\end{matrix};1\right]}{\alpha_2-\alpha_1+n} & \dfrac{{}_4F_3\left[\begin{matrix}-n,\alpha_1+\beta+3n+1,\alpha_1-\alpha_2-n+1,\alpha_1-\alpha_3-n+1\\ \alpha_1-\alpha_2+1,\alpha_1-\alpha_3+1,\alpha_1+\beta+N+2\end{matrix};1\right]}{n(\alpha_1+\beta+3n+1)}\\
{}_4F_3\left[\begin{matrix}-n,\alpha_2+\beta+3n+2,\alpha_2-\alpha_1-n,\alpha_2-\alpha_3-n+1\\ \alpha_2-\alpha_1+1,\alpha_2-\alpha_3+1,\alpha_2+\beta+N+2\end{matrix};1\right] & \dfrac{{}_4F_3\left[\begin{matrix}-n+1,\alpha_2+\beta+3n+1,\alpha_2-\alpha_1-n,\alpha_2-\alpha_3-n+1\\ \alpha_2-\alpha_1+1,\alpha_2-\alpha_3+1,\alpha_2+\beta+N+2\end{matrix};1\right]}{\alpha_2+\beta+3n+1}
\end{vmatrix}}{\begin{vmatrix}
\dfrac{{}_4F_3\left[\begin{matrix}-n,\alpha_1+\beta+3n+3,\alpha_1-\alpha_2-n,\alpha_1-\alpha_3-n\\ \alpha_1-\alpha_2+1,\alpha_1-\alpha_3+1,\alpha_1+\beta+N+2\end{matrix};1\right]}{\alpha_3-\alpha_1+n} & \dfrac{{}_4F_3\left[\begin{matrix}-n,\alpha_1+\beta+3n+2,\alpha_1-\alpha_2-n,\alpha_1-\alpha_3-n+1\\ \alpha_1-\alpha_2+1,\alpha_1-\alpha_3+1,\alpha_1+\beta+N+2\end{matrix};1\right]}{(\alpha_1+\beta+3n+2)}\\
\dfrac{{}_4F_3\left[\begin{matrix}-n,\alpha_2+\beta+3n+3,\alpha_2-\alpha_1-n,\alpha_2-\alpha_3-n\\ \alpha_2-\alpha_1+1,\alpha_2-\alpha_3+1,\alpha_2+\beta+N+2\end{matrix};1\right]}{\alpha_3-\alpha_2+n} & \dfrac{{}_4F_3\left[\begin{matrix}-n,\alpha_2+\beta+3n+2,\alpha_2-\alpha_1-n,\alpha_2-\alpha_3-n+1\\ \alpha_2-\alpha_1+1,\alpha_2-\alpha_3+1,\alpha_2+\beta+N+2\end{matrix};1\right]}{\alpha_2+\beta+3n+2}
\end{vmatrix}}
\end{aligned}
$$

$$
\begin{aligned}
\frac{\tau^A_{2,3n+2}}{\tau^A_{2,3n+3}} = {}&\frac{(N-3n-2)_2(\beta+3n+2)_2(\alpha_1+\beta+3n+2)(\alpha_2+\beta+3n+2)(\alpha_3+\beta+3n+2)_2}{(\alpha_1+\beta+4n+3)_3(\alpha_2+\beta+4n+3)_2(\alpha_3+\beta+4n+2)_3}\\
&\times(\alpha_3-\alpha_1+n)(\alpha_3-\alpha_2+n)\frac{\begin{vmatrix}
\dfrac{{}_4F_3\left[\begin{matrix}-n,\alpha_1+\beta+3n+3,\alpha_1-\alpha_2-n,\alpha_1-\alpha_3-n\\ \alpha_1-\alpha_2+1,\alpha_1-\alpha_3+1,\alpha_1+\beta+N+2\end{matrix};1\right]}{\alpha_3-\alpha_1+n} & \dfrac{{}_4F_3\left[\begin{matrix}-n,\alpha_1+\beta+3n+2,\alpha_1-\alpha_2-n,\alpha_1-\alpha_3-n+1\\ \alpha_1-\alpha_2+1,\alpha_1-\alpha_3+1,\alpha_1+\beta+N+2\end{matrix};1\right]}{(\alpha_1+\beta+3n+2)}\\
\dfrac{{}_4F_3\left[\begin{matrix}-n,\alpha_2+\beta+3n+3,\alpha_2-\alpha_1-n,\alpha_2-\alpha_3-n\\ \alpha_2-\alpha_1+1,\alpha_2-\alpha_3+1,\alpha_2+\beta+N+2\end{matrix};1\right]}{\alpha_3-\alpha_2+n} & \dfrac{{}_4F_3\left[\begin{matrix}-n,\alpha_2+\beta+3n+2,\alpha_2-\alpha_1-n,\alpha_2-\alpha_3-n+1\\ \alpha_2-\alpha_1+1,\alpha_2-\alpha_3+1,\alpha_2+\beta+N+2\end{matrix};1\right]}{\alpha_2+\beta+3n+2}
\end{vmatrix}}{\begin{vmatrix}
\dfrac{{}_4F_3\left[\begin{matrix}-n-1,\alpha_1+\beta+3n+4,\alpha_1-\alpha_2-n,\alpha_1-\alpha_3-n\\ \alpha_1-\alpha_2+1,\alpha_1-\alpha_3+1,\alpha_1+\beta+N+2\end{matrix};1\right]}{n+1} & \dfrac{{}_4F_3\left[\begin{matrix}-n,\alpha_1+\beta+3n+3,\alpha_1-\alpha_2-n,\alpha_1-\alpha_3-n\\ \alpha_1-\alpha_2+1,\alpha_1-\alpha_3+1,\alpha_1+\beta+N+2\end{matrix};1\right]}{\alpha_1+\beta+3n+3}\\
\dfrac{{}_4F_3\left[\begin{matrix}-n,\alpha_2+\beta+3n+4,\alpha_2-\alpha_1-n-1,\alpha_2-\alpha_3-n\\ \alpha_2-\alpha_1+1,\alpha_2-\alpha_3+1,\alpha_2+\beta+N+2\end{matrix};1\right]}{\alpha_1-\alpha_2+n+1} & \dfrac{{}_4F_3\left[\begin{matrix}-n,\alpha_2+\beta+3n+3,\alpha_2-\alpha_1-n,\alpha_2-\alpha_3-n\\ \alpha_2-\alpha_1+1,\alpha_2-\alpha_3+1,\alpha_2+\beta+N+2\end{matrix};1\right]}{\alpha_2+\beta+3n+3}
\end{vmatrix}}
\end{aligned}
$$

Now we can write the other lower bidiagonal entries as follows. Note that

$$
\begin{aligned}
L_{1,n+1} &= -\frac{A_n^{(1)}(0)}{A_{n+1}^{(1)}(0)}, && n \in \{0,1,\dots,N-4\},\\
L_{2,n+1} &= -\frac{1}{L_{1,n+2}}\frac{\tau_{2,n}^A}{\tau_{2,n+1}^A} && n \in \{0,1,\dots,N-5\}\\
L_{3,n+1} &= -\frac{b_{n+3}^3}{L_{2,n+2}L_{1,n+3}U_{1,n-1}} = -\frac{b_{n+3}^3}{U_{1,n-1}}\frac{\tau_{2,n+2}^A}{\tau_{2,n+1}^A} && n \in \{0,1,\dots,N-6\}.
\end{aligned}
$$

Hence, using

$$
L_{2,3n+1} = -\frac{1}{L_{1,3n+2}}\frac{\tau_{2,3n}^A}{\tau_{2,3n+1}^A}, \quad L_{2,3n+2} = -\frac{1}{L_{1,3n+3}}\frac{\tau_{2,3n+1}^A}{\tau_{2,3n+2}^A}, \quad L_{2,3n+3} = -\frac{1}{L_{1,3n+4}}\frac{\tau_{2,3n+2}^A}{\tau_{2,3n+3}^A}
$$

we obtain the L_2's. Finally, from

$$
L_{3,3n+1} = -\frac{b_{3n+3}^3}{U_{1,3n-1}}\frac{\tau_{2,3n+2}^A}{\tau_{2,3n+1}^A}, \quad L_{3,3n+2} = -\frac{b_{3n+4}^3}{U_{1,3n}}\frac{\tau_{2,3n+3}^A}{\tau_{2,3n+2}^A}, \quad L_{3,3n+3} = -\frac{b_{3n+5}^3}{U_{1,3n+1}}\frac{\tau_{2,3n+4}^A}{\tau_{2,3n+3}^A}.
$$

we obtain the L_3's. □

Declarations

Competing interests. The authors declare that they have no competing interests.

Author contributions. All authors contributed to the conception of the work, the development of the theoretical results, and the writing of the manuscript. All authors read and approved the final manuscript.

Use of artificial intelligence. Artificial intelligence tools were used solely to improve the presentation (e.g. language and readability). All AI-assisted edits were reviewed and validated by the authors, who remain fully responsible for the content of the manuscript.

Data availability. No datasets were generated or analysed during the current study.

Code availability. No code was used or generated for the current study.

Ethics approval. Not applicable.

Consent to participate. Not applicable.

Consent for publication. Not applicable.

Acknowledgments

AB was financially supported by the Fundação para a Ciência e a Tecnologia (Portuguese Foundation for Science and Technology) under the scope of the projects UID/00324/2025 (Centre for Mathematics of the University of Coimbra).

AF acknowledges CIDMA Center for Research and Development in Mathematics and Applications (University of Aveiro) and the Portuguese Foundation for Science and Technology (FCT) within project UID/04106/2025.

MM acknowledges Spanish "Agencia Estatal de Investigación" research projects [PID2021-122154NB-I00], *Ortogonalidad y Aproximación con Aplicaciones en Machine Learning y Teoría de la Probabilidad* and [PID2024-155133NB-I00], *Ortogonalidad, aproximación e integrabilidad: aplicaciones en procesos estocásticos clásicos y cuánticos.*

References

[1] Carlos Álvarez-Fernández, Ulises Fidalgo Prieto, and Manuel Mañas, *Multiple orthogonal polynomials of mixed type: Gauss–Borel factorization and the multi-component 2D Toda hierarchy*, Advances in Mathematics **227** (2011), 1451–1525.

[2] Carlos Álvarez-Fernández, Ulises Fidalgo Prieto, and Manuel Mañas, *The multicomponent 2D Toda hierarchy: Generalized matrix orthogonal polynomials, multiple orthogonal polynomials and Riemann–Hilbert problems*, Inverse Problems **26** (2010), 055009.

[3] George E. Andrews, Roger Askey, and Ranjan Roy, *Special Functions*, Cambridge University Press, Cambridge, 1999.

[4] Jorge Arvesu, Jonathan Coussement & Walter Van Assche, *Some discrete multiple orthogonal polynomials*, Journal of Computational and Applied Mathematics **153** (2003) 19–45.

[5] Bernhard Beckerman, Jonathan Coussement & Walter Van Assche, *Multiple Wilson and Jacobi–Piñeiro polynomials*, Journal of Approximation Theory **132** (2005) 155–181.

[6] Amílcar Branquinho, Ana Foulquié-Moreno, and Manuel Mañas, *Multiple orthogonal polynomials: Pearson equations and Christoffel formulas*, Analysis and Mathematical Physics **12** (2022), paper 129.

[7] Amílcar Branquinho, Ana Foulquié-Moreno, and Manuel Mañas, *Bidiagonal factorization of tetradiagonal matrices and Darboux transformations*, Analysis and Mathematical Physics **13** (2023), article number 42.

[8] Amílcar Branquinho, Ana Foulquié-Moreno, and Manuel Mañas, *Spectral theory for bounded banded matrices with positive bidiagonal factorization and mixed multiple orthogonal polynomials*, Advances in Mathematics **434** (2023), 109313.

[9] Amílcar Branquinho, Juan EF Díaz, Ana Foulquié-Moreno, and Manuel Mañas, *Hahn multiple orthogonal polynomials of type I: Hypergeometric expressions*, Journal of Mathematical Analysis and Applications **528** (2023) 1277471.

[10] Amílcar Branquinho, Juan EF Díaz, Ana Foulquié-Moreno, and Manuel Mañas, *Hypergeometric expressions for type I Jacobi-Piñeiro orthogonal polynomials with arbitrary number of weights*, Proceedings of the American Mathematical Society **B11** (2024) 200–210.

[11] Amílcar Branquinho, Juan EF Díaz, Ana Foulquié-Moreno, and Manuel Mañas, *Classical multiple orthogonal polynomials for arbitrary number of weights and their explicit representation*, Studies in Applied Mathematics **154** (2025) e70033.

[12] Amílcar Branquinho, Juan EF Díaz, Ana Foulquié-Moreno, and Manuel Mañas, *Finite Markov chains and multiple orthogonal polynomials*, Journal of Computational and Applied Mathematics **463** (2025) 116485.

[13] Amílcar Branquinho, Juan EF Díaz, Ana Foulquié-Moreno, and Manuel Mañas, *Bidiagonal factorization of the recurrence matrix for the multiple Hahn orthogonal polynomials,* Linear Algebra and its Applications **721** (2025) 5-25.

[14] Amilcar Branquinho, Juan EF Díaz, Ana Foulquié-Moreno, Manuel Mañas, and Carlos Álvarez-Fernández, *Jacobi-Piñeiro random walks*, Revista de la Real Academia de Ciencias Exactas, Físicas y Naturales. Serie A. Matemáticas **118** (2024), article number 15.

[15] Amílcar Branquinho, Juan EF Díaz, Ana Foulquié-Moreno, Manuel Mañas, and Thomas Wolfs, *Integral and hypergeometric representations for multiple orthogonal polynomials*, International Mathematics Research Notices **2025** (12) (2025) rnaf168.

[16] Shaun M. Fallat and Charles R. Johnson, *Totally Nonnegative Matrices*, Princeton Series in Applied Mathematics, Princeton University Press, Princeton, 2011.

[17] Manuel Mañas and Miguel Rojas, *General Christoffel perturbations for multiple orthogonal polynomials of the mixed type on the step-line*, Numerical Algorithms (2025), doi:10.1007/s11075-025-02230-6.

[18] Manuel Mañas and Miguel Rojas. *General Geronimus perturbations for mixed multiple orthogonal polynomials*, Analysis and Mathematical Physics **15** (2025) article number 50 .

[19] Peter J. Olver. *On multivariate interpolation*, Studies in Applied Mathematics **116** (2006) 135-240.

[20] Allan Pinkus, *Totally Positive Matrices*, Cambridge Tracts in Mathematics **181**, Cambridge University Press, Cambridge, 2010.

[21] Lucy J. Slater,*Generalized Hypergeometric Functions*, Cambridge University Press, Cambridge, 2008.