FAT LIE THEORY

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Abstract. We discuss a new point of view of representation theory of Lie groupoids and algebroids: fat Lie theory. The category of fat extensions is introduced, as well as the category of abstract 2-term representations up to homotopy (ruths) — the intrinsic objects behind usual (split) 2-term ruths. We obtain a one-to-one correspondence between them, and relate to the well-known equivalence between 2-term ruths and VB-groupoids/algebroids. On the other hand, we show that fat extensions of groupoids correspond to general linear PB-groupoids. The differentiation procedure of fat extensions is discussed, as well as the functorial aspects of all mentioned correspondences. In particular, we upgrade the one-to-one correspondence between general linear PB-groupoids and VB-groupoids of Cattafi and Garmendia to an equivalence of categories. Fat extensions are intimately related to another notion we introduce: core extensions. We show that they correspond to vertically/horizontally core-transitive double groupoids, generalising work by Brown, Jotz-Lean and Mackenzie. This way, we also realise regular fat extensions as general linear double groupoids.

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Introduction

The representation theory of Lie groupoids and algebroids has been widely studied by now, and many different viewpoints have been proposed; see, for example, [ELW99, CF05, GSM10, AAC12, AAC13, GSM17, Wol17, CMS20, dHS19, Bou22, CG26]. The adjoint representation can be seen as a 2-term representation up to homotopy [AAC12, AAC13] (we abbreviate representation up to homotopy to ruth from now on), and as a VB-groupoid/algebroid [Pra74, Pra88, Mac05] through the tangent bundle. The two points of view are related in a very precise way, for there is an equivalence of categories between the categories of 2-term ruths and VB-groupoids/algebroids that sends the adjoint ruth of a groupoid/algebroid to (an isomorphic copy of) the tangent bundle of the groupoid/algebroid [GSM10, GSM17, dHO20].

In this work, we propose another point of view, axiomatising a structure that has been around, but has not yet been explicitly mentioned in the literature. We introduce a category of *fat extensions*, both for Lie groupoids and for Lie algebroids, that is equivalent to the category of 2-terms ruths and of VB-groupoids/algebroids. The structure has indeed been around in different contexts, for the main example is the first *jet groupoid* or first *jet algebroid*, together with its canonical cochain complex representation; see, for example, [CF05, Bla06, Sal13, CSS15, Bla16]. But in its generality presented here, it seems that the type of structure we will consider appeared first in [GSM10] as the *fat algebroid* of a VB-algebroid, and in [GSM17] as the *fat groupoid* of a VB-groupoid. Especially the fat algebroid and its generalisations have been studied in different contexts [She12, DE19, Pap21, MP21, LPV23, She23, LBM25]. The fat groupoid appears, for example, in [LPV23, LBM25, CG26].

The main goal of this work is to establish equivalences between the category of fat extensions and the following other models presented in the literature:

$$\begin{tikzcd}
& \{\text{Fat extensions}\} \arrow[dl, leftrightarrow] \arrow[d, leftrightarrow] \arrow[dr, leftrightarrow] & \\
\{\text{VB-groupoids}\} \arrow[r, leftrightarrow] & \{\text{General linear PB-groupoids}\} \arrow[r, leftrightarrow] & \{\text{2-term ruths}\}
\end{tikzcd}$$

We also explain the connection to [Wol17] and [dHS19]. The discussion about the infinitesimal analogue of the correspondences is kept shorter, and we only briefly comment on the notion of "general linear PB-algebroid". In turn, we also discuss the Lie functor — the differentiation procedure. The work presented here reveals many new concepts that become directly related to the representation theory of Lie groupoids and algebroids. On the other hand, we motivate throughout that, for many purposes, fat Lie theory is a natural and useful perspective on the representation theory in Lie theory.

We start the motivation of this work through two tangential projects that will be mentioned several times. One is joint work with João Nuno Mestre and Luca Vitagliano (to appear in [MOV]) and one is joint work with Ioan Mărcuţ and João Nuno Mestre (to appear in [MMO]).

The topic of [MOV] is to develop further the theory of (infinitesimally) multiplicative tensors on Lie groupoids and algebroids. In that project, we consider tensor products of VB-groupoids and VB-algebroids, as well as their associated linear cohomology theory (see also [BCO09, BC12, CD17, BD19, DE19]). While working on that, it became clear that (infinitesimally) multiplicative tensors can be thought of as natural objects defined on the fat groupoid and the fat algebroid. In fact, there are at least two natural ways to take tensor products of fat extensions, and one in particular will show up in this work. Namely, morphisms of fat extensions can naturally be viewed as, what we call here, *invariant* cochains. Moreover, from the description presented in this work, it is immediate that maps of fat extensions define 1-cocycles in a canonically defined cochain complex — a 3-term ruth. Such reformulations to the theory of fat extensions show that the language is especially well-suited for such representation theoretic aspects like taking tensor products. While we make these points more precise, especially when discussing the functorial aspects, we discuss the relationship between tensor products of VB-groupoids, 2-term ruths and fat extensions in greater detail in [MOV].

Although we summarise the results already in this work, we develop the theory of *abstract ruths* for Lie groupoids in [MMO] (this should be compared to the infinitesimal counterpart discussed in [Meh14]). The main goal of that project, however, is to showcase different perspectives on the construction of certain homotopy operators for ruths, and how to do so *explicitly*. This should be compared to [TV25]. General homotopy theoretic methods (perturbation theory) are used to construct the desired homotopy operators for ruths, and this is especially satisfactory when understood abstractly. However, for the applications we have in mind, it is important to understand the analytical properties of the homotopy operators. Therefore, it is useful to pass to *split* ruths (which become ruths in the "usual sense"). Understanding the interplay between abstract and split ruths then becomes crucial. We start this work with a summary on abstract ruths, and we put an emphasis on abstract 2-term ruths.

Motivation for fat Lie theory is presented already in the literature: usually, when the fat groupoid or algebroid is used, it is because the language of fat extensions helps in thinking about the problem at hand. Take, for example, the appearance of the fat groupoid in [CG26]. The *frame bundle* of a VB-groupoid, one of the main objects of study in [CG26], is defined using the fat groupoid. The reason is simple: the structure of the frame bundle in terms of fat extensions uses only actions that are naturally associated to the fat groupoid and its canonical representations. Another example is by reinterpretation of [AAC11, MP21]. Namely, the *Weil algebra* of a Lie algebroid, the infinitesimal counterpart of the *Bott-Shulman-Stasheff complex* of a Lie groupoid [BSS76], is efficient to write down using the jet algebroid and its canonical representations. The conditions appearing in the definition of the Weil algebra of [AAC11] are natural in terms of the jet algebroid. The work [MP21] presents generalisations, and this has to do with possible generalisations of this work to the setting of LA-groupoids (or quasi LA-groupoids; see [ÁC24]) and double algebroids (see also [JL19], [Gra25a] and [FM23]). A clear future direction of this work is therefore to describe "enriched" versions of the equivalences described above.

On the other hand, we will make first steps towards "higher" versions of the equivalences discussed above. In [dHT25], a higher version of the equivalence between VB-groupoids and 2-term ruths is discussed. More precisely, it is proved that the categories of so-called higher vector bundles (VB-n-groupoids), equipped with a normal weakly flat cleavage, and split $(n+1)$-term ruths are equivalent. Our reformulation of ruths seems to suggest that the appearance of "normal weakly flat cleavages" has to do with the fact that split ruths are considered instead of abstract ruths. So, a clear future direction of this work is to use the present work as a stepping stone towards formulating the correct categories and establish the equivalences above in such a "higher" generality. This also relates to [MOV], for tensor products of VB-groupoids and VB-algebroids should be considered in such a higher context.

We now give a description of the contents of the work.

Description of the work. Before we explain the contents, let us give a brief introduction to the fat extension of a VB-groupoid V_G. The fat groupoid

$$\widehat{V}_G \rightrightarrows M$$

can be seen as the Lie groupoid of "pointwise bihorizontal distributions" or "pointwise linear bisections". That is, the bisections of $\widehat{V}_G$ comprise the *linear bisections* of V_G. The groupoid structure of $\widehat{V}_G$ is reminiscent of the usual group structure of bisections of a Lie groupoid. Moreover, the canonical action of the group of bisections on a groupoid defines a linear groupoid action of $\widehat{V}_G$ on V_G. The restriction of this action to the (core) 2-term cochain complex $C_M \to V_M$ of V_G, which sits naturally inside V_G as an action VB-groupoid $C_M \ltimes V_M$, reveals the structure of a *cochain complex representation* of $\widehat{V}_G$ on $C_M \to V_M$. The short exact sequence of VB-groupoids

$$\begin{array}{ccccccccc} 1 & \longrightarrow & C_M \ltimes V_M & \longrightarrow & V_G & \longrightarrow & 0_G & \longrightarrow & 1 \\ & & \downarrow\downarrow & & \downarrow\downarrow & & \downarrow\downarrow & & \\ 1 & \longrightarrow & V_M & \longrightarrow & V_M & \longrightarrow & 0_M & \longrightarrow & 1 \end{array}$$

descends, by passing to pointwise linear bisections, to a short exact sequence of Lie groupoids

$$1 \longrightarrow \mathrm{H}(V_M, C_M) \longrightarrow \widehat{V}_G \longrightarrow G \longrightarrow 1.$$

On the one hand, this reveals that $\widehat{V}_G$ is an extension of G by the *bundle of invertible homotopies*

$$\mathrm{H}(V_M, C_M) := \{h_x \in \mathrm{Hom}(V_M, C_M) \mid 1 + \partial h \text{ is invertible}\}.$$

On the other hand, it shows that the two pieces of data on $\widehat{V}_G$ defined above — the cochain complex representation and the extension — are compatible in a precise way (we come back to this later). This identifies $\widehat{V}_G$ as a *fat extension* of G over $C_M \to V_M$.

We now discuss the contents of this work. As mentioned above, our first goal in Section 2 is to explain an abstract point of view on ruths of Lie groupoids. Our viewpoint is related to the remark made after Definition 3.9 in [AAC13] and also with the approach on ruths of Lie groupoids taken in [Ste19]. However, our approach is to view ruths of Lie groupoids as intrinsically defined objects. Namely, as in [Meh14] in the case of Lie algebroids, we view a ruth as a finitely generated projective differential graded module — see Definition 2.1. The reason for including the discussion is that, with this point of view on ruths, the equivalence of categories described in [GSM17, dHO20] can be understood canonically as the functor

$$\{\text{VB-groupoids}\} \to \{\text{Abstract 2-term ruths}\} \qquad V_G \mapsto C_{\mathrm{VB}} V_G^* \cong C_{\mathrm{proj}} V_G$$

using the VB-complex from [CD17] or the projectable complex from [GSM17, CMS20]. Although a more complete discussion will appear in [MMO], we believe the point of view we take is useful, and clarifies that all equivalences appearing in this work are canonical and come with a "split" version. At least in the 2-term case, these split categories are equivalent to the intrinsic models via the forgetful functor. We will comment on this several times.

After the introduction on abstract ruths, we formulate the category of fat extensions in Section 3 using the language of VB-groupoids. That is, we study properties of the fat groupoid of a VB-groupoid, and we will explain what structure turns it into a fat extension — see Definition 3.16. We also explain that we can encode the data of a fat extension using the *fat category* instead. We call such objects *fat category extensions* — see Definition 3.22. Fat category extensions are similar to fat extensions, but they are the intrinsic models for *weak representations* (see [Wol17]). Having both fat extensions and fat category extensions in mind will turn out to be useful. Afterwards, we will describe the typical constructions of VB-groupoids in terms of fat extensions. For example, the dual of a fat extension (see Definition 3.45) dualises simply its cochain complex representation, and the analogue of fiber products of VB-groupoids become a type of space of "invertible blockmatrices" for fat extensions (see Definition 3.37). In essence, such phenomena seem to make the theory of fat extensions suited for considering (infinitesimally) multiplicative tensors, for example.

This introduction to the theory of fat extensions, together with the initial step of reformulating the theory of ruths of groupoids, will allow us to formulate the canonical one-to-one correspondences of categories

$$\{\text{VB-groupoids}\} \longleftrightarrow \{\text{Fat extensions}\} \longrightarrow \{\text{Abstract 2-term ruths}\}.$$

The functorial aspects of these equivalences are also discussed, but delayed until Section 7. As mentioned above, the step from VB-groupoids to abstract 2-term ruths is to associate to a VB-groupoid either its VB-complex or its projectable complex. The step from VB-groupoids to fat extensions is discussed in Section 4. Associating an abstract 2-term ruth of a fat extension is by taking its *invariant complex* (see Definition 4.14). In other words, we reformulate the abstract 2-term ruth of a VB-groupoid using the fat groupoid (see Proposition 4.17). The complex turns out to be a type of *relative complex* as introduced in [Sal24]. When interpreting the jet groupoid as a fat extension, the result is the adjoint ruth. This complex can be seen as the deformation complex of the Lie groupoid (see [CMS20]), and it is the global counterpart of the deformation complex of a Lie algebroid in terms of the jet algebroid (see [She12]). A small section, Section 10, is dedicated to this new description of the deformation complex.

We then proceed: in [CG26], it is shown that there is a one-to-one correspondence between *general linear PB-groupoids* (PB-groupoids with structural strict Lie 2-groupoid the general linear strict Lie 2-groupoid) and VB-groupoids. The fat groupoid is used explicitly in [CG26]; in hindsight, the fat groupoid shows up in [CG26] because of passing through the correspondence with the category of fat extensions:

$$\{\text{VB-groupoids}\} \longrightarrow \{\text{Fat extensions}\} \longrightarrow \{\text{General linear PB-groupoids}\}.$$

We will go through these correspondences in detail in Section 5, and slightly reformulate the correspondence between VB-groupoids and general linear PB-groupoids. Moreover, part of this work is to formulate the category of general linear PB-groupoids that turns the above correspondences into equivalences of categories. This is done in Section 7. The step

$$\{\text{General linear PB-groupoids}\} \to \{\text{Abstract 2-term ruths}\}$$

produces from a PB-groupoid a type of complex of GL-equivariant cochains.

Now, this relation between general linear PB-groupoids and fat extensions is rather similar to the discussion of [BM92] (and [JLM21]), where double groupoids (double algebroids) are discussed and how double transitive groupoids (double transitive algebroids) are equivalently described by so-called *core diagrams*. The suspicion seems justified, for general linear PB-groupoids sometimes come with a *gauge double groupoid*. However, the double groupoids in question are not double transitive groupoids. Still, a generalisation of [BM92] shows that horizontally or vertically *core-transitive* double groupoids are equivalently described by, what we call here, *core extensions* (see Definition A.3). The generalisation should be seen as a clarification of the concluding remarks in [BM92]. Core extensions are generalisations of core diagrams of [BM92], but they are also generalisations of so-called *crossed modules* of Lie groupoids as in [Mac89, And05, LGW08]. The proof of the generalisation of the main result of [BM92] is postponed to Appendix A. There we also show that, as expected, core extensions of general linear double groupoids are fat extensions. In the main body, in Section 5, we do explain what the concepts mentioned above mean.

After the discussion on the many equivalent ways of organising the information of the above categories in the setting of Lie groupoids, we turn to the infinitesimal side of things in Section 8. That discussion is kept brief, but the essential points will be made. In particular, a new model for the linear complex of a VB-algebroid will be discussed in terms of an invariant complex of the fat algebroid (see Definition 8.19). The discussion on the equivalences between 2-term ruths, VB-algebroids and fat extensions is short, but rather complete. Although clear indications will be given about the steps to "PB-algebroids" and "core-transitive double algebroids", these concepts are not fully developed and deserve a more thorough treatment elsewhere. The concept of core extension for Lie algebroids is also introduced (see Definition 8.27). After that, in Section 9, we discuss the Fat Lie theory to differentiate from global fat extensions to infinitesimal fat extensions. Part of that discussion appears in [DE19], but we elaborate on some aspects. We also write the Van Est map of [AAS11, CD17] in terms of invariant cochains.

Our discussion in Appendix A points towards an interesting new direction: can we describe PB-groupoids/algebroids in terms of "core data"? With an eye on defining geometric structures on Lie groupoids and algebroids, this seems an interesting direction that will be pursued elsewhere.

The content of the sections is as follows:

- In Section 1 we recall some of the basic structure behind VB-groupoids. For the reader that is new to this topic, see, for example, [Pra74, Pra88, Mac05, GSM10, GR09, BCdH16, GSM17, GGR22]
- In Section 2 we explain our point of view on abstract ruths. We give a rather complete summary, but to keep the discussion short, the proofs of the statements presented there are kept to a minimum. The equivalence between VB-groupoids and abstract 2-term ruths is Theorem 2.20. In Proposition 2.19 we also demonstrate how abstract ruths can be "split" to recover the usual notion of ruths.
- In Section 3 we give the necessary background on the structure behind the fat groupoid of a VB-groupoid. In particular, we give an account on the bundle of invertible homotopies (see Section 3.2). The main definition of fat (category) extension of groupoids will be discussed here (Definition 3.16 and Definition 3.22), and so will examples of fat extensions (see Section 3.6).
- In Section 4 we explain the one-to-one correspondence between VB-groupoids and fat extensions (Proposition 4.5) and we do the same for split 2-term ruths and fat extensions (see Section 4.3, but also Section 6.2) as well as for abstract 2-term ruths and fat extensions (See Corollary 4.11 and Proposition 4.12).
- In Section 5 we define (general linear) PB-groupoids. Our setup is slightly different from [CG26], but we comment on the differences. In the rest of the section, we set the correspondences between general linear PB-groupoids and fat extensions (See Proposition 5.21), as well as between fat extensions and general linear double groupoids (in the regular case; the details for double groupoids are left for Appendix A). Here, we also present the model of 2-term ruths in terms of general linear PB-groupoids (Proposition 5.28). Since our definition of general linear PB-groupoid is slightly different than in [CG26], we also show how general linear PB-groupoids correspond to VB-groupoids (Proposition 5.24).

- In Section 6 we make some other types of comments on the correspondences. Firstly, the vanishing theorem for 2-term ruths of [AAC13, CD17] is proved using the invariant complex of a fat extension (Proposition 6.1). Secondly, the "split" category of fat extensions is introduced. That is, we define the notion of *fat splitting* (see Definition 6.2), which corresponds to a cleavage on the level of VB-groupoids (see Proposition 6.3). Thirdly, we explain the connection to the work [dHS19] (see Section 6.3).
- In Section 7 we go into the functorial aspects. We introduce the definition of a fat extension through the language of VB-groupoids. That way, we will naturally set the equivalence of categories between VB-groupoids and fat extensions (Theorem 7.13). We also comment on the equivalence of categories between split 2-term ruths and fat extensions. Afterwards, using the language of VB-groupoids again, we will introduce morphisms of general linear PB-groupoids, and show that the categories of VB-groupoids and general linear PB-groupoids are equivalent (Theorem 7.22). We will then also comment on the equivalence between general linear PB-groupoids and fat extensions (Theorem 7.24).
- In Section 8 we explain the infinitesimal picture. That is, we introduce fat extensions using the language of VB-algebroids, we give an abstract definition of fat extension, and then we go through the correspondences between abstract and split 2-term ruths, VB-algebroids and fat extensions (Proposition 8.26 and Theorem 8.32). Except for a small comment, "general linear PB-algebroids" and "general linear double algebroids" are not discussed.
- In Section 9 we explain how, abstractly, fat extensions of groupoids differentiate, in a functorial way, to fat extensions of algebroids (Proposition 9.1). Then we explicitly work out the details for the case of the fat groupoid of a VB-groupoid (see Section 9.3; for this step, see also [DE19]). After that, we write the Van Est map for invariant cochains in terms of the fat groupoid of a VB-groupoid (Theorem 9.10).
- In Section 10 we make a small comment that we get a new model for the deformation cohomology of a Lie groupoid. In particular, we explain there that deformations of Lie groupoids give rise to 2-cocycles (this is entirely analogous to the construction of [CMS20]).
- In Appendix A we introduce the notion of vertical/horizontal core extensions and prove that they correspond to vertically/horizontally core-transitive double groupoids (Theorem A.9). There is also a discussion on PB-groupoids that come with a gauge double groupoid. Proposition A.21 shows that regular fat extensions correspond to a type of core extension.

1. Notation for VB-groupoids

We develop here the relevant notation for VB-groupoids that we use.

1.1. **Notation for groupoids.** We denote the structure maps of a groupoid $G \rightrightarrows M$ by $(\mathbf{s}, \mathbf{t}, \mathbf{m}, \mathbf{i}, \mathbf{u})$, but we often simplify the notation to

$$\mathbf{m}(g, g') = gg' \qquad \mathbf{i}(g) = g^{-1} \qquad \mathbf{u}(x) = 1_x.$$

We also use the division map of G, denoted by

$$\overline{\mathbf{m}}(g, g') = g(g')^{-1}.$$

The groupoid G has an underlying simplicial manifold (the nerve):

$$G^{(\bullet)} : \cdots \; G^{(2)} \; \substack{\longrightarrow \\ \longleftarrow \\ \longrightarrow \\ \longleftarrow \\ \longrightarrow} \; G \; \substack{\longrightarrow \\ \longleftarrow \\ \longrightarrow} \; M.$$

We denote the face maps by[1]

$$d_k(g_1,\dots,g_\bullet) \coloneqq \begin{cases} (g_2,\dots,g_\bullet) & \text{if } k=0 \\ (g_1,\dots,g_kg_{k+1},\dots,g_\bullet) & \text{if } 1\le k\le \bullet-1 \\ (g_1,\dots,g_{\bullet-1}) & \text{if } k=\bullet \end{cases}$$

(for $\bullet = 1$, $d_0 = \mathbf{s}$ and $d_1 = \mathbf{t}$) and the degeneracy maps by[2]

$$u_k(g_1,\dots,g_\bullet) \coloneqq (g_1,\dots,g_k,1_{\mathbf{s}g_k},g_{k+1},\dots,g_\bullet) \text{ for all } 0\le k\le \bullet.$$

We put $C^\bullet G = C^\infty G^{(\bullet)}$ and

$$\delta \coloneqq (-1)^\bullet \textstyle\sum_{k=0}^{\bullet+1}(-1)^k d_k^*.$$

Then CG becomes a differential graded algebra with δ and the product[3]

$$f\cdot f'(g,g') \coloneqq (-1)^{\bullet\cdot\bullet'} f(g)f'(g'),$$

where $f\in C^\bullet G$, $f'\in C^{\bullet'}G$ and $(g,g')\in G^{(\bullet+\bullet')}$. We will also use the normalised (sub)complex

$$(C^\bullet G)_{\mathrm{N}} \coloneqq \{f\in C^\bullet G \mid \text{for all } 0\le \ell\le \bullet-1, v_\ell f = 0\}$$

where $v_\ell = u_\ell^*$. Importantly, the inclusion

$$(CG)_{\mathrm{N}} \hookrightarrow CG$$

is a quasi-isomorphism (see [EM47] for groups; see also Remark 2.11).

Lastly, we write

$$\mathbf{s}^{(\bullet)}(g_1,\dots,g_\bullet) \coloneqq \mathbf{s}g_\bullet \qquad \mathbf{t}^{(\bullet)}(g_1,\dots,g_\bullet) \coloneqq \mathbf{t}g_1 \qquad \mathrm{pr}_k^{(\bullet)}(g_1,\dots,g_\bullet) \coloneqq g_k$$

and if $E_M \to M$ is a vector bundle, we write

$$C^\bullet(G;E) \coloneqq \Gamma(\mathbf{t}^{(\bullet)})^* E_M.$$

1.2. **Notation for VB-groupoids.** We denote a VB-groupoid over $G \rightrightarrows M$ by $V_G \rightrightarrows V_M$:

$$\begin{tikzcd} V_G \arrow[r, shift left] \arrow[r, shift right] \arrow[d] & V_M \arrow[d] \\ G \arrow[r, shift left] \arrow[r, shift right] & M \end{tikzcd}$$

The structure maps for V_G and for G are denoted by the same symbols. We then write, for all $g\in G$,

$$\mathbf{s}_g : (V_G)_g \to (V_M)_{\mathbf{s}g} \qquad \mathbf{t}_g : (V_G)_g \to (V_M)_{\mathbf{t}g}$$

for the induced linear maps on fibers. The scalar multiplication of V_G and of V_M are both denoted by h^λ, but we often simplify to

$$h^\lambda v = \lambda v.$$

For all $\lambda\in\mathbb{R}_{>0}$ and $(v,w)\in V_G^{(2)}$ we then have

$$\lambda(v\cdot w) = (\lambda v)\cdot(\lambda w).$$

[1] That is, d_k "skips" the k-th basepoint.

[2] That is, u_k "repeats" the k-th basepoint.

[3] If we view f and g as having degree $\bullet$, and f' and g' as having degree $\bullet'$, we can interpret the sign as a Koszul sign rule. Similarly, our sign convention for δ can be explained this way.

That is, scalar multiplication by $\lambda \in \mathbb{R}_{>0}$ is a groupoid isomorphism $V_G \to V_G$ (see, for example, [BCdH16]).

For the core of the VB-groupoid we always use the right-core

$$C_M := \ker \mathbf{s}|_M$$

and we denote

$$\partial := C_M \xrightarrow{\mathbf{t}} V_M$$

for the 2-term complex associated to V_G.

Writing

$$\ell_g : (C_M)_{\mathbf{s}g} \xrightarrow{-0_g \cdot c^{-1}} (V_G)_g \qquad r_g : (C_M)_{\mathbf{t}g} \xrightarrow{c \cdot 0_g} (V_G)_g,$$

the dual VB-groupoid $V_G^* \rightrightarrows C_M^*$ has source and target map given by $\mathbf{s} = \ell_g^*$ and $\mathbf{t} = r_g^*$, and the other structure maps are determined by

$$\mathbf{m}(\varphi_g, \varphi_h)(v \cdot w) = \varphi_g v + \varphi_h w \qquad \mathbf{i}(\varphi_g)(v^{-1}) = -\varphi_g v \qquad \mathbf{u}\varphi_x = \varphi_x|_{(C_M)_x}.$$

In particular, V_G^* has

$$\partial^* : V_M^* \xrightarrow{\mathbf{t}^*} C_M^*$$

as its associated 2-term complex.

Lastly, we mean by a cleavage Σ a unital splitting of the exact sequence

$$\begin{tikzcd} 0 \arrow[r] & \mathbf{t}^*C_M \arrow[r, "r_g"] & V_G \arrow[r, "\mathbf{s}"] & \mathbf{s}^*V_M \arrow[l, bend left, "\Sigma"] \arrow[r] & 0. \end{tikzcd}$$

The cleavage Σ induces a unital splitting $\overline{\Sigma}$

$$\begin{tikzcd} 0 \arrow[r] & \mathbf{s}^*C_M \arrow[r, "\ell_g"] & V_G \arrow[r, "\mathbf{t}"] & \mathbf{t}^*V_M \arrow[l, bend left, "\overline{\Sigma}"] \arrow[r] & 0 \end{tikzcd}$$

given by $\overline{\Sigma}_g v = (\Sigma_{g^{-1}} v)^{-1}$. The duals of these exact sequences are the corresponding exact sequences for the target and the source map, respectively, of the dual VB-groupoid.

1.3. **The fat groupoid and the fat extension of a VB-groupoid.** One of the main points of this work is the observation that, equivalently, we can describe a VB-groupoid $V_G \rightrightarrows V_M$ through its so-called fat groupoid $\widehat{V}_G \rightrightarrows M$, where

$$\widehat{V}_G = \{H_g : (V_M)_{\mathbf{s}g} \to (V_G)_g \mid \mathbf{s}_g H_g = 1 \text{ and } \mathbf{t}_g H_g \text{ is a linear isomorphism}\}$$

(see [GSM17]). Although we leave the full discussion for Section 3, here is a summary. A VB-groupoid turns out to be equivalent to the data of, what we call, a *fat extension*. In short, the fat extension is defined out of the Lie groupoid map

$$\widehat{V}_G \to G \qquad H_g \mapsto g,$$

which is a surjective submersion. The kernel of this map is a bundle of Lie groups, denoted $\mathrm{H}(V_M, C_M) \subset \mathrm{Hom}(V_M, C_M)$, and called the *bundle of invertible homotopies* for reasons that will be clarified in Section 3.2. Its structure of bundle of Lie groups only depends on the cochain complex $C_M \to V_M$, which, in turn, is canonically a cochain complex representation of $\mathrm{H}(V_M, C_M)$. But $C_M \to V_M$ is also canonically a cochain complex representation of $\widehat{V}_G$, and the short exact sequence of Lie groupoids over M

$$1 \longrightarrow \mathrm{H}(V_M, C_M) \longrightarrow \widehat{V}_G \longrightarrow G \longrightarrow 1$$

is compatible with these cochain complex representations. By this we mean that, on the one hand, the inclusion $\mathrm{H}(V_M, C_M) \hookrightarrow \widehat{V}_G$ is equivariant with respect to the cochain complex representations and, on the other hand, the two natural conjugation actions of $\widehat{V}_G$ on $\mathrm{H}(V_M, C_M)$ agree. Together, the cochain complex representation of $\widehat{V}_G$ on $C_M \to V_M$, the exact sequence above, and the compatibility between these two structures that we described is called the fat extension associated to V_G. So, the extension is of a very particular form, and our abstract definition of fat extension keeps track of this form.

1.4. **The linear complex and the VB-subcomplex.** The components of the nerve of a VB-groupoid V_G are vector bundles: for all $\bullet \geq 0$,

$$V_G^{(\bullet)} \to G^{(\bullet)}$$

is a vector bundle (with structure defined "componentwise"). The simplicial maps of V_G are vector bundle maps. In other words, the nerve of V_G is a simplicial vector bundle over the nerve of G. In particular, the 1-homogeneous cochains[4]

$$C^\bullet_{\mathrm{lin}} V_G = \{ f \in C^\infty V_G^{(\bullet)} \mid \text{ for all } \lambda \in \mathbb{R}_{>0},\ \lambda^* f = \lambda \cdot f \} \subset C^\bullet V_G \tag{1.1}$$

(i.e. the linear functions on the nerve) form a subcomplex called the linear complex.

The VB-complex is a subcomplex of the linear complex given by[5]

$$C^\bullet_{\mathrm{VB}} V_G = \{ f \in C^\bullet_{\mathrm{lin}} V_G \mid f(0_{g_1}, v_2, \dots, v_\bullet) = 0 \text{ and } \delta f(0_{g_1}, v_2, \dots, v_{\bullet+1}) = 0 \}.$$

Using this notation, the inclusion $C_{\mathrm{VB}} V_G \hookrightarrow C_{\mathrm{lin}} V_G$ is a quasi-isomorphism (see [CD17, dHO20] and Remark 2.13). Moreover, $C_{\mathrm{VB}} V_G^*$ is isomorphic to the complex of projectable cochains

$$C^\bullet_{\mathrm{proj}} V_G = \{ c : G^{(\bullet)} \to V_G \mid c(g_1, \dots, g_\bullet) \in (V_G)_{g_1} \text{ and } \mathbf{s} c(g_1, \dots, g_\bullet) = \mathbf{s} c(1_{\mathbf{t} g_2}, g_2, \dots, g_\bullet) \}$$

with differential given by

$$\delta = (-1)^\bullet (-\overline{\mathbf{m}}(d_1^*, d_0^*) + \textstyle\sum_{k=2}^{\bullet+1} (-1)^k d_k^*).$$

Without giving explicitly the differential, this complex appeared in [GSM17]. For $V_G = TG$, we have $C_{\mathrm{proj}} V_G = C_{\mathrm{def}} G$, and this complex appeared in [CMS20].

In terms of a representation up to homotopy (we write ruth for short), $C_{\mathrm{VB}} V_G^*$ is isomorphic to the ruth $\mathcal{E}_G$ associated to V_G

$$\mathcal{E}_G^\bullet = C^\bullet(G; C_M) \oplus C^{\bullet-1}(G; V_M)$$

with differential given by[6]

$$\delta = \partial + R_1 + R_2 + (-1)^\bullet \textstyle\sum_{k=1}^{\bullet+1} (-1)^k d_k^*.$$

---

[4] We wrote $\mathbb{R}_{>0}$ for the open interval $(0, \infty) \subset \mathbb{R}$.

[5] The conditions for $f \in C^\bullet_{\mathrm{lin}} V_G$ to be an element of $C^\bullet_{\mathrm{VB}} V_G$ can be rewritten as

$$f(0_{g_1}, v_2, \dots, v_\bullet) = 0 \text{ and } f(0_{g_1} \cdot v_2, \dots, v_{\bullet+1}) = f(v_2, \dots, v_{\bullet+1}).$$

[6] The structure of the ruth $\mathcal{E}_G$ is denoted by ∂, R_1, R_2. Defining this structure out of V_G uses a cleavage; we will recall the construction in Section 6.2.

We view the dual ruth $\mathcal{E}_G^*$ as being concentrated in positive degrees as well (this is a non-standard convention; we come back to this in Remark 2.9).

We describe a new model for the VB-complex in Section 4, which comes from understanding VB-groupoids as a fat extension. In the next section, Section 2, we adopt a point of view on ruths which allows us to see all the different models of VB-complexes canonically as ruths. The model $\mathcal{E}_G$ will then be called a split ruth.

2. Abstract representations up to homotopy of Lie groupoids

This section is kept short for it will appear in [MMO]. Still, we would like to emphasise the main points. We think the point of view we take is useful, and we stress on the natural appearance of the fat groupoid in the main example (see Proposition 2.19).

2.1. **The definition of an abstract ruth.**

We take the following approach to ruths of Lie groupoids (for Lie algebroids, see [Meh14]):

Definition 2.1. An *abstract representation up to homotopy* (ruth) $(\mathcal{E}_G)_{\mathrm{N}}$ is a finitely generated projective right differential graded $(CG)_{\mathrm{N}}$-module. In particular, the differential

$$\delta : (\mathcal{E}_G^\bullet)_{\mathrm{N}} \to (\mathcal{E}_G^{\bullet+1})_{\mathrm{N}}$$

satisfies the Leibniz rule: for all $c \in (\mathcal{E}_G)_{\mathrm{N}}$ and $f \in (CG)_{\mathrm{N}}$,

$$\delta(c \cdot f) = \delta(c) \cdot f + (-1)^c c \cdot \delta f.$$

Remark 2.2. Using the inclusion

$$(CG)_{\mathrm{N}} \hookrightarrow CG$$

an abstract ruth $(\mathcal{E}_G)_{\mathrm{N}}$ gives rise to a finitely generated projective differential graded CG-module

$$\mathcal{E}_G := (\mathcal{E}_G)_{\mathrm{N}} \otimes_{(CG)_{\mathrm{N}}} CG.$$

Here, the elements of $\mathcal{E}_G$ are graded by total degree and generated by pure tensors

$$c \otimes f \in \mathcal{E}_G$$

where $c \in (\mathcal{E}_G)_{\mathrm{N}}$ and $f \in CG$. Whenever $c \in (\mathcal{E}_G)_{\mathrm{N}}$, $f \in C_N G$ and $f' \in CG$, we have

$$(c \cdot f) \otimes f' = c \otimes (f \cdot f')$$

and the differential on $\mathcal{E}_G$ is such that

$$\delta(c \otimes f) = \delta(c) \otimes f + (-1)^c c \otimes \delta f.$$

We will often work at the level of CG-modules, but it is important to incorporate a subcomplex of normalised cochains $(\mathcal{E}_G)_{\mathrm{N}}$ of $\mathcal{E}_G$ (in this way). We will elaborate on this point in Section 2.3.

To clarify: a CG-module $\mathcal{E}_G$ is finitely generated projective if we can find a split short exact sequence of CG-modules[7]

$$0 \longrightarrow \mathcal{R} \longrightarrow \bigoplus_{k\in I} CG[-\ell_k] \longrightarrow \mathcal{E}_G \longrightarrow 0$$

where I is a finite indexing set, and all ℓ_k are non-negative. Alternatively, this is equivalent to the existence of a CG-module $\mathcal{R}$ such that $\mathcal{R} \oplus \mathcal{E}_G$ is a free CG-module.

[7]Here, $C^\bullet G[-\ell] = C^{\bullet-\ell}G$, so $CG[-\ell]$ is concentrated in degrees $\bullet \geq \ell$.

Example 2.3. The main example we consider in this work is the VB-complex, or the projectable complex, of a VB-groupoid V_G:

$$C_{\mathrm{VB}}V_G \cong C_{\mathrm{proj}}V_G^*.$$

(see Section 2.5). Notice that the normalised complex is the subcomplex

$$(C^\bullet_{\mathrm{VB}}V_G)_{\mathrm{N}} = \bigcap_{\ell=0}^{\bullet-1} \ker \nu_\ell,$$

where $\nu_\ell = u_\ell^*$ for all $0 \le \ell \le \bullet - 1$ (here, u_ℓ are the degeneracy maps of V_G). For $C_{\mathrm{proj}}V_G$, ν_0 takes the form ($\mathbf{u}$ and $\mathbf{s}$ denote the unit and source map of V_G, respectively)

$$\nu_0 : C^\bullet_{\mathrm{proj}}V_G \to C^{\bullet-1}_{\mathrm{proj}}V_G \qquad \nu_0 c(g_2, \cdots, g_\bullet) = c(1_{\mathbf{t}g_2}, g_2, \cdots, g_\bullet) - \mathbf{u}_{1_{\mathbf{t}g_2}}\mathbf{s}_{1_{\mathbf{t}g_2}} c(1_{\mathbf{t}g_2}, g_2, \cdots, g_\bullet)$$

and $\nu_\ell = u_\ell^*$ for all $1 \le \ell \le \bullet - 1$ (here, u_ℓ are the degeneracy maps of G).

2.2. **Locally free sheaves of CG-modules.** In fact, we conjecture that a type of Serre-Swan theorem relates higher vector bundles with abstract ruths (see Remark 2.21). A weak version at least holds: the global sections functor sets an equivalence of categories between "locally free sheaves of CG-modules" and finitely generated projective CG-modules. We discuss next how we think of a (locally free) sheaf of CG-modules.

Remark 2.4. We can view CG as a graded sheaf (over M) of right CG-modules as follows. For each open set U of M, we set

$$G^{(\bullet)}U = (\mathbf{t}^{(\bullet)})^{-1}U = \{(g_1, \dots, g_\bullet) \in G^{(\bullet)} \mid \mathbf{t}g_1 \in U\}.$$

We then write

$$C^\bullet G(U) = C^\infty(G^{(\bullet)}U).$$

The product of CG defines the structure of right CG-module on this sheaf. That is, for $f \in C^\bullet G(U)$ and $f' \in C^\bullet G$, we define $f \cdot f' \in C^{\bullet+\bullet'}G(U)$ by

$$(f \cdot f')(g, g') \coloneqq (-1)^{\bullet\cdot\bullet'} f(g) \cdot f'(g').$$

When we view CG as a sheaf in this way, the differential is only defined on the global sections. There are ways to think of the differential as being local which is interesting for developing further representation theory for non-Hausdorff Lie groupoids. This will be discussed in [MMO].

Consider now a graded sheaf of CG-modules $\mathcal{E}_G$. That is, for all open sets $U \subset M$, we have a right CG-module $\mathcal{E}_G(U)$ (altogether satisfying sheaf-like properties). Locally free means that, for all $x \in M$, there is an open subset $U \ni x$ of M such that

$$\mathcal{E}_G(U) \cong \bigoplus_{k\in I} CG[-\ell_k](U)$$

as sheaves of CG-modules, where I is a finite indexing set and all ℓ_k are non-negative. The following result is a first step towards a type of Serre-Swan theorem.

Proposition 2.5. *The global sections functor establishes an equivalence of categories*

$$\{\text{locally free sheaves of } CG\text{-modules}\} \xrightarrow{\sim} \{\text{finitely generated projective } CG\text{-modules}\}.$$

Although the proof is relatively straightforward, we discuss the proof only in [MMO]. In fact, the main ingredients of the proof will be discussed next.

Remark 2.6. Notice that CG is canonically filtered by (differential, left and right) ideals:

$$\mathcal{I}_k := \bigoplus_{\bullet \geq k} C^\bullet G.$$

We write $\mathcal{I}_k^\bullet = \mathcal{I}_k \cap C^\bullet G$. These ideals canonically define a filtration on a locally free sheaf of CG-modules / finitely generated projective CG-module $\mathcal{E}_G$:

$$\mathcal{F}_k := \mathcal{E}_G \cdot \mathcal{I}_k = \bigoplus_{\bullet \geq k} \{c \in \mathcal{E}_G^\bullet \mid c = \textstyle\sum_m c_m \cdot f_m \text{ for some } c_m \in \mathcal{E}_G^{\bullet - \bullet'} \text{ and } f_m \in \mathcal{I}_k^{\bullet'}\}.$$

We write $\mathcal{F}_k^\bullet = \mathcal{F}_k \cap \mathcal{E}_G^\bullet$.

Proposition 2.7. *Let $\mathcal{E}_G$ be a ruth. For all $\bullet \geq 0$,*

$$\Gamma E_M^\bullet = \mathcal{F}_0^\bullet / \mathcal{F}_1^\bullet$$

forms the space of sections of a vector bundle $E_M^\bullet$ over M. Moreover, the differential δ descends to a cochain complex of vector bundles:

$$E_M^0 \xrightarrow{\partial} E_M^1 \xrightarrow{\partial} \cdots \xrightarrow{\partial} E_M^{n-1} \xrightarrow{\partial} E_M^n$$

(We assume that $E_M^{\bullet > n} = 0$; in this case we call $\mathcal{E}_G$ an $(n+1)$-term abstract ruth).

Remark 2.8. The quotient map $(CG)_{\mathrm{N}} \to (CG)_{\mathrm{N}}/\mathcal{I}_1 = C^\infty M$ comes from the unit map

$$1_M \hookrightarrow G$$

viewed as a groupoid map (here, 1_M denotes the unit groupoid). Now, we can always pullback an abstract ruth along a Lie groupoid map, and what Proposition 2.7 shows is that the resulting abstract ruth $\mathcal{E}_G|_M$ can be seen as a complex of vector bundles.

Remark 2.9. Notice that we are assuming that (abstract) ruths are non-negatively graded. In particular, when dualising an abstract ruth, we shift it to non-negative degrees. To be clear, given an $(n+1)$-term abstract ruth $\mathcal{E}_G$,

$$(\mathcal{E}_G^*)^k := \mathrm{Hom}^k_{CG\text{-lin}}(\mathcal{E}_G, CG) = \{\varphi : \mathcal{E}_G \to CG[k] \mid \text{for all } f \in CG,\ \varphi(c \cdot f) = \varphi(c) \cdot f\}$$

is the dual ruth, which is concentrated in non-positive degrees. The differential is given by[8]

$$\delta^* \varphi := [\delta, \varphi]$$

and it is a right CG-module by setting ($\iota := (-1)^{\frac{\bullet(\bullet+1)}{2}} \mathbf{i}^*$ is pullback by the inversion map[9])

$$(\varphi \cdot f)c := (-1)^{\varphi \cdot f} (\iota f) \cdot (\varphi c).$$

We will, however, consider $\mathcal{E}_G^*[-n]$, which introduces a sign $(-1)^n$ in the equation for δ. Instead of writing $\mathcal{E}_G^*[-n]$, we simply write $\mathcal{E}_G^*$. So, the dual construction we consider is a "shifted duality" and in the 2-term case it aligns with the dualisation of VB-groupoids. For higher terms, we expect there is a precise relation with [RZ25] (see Remark 2.21).

Example 2.10. As mentioned implicitly above, we have an isomorphism of abstract ruths

$$C_{\mathrm{VB}} V_G \cong (C_{\mathrm{VB}} V_G^*)^*$$

which, explicitly, is given by sending $f \in C_{\mathrm{VB}} V_G$ to

$$\varphi_f(c)(g_1, \cdots, g_{f+c-1}) := (-1)^{(f-1)c + \frac{f(f+1)}{2}} c(f((-)_{g_f^{-1}}, g_{f-1}^{-1}, \ldots, g_1^{-1})^{-1}, g_{f+1}, \ldots, g_{f+c-1}).$$

[8]Here and later, the notation $[\cdot, \cdot]$ is used for the graded commutator. In this case, $[\delta, \varphi]$ means $\delta\varphi - (-1)^\varphi \varphi\delta$.

[9]Notice that $[\delta, \iota] = \delta\iota - \iota\delta = 0$.

2.3. **The normalised complex of an abstract ruth.** Here, we elaborate on our choice to define abstract ruths as in Definition 2.1 as $(CG)_{\mathrm{N}}$-modules.

Remark 2.11. The cosimplicial version of the Dold-Kan correspondence [Dol58, Kan58] sets an equivalence of categories between (non-negatively graded) cochain complexes and cosimplicial objects (in an abelian category). The cosimplicial object (say, as a cosimplicial vector space)

$$C^\infty M \rightleftarrows C^1G \rightleftarrows C^2G \cdots : C^\bullet G$$

corresponds this way to the cochain complex $(CG)_{\mathrm{N}}$. A crucial point is that the inclusion

$$(CG)_{\mathrm{N}} \hookrightarrow CG$$

is a quasi-isomorphism, and especially the way in which it is. In short, the degeneracy maps provide homotopy data

$$\eta_m := (-1)^{m+1}\nu_{\bullet-m-1} = (-1)^{m+1}u^*_{\bullet-m-1} \qquad \pi_m := 1-[\delta,\eta_m]$$

that show that each inclusion in the filtration

$$(C^\bullet G)_{\mathrm{N}} \underset{\pi_{\bullet-1}}{\rightleftarrows} \bigcap_{\ell=1}^{\bullet-1}\ker\nu_\ell \underset{\pi_{\bullet-2}}{\rightleftarrows} \cdots \underset{\pi_1}{\rightleftarrows} \ker\nu_{\bullet-1} \underset{\pi_0}{\rightleftarrows} C^\bullet G$$

is a quasi-isomorphism.[10] The VB-complex $C_{\mathrm{VB}}V_G$ of a VB-groupoid V_G is similarly filtered

$$(C^\bullet_{\mathrm{VB}}V_G)_{\mathrm{N}} \underset{\pi_{\bullet-1}}{\rightleftarrows} \bigcap_{\ell=1}^{\bullet-1}\ker\nu_\ell \underset{\pi_{\bullet-2}}{\rightleftarrows} \cdots \underset{\pi_1}{\rightleftarrows} \ker\nu_{\bullet-1} \underset{\pi_0}{\rightleftarrows} C^\bullet_{\mathrm{VB}}V_G$$

and each inclusion is a quasi-isomorphism by the same argument as above. However, realising

$$C_{\mathrm{VB}}V_G \cong (C_{\mathrm{VB}}V_G)_{\mathrm{N}} \otimes_{(CG)_{\mathrm{N}}} CG,$$

this fact can be seen as a consequence of the above argument for $(CG)_{\mathrm{N}} \hookrightarrow CG$.

By virtue of the way in which we defined abstract ruths, we have the following general statement.

Proposition 2.12. *Given an abstract ruth $(\mathcal{E}_G)_{\mathrm{N}}$, the inclusion*

$$(\mathcal{E}_G)_{\mathrm{N}} \hookrightarrow \mathcal{E}_G$$

is a $((CG)_{\mathrm{N}} \hookrightarrow CG)$-linear quasi-isomorphism.

So, if we wish to view an abstract ruth as a CG-module, Definition 2.1 automatically incorporates a sensible notion of "normalised subcomplex". In the next section (Section 2.4) we show how to recover the usual notion of a ruth (called split ruth in this work). We conclude with two remarks.

Remark 2.13. Observe the similarities of the above discussion with the result that the inclusion

$$C_{\mathrm{VB}}V_G \hookrightarrow C_{\mathrm{lin}}V_G$$

is a quasi-isomorphism of complexes (see [CD17, dHO20]). As made precise in [dHO20], this fact can be proven via the same strategy explained above. Indeed, $C_{\mathrm{lin}}V_G$ is canonically filtered

$$C^\bullet_{\mathrm{VB}}V_G \underset{\pi_{\bullet-1}}{\rightleftarrows} \mathcal{K}^\bullet_{\bullet-1} \underset{\pi_{\bullet-2}}{\rightleftarrows} \cdots \underset{\pi_1}{\rightleftarrows} \mathcal{K}^\bullet_1 \underset{\pi_0}{\rightleftarrows} C^\bullet_{\mathrm{lin}}V_G$$

[10] This argument works for general cosimplicial objects.

by the subcomplexes

$$\mathcal{K}_m^\bullet = \{f \in C^\bullet_{\mathrm{lin}}V_G \mid \begin{array}{l} f(0_{g_1}, \ldots, 0_{g_{\bullet-m+1}}, v_{\bullet-m+2}, \ldots, v_\bullet) = 0 \\ \delta f(0_{g_1}, \cdots, 0_{g_{\bullet-m+1}}, v_{\bullet-m+2}, \ldots, v_{\bullet+1}) = 0 \end{array} \}$$

and the homotopy data is defined via a (unital) cleavage Σ. Explicitly, given $f \in K_m^\bullet$,

$$\eta_m f(v_1, \ldots, v_{\bullet-1}) \coloneqq (-1)^\bullet f(\Sigma_{g_{\bullet-1}^{-1}\cdots g_1^{-1}} \mathbf{t}_{g_1} v_1, v_1, \ldots, v_{\bullet-1})$$

and then, as before, we set $\pi_m \coloneqq 1 - [\delta, \eta_m]$.

Remark 2.14. Given a VB-groupoid $V_G \rightrightarrows V_M$, the cosimplicial object

$$C^\infty_{\mathrm{lin}}V_M \;\substack{\longrightarrow\\ \longleftarrow\\ \longrightarrow}\; C^1_{\mathrm{lin}}V_G \;\substack{\longrightarrow\\ \longleftarrow\\ \longrightarrow\\ \longleftarrow\\ \longrightarrow}\; C^2_{\mathrm{lin}}V_G \;\cdots : C^\bullet_{\mathrm{lin}}V_G$$

is related to the cochain complex $(C_{\mathrm{lin}}V_G)_{\mathrm{N}}$ via the Dold-Kan correspondence. But, usually, we can not view $C_{\mathrm{VB}}V_G \subset C_{\mathrm{lin}}V_G$ as a cosimplicial subobject even though

$$(C_{\mathrm{VB}}V_G)_{\mathrm{N}} \subset (C_{\mathrm{lin}}V_G)_{\mathrm{N}}$$

is a subcomplex. Instead, $(C_{\mathrm{VB}}V_G)_{\mathrm{N}}$ corresponds to the cosimplicial subobject $C_{\mathrm{VB,cs}}V_G$ consisting of those $f \in C^\bullet_{\mathrm{lin}}V_G$ for which

$$f(0_{g_1} \cdot c^{-1}, v_2, \ldots, v_\bullet) = f(c^{-1}, v_2, \ldots, v_\bullet),$$

and such that, for all $2 \le k \le \bullet$,

$$f(0_{g_1}, \ldots, 0_{g_{k-1}}, 0_{g_k} \cdot c^{-1}, v_{k+1}, \ldots, v_\bullet) = f(0_{g_1}, \ldots, 0_{g_{k-1}g_k}, c^{-1}, v_{k+1}, \ldots, v_\bullet).$$

Indeed, we have

$$(C_{\mathrm{VB,cs}}V_G)_{\mathrm{N}} = (C_{\mathrm{VB}}V_G)_{\mathrm{N}}.$$

Notice that, $C_{\mathrm{VB,\,cs}}V_G \subset C_{\mathrm{lin}}V_G$ is not a CG-submodule and, for all $\bullet \ge 1$, $C^\bullet_{\mathrm{VB,\,cs}}V_G \subset C^\bullet_{\mathrm{lin}}V_G$ is not a $C^\infty G^{(\bullet)}$-submodule in general.

2.4. **Split ruths.** Let $\mathcal{E}_G = (\mathcal{E}_G)_{\mathrm{N}} \otimes_{(CG)_{\mathrm{N}}} CG$ be an abstract ruth. There is a non-canonical, filtration preserving, CG-linear isomorphism of $\mathcal{E}_G^\bullet$ with the sum of the $\bullet$-antidiagonal of

$$\begin{array}{ccc} C^2(G; E_M^0) & C^2(G; E_M^1) & C^2(G; E_M^2) \\ \\ C^1(G; E_M^0) & C^1(G; E_M^1) & C^1(G; E_M^2) \\ \\ C^0(G; E_M^0) & C^0(G; E_M^1) & C^0(G; E_M^2) \end{array}$$

We consider the bigraded object above to be filtered by the rows: $\mathcal{F}_k$ consists of those elements of $\mathcal{E}_G$ that lie in a row $\ge k$. We can view a component in degree $(k, \bullet - k)$[11] as

$$\mathcal{F}_k^\bullet / \mathcal{F}_{k+1}^\bullet = C^k(G; E_M^{\bullet-k}).$$

Now, δ decomposes into maps[12]

$$\delta_\ell \coloneqq C^k(G; E_M^{\bullet-k}) \to C^{k+\ell}(G; E_M^{\bullet-k+1-\ell})$$

[11]We use the matrix convention, so k denotes the row and $\bullet - k$ the column.

[12]Notice that $\partial = \delta_0$ is determined by the previously defined ∂.

that diagrammatically are given as follows:[13]

$$\begin{tikzcd}
C^2(G;E^0_M) \arrow[r] & C^2(G;E^1_M) \arrow[r] & C^2(G;E^2_M) \\
C^1(G;E^0_M) \arrow[u] \arrow[r, dotted] & C^1(G;E^1_M) \arrow[u] \arrow[r, dotted] & C^1(G;E^2_M) \arrow[u] \\
C^0(G;E^0_M) \arrow[u, "\delta_1"] \arrow[r, "\partial"] & C^0(G;E^1_M) \arrow[u] \arrow[r] \arrow[uul, "\delta_2"'] & C^0(G;E^2_M) \arrow[u] \arrow[uul]
\end{tikzcd}$$

While this seems to realise the abstract ruth as a ruth in the usual sense, this is only true if the splitting preserves normalised cochains.

Definition 2.15. A *splitting* of an $(n+1)$-term abstract ruth $\mathcal{E}_G = (\mathcal{E}_G)_{\mathrm{N}} \otimes_{(CG)_{\mathrm{N}}} CG$ is a CG-linear isomorphism

$$\mathcal{E}_G^\bullet \cong C(G;E_M)^\bullet = \bigoplus_{k=0}^\bullet C^k(G;E_M^{\bullet-k})$$

that, for all $\bullet \leq n$, induces the identity map on $\Gamma E_M^\bullet$. The splitting is called *normalised* or *unital* if it preserves normalised cochains.

Remark 2.16. A (unital) splitting of an $(n+1)$-term abstract ruth $\mathcal{E}_G$ is automatically filtration preserving. Moreover, it is equivalently described by a choice of sections of the n projections

$$(\mathcal{E}_G^{1\leq\bullet\leq n})_{\mathrm{N}} \to \Gamma E_M^\bullet.$$

We adopt the following terminology if we want to emphasise the difference:

Definition 2.17. A *split ruth* is a graded vector bundle $E_M^{\bullet\geq 0}$ together with a differential $\delta = \sum_{\ell\geq 0}\delta_\ell$ on the graded CG-module

$$\mathcal{E}_G^\bullet = C(G;E_M)^\bullet = \bigoplus_{k=0}^\bullet C^k(G;E_M^{\bullet-k})$$

turning it into a differential graded CG-module, i.e. such that the Leibniz rule

$$\delta(c\cdot f) = \delta c\cdot f + (-1)^c c\cdot \delta f$$

is satisfied. We assume $\mathcal{E}_G$ is *unital*, meaning that δ preserves normalised cochains.

Remark 2.18. Given a split ruth, we can define maps $R = (R_\ell)_{\ell\geq 0}$, where

$$R_\ell \in \Gamma\mathrm{End}^{\ell,1-\ell}E_M \qquad \mathrm{End}^{\ell,\ell'}E_M := \mathrm{Hom}((\mathbf{s}^{(\ell)})^*E_M^\bullet, (\mathbf{t}^{(\ell)})^*E_M^{\bullet+\ell'})$$

are defined as

$$R_{\ell\neq 1} := \delta_\ell \qquad R_1 := \delta_1 - (-1)^\bullet \textstyle\sum_{k=1}^{\bullet+1}(-1)^k d_k^*.$$

This uses the Leibniz rule and the fact that all CG-linear maps

$$C^\bullet(G;E_M^{\bullet'}) \to C^{\bullet+\ell}(G;E_M^{\bullet'+\ell'})$$

(so of bidegree (ℓ,ℓ')) come uniquely from elements of $\mathrm{End}^{\ell,\ell'}E_M$ (see [AAC13]). The correspondence is such that the Koszul sign rule applies: given $R_{\ell,\ell'} \in \mathrm{End}^{\ell,\ell'}E_M$, the associated CG-linear map is given by[14]

$$R_{\ell,\ell'}\cdot f(g,g') = (-1)^{\ell\cdot f} R_{\ell,\ell'}(g) f(g')$$

[13] We only drew the maps ∂, δ_1 and δ_2.

[14] As mentioned in Section 1, we think of $g \in G^{(\ell)}$ as having degree ℓ.

That $\delta^2 = 0$ then translates into the Maurer-Cartan equation

$$\delta_{\text{ruth}} R + \tfrac{1}{2}[R, R]_{\text{ruth}} = 0.$$

The Maurer-Cartan equation can be taken literally as follows: the dgLa of ruths, that controls deformations, is given by

$$C^\bullet_{\text{ruth}} := \bigoplus\nolimits_{\ell \geq 0} \operatorname{End}^{\ell, \bullet - \ell} E_M$$

together with the graded Lie bracket and differential defined through the differential graded algebra[15]

$$R_\ell \cdot R_{\ell'}(g, g') = (-1)^{\ell \cdot R_{\ell'}} R_\ell(g) R_{\ell'}(g') \qquad \delta_{\text{ruth}} = (-1)^\bullet \textstyle\sum_{k=1}^{\ell} (-1)^k d_k^*$$

That is, the graded Lie bracket $[\cdot, \cdot]_{\text{ruth}}$ is the commutator. This differential graded algebra appeared in [AACD11]. The deformation complex (which is a dgLa) of a ruth $R = (R_\ell)$ is then

$$C^\bullet_{\text{def}} R = C^\bullet_{\text{ruth}}$$

together with the same graded Lie bracket, but with

$$\delta_R := \delta_{\text{ruth}} + [R, \cdot].$$

(see also [LP20]). Notice that the Maurer-Cartan equation above gives rise to *non-unital ruths*; we didn't take into account the condition that δ preserves normalised cochains. It is readily verified that this happens precisely when $R_1(1) = 1$ and $u_k^* R_{\ell \geq 2} = 0$ for all degeneracies u_k.

2.5. **Towards a Serre-Swan theorem.** Definitions of ruths as finitely generated projective differential graded CG-modules can already be found in a remark in [AAC13] (after Definition 3.9 there), and in [Ste19]. However, it seems that in the literature ruths of Lie groupoids have not properly been studied as "abstract" or "canonical" objects. The recognition of a normalised complex as being (part of) the structure of a ruth is important for achieving this. That abstract ruths appear canonically in examples becomes especially helpful in more complicated examples, but we observe here that the VB-complex of a VB-groupoid is a valid (universal) example of a 2-term abstract ruth. In essence, this was proved in [GSM17].

Proposition 2.19. *Let V_G be a VB-groupoid. Then $C_{\mathrm{VB}} V_G$ is a 2-term abstract ruth. Moreover, (unital) cleavages correspond to (unital) splittings.*

Proof. That $C_{\mathrm{VB}} V_G$ is finitely generated and projective as a CG-module can be seen by picking a cleavage Σ, for Σ sets an isomorphism

$$C^\bullet_{\mathrm{VB}} V_G \cong C^\bullet(G; V_M^*) \oplus C^{\bullet-1}(G; C_M^*) \qquad f \mapsto (f_{V_M}, f_{C_M}),$$

where[16]

$$f_{V_M}(g_1, \ldots, g_\bullet)v := f(\overline{\Sigma}_{g_1} v, g_2, \ldots, g_\bullet) \qquad f_{C_M}(g_2, \ldots, g_\bullet)c := f(-c^{-1}, g_2, \ldots, g_\bullet).$$

Given a (unital) splitting of the abstract ruth $C_{\mathrm{VB}} V_G$, the (unital) cleavage $\overline{\Sigma}$ (a section of $\mathbf{t}$) is recovered as the unique linear map for which

$$C^1_{\mathrm{VB}} V_G \cong C^1(G; V_M^*) \oplus \Gamma C_M^* \xrightarrow{\text{pr}} C^1(G; V_M^*) \qquad f \mapsto f \circ \overline{\Sigma}.$$

[15] Notice that d_0 and $d_{\bullet+1}$ are not appearing in the equation for δ_{ruth}.

[16] We use here that, if $f \in C^\bullet_{\mathrm{VB}} V_G$, then $f(v_{g_1}, \ldots, v_{g_\bullet}) \in \mathbb{R}$ only depends on $(v_1, g_2, \ldots, g_\bullet)$.

That $\overline{\Sigma}$ is a splitting follows by considering $f = f_{V_M} \cdot f'$ for all $f_{V_M} \in \Gamma V_M^*$ and $f' \in C^1G$. If the ruth splitting is unital, then for all normalised f,

$$f \circ \overline{\Sigma}_1 = 0.$$

So, $\overline{\Sigma}_1$ maps into the units of V_G, and therefore it is the unit map. This proves the statement. □

We will discuss a slightly different proof, which shows directly that $C_{\mathrm{VB}}V_G$ can be seen as a locally free sheaf of CG-modules. The main point is that, locally, we can use a special type of cleavage H, one for which all maps $\mathbf{t}_g H_g$ are invertible. We can see such sections as local sections of the surjective submersion[17]

$$\widehat{V}_G \to G \qquad H_g \mapsto g.$$

That is, H is a smooth family of linear maps

$$g \mapsto (V_M)_{\mathbf{s}g} \xrightarrow{H_g} (V_G)_g$$

such that, for all $g \in G$, $\mathbf{s}_g H_g = 1$ and $\mathbf{t}_g H_g$ is a linear isomorphism.

Now, in the above isomorphism using the cleavage Σ, we used the linear isomorphisms

$$(C_M)_{\mathbf{s}g} \oplus (V_M)_{\mathbf{t}g} \to (V_G)_g \qquad (c, v) \mapsto \ell_g c + \overline{\Sigma}_g v.$$

Given a local section H of $\widehat{V}_G \to G$, we can instead use the linear isomorphism

$$(C_M)_{\mathbf{s}g} \oplus (V_M)_{\mathbf{t}g} \to (V_G)_g \qquad (c, v) \mapsto \ell_g c + H_g(\mathbf{t}_g H_g)^{-1} v.$$

In fact, this observation leads to a whole new description of $C_{\mathrm{VB}}V_G$ in terms of the fat groupoid $\widehat{V}_G$. This will be made more precise in Section 4.5.

Alternative proof of Proposition 2.19. As said before, we will show that $C_{\mathrm{VB}}V_G$ is locally free as a sheaf of CG-modules (see Proposition 2.5). In fact, given $U \subset M$ and a local section H of $\widehat{V}_G \to G$ defined over U, then H defines an isomorphism

$$C^\bullet_{\mathrm{VB}}V_G(U) \cong C^\bullet(G; V_M^*)(U) \oplus C^{\bullet-1}(G; C_M^*)(U) \qquad f \mapsto (f_{V_M}, f_{C_M})$$

where

$$f_{V_M}(g_1, \ldots, g_\bullet)v := f(H_{g_1}(\mathbf{t}H_{g_1})^{-1}v, g_2, \ldots, g_\bullet) \qquad f_{C_M}(g_2, \ldots, g_\bullet)c := f(-c^{-1}, g_2, \ldots, g_\bullet).$$

For the inverse map, we construct out of $f_{V_M} \in C^1(G; V_M^*)$ a linear function $f \in C^1_{\mathrm{lin}}V_G$ by setting

$$f(v) := f_{V_M}(g)\mathbf{t}_g v.$$

Since $C^1(G; V_M^*) \cong \Gamma V_M \otimes C^\infty G$, we see that every such function is locally generated by sections of V_M^* (sitting in degree 0). Now, given a local section H of $\widehat{V}_G \to G$, a section f_{C_M} of C_M^* gives rise to the linear function $f \in C^1_{\mathrm{lin}}V_G$ defined as

$$f(v) := f_{C_M}\omega_g^{H_g} v,$$

where

$$\omega_g^{H_g} : (V_G)_g \to (C_M)_{\mathbf{s}g} \qquad v \mapsto r_g((H_g(\mathbf{t}_g H_g)^{-1}\mathbf{t}_g v)^{-1} - v^{-1}).$$

[17]See Section 1 for a short introduction to the fat groupoid $\widehat{V}_G$. In Section 3 we give a more thorough introduction

It is readily verified that both types of linear functions $f \in C^1_{\mathrm{lin}} V_G$ are elements of $C^1_{\mathrm{VB}} V_G$. Similarly, or extending CG-linearly, we obtain elements $C^\bullet_{\mathrm{VB}} V_G$ out of (local) sections of $C^\bullet(G; V_M^*)$ and out of (local) sections of $C^{\bullet-1}(G; C_M^*)$. Now, given $f \in C^1_{\mathrm{VB}} V_G$, we have

$$f(v) = f_{C_M} \omega_g^{H_g} v + f_{V_M}(g)\mathbf{t}_g v,$$

and similarly we can decompose elements of $C^\bullet_{\mathrm{VB}} V_G$. This shows that the above maps are inverse to each other, so $C_{\mathrm{VB}} V_G$ is locally free as a sheaf of CG-modules. □

The equivalence of categories between VB-groupoids and 2-term ruths can now be understood through the assignment

$$V_G \mapsto C_{\mathrm{VB}} V_G^* \coloneqq C_{\mathrm{VB}}(V_G^*) \cong (C_{\mathrm{VB}} V_G)^*.$$

The assignment on morphisms uses, for $\Phi : V_G \to V_H$ a VB-groupoid map, the dual map

$$\Phi^* : V_H^* \to V_G^*$$

and then the pullback map

$$(\Phi^*)^*_{\mathrm{VB}} : C_{\mathrm{VB}} V_G^* \to C_{\mathrm{VB}} V_H^*.$$

(or this pullback map first and then the dual map of ruths). This defines a functor, and it is an equivalence of categories [GSM17, dHO20]:

Theorem 2.20. *The functor*

$$\{\text{VB-groupoids}\} \to \{\text{Abstract 2-term ruths}\} \qquad V_G \mapsto C_{\mathrm{VB}} V_G^* \qquad \Phi \mapsto (\Phi^*)^*_{\mathrm{VB}}$$

sets an equivalence of categories.

Remark 2.21. Perhaps we should identify Theorem 2.20 truly as a Serre-Swan type theorem. That is, we can think of $C_{\mathrm{VB}} V_G^*$ (or $C_{\mathrm{proj}} V_G$) as a type of "space of sections" of the VB-groupoid V_G. A current work in progress is to understand generalisations of this fact. Motivated by [dHT25], we expect to set a canonical Serre-Swan correspondence between higher vector bundles and abstract ruths. In doing so, we believe it is important to investigate the link between the choice of a "normal pre-cleavage" (or a normal weakly flat cleavage) for higher vector bundles in [dHT25] to the above unital splittings of abstract ruths. Indeed, in [dHT25] it is shown that the choice of a normal weakly flat cleavage of a higher vector bundle (that admits such a cleavage) defines a ruth, and the normal and weakly flat conditions are used to show that this ruth is unital.

Remark 2.22. Related to the above remark is how the tensor product constructions of VB-groupoids and 2-term ruths relate (for the latter, see [AACD11]). This is part of the work in progress [MOV].

2.6. **Cochain complex representations.** We end this section by mentioning a special type of 2-term ruth that is central to this work:

Definition 2.23. Let $G \rightrightarrows M$ be a Lie groupoid. A *cochain complex representation*, or strict ruth, is a split ruth $\mathcal{E}_G$ such that $R_{\ell \geq 2} = 0$.

A cochain complex representation of G on $C_M \to V_M$ is a pair of representations of G on C_M and V_M, such that the map $C_M \to V_M$ is equivariant with respect to the actions. We write the abstract ruth associated to a cochain complex representation by

$$C(G; C_M \to V_M) = C_{\mathrm{VB}}(G \ltimes (V_M^* \to C_M^*)).$$

There are at least two more ways of describing a cochain complex representation that we want to discuss.

Remark 2.24. Given two vector bundles C_M and V_M, there is a type of general linear groupoid given by

$$\mathrm{GL}(C_M, V_M) \coloneqq \mathrm{GL}\, C_M \times_{M\times M} \mathrm{GL}\, V_M$$

which we can view as a fiber product of Lie groupoids.[18] Now, the differential induces a map

$$[\cdot, \partial] : \mathrm{GL}(C_M, V_M) \to \mathrm{Hom}(\mathrm{pr}_1^* C_M, \mathrm{pr}_2^* V_M) \qquad \Phi = (\Phi^{C_M}, \Phi^{V_M}) \mapsto \Phi^{V_M}\partial - \partial\Phi^{C_M}.$$

We can interpret the preimage of zero

$$\mathrm{Aut}(C_M \to V_M) \coloneqq \{\Phi \in \mathrm{GL}(C_M, V_M) \mid \partial\Phi^{C_M} = \Phi^{V_M}\partial\}$$

as cochain isomorphisms

$$\begin{tikzcd} (C_M)_x \arrow[r, "\partial"] \arrow[d, "\Phi^{C_M}"] & (V_M)_x \arrow[d, "\Phi^{V_M}"] \\ (C_M)_y \arrow[r, "\partial"] & (V_M)_y \end{tikzcd}$$

However, while $\mathrm{Aut}(C_M \to V_M)$ is a subgroupoid of $\mathrm{GL}(C_M, V_M)$, it does not define a smooth Lie subgroupoid in general. In fact, the above commutator map is the restriction of a linear map, and then it is readily verified that it has constant rank precisely when ∂ has constant rank. In any case, we can think of a cochain complex representation as a Lie groupoid map

$$G \to \mathrm{GL}(C_M, V_M) \dashrightarrow \mathrm{Hom}(\mathrm{pr}_1^* C_M, \mathrm{pr}_2^* V_M)$$

such that postcomposing with the map $[\cdot, \partial]$ is zero.

Remark 2.25. Another way to organise the information of a cochain complex representation of a Lie groupoid G on $C_M \to V_M$ is as a Lie groupoid map in the following way. Consider first the general linear groupoid from Remark 2.24:

$$\mathrm{GL}(C_M, V_M) = \mathrm{GL}\, C_M \times_{M\times M} \mathrm{GL}\, V_M.$$

This groupoid acts (on the right) on "differentials" $\mathrm{Hom}(C_M, V_M)$ by conjugating the differentials: given $\Psi_{y,x} = (\Psi^{C_M}_{y,x}, \Psi^{V_M}_{y,x}) \in \mathrm{GL}(C_M, V_M)$ and $\partial_y \in \mathrm{Hom}(C_M, V_M)$,

$$\partial_y \cdot \Psi_{y,x} \coloneqq (\Psi^{V_M}_{y,x})^{-1}\partial_y \Psi^{C_M}_{y,x}.$$

We write the resulting action groupoid by

$$\mathrm{GL}(C_M, V_M)_M \coloneqq \mathrm{Hom}(C_M, V_M) \rtimes \mathrm{GL}(C_M, V_M).$$

A cochain complex representation of G on $C_M \to V_M$ can now be seen as a groupoid map

$$G \to \mathrm{GL}(C_M, V_M)_M$$

whose basemap is the differential:

$$\partial : M \to \mathrm{Hom}(C_M, V_M).$$

If we postcompose this map with the groupoid projection $\mathrm{GL}(C_M, V_M)_M \to \mathrm{GL}(C_M, V_M)$, then we recover the map we considered in Remark 2.24.

[18] We think here of $M \times M$ as the pair groupoid, and we use the (surjective) anchor maps.

3. The fat extension of a VB-groupoid

We start this section by introducing the fat groupoid of a VB-groupoid. In this generality, it seems to have first appeared in [GSM17].

3.1. **The fat groupoid.** Let $V_G \rightrightarrows V_M$ be a VB-groupoid. Consider the surjective linear map

$$\mathrm{Hom}(\mathbf{s}^* V_M, V_G) \longrightarrow \mathrm{End}\, V_M \qquad H_g \mapsto \mathbf{s}_g H_g$$

over $\mathbf{s} : G \longrightarrow M$, and the preimage of the identity section

$$\widehat{V}_G^{\mathrm{cat}} = \{H_g : (V_M)_{\mathbf{s}g} \longrightarrow (V_G)_g \mid \mathbf{s}_g H_g = 1\}.$$

Then $\widehat{V}_G^{\mathrm{cat}}$ is an affine bundle over G[19] and it is a Lie category (without boundary) over M (a thorough discussion on Lie categories is presented in [Gra25b]). The structure maps are given as follows: we set $\widehat{\mathbf{s}} H_g = \mathbf{s}g$, $\widehat{\mathbf{t}} H_g = \mathbf{t}g$ and

$$(H_g \cdot H_{g'})v = H_g(\mathbf{t}_{g'} H_{g'} v) \cdot H_{g'} v.$$

The invertible elements of this Lie category are given by the following open subset of $\widehat{V}_G^{\mathrm{cat}}$:

$$\widehat{V}_G = \{H_g \in \widehat{V}_G^{\mathrm{cat}} \mid \mathbf{t}_g H_g \text{ is a linear isomorphism}\}$$

which becomes a Lie groupoid with

$$(H_g)^{-1} v = (H_g(\mathbf{t}_g H_g)^{-1} v)^{-1}.$$

Definition 3.1. Given a VB-groupoid V_G, we call $\widehat{V}_G \rightrightarrows M$ the fat groupoid (of V_G).

We can think of the elements of $\widehat{V}_G$ as coming from (fiberwise) linear bisections of V_G.

Remark 3.2. Notice that there is a one-to-one correspondence between bisections of $\widehat{V}_G$ and linear bisections of V_G. Indeed, a linear bisection σ of V_G is a bisection

$$V_M \longrightarrow V_G \qquad x \mapsto \sigma x$$

that is a vector bundle map over a bisection σ of G.[20] We can therefore, equivalently, view σ as the bisection

$$M \longrightarrow \widehat{V}_G \qquad x \mapsto (V_M)_x \xrightarrow{\sigma x} (V_G)_{\sigma x}$$

of $\widehat{V}_G$. There are also "core bisections" of V_G (over the identity bisection of G): given $\chi \in \Gamma C_M$, we can construct a bisection σ_χ of V_G by setting

$$\sigma_\chi(v_x) = 1_{v_x} + \chi(x).$$

Applying the target to this expression gives the translation $v_x \mapsto v_x + \partial\chi(x)$.

Remark 3.3. In [GSM17] the fat groupoid is denoted by $\widehat{G}$. However, we prefer to keep track of the dependence on V_G in the notation.

We end the section by mentioning a few examples. In Section 3.6 we discuss more examples and go into more detail.

[19] The fiber over g is isomorphic to $\mathrm{Hom}((V_M)_{\mathbf{s}g}, (C_M)_{\mathbf{t}g})$.

[20] Alternatively, it is a 0-homogeneous bisection, i.e. $\lambda^*\sigma(= \lambda^{-1} \circ \sigma \circ \lambda) = \sigma$.

Example 3.4. Probably the most well-known example of a "fat groupoid" is the jet groupoid J^1G:

$$J^1G = \{j^1_x\sigma \mid \sigma \in \mathrm{Bis}_{\mathrm{loc}}G\}.$$

In words, its elements consist of all possible values of 1-jets of local bisections σ. The product is induced by the usual product of (local) bisections. We can realise J^1G as the fat groupoid $\widehat{TG}$ of the VB-groupoid TG. Indeed, every 1-jet of a local bisection takes the form

$$H_g : T_{\mathbf{s}g}M \to T_gG,$$

where $d\mathbf{s}_g H_g = 1$ and $d\mathbf{t}_g H_g$ is a linear isomorphism. On the other hand, we can integrate such a linear map H_g to a local bisection σ with the property that $j^1_{\mathbf{s}g}\sigma = H_g$. This follows from standard proofs on the existence of local bisections through elements of G.

Example 3.5. Let $H \rightrightarrows N$ be a Lie subgroupoid of $G \rightrightarrows M$. The jet groupoid J^1G contains the Lie subgroupoid

$$J^1_HG := \{j^1_y\sigma \in J^1G \mid y \in N, \mathrm{im}\,\sigma|_N \subset H\} \rightrightarrows N.$$

This subgroupoid is equal to J^1H only if $N = M$. How are J^1H and J^1G related if $N \neq M$? For now, we only observe that J^1_HG and J^1H are related via a restriction map

$$J^1_HG \to J^1H.$$

A similar phenomenon appears when considering $J^1_{\mathcal{N}}(G,H)$, the fat groupoid of

$$\mathcal{N}(G,H) \rightrightarrows \mathcal{N}(M,N),$$

the normal bundle, which is a VB-groupoid over H.[21] Namely, there is again a natural map

$$J^1_HG \to J^1_{\mathcal{N}}(G,H).$$

In Section 3.6, we will further clarify the relation between J^1H, J^1G and $J^1_{\mathcal{N}}(G,H)$.

Example 3.6. Let $E_M \to M$ be a vector bundle. The general linear groupoid

$$\mathrm{GL}\ E_M = \{\Psi_{y,x} : E_x \xrightarrow{\sim} E_y \text{ is a linear isomorphism}\}.$$

is the fat groupoid of the pair groupoid $E_M \times E_M \rightrightarrows E_M$, seen as a VB-groupoid over $M \times M \rightrightarrows M$.

The last example shows the extension of the "classical" analogues of the correspondences we aim to describe in Sections 4 and 5:

$$\begin{tikzcd}
\{\text{f.g.p. } C^\infty M\text{-mods}\} & \{\text{Vector bundles}\} \arrow[l, "\text{Serre-Swan}"'] \arrow[r, "\text{Fr}"] \arrow[d, "\text{GL}"] & \{\text{Principal GL-bundles}\} \arrow[dl, "\text{Gauge groupoid}"] \\
& \{\text{fat extensions of } M \times M \text{ over } 1_{E_M}\} \arrow[ul] &
\end{tikzcd}$$

The goal of the coming two sections (Sections 3.2 and 3.3) is to discuss two essential pieces of information: the bundle of invertible homotopies, and the canonical representations of the fat groupoid. This is enough to understand the basic structure of fat extensions (see Section 3.4). In the sections afterwards we look at several consequences of the structure behind the fat groupoid and discuss, in this order, the fat category (Section 3.5), examples (Section 3.6), other models and dualisation of the fat groupoid (Section 3.7) and, lastly, the fat pairing (Section 3.8).

[21]The Lie groupoid structure of the normal bundle $\mathcal{N}(G,H)$ is induced by that of TG.

3.2. **The bundle of invertible homotopies of a 2-term complex.** As mentioned before, if V_G is a VB-groupoid, then the groupoid map

$$\widehat{V}_G \to G \qquad H_g \mapsto g$$

is a surjective submersion. The kernel of this map is a bundle of Lie groups, and it can be identified with the open set of $\mathrm{Hom}(V_M, C_M)$ given by

$$\mathrm{H}(V_M, C_M) = \{h_x : (V_M)_x \to (C_M)_x \mid 1 + \partial h_x \text{ is invertible}\}$$

(via $h_x \mapsto 1 + h_x$). The unit is the zero map and

$$h_x \cdot h'_x = h_x + h'_x + h_x \partial h'_x \qquad h_x^{-1} = -h_x(1 + \partial h_x)^{-1} = -(1 + h_x\partial)^{-1} h_x.$$

Elements $h_x \in \mathrm{H}(V_M, C_M)$ are homotopies, for they show that the isomorphism[22]

$$1 + [\partial, h_x]$$

(from $C_M \to V_M$ to itself) is homotopic to the identity map. The product can then be interpreted as a "composition of homotopies" (see also Section 4.3):

Lemma 3.7. *Suppose we are given complexes C_1, C_2 and C_3 (with differentials ∂) with the following homotopy data: let*[23]

$$\begin{tikzcd}
C_1 \arrow[r, shift left, "\Phi"] \arrow[r, shift right, "\Psi"'] \arrow[d] & C_2 \arrow[r, shift left, "\Phi"] \arrow[r, shift right, "\Psi"'] \arrow[d] & C_3 \arrow[d] \\
C_1 \arrow[ur, "\eta"] \arrow[r, shift left, "\Phi"] \arrow[r, shift right, "\Psi"'] & C_2 \arrow[ur, "\eta"] \arrow[r, shift left, "\Phi"] \arrow[r, shift right, "\Psi"'] & C_3
\end{tikzcd}$$

be maps satisfying

$$[\partial, \Phi] = 0 \qquad [\partial, \Psi] = 0 \qquad \Phi - \Psi = [\partial, \eta].$$

Then

$$\Phi_{32}\Phi_{21} - \Psi_{32}\Psi_{21} = [\partial, \eta_{32}\Phi_{21} + \Psi_{32}\eta_{21}] = [\partial, \eta_{32}\Psi_{21} + \Phi_{32}\eta_{21}].$$

Moreover, $[\partial, \eta_{32}\eta_{21}] = 0$ if and only if

$$\eta_{32}\Phi_{21} + \Psi_{32}\eta_{21} = \eta_{32}\Psi_{21} + \Phi_{32}\eta_{21}.$$

In that case, we call this map the composition of homotopies (of η_{32} and η_{21}).

The composition of homotopies of $\mathrm{H}(V_M, C_M)$ is well-defined (and is equal to either side of the above formula). This composition actually defines a structure of Lie category on $\mathrm{Hom}(V_M, C_M)$,[24] and $\mathrm{H}(V_M, C_M)$ form precisely the invertible elements of this Lie category. Notice that the structure of $\mathrm{H}(V_M, C_M)$ as a bundle of Lie groups is independent of the fat groupoid $\widehat{V}_G$.

Definition 3.8. Let

$$C_M \xrightarrow{\partial} V_M$$

be a 2-term complex. We call the bundle of Lie groups

$$\mathrm{H}(V_M, C_M) = \{h_x : (V_M)_x \to (C_M)_x \mid 1 + \partial h_x \text{ is invertible}\}$$

[22]Notice that we have defined two maps, $1 + \partial h_x : (V_M)_x \to (V_M)_x$ and $1 + h_x\partial : (C_M)_x \to (C_M)_x$. The latter map is a linear isomorphism by the five-lemma.

[23]We use a subscript ji to denote the relevant map $C_i \to C_j$.

[24]More precisely, $\mathrm{Hom}(V_M, C_M)$ is a bundle of Lie monoids.

the *bundle of invertible homotopies* of the 2-term complex $C_M \to V_M$.

Remark 3.9. Although we write elements of $\mathrm{H}(V_M, C_M)$ as h_x, its product shows that a good way to think of the elements h_x is as $1 + h_x$. This shows that the product of $\mathrm{H}(V_M, C_M)$ is determined by a structure of Lie category on $\mathrm{Hom}(V_M, C_M)$:

$$h_x \cdot h'_x := h_x \partial h'_x.$$

This product defines a Lie category, and we will elaborate on its structure in Section 3.5.

That the elements of $\mathrm{H}(V_M, C_M)$ give rise to isomorphisms of the complex $C_M \to V_M$ can be interpreted as a cochain complex representation of $\mathrm{H}(V_M, C_M)$ on $C_M \to V_M$:

$$\mathrm{H}(V_M, C_M) \to \mathrm{GL}(C_M, V_M)_M \qquad h_x \mapsto (1 + h_x\partial, 1 + \partial h_x).$$

This canonical representation of $\mathrm{H}(V_M, C_M)$ plays a central role in this work.

Before we move on, notice that elements $h_x \in \mathrm{Hom}(V_M, C_M)$ formally have an inverse:

$$(1 + \partial h_x)^{-1} = \textstyle\sum_{k\geq 0}(-1)^k(\partial h_x)^k.$$

In particular, if ∂h_x is nilpotent, i.e. $(\partial h_x)^k = 0$ for some k, then $1 + \partial h_x$ has a smooth inverse.

Remark 3.10. With cohomological perturbation theory in mind (see, for example, [Cra04, LBM15, MS20] and references therein), we can think of ∂ as a perturbation of the zero differential. The identity map

$$\begin{tikzcd} (C_M)_x \arrow[d, "="] & (V_M)_x \arrow[dl, dotted, "h_x"'] \arrow[d, "="] \\ (C_M)_x & (V_M)_x \end{tikzcd}$$

can then be perturbed, using $h_x \in \mathrm{H}(V_M, C_M)$, to an isomorphism

$$\begin{tikzcd} (C_M)_x \arrow[r, "\partial"] \arrow[d, "\varphi_x"] & (V_M)_x \arrow[dl, dotted] \arrow[d, "\varphi_x"] \\ (C_M)_x \arrow[r, "\partial"] & (V_M)_x \end{tikzcd}$$

In fact, we can view $\mathrm{H}(V_M, C_M)$ as a level set (of a section of $\mathrm{Hom}(C_M, V_M) \to M$; the differential) of the canonical projection

$$\mathrm{Pert}(C_M, V_M) \to \mathrm{Hom}(C_M, V_M)$$

where

$$\mathrm{Pert}(C_M, V_M) = \{(\partial, h) \in \mathrm{Hom}(C_M, V_M) \oplus \mathrm{Hom}(V_M, C_M) \mid 1 + \partial h \text{ is invertible}\}.$$

We can also view $\mathrm{Pert}(C_M, V_M)$ as a bundle of Lie groups, with the same product as we defined for $\mathrm{H}(C_M, V_M)$. This groupoid structure becomes very relevant in Section 5.

Heuristically, a "small" perturbation ∂ comes with fewer restrictions on h_x, while a "big" perturbation imposes more restrictions on h_x. For example, if $\partial = 0$, then $\mathrm{H}(V_M, C_M) = \mathrm{Hom}(V_M, C_M)$. But if ∂ is of full rank, then $h_x \in \mathrm{H}(V_M, C_M)$ is determined by isomorphisms φ_x and left or right inverses of ∂.

The last observation of the above remark can be made much more precise as follows.

Proposition 3.11. *Let $\mathbb{R}^k \xrightarrow{d} \mathbb{R}^\ell$ be a linear map. The Lie group*
$$\mathrm{H}(\mathbb{R}^\ell, \mathbb{R}^k) = \{h : \mathbb{R}^\ell \to \mathbb{R}^k \mid 1 + dh \text{ is invertible}\}$$
embeds into $\mathrm{GL}(k+\ell)$ *via*
$$\mathrm{H}(\mathbb{R}^\ell, \mathbb{R}^k) \to \mathrm{GL}(k+\ell) \qquad h \mapsto \begin{pmatrix} 1 & h \\ 0 & 1+dh \end{pmatrix}.$$
In fact, if $r = \operatorname{rank} d$, then $\mathrm{H}(\mathbb{R}^\ell, \mathbb{R}^k)$ is isomorphic to the semidirect product associated to the action of $\mathrm{GL}(r)$ on upper diagonal blockmatrices[25] *given by*
$$g \cdot \begin{pmatrix} 1 & h_1 & h_2 \\ 0 & 1 & h_3 \\ 0 & 0 & 1 \end{pmatrix} = \begin{pmatrix} 1 & h_1 g^{-1} & h_2 \\ 0 & 1 & g h_3 \\ 0 & 0 & 1 \end{pmatrix}.$$
In particular, if $C_M \xrightarrow{\partial} V_M$ is a 2-term complex, then ∂ is regular if and only if $\mathrm{H}(V_M, C_M)$ is a Lie group bundle.[26]

Proof. The first statement is readily verified. To prove the second statement, we conjugate d so that $d : \mathbb{R}^r \oplus \mathbb{R}^{k-r} \to \mathbb{R}^r \oplus \mathbb{R}^{\ell-r}$ is in the canonical form $(x, y) \mapsto (x, 0)$.[27] The subgroup of $\mathrm{GL}(k+\ell)$ corresponding to $\mathrm{H}(\mathbb{R}^k, \mathbb{R}^\ell)$ consists now of invertible matrices of the form
$$\begin{pmatrix} 1 & 0 & h_4 & h_3 \\ 0 & 1 & h_1 & h_2 \\ 0 & 0 & 1+h_4 & h_3 \\ 0 & 0 & 0 & 1 \end{pmatrix}.$$
Notice that h_1, h_2 and h_3 can be taken arbitrary. The constraint on h_4 is that $1 + h_4$ is invertible. Now, the map that sends a matrix as above to $1 + h_4$ is a Lie group homomorphism onto $\mathrm{GL}(r)$ that is a surjective submersion. Its kernel consist of elements of the form
$$\begin{pmatrix} 1 & 0 & 0 & h_3 \\ 0 & 1 & h_1 & h_2 \\ 0 & 0 & 1 & h_3 \\ 0 & 0 & 0 & 1 \end{pmatrix},$$
and this group is isomorphic to the group of upper triangular blockmatrices (by eliminating, from such elements, the first row and column). A direct computation shows that the (conjugation) action that defines the resulting semidirect product is given as in the statement.

The last statement follows by the constant rank theorem: if ∂ is regular, then local frames adapted to ∂ trivialise $\mathrm{H}(V_M, C_M)$ locally. □

Remark 3.12. Notice that, if ∂ is regular, we can split (non-canonically)
$$V_M \cong \operatorname{coker} \partial \oplus \operatorname{im} \partial \qquad C_M \cong \ker \partial \oplus \operatorname{im} \partial.$$
Therefore, there is a global non-canonical isomorphism between $\mathrm{H}(V_M, C_M)$ and a type of semidirect product as in the above statement. Indeed, we can form a semidirect product (of Lie group

[25] The non-trivial blocks are of the form $(k-r) \times r$, $(k-r) \times (\ell - r)$ and $r \times (\ell - r)$.

[26] A Lie group bundle is a locally trivial bundle of Lie groups.

[27] If we conjugate d to AdB, then the two models of $\mathrm{H}(\mathbb{R}^\ell, \mathbb{R}^k)$ are isomorphic via $h \mapsto B^{-1} h A^{-1}$.

bundles) of the isotropy Lie group bundle of $\mathrm{GL}(\operatorname{im}\partial)$, and the Lie group bundle of upper triangular blockmatrices, where now

$$h_1 \in \operatorname{Hom}(\operatorname{im}\partial, \ker\partial) \qquad h_2 \in \operatorname{Hom}(\operatorname{coker}\partial, \ker\partial) \qquad h_3 \in \operatorname{Hom}(\operatorname{coker}\partial, \operatorname{im}\partial).$$

In general, given a 2-term complex $C_M \to V_M$, we can represent elements of $\mathrm{H}(V_M, C_M) \ni h_x$ as blockmatrices

$$\begin{pmatrix} 1 & h_x \\ 0 & 1 + \partial h_x \end{pmatrix} \in \mathrm{GL}(C_M \oplus V_M).$$

Another way is to use instead

$$\begin{pmatrix} 1 + h_x\partial & h_x \\ 0 & 1 \end{pmatrix} \in \mathrm{GL}(C_M \oplus V_M)$$

which is related to the previous matrix as follows:

$$\begin{pmatrix} 1 + h_x\partial & h_x \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} 1 & h_x \\ 0 & 1 + \partial h_x \end{pmatrix} \begin{pmatrix} 1 + h_x\partial & h_x\partial h_x(1+\partial h_x)^{-1} \\ 0 & (1+\partial h_x)^{-1} \end{pmatrix}.$$

Similarly, we can depict elements of $\mathrm{Pert}(C_M, V_M)$, but such elements "carry a differential".

Under the identification of $\operatorname{Hom}(V_M, C_M)$ with $\operatorname{Hom}(C_M^*, V_M^*)$ (by taking the dual), the structure of $\mathrm{H}(V_M, C_M)$ takes its "opposite" form:

$$h_x \cdot_{\mathrm{op}} h'_x := h'_x \cdot h_x = h_x + h'_x + h'_x\partial h_x \qquad (h_x)^{-1}_{\mathrm{op}} := h_x^{-1} = -h_x(1+\partial h_x)^{-1}.$$

As happens for groups, bundles of groups are isomorphic to their opposites via the inverse:

Proposition 3.13. *Let $C_M \to V_M$ be a 2-term complex. The canonical map*

$$\mathrm{H}(V_M, C_M) \to \mathrm{H}(C_M^*, V_M^*) \qquad h_x \mapsto (h_x^{-1})^*$$

is an isomorphism of bundles of Lie groups.

In the coming section, Section 3.3, we define the well-known canonical cochain complex representation of the fat groupoid $\widehat{V}_G$ of a VB-groupoid V_G. This cochain complex representation is compatible with the exact sequence of Lie groupoids[28]

$$1 \longrightarrow \mathrm{H}(V_M, C_M) \longrightarrow \widehat{V}_G \longrightarrow G \longrightarrow 1$$

in two ways. One is that the inclusion $\mathrm{H}(V_M, C_M) \hookrightarrow \widehat{V}_G$ is equivariant with respect to the canonical cochain complex representations. Another is that the conjugation action of $\widehat{V}_G$ on $\mathrm{H}(V_M, C_M)$ is induced by the representation on $\operatorname{Hom}(V_M, C_M)$. This is precisely what encodes the fat extension associated to V_G (see the next Section 3.4 and Section 4).

[28]The short exact sequence can be seen as a restriction of the short exact sequence of Lie categories

$$1 \longrightarrow \operatorname{Hom}(V_M, C_M) \longrightarrow \widehat{V}_G^{\mathrm{cat}} \longrightarrow G \longrightarrow 1.$$

3.3. **Canonical representations of the fat groupoid.** The action of V_G on itself induces an action of $\widehat{V}_G$ on V_G: given $H_g \in \widehat{V}_G$ and $v \in (V_G)_{g'}$ with $(g, g') \in G^{(2)}$,

$$H_g \cdot v := H_g(\mathbf{t}_g v) \cdot v.$$

In turn, this induces actions of $\widehat{V}_G$ on C_M and V_M: given $H_g \in \widehat{V}_G$, $c \in (C_M)_{\mathbf{s}g}$ and $v \in (V_M)_{\mathbf{s}g}$,

$$H_g \cdot c := r_{g^{-1}}(H_g(\partial c) \cdot c) \qquad H_g \cdot v := \mathbf{t}_g(H_g v). \tag{3.1}$$

Remark 3.14. More generally, a linear action of V_G on a vector bundle $V_P \to P$ induces an action of $\widehat{V}_G$ on V_P. By linear, we mean that the action map

$$V_G \ltimes V_P \longrightarrow V_P$$

is linear (over an action of G on P). A more thorough discussion (including a discussion on VB-Morita equivalences in terms of bibundles, the notion of Morita equivalence in terms of ruths, and the notion of Morita equivalence in terms of fat extensions) will appear in [MMO].

It is readily verified that the 2-term complex

$$\partial : C_M \xrightarrow{\mathbf{t}} V_M$$

is equivariant with respect to the actions, so we can see $C_M \to V_M$ as a cochain complex representation of $\widehat{V}_G$. In turn, there is a dual cochain complex representation on[29]

$$\partial^* : V_M^* \xrightarrow{\mathbf{t}^*} C_M^*$$

and the representations can also be put together to give a representation on $\mathrm{Hom}(V_M, C_M) = V_M^* \otimes C_M$. To be clear, $\widehat{V}_G$ acts on $\mathrm{Hom}(V_M, C_M)$ via the representation of C_M and via the representation of V_M. Denoting these actions by $\cdot_{C_M}$ and $\cdot_{V_M}$, respectively, the conjugation action is given by

$$H_g \cdot h_{\mathbf{s}g} = H_g \cdot_{V_M} h_{\mathbf{s}g} \cdot_{C_M} H_g^{-1}.$$

When using this action, we use the notation from the right hand side to avoid confusion. This representation descends to an action on $\mathrm{H}(V_M, C_M) \subset \mathrm{Hom}(V_M, C_M)$.[30]

Proposition 3.15. *The inclusion map*

$$\mathrm{H}(V_M, C_M) \hookrightarrow \widehat{V}_G \qquad h_x \mapsto 1 + h_x$$

is equivariant with respect to the canonical cochain complex representations on $C_M \to V_M$. Moreover, the conjugation action of $\widehat{V}_G$ on $\mathrm{H}(V_M, C_M)$, induced by the representation on $\mathrm{Hom}(V_M, C_M)$, coincides with the conjugation action of $\widehat{V}_G$ on $\mathrm{H}(V_M, C_M)$ by viewing $\mathrm{H}(V_M, C_M)$ as a normal subgroupoid of $\widehat{V}_G$.

Proof. The first statement is readily verified. For the second statement we do a direct computation:

$$\begin{aligned}(H_g \cdot (1 + h_{\mathbf{s}g}) \cdot H_g^{-1})v &= H_g(1 + \partial h_{\mathbf{s}g})(H_g^{-1} \cdot v) \cdot (H_g^{-1} \cdot v + h_{\mathbf{s}g}(H_g^{-1} \cdot v)) \cdot H_g^{-1}(v) \\ &= H_g(1 + \partial h_{\mathbf{s}g})(H_g^{-1} \cdot v) \cdot (H_g^{-1}(v) + h_{\mathbf{s}g}(H_g^{-1} \cdot v) \cdot 0_{g^{-1}}) \\ &= v + H_g(\partial h_{\mathbf{s}g}(H_g^{-1} \cdot v)) \cdot h_{\mathbf{s}g}(H_g^{-1} \cdot v) \cdot 0_{g^{-1}}\end{aligned}$$

[29]Recall that, with our convention, V_M^* sits in degree 0 and C_M^* sits in degree 1.

[30]For $h_{\mathbf{s}g} \in \mathrm{H}(V_M, C_M)$ and $H_g \in \widehat{V}_G$, the map $\mathbf{t}_g(H_g \cdot h_{\mathbf{s}g})$ is given by $(\mathbf{t}_g H_g)(1 + \partial h_{\mathbf{s}g})$.

$$= (1 + H_g \cdot_{V_M} h_{\mathbf{s}g} \cdot_{C_M} H_g^{-1})v.$$

This proves the statement. □

3.4. **Fat extensions.** As we discussed now the structure behind the fat groupoid of a VB-groupoid that makes it into a *fat extension*, we already put here the abstract definition:

Definition 3.16. Let $G \rightrightarrows M$ be a Lie groupoid, and consider a 2-term complex

$$C_M \xrightarrow{\partial} V_M.$$

A *fat extension* of G, with underlying complex $C_M \to V_M$, consists of a Lie groupoid

$$F_G \rightrightarrows M,$$

that comes with a cochain complex representation on $C_M \to V_M$, and a short exact sequence of Lie groupoids (over M)

$$1 \longrightarrow \mathrm{H}(V_M, C_M) \longrightarrow F_G \longrightarrow G \longrightarrow 1.$$

Moreover, the cochain complex representation of F_G on $C_M \to V_M$ restricts to the canonical cochain complex representation of $\mathrm{H}(V_M, C_M)$, and the two natural conjugation actions of F_G on $\mathrm{H}(V_M, C_M)$ agree. That is, denoting by $\cdot_{V_M}$ and $\cdot_{C_M}$ the actions of F_G on $\mathrm{H}(V_M, C_M)$ induced by the representations, we have that, for all $H_g \in F_G$ and $h_{\mathbf{s}g} \in \mathrm{H}(V_M, C_M)$,

$$H_g \cdot h_{\mathbf{s}g} \cdot H_g^{-1} = H_g \cdot_{V_M} h_{\mathbf{s}g} \cdot_{C_M} H_g^{-1}.$$

We will simply write F_G for a fat extension. The underlying groupoid F_G is called the fat groupoid of the fat extension.

To understand the structure and examples of fat extensions, it is helpful to keep the reference with VB-groupoids, and so, for now, we study the above structure in terms of VB-groupoids.

3.5. **Fat category extensions.** Before we move to the examples, we observe that $\widehat{V}_G^{\mathrm{cat}}$ can be recovered from $\widehat{V}_G$ and its cochain complex representation. In fact, this leads to yet another equivalent way to encode the structure of a fat extension in terms of a *fat category extension*. This defines the intrinsic notion of weak representation as defined in [Wol17].

To start, observe that $\mathrm{Hom}(\mathbf{s}^*V_M, \mathbf{t}^*C_M) \rightrightarrows M$ is a Lie category with $\mathbf{s}h_g = \mathbf{s}g$, $\mathbf{t}h_g = \mathbf{t}g$ and

$$h_g \cdot h_{g'} := h_g \partial h_{g'}.$$

This is not a "VB-category" over G, because this product is not linear but bilinear. The fat groupoid $\widehat{V}_G$ acts on the vector bundle $\mathrm{Hom}(\mathbf{s}^*V_M, \mathbf{t}^*C_M)$, both on the left and on the right, via the two representations on C_M and V_M. We denote these two actions by $\cdot_{C_M}$ and $\cdot_{V_M}$ for clarity. We then get a groupoid structure on

$$\mathrm{Hom}(\mathbf{s}^*V_M, \mathbf{t}^*C_M) \times_G \widehat{V}_G \rightrightarrows M$$

by setting $\mathbf{s}(h_g, H_g) = \mathbf{s}g$, $\mathbf{t}(h_g, H_g) = \mathbf{t}g$ and

$$(h_g, H_g) \cdot (h_{g'}, H_{g'}) := (h_g \cdot h_{g'} + H_g \cdot_{C_M} h_{g'} + h_g \cdot_{V_M} H_{g'}, H_g \cdot H_{g'}).$$

The connection with $\widehat{V}_G^{\mathrm{cat}}$ becomes clear when associating to (h_g, H_g) the map

$$r_g h_g + H_g : (V_M)_{\mathbf{s}g} \to (V_G)_g \qquad v \mapsto r_g h_g v + H_g v$$

which satisfies $\mathbf{s}_g(r_g h_g + H_g) = 1$.

Proposition 3.17. *The map*

$$\mathrm{Hom}(\mathbf{s}^*V_M, \mathbf{t}^*C_M) \times_G \widehat{V}_G \to \widehat{V}_G^{\mathrm{cat}} \qquad (h_g, H_g) \mapsto r_g h_g + H_g$$

is a surjective submersion of Lie categories. Moreover, the embedding

$$\mathrm{H}(V_M, C_M) \hookrightarrow \mathrm{Hom}(\mathbf{s}^*V_M, \mathbf{t}^*C_M) \times_G \widehat{V}_G \qquad h_x \mapsto (-h_x, 1 + h_x)$$

defines free and proper left and right actions whose quotients are both given by $\widehat{V}_G^{\mathrm{cat}}$, and the above map defines both quotient maps.

Proof. To see that the first map is a surjective submersion follows, for example, by constructing a local section from a local section H of $\widehat{V}_G \to G$:

$$\widehat{V}_G^{\mathrm{cat}} \to \mathrm{Hom}(\mathbf{s}^*V_M, \mathbf{t}^*C_M) \times_G \widehat{V}_G \qquad H'_g \mapsto (r_{g^{-1}}(H'_g - H_g), H_g).$$

The second statement follows by realising that, whenever

$$r_g h_g + H_g = r_g h'_g + H'_g$$

there is a unique element $h_{\mathbf{t}g} \in \mathrm{H}(V_M, C_M)$ such that

$$(-h_{\mathbf{t}g}, 1 + h_{\mathbf{t}g}) \cdot (h_g, H_g) = (h'_g, H'_g),$$

namely, the unique element $h_{\mathbf{t}g}$ such that $(1 + h_{\mathbf{t}g}) \cdot H_g = H'_g$ (and similarly for the right action). This proves the statement. □

Remark 3.18. The above statement describes an "exact sequence of Lie categories"

$$1 \longrightarrow \mathrm{H}(V_M, C_M) \longrightarrow \mathrm{Hom}(\mathbf{s}^*V_M, \mathbf{t}^*C_M) \times_G \widehat{V}_G \longrightarrow \widehat{V}_G^{\mathrm{cat}} \longrightarrow 1.$$

Namely, while it is true that $\mathrm{H}(V_M, C_M)$ is the kernel, in the sense that it comprises the elements mapping to units in $\widehat{V}_G$, it is the kernel in a more general, more appropriate sense. Being more precise here leads to the above statement: whenever two elements coincide in the image of the map

$$\mathrm{Hom}(\mathbf{s}^*V_M, \mathbf{t}^*C_M) \times_G \widehat{V}_G \to \widehat{V}_G^{\mathrm{cat}}$$

they are related under the equivalence obtained from the inclusion by $\mathrm{H}(V_M, C_M)$. That is, the action map from $\mathrm{H}(V_M, C_M) \times_M (\mathrm{Hom}(\mathbf{s}^*V_M, \mathbf{t}^*C_M) \times_G \widehat{V}_G)$ to

$$(\mathrm{Hom}(\mathbf{s}^*V_M, \mathbf{t}^*C_M) \times_G \widehat{V}_G) \times_{\widehat{V}_G^{\mathrm{cat}}} (\mathrm{Hom}(\mathbf{s}^*V_M, \mathbf{t}^*C_M) \times_G \widehat{V}_G)$$

is an isomorphism (and similarly for the right action), meaning that the action is free and proper.

Remark 3.19. From the previous statement, or rather its proof, we saw that $\widehat{V}_G^{\mathrm{cat}}$ is a quotient with respect to the (left) action of $\mathrm{H}(V_M, C_M)$ on $\mathrm{Hom}(\mathbf{s}^*V_M, \mathbf{t}^*C_M) \times_G \widehat{V}_G$ given by

$$h_{\mathbf{t}g} \cdot (h_g, H_g) := (h_g - h_{\mathbf{t}g} \cdot H_g, (1 + h_{\mathbf{t}g}) \cdot H_g).$$

Remark 3.20. The Lie category $\widehat{V}_G$ also has $C_M \to V_M$ as a cochain complex representation. By representation we mean here a smooth functor to the endomorphism Lie category

$$\mathrm{End}\, E_M := \{E_x \to E_y \mid x, y \in M\}$$

where E_M is a vector bundle. It is defined exactly as in (3.1).

The above discussion shows that the fat category of a fat extension can be defined as follows:

Definition 3.21. Let F_G be a fat extension. Consider the Lie category

$$\mathrm{Hom}(\mathbf{s}^*V_M, \mathbf{t}^*C_M) \times_G F_G \rightrightarrows M$$

The fat category $F_G^{\mathrm{cat}} \rightrightarrows M$ (of F_G) is defined as the quotient of

$$\mathrm{Hom}(\mathbf{s}^*V_M, \mathbf{t}^*C_M) \times_G F_G$$

by the action induced by the inclusion[31]

$$\mathrm{H}(V_M, C_M) \hookrightarrow \mathrm{Hom}(\mathbf{s}^*V_M, \mathbf{t}^*C_M) \times_G F_G \qquad h_x \mapsto (-h_x, h_x).$$

Now, apart from the cochain complex representation, the fat category $\widehat{V}_G^{\mathrm{cat}}$ comes with a type of "comparison" map

$$h : \widehat{V}_G^{\mathrm{cat}} \times_G \widehat{V}_G^{\mathrm{cat}} \to \mathrm{Hom}(\mathbf{s}^*V_M, \mathbf{t}^*C_M) \qquad (H_g, H'_g) \mapsto r_{g^{-1}}(H'_g - H_g)$$

that is equivariant for both $\widehat{V}_G^{\mathrm{cat}}$ actions (considering the diagonal action on $\widehat{V}_G^{\mathrm{cat}} \times_G \widehat{V}_G^{\mathrm{cat}}$). It can be interpreted as a homotopy, for, given $H_g, H'_g \in F_G^{\mathrm{cat}}$, $c \in (C_M)_{\mathbf{s}g}$ and $v \in (V_M)_{\mathbf{s}g}$, we have

$$h(H_g, H'_g)\partial c = H'_g \cdot c - H_g \cdot c \qquad \partial h(H_g, H'_g)v = H'_g \cdot v - H_g \cdot v.$$

Moreover, the map h restricts to the "canonical" comparison map of $\mathrm{Hom}(V_M, C_M)$. In particular, it restricts to the zero map on the diagonal embedding

$$\widehat{V}_G^{\mathrm{cat}} \hookrightarrow \widehat{V}_G^{\mathrm{cat}} \times_G \widehat{V}_G^{\mathrm{cat}}.$$

This defines the structure of a fat category extension:

Definition 3.22. Let $G \rightrightarrows M$ be a Lie groupoid, and consider a 2-term complex

$$C_M \xrightarrow{\partial} V_M.$$

A *fat category extension* of G, with underlying complex $C_M \to V_M$, consists of a Lie category

$$F_G^{\mathrm{cat}} \rightrightarrows M,$$

that comes with a cochain complex representation on $C_M \to V_M$, a short exact sequence of Lie categories (over M)[32]

$$1 \longrightarrow \mathrm{Hom}(V_M, C_M) \longrightarrow F_G^{\mathrm{cat}} \longrightarrow G \longrightarrow 1$$

and a smooth map

$$h : F_G^{\mathrm{cat}} \times_G F_G^{\mathrm{cat}} \to \mathrm{Hom}(\mathbf{s}^*V_M, \mathbf{t}^*C_M).$$

The cochain complex representation of F_G^{cat} on $C_M \to V_M$ restricts to the canonical cochain complex representation of $\mathrm{Hom}(V_M, C_M)$. Moreover, h restricts to the canonical comparison map of $\mathrm{Hom}(V_M, C_M)$ (taking the difference). Lastly, h is multiplicative, meaning that, for all $H_g, H'_g \in F_G^{\mathrm{cat}}$, $c \in (C_M)_{\mathbf{s}g}$ and $v \in (V_M)_{\mathbf{s}g}$,

$$h(H_g, H'_g)\partial c = H'_g \cdot c - H_g \cdot c \qquad \partial h(H_g, H'_g)v = H'_g \cdot v - H_g \cdot v$$

and it is equivariant with respect to the canonical left/right (diagonal) actions of F_G^{cat}.

[31] Notice that here we identify $h_x \in \mathrm{H}(V_M, C_M)$ with its image in F_G. If $F_G = \widehat{V}_G$, then this element equals $1 + h_x$.

[32] Recall that this means that $\mathrm{Hom}(V_M, C_M)$ defines a free and proper action on F_G^{cat}.

Remark 3.23. The above definition can be seen to define the intrinsic object behind a "weak representation" of [Wol17]. Namely, as was already observed in [GSM17], a splitting Σ of a VB-groupoid V_G can be seen as a section

$$\Sigma : G \to \widehat{V}_G^{\mathrm{cat}}$$

of the projection $V_G^{\mathrm{cat}} \to G$ (that is not necessarily a functor). Therefore, such a map should be seen as "splitting" a fat category extension. Decomposing $\widehat{V}_G^{\mathrm{cat}}$ using Σ, this leads precisely to the axioms of a weak representation as in [Wol17]. There, it is proved that the category of weak representations is equivalent to the category of (split) 2-term ruths and to the category of VB-groupoids.

Remark 3.24. As we will see in Section 7, the comparison map

$$h : F_G^{\mathrm{cat}} \times_G F_G^{\mathrm{cat}} \to \mathrm{Hom}(\mathbf{t}^* V_M, \mathbf{s}^* C_M)$$

from above can be seen as the "identity map" of the fat category extensions F_G^{cat}.

From the discussion in terms of VB-groupoids it is already clear that we can pass from a fat category extension F_G^{cat} to F_G by taking the invertible elements of F_G^{cat}. Indeed, the properties of the map h imply that, for all $H_g \in F_G$ and $h_{\mathbf{s}g} \in \mathrm{H}(V_M, C_M)$,

$$H_g \cdot h_{\mathbf{s}g} \cdot H_g^{-1} = h(H_g \cdot h_{\mathbf{s}g} \cdot H_g^{-1}, 0_{\mathbf{t}g}) = H_g \cdot_{C_M} h_{\mathbf{s}g} \cdot_{V_M} H_g^{-1}.$$

So, to end this section, we conclude:

Proposition 3.25. *The assignment*

$$\{\text{Fat category extensions}\} \to \{\text{Fat extensions}\} \qquad F_G^{\mathrm{cat}} \mapsto F_G$$

sets a one-to-one correspondence, and the assignment

$$\{\text{Fat extensions}\} \to \{\text{Fat category extensions}\} \qquad F_G \mapsto F_G^{\mathrm{cat}}$$

is inverse to it (up to canonical isomorphisms).

We now move to a more detailed discussion on examples of fat extensions.

3.6. **Examples of fat extensions.** In this section we describe, for well-known constructions of VB-groupoids, the associated fat extensions

$$1 \longrightarrow \mathrm{H}(V_M, C_M) \longrightarrow \widehat{V}_G \longrightarrow G \longrightarrow 1.$$

Naturally, we will see how to define the constructions on an abstract level for fat extensions.

To start, recall that we can view 2-term cochain complexes as VB-groupoids (via the Dold-Kan correspondence). The fat extension associated to this VB-groupoid keeps track of the cochain complex itself, but it also comes with a fat groupoid, namely, the bundle of invertible homotopies:

Example 3.26. Notice that $\mathrm{H}(V_M, C_M)$, together with its canonical cochain complex representation $C_M \to V_M$, itself comes with a short exact sequence:

$$1 \longrightarrow \mathrm{H}(V_M, C_M) \longrightarrow \mathrm{H}(V_M, C_M) \longrightarrow 1_M \longrightarrow 1,$$

where 1_M denotes the unit groupoid of M. This exact sequence appears when considering

$$C_M \ltimes V_M \rightrightarrows V_M$$

as a VB-groupoid over 1_M; here, C_M acts on V_M via the differential:

$$c \cdot v = \partial c + v.$$

Its underlying 2-term complex is $C_M \to V_M$, and its fat groupoid is $\mathrm{H}(V_M, C_M)$. It is now readily verified that the associated exact sequence is the one written above.

Given a single vector bundle E_M, recall that there is a slightly different fat groupoid we can associate to it: the general linear groupoid (see 3.6).

Example 3.27. In Example 3.6 we already mentioned that GL E_M is the fat groupoid of the pair groupoid $E_M \times E_M \rightrightarrows E_M$. In this case,

$$\mathrm{H}(E_M, E_M) = \mathrm{Aut}\, E_M = \{\Psi_x : (E_M)_x \xrightarrow{\sim} (E_M)_x\}$$

and the representation on $E_M \xrightarrow{=} E_M$ is the canonical one: given $\Psi_{y,x} \in \mathrm{GL}\, E_M$ and $e \in (E_M)_x$,

$$\Psi_{y,x} \cdot e \coloneqq \Psi_{y,x} e.$$

Remark 3.28. As mentioned in Remark 2.24, a differential $C_M \to V_M$ defines the map

$$[\cdot, \partial] : \mathrm{GL}(C_M, V_M) = \mathrm{GL}\, C_M \times_{M\times M} \mathrm{GL}\, V_M \to \mathrm{Hom}(\mathrm{pr}_1^* C_M, \mathrm{pr}_2^* V_M).$$

We can think of this map as a type of comparison map. In this case, the zeros of the above map are exactly the cochain maps between $C_M \to V_M$ and itself. In Section 7, we will see that we can think of this as a map of fat extensions between GL C_M and GL V_M. On the level of VB-groupoids, this map corresponds to the VB-groupoid map

$$\partial \times \partial : C_M \times C_M \to V_M \times V_M.$$

We now discuss the continuation of Example 3.5. More precisely, we will come to see what a *fat subextension* and a *fat quotient extension* should be abstractly. The following bundles of Lie groups will play an important role.

Definition 3.29. Let $C_M \to V_M$ be a 2-term complex, and $C_N \to V_N$ a subcomplex. The *restricted bundle of invertible homotopies* is the subbundle of of Lie groups of $\mathrm{H}(V_M, C_M)$ given by

$$\mathrm{H}(V_M, C_M)_{(V_N, C_N)} \coloneqq \{h_y \in \mathrm{H}(V_M, C_M) \mid y \in N, h_y|_{(V_N)_y} \subset (C_N)_y\}.$$

Example 3.30. Recall that, given a Lie groupoid G and a subgroupoid H, J^1G has the subgroupoid

$$J^1_H G = \{j^1_y \sigma \in J^1 G \mid y \in N, \mathrm{im}\, \sigma|_N \subset H\}$$

which played a role in the relation between J^1H, J^1G and $J^1_N(G, H)$ (see Example 3.5). In general, given a VB-groupoid V_G over $C_M \to V_M$ and a VB-subgroupoid V_H over $C_N \to V_N$, $\widehat{V}_G$ has

$$\widehat{V}_{G,H} \coloneqq \{H_h \in \widehat{V}_G \mid h \in H, \mathrm{im}\, H_h|_{(V_N)_{sh}} \subset (V_H)_h\}$$

as a subgroupoid. It fits into an extension

$$1 \longrightarrow \mathrm{H}(V_M, C_M)_{(V_N, C_N)} \longrightarrow \widehat{V}_{G,H} \longrightarrow H \longrightarrow 1$$

and $\widehat{V}_{G,H}$ comes with a cochain complex representation on

$$C_M|_N \to V_M|_N.$$

Moreover, the subcomplex $C_N \to V_N$ is also a cochain complex representation of $\widehat{V}_{G,H}$, and both cochain complex representations restrict to $\mathrm{H}(V_M, C_M)_{(V_N, C_N)}$. So, while not classifying as a fat extension, $\widehat{V}_{G,H}$ carries at least a very similar structure. In turn, it induces two actual fat extensions: one recovers $\widehat{V}_H$ as a quotient:

$$1 \longrightarrow \mathrm{H}(V_M, C_M)_{(C_N, 0_N)} \longrightarrow \widehat{V}_{G,H} \longrightarrow \widehat{V}_H \longrightarrow 1$$

and one recovers $\widehat{V_G/V_H}$[33] as a quotient:

$$1 \longrightarrow \mathrm{H}(V_M, C_M)_{(V_M|_N, C_N)} \longrightarrow \widehat{V}_{G,H} \longrightarrow \widehat{V_G/V_H} \longrightarrow 1.$$

To be clear, also the cochain complex representations of $\widehat{V}_H$ and $\widehat{V_G/V_H}$ are induced by the cochain complex representations of $\widehat{V}_{G,H}$.

From the previous example we can infer how to think of fat subextensions and fat quotient extensions abstractly using the following intermediate definition.

Definition 3.31. Let F_G be a fat extension over $C_M \to V_M$. Moreover, let $H \rightrightarrows N$ be a Lie subgroupoid of G, and let $C_N \to V_N$ be a subcomplex of $C_M \to V_M$. A *restricted fat extension* of F_G consists of a Lie subgroupoid

$$F_{G,H} \rightrightarrows N$$

of F_G that comes with a cochain complex representation on $C_M|_N \to V_M|_N$, which further restricts to $C_N \to V_N$, and a short exact sequence[34]

$$1 \longrightarrow \mathrm{H}(V_M, C_M)_{(V_N, C_N)} \longrightarrow F_{G,H} \longrightarrow H \longrightarrow 1.$$

Moreover, the cochain complex representation of $F_{G,H}$ on $C_M|_N \to V_M|_N$ restricts to the canonical cochain complex representation of $\mathrm{H}(V_M, C_M)_{(V_N, C_N)}$.

We now define:

Definition 3.32. Given a restricted fat extension $F_{G,H}$ of a fat extension F_G, we call

$$1 \longrightarrow \mathrm{H}(V_N, C_N) \longrightarrow F_G/\mathrm{H}(V_M, C_M)_{(V_N, 0_N)} \longrightarrow H \longrightarrow 1$$

the *fat subextension* of $F_{G,H}$, denoted F_H, and

$$1 \longrightarrow \mathrm{H}(V_M|_N/V_N, C_M|_N/C_N) \longrightarrow F_G/\mathrm{H}(V_M, C_M)_{(V_M|_N, C_N)} \longrightarrow H \longrightarrow 1$$

the *fat quotient extension* of $F_{G,H}$, denoted F_G/F_H.

Remark 3.33. We can compare elements of $\widehat{V}_H$ and $\widehat{V}_G$, and we can think of this as a map

$$\iota : \widehat{V}_H \times_G \widehat{V}_G \to \mathrm{Hom}(\mathbf{s}^* V_N, \mathbf{t}^* C_M) \qquad (H_{h,1}, H_{h,2}) \mapsto r_{h^{-1}}(H_{h,1} - H_{h,2}).$$

We can also compare elements of $\widehat{V}_G$ and $\widehat{V_G/V_H}$ in a similar way

$$\pi : \widehat{V}_G \times_G \widehat{V_G/V_H} \to \mathrm{Hom}(\mathbf{s}^* V_M, \mathbf{t}^* C_M|_N/C_N).$$

Given $H_{h,2} \in \widehat{V}_{G,H}$, there is a unique $H_{h,1} \in \widehat{V}_H$ such that

$$\iota(H_{h,1}, H_{h,2}) = 0,$$

and, similarly, given $H_{h,1} \in \widehat{V}_{G,H}$, there is a unique $H_{h,2} \in \widehat{V_G/V_H}$ such that

$$\pi(H_{h,1}, H_{h,2}) = 0.$$

[33] We wrote V_G/V_H for $V_G|_H/V_H$.

[34] Again, we wrote

$$\mathrm{H}(V_M, C_M)_{(V_N, C_N)} \coloneqq \{h_y \in \mathrm{H}(V_M, C_M) \mid y \in N, \operatorname{im} h_y|_{(V_N)_y} \subset (C_N)_y\}.$$

The map ι should be interpreted as the "inclusion" map and π as the "projection" map. These are examples of maps of fat extensions that we will discuss in Section 7.

In the the next example we explain how the fiber product of VB-groupoids looks like in terms of fat extensions. Particular examples of interest are the pullback and direct sum constructions, but we first discuss the more general case.

Example 3.34. Let

$$G_1 \to G \leftarrow G_2$$

be Lie groupoid maps that cleanly intersect, and let V_{G_1} and V_{G_2} be VB-groupoids over G_1 and G_2, respectively. Then we can form the fiber product of VB-groupoids (see [BCdH16]):

$$V_{G_1} \times_G V_{G_2} \rightrightarrows V_{M_1} \times_M V_{M_2}.$$

This is a VB-groupoid over $G_1 \times_G G_2 \rightrightarrows M_1 \times_M M_2$ whose structure is defined "componentwise". To understand how the fat groupoid of this VB-groupoid looks like, notice that an element of it can be decomposed into four maps that we organise into a matrix as follows:

$$\begin{pmatrix} H_{g_1} & h_{12} \\ h_{21} & H_{g_2} \end{pmatrix} = \begin{pmatrix} H_{g_1} & r_{g_1} h_{12} \\ r_{g_2} h_{21} & H_{g_2} \end{pmatrix} : (V_{M_1})_{\mathbf{s}g_1} \times (V_{M_2})_{\mathbf{s}g_2} \to (V_{G_1})_{g_1} \times (V_{G_2})_{g_2}.$$

Here, $H_{g_1} \in \widehat{V}^{\mathrm{cat}}_{G_1}$, $H_{g_2} \in \widehat{V}^{\mathrm{cat}}_{G_2}$ and

$$h_{12} \in \mathrm{Hom}((V_{M_2})_{\mathbf{s}g_2}, (C_{M_1})_{\mathbf{t}g_1}) \qquad h_{21} \in \mathrm{Hom}((V_{M_1})_{\mathbf{s}g_1}, (C_{M_2})_{\mathbf{t}g_2}).$$

In fact, without extra conditions, such matrices describe exactly

$$(V_{G_1} \widehat{\times}_G V_{G_2})^{\mathrm{cat}} \rightrightarrows M_1 \times_M M_2.$$

When interpreted in a specific way, the product of this Lie category can really be seen as matrix multiplication:

$$\begin{pmatrix} H_{g_1} & h_{12} \\ h_{21} & H_{g_2} \end{pmatrix} \cdot \begin{pmatrix} H_{g_1'} & h_{12}' \\ h_{21}' & H_{g_2'} \end{pmatrix} = \begin{pmatrix} H_{g_1} \cdot H_{g_1'} + h_{12} \cdot h_{21}' & H_{g_1} \cdot_{C_M} h_{12}' + h_{12} \cdot_{V_M} H_{g_2'} \\ h_{21} \cdot_{V_M} H_{g_1'} + H_{g_2} \cdot_{C_M} h_{21}' & h_{21} \cdot h_{12}' + H_{g_2} \cdot H_{g_2'} \end{pmatrix}.$$

We used here the notion from Section 3.5, but it is also readily verified by a direct computation what the different products are supposed to represent. We obtain the invertible elements of this category, the elements of the fat groupoid $V_{G_1} \times_G V_{G_2}$, by applying the target to each entry and asking the result to be invertible:

$$\begin{pmatrix} \mathbf{t}_{g_1} H_{g_1} & \partial h_{12} \\ \partial h_{21} & \mathbf{t}_{g_2} H_{g_2} \end{pmatrix} : (V_{M_1})_{\mathbf{s}g_1} \times (V_{M_2})_{\mathbf{s}g_2} \xrightarrow{\sim} (V_{M_1})_{\mathbf{t}g_1} \times (V_{M_2})_{\mathbf{t}g_2}.$$

Notice that

$$\mathrm{H}(V_{M_1} \times_M V_{M_2}, C_{M_1} \times_M C_{M_2}) = \{ \begin{pmatrix} h_1 & h_{12} \\ h_{21} & h_2 \end{pmatrix} \mid \begin{pmatrix} 1 + \partial h_1 & h_{12} \\ h_{21} & 1 + \partial h_2 \end{pmatrix} \text{ is a linear isomorphism} \}$$

where

$$h_1 \in \mathrm{Hom}(V_{M_1}, C_{M_1}) \quad h_{12} \in \mathrm{Hom}(V_{M_2}, C_{M_1}) \quad h_{21} \in \mathrm{Hom}(V_{M_1}, C_{M_2}) \quad h_2 \in \mathrm{Hom}(V_{M_2}, C_{M_2}).$$

The diagonal blockmatrices (those for which h_{12} and h_{21} are both zero) form the fiber product of the fat groupoids of V_{G_1} and V_{G_2}:

$$\widehat{V}_{G_1} \times_G \widehat{V}_{G_2} \rightrightarrows M_1 \times_M M_2$$

so the structure is defined componentwise. This Lie groupoid fits into an exact sequence

$$1 \longrightarrow \mathrm{H}(V_{M_1}, C_{M_1}) \times_M \mathrm{H}(V_{M_2}, C_{M_2}) \longrightarrow \widehat{V}_{G_1} \times_G \widehat{V}_{G_2} \longrightarrow G_1 \times_G G_2 \longrightarrow 1.$$

We continue this line of thought in the next example.

As an application of the previous example, we describe the fat groupoid of a direct sum of VB-groupoids and of a pullback VB-groupoid.

Example 3.35. To describe the direct sum, suppose we are given two VB-groupoids $V_{G,1} \rightrightarrows V_{M,1}$ and $V_{G,2} \rightrightarrows V_{M,2}$ over the same base groupoid G. Then the fat groupoid of

$$V_{G,1} \oplus V_{G,2} \rightrightarrows V_{M,1} \oplus V_{M,2}$$

consist of matrices

$$\begin{pmatrix} H_{g,1} & h_{12} \\ h_{21} & H_{g,2} \end{pmatrix} : (V_{M,1})_{\mathbf{s}g} \oplus (V_{M,2})_{\mathbf{s}g} \to (V_{G,1})_g \oplus (V_{G,2})_g.$$

Here, $H_{g,1} \in \widehat{V}^{\mathrm{cat}}_{G,1}$, $H_{g,2} \in \widehat{V}^{\mathrm{cat}}_{G,2}$ and

$$h_{12} \in \mathrm{Hom}((V_{M,2})_{\mathbf{s}g}, (C_{M,1})_{\mathbf{t}g}) \qquad h_{21} \in \mathrm{Hom}((V_{M,1})_{\mathbf{s}g}, (C_{M,2})_{\mathbf{t}g}).$$

The diagonal blockmatrices constitute the fiber product of fat groupoids $\widehat{V}_{G,1} \times_G \widehat{V}_{G,2}$. This groupoid fits into an exact sequence of the form

$$1 \longrightarrow \mathrm{H}(V_{M,1}, C_{M,1}) \times_M \mathrm{H}(V_{M,2}, C_{M,2}) \longrightarrow \widehat{V}_{G,1} \times_G \widehat{V}_{G,2} \longrightarrow G \longrightarrow 1.$$

This groupoid has four natural 3-term cochain complex representations by taking tensor products. For example, on

$$\mathrm{Hom}(V_{M,1}, C_{M,2}) \longrightarrow \mathrm{Hom}(V_{M,1}, V_{M,2}) \oplus \mathrm{Hom}(C_{M,1}, C_{M,2}) \longrightarrow \mathrm{Hom}(C_{M,1}, V_{M,2}).$$

The actions are defined "componentwise". Essentially, $\widehat{V}_{G,1} \times_G \widehat{V}_{G,2}$ together with its representations on $\mathrm{Hom}(V_{M,1}, C_{M,2})$ and on $\mathrm{Hom}(V_{M,2}, C_{M,1})$ are what constitutes the product of the upper/lower triangular blockmatrices.

From [BD19] we know that multiplicative tensors define homogeneous cochains on Whitney sums of VB-groupoids. In the upcoming work [MOV] we will argue that multiplicative tensors on VB-groupoids $V_{G,1}, \dots, V_{G,n}$ appear naturally as objects defined on $\widehat{V}_{G,1} \times_G \cdots \times_G \widehat{V}_{G,n}$ together with its canonical representations. A first instance of this can be found in Section 7 where we introduce morphisms of fat extensions.

Example 3.36. To describe the pullback, let $V_{G_2} \rightrightarrows V_{M_2}$, be a VB-groupoid over G_2 and $\Phi : G_1 \to G_2$ a groupoid map. The fat groupoid of the pullback VB-groupoid $\Phi^* V_{G_2}$ is

$$\Phi^* \widehat{V}_{G_2} = G_1 \times_{G_2} \widehat{V}_{G_2} \rightrightarrows M_1$$

where the multiplication of $\Phi^* \widehat{V}_{G_2}$ is defined componentwise. If Φ is the inclusion of a Lie subgroupoid H, we simply write

$$\widehat{V}_G|_H = \{H_h \in \widehat{V}_G \mid h \in H\}.$$

To conclude, we formulate the abstract definition of the above examples for fat extensions.

Definition 3.37. Let F_{G_1} and F_{G_2} be fat extensions of $G_1 \rightrightarrows M_1$ and $G_2 \rightrightarrows M_2$, over $C_{M_1} \to V_{M_1}$ and $C_{M_2} \to V_{M_2}$, respectively, and let

$$G_1 \to G \leftarrow G_2$$

be two Lie groupoid maps that cleanly intersect. The *fat extension of invertible blockmatrices* of F_{G_1} and F_{G_2} is the following fat extension. The fat groupoid form the invertible elements of the following fat category:

$$\left\{ \begin{pmatrix} H_{g_1} & h_{12} \\ h_{21} & H_{g_2} \end{pmatrix} \mid H_{g_1} \in F^{\mathrm{cat}}_{G_1}, h_{12} \in \mathrm{Hom}(\mathbf{s}^* V_{M_2}, \mathbf{t}^* C_{M_1}), h_{21} \in \mathrm{Hom}(\mathbf{s}^* V_{M_1}, \mathbf{t}^* C_{M_2}), H_{g_2} \in F^{\mathrm{cat}}_{G_2} \right\}$$

whose product is defined as

$$\begin{pmatrix} H_{g_1} & h_{12} \\ h_{21} & H_{g_2} \end{pmatrix} \cdot \begin{pmatrix} H_{g_1'} & h_{12}' \\ h_{21}' & H_{g_2'} \end{pmatrix} = \begin{pmatrix} H_{g_1} \cdot H_{g_1'} + h_{12} \cdot h_{21}' & H_{g_1} \cdot_{C_M} h_{12}' + h_{12} \cdot_{V_M} H_{g_2'} \\ h_{21} \cdot_{V_M} H_{g_1'} + H_{g_2} \cdot_{C_M} h_{21}' & h_{21} \cdot h_{12}' + H_{g_2} \cdot H_{g_2'} \end{pmatrix}.$$

Here, $h_{12} \cdot h_{21}' = h_{12} \partial h_{21}'$, $h_{21} \cdot h_{12}' = h_{21} \partial h_{12}'$, and $\cdot_{V_M}$ and $\cdot_{C_M}$ are the actions induced by the representations. The cochain complex representation on

$$\partial \times \partial : C_{M_1} \times_M C_{M_2} \to V_{M_1} \times_M V_{M_2}$$

is given by matrix application: given $(c_1, c_2) \in (C_M)_{\mathbf{s}g_1} \times (C_M)_{\mathbf{s}g_2}$ and $(v_1, v_2) \in (V_M)_{\mathbf{s}g_1} \times (V_M)_{\mathbf{s}g_2}$,

$$\begin{pmatrix} H_{g_1} & h_{12} \\ h_{21} & H_{g_2} \end{pmatrix} \begin{pmatrix} c_1 \\ c_2 \end{pmatrix} = \begin{pmatrix} H_{g_1} \cdot c_1 + h_{12} \partial c_2 \\ h_{21} \partial c_1 + H_{g_2} \cdot c_2 \end{pmatrix} \qquad \begin{pmatrix} H_{g_1} & h_{12} \\ h_{21} & H_{g_2} \end{pmatrix} \begin{pmatrix} v_1 \\ v_2 \end{pmatrix} = \begin{pmatrix} H_{g_1} \cdot v_1 + \partial h_{12} v_2 \\ \partial h_{21} v_1 + H_{g_2} \cdot v_2 \end{pmatrix}.$$

In the last sections, Sections 3.7 and 3.8, we make remarks on other models of the fat groupoid of a VB-groupoid. In particular, we will see that dualising a fat extension should be seen as dualising its cochain complex representation.

3.7. **Other models for the fat groupoid of a VB-groupoid.** The fat groupoid of a VB-groupoid V_G appeared in [GSM17] as follows. Consider

$$\widetilde{V}^{\mathrm{cat}}_G = \{\widetilde{H}_g \subset (V_G)_g \mid (V_G)_g = \ker \mathbf{s}_g \oplus \widetilde{H}_g\}$$

and the following open subset of it:

$$\widetilde{V}_G = \{\widetilde{H}_g \in \widetilde{V}^{\mathrm{cat}}_G \mid (V_G)_g = \ker \mathbf{t}_g \oplus \widetilde{H}_g\}.$$

Then $\widetilde{V}^{\mathrm{cat}}_G$ is a Lie category over M with $\widetilde{\mathbf{s}}\widetilde{H}_g = \mathbf{s}g$, $\widetilde{\mathbf{t}}\widetilde{H}_g = \mathbf{t}g$, and

$$\widetilde{H}_g \cdot \widetilde{H}_h = \mathrm{im}\, \mathbf{m}(\widetilde{H}_g, \widetilde{H}_h) = \{v_g \cdot v_h \in (V_G)_{gh} \mid v_g \in \widetilde{H}_g, v_h \in \widetilde{H}_h\},$$

and $\widetilde{V}_G$ form the invertible elements with

$$(\widetilde{H}_g)^{-1} = \mathrm{im}\, \mathbf{i}(\widetilde{H}_g).$$

It is rather straightforward to see that the canonical map

$$\widehat{V}^{\mathrm{cat}}_G \xrightarrow{\mathrm{im}} \widetilde{V}^{\mathrm{cat}}_G$$

is an isomorphism of Lie categories (that therefore restricts to an isomorphism of $\widehat{V}_G$ to $\widetilde{V}_G$). Still, we discuss the proof, because some interesting things appear. For example, it becomes clear that the fat groupoid $\widehat{V^*_G}$ of the dual VB-groupoid V^*_G is isomorphic to $\widehat{V}_G$ as groupoids.

Notice that we can think of elements of $\widehat{V^*_G}$ as (the dual of) maps

$$\omega_g : (V_G)_g \to (C_M)_{\mathbf{s}g}$$

such that precomposing with ℓ_g gives the identity map, and precomposing with r_g gives a linear isomorphism.[35] The product of two elements ω_g and ω_h is given by

$$(\omega_g \cdot \omega_h)(v_g \cdot v_h) = \omega_h(r_h\omega_g(v_g)) + \omega_h(v_h)$$

(here $(v_g, v_h) \in V_G^{(2)}$ is a composable element) and the inverse of ω_g is

$$\omega_g^{-1}(v_g^{-1}) = -(\omega_g r_g)^{-1}(\omega_g v_g).$$

Proposition 3.38. *The maps*

$$\widehat{V}_G \to \widetilde{V}_G \leftarrow \widehat{V}_G^* \qquad H_g \mapsto \operatorname{im} H_g = \ker \omega_g \leftarrow\!\shortmid \omega_g$$

are groupoid isomorphisms.

Proof. We construct the inverse of the first map. To do this, take for every $g \in G$ an $H_{g,1} \in \widehat{V}_G$. Given $\widetilde{H}_g \in \widetilde{V}_G$, we can then write, for all $v \in (V_M)_{\mathbf{s}g}$,

$$H_{g,1}v = v_{\mathbf{s},1} + H_g v$$

where $v_{\mathbf{s},1} \in \ker \mathbf{s}_g$ and $H_g v \in \widetilde{H}_g$. We claim that the map

$$\widetilde{V}_G \to \widehat{V}_G \qquad \widetilde{H}_g \mapsto H_g$$

is a well-defined smooth inverse of the map $\widehat{V}_G \to \widetilde{V}_G$. To see that it is well-defined, take another $H_{g,2} \in \widehat{V}_G$ and write as above, for all $v \in (V_M)_{\mathbf{s}g}$,

$$H_{g,2}v = v_{\mathbf{s},2} + H'_g v.$$

Then it follows that

$$H'_g v - H_g v \in \ker \mathbf{s}_g \cap \widetilde{H}_g = 0.$$

A similar argument shows that $\mathbf{t}_g H_g$ must be invertible: if $\mathbf{t}_g H_g v = 0$, then

$$H_g v \in \ker \mathbf{t}_g \cap \widetilde{H}_g = 0.$$

The map is smooth, for once we work locally, we can take $H_{g,1}$ to be in the image of a local section of the projection $\widehat{V}_G \to G$. For a proof that the maps are groupoid maps, we refer to the proof of Proposition 3.40. □

Remark 3.39. Notice that, in the above proof, the auxiliary elements $H_{g,1} \in \widehat{V}_G$ that are used to define the inverse map $\widetilde{V}_G \to \widehat{V}_G$ can be taken in $\widehat{V}_G^{\mathrm{cat}}$ as well. In particular, we could also use a cleavage Σ to define, for all $v \in (V_M)_{\mathbf{s}g}$, the inverse through the equation

$$\Sigma_g v = v_{\mathbf{s}} + H_g v$$

where $v_{\mathbf{s}} \in \ker s_g$ and $H_g v \in \widetilde{H}_g$.

The isomorphism $\widehat{V}_G \to \widetilde{V}_G \to \widehat{V}_G^*$ can be understood through a pairing that we discuss next.

[35] As before, we wrote

$$\ell_g : (C_M)_{\mathbf{s}g} \xrightarrow{-0_g \cdot c^{-1}} (V_G)_g \qquad r_g : (C_M)_{\mathbf{t}g} \xrightarrow{c \cdot 0_g} (V_G)_g.$$

3.8. **The fat pairing.** We first look at the isomorphism $\widehat{V}_G \to \widetilde{V}_G \to \widehat{V}^*_G$ more carefully.

Proposition 3.40. *Let V_G be a VB-groupoid and consider $\widehat{V}_G$ and $\widehat{V}^*_G$. For $H_g \in \widehat{V}_G$, write*

$$\omega_g^{H_g} : (V_G)_g \to (C_M)_{\mathbf{s}g} \qquad v \mapsto r_g(H_g^{-1}\mathbf{t}_g v - v^{-1}).$$

The map

$$\widehat{V}_G \to \widehat{V}^*_G \qquad H_g \mapsto \omega_g^{H_g}$$

is a Lie groupoid isomorphism. Moreover, the map restricts to the canonical isomorphism $\mathrm{H}(V_M, C_M) \cong \mathrm{H}(C^*_M, V^*_M)$ *from Proposition 3.13, so the diagram*

$$\begin{tikzcd}
1 \arrow[r] & \mathrm{H}(V_M, C_M) \arrow[r] \arrow[d, "\cong"] & \widehat{V}_G \arrow[r] \arrow[d, "\cong"] & G \arrow[r] \arrow[d, "="] & 1 \\
1 \arrow[r] & \mathrm{H}(C^*_M, V^*_M) \arrow[r] & \widehat{V}^*_G \arrow[r] & G \arrow[r] & 1
\end{tikzcd}$$

commutes.

Proof. Notice that $\omega_g^{H_g}$ is the unique element $\omega_g \in \widehat{V}^*_G$ with the property that $\omega_g H_g = 0$.[36] To see that the map $\widehat{V}_G \to \widehat{V}^*_G$ is well-defined, observe that

$$\omega_g^{H_g} r_g = r_g(H_g^{-1}(\partial c) + \ell_g(\partial c - c)) = H_g^{-1} \cdot c$$

has inverse $c \mapsto H_g \cdot c$.

We compute the inverse map $\omega_g \mapsto H_g^{\omega_g}$ explicitly. Pick for this some $H'_g \in \widehat{V}_G$. Then $H_g^{\omega_g}$ is the unique element $H_g \in \widehat{V}_G$ such that

$$H'_g v = \ell_g \omega_g(H'_g v) + H_g(v + \partial\omega_g(H'_g v)).$$

In fact, given $v \in (V_M)_{\mathbf{s}g}$, writing $h_g v \in (C_M)_{\mathbf{t}g}$ for the inverse of $-\omega_g(H'_g v)$ under $\omega_g r_g$, we have

$$H_g^{\omega_g} v = H'_g v + r_g h_g v.$$

We end the proof by showing that $\widehat{V}_G \to \widehat{V}^*_G$ is a groupoid map. For $H_g, H_{g'} \in \widehat{V}_G$ composable,

$$\omega_g^{H_g} \cdot \omega_{g'}^{H_{g'}}(v_g \cdot v_{g'}) = (H_{g'}^{-1}((\mathbf{t}_g H_g)^{-1}\mathbf{t}_g v_g) + 0_{(g')^{-1}g^{-1}} \cdot (v_g - H_g(\mathbf{t}_g H_g)^{-1}\mathbf{t}_g v_g) - v_{g'}^{-1}) \cdot 0_{g'},$$

while

$$\omega_{gg'}^{H_{gg'}}(v_g \cdot v_{g'}) = (H_{g'}^{-1} H_g^{-1}\mathbf{t}_g v_g - v_{g'}^{-1} v_g^{-1}) \cdot 0_g \cdot 0_{g'}.$$

If we replace 0_g with $\pm v_g \pm H_g(\mathbf{t}_g H_g)^{-1}\mathbf{t}_g v_g$ in the last equation, we see that the two expressions agree. This proves the statement. □

Remark 3.41. Given H_g and $\omega_g = \omega_g^{H_g}$, we constructed in the above proof a map h_g using an auxiliary element $H'_g \in \widehat{V}_G$:

$$h_g : (V_M)_{\mathbf{s}g} \to (C_M)_{\mathbf{t}g} \qquad v \mapsto r_{g^{-1}}(\ell_g + H_g\partial)\omega_g(H'_g v) = r_{g^{-1}}(H'_g v - H_g v).$$

[36] If ω'_g is another element of $\widehat{V}^*_G$ such that $\omega'_g H_g = 0$, then $\omega_g^{H_g} = \omega'_g$ on $\operatorname{im} H_g$, but also on $\ker \mathbf{t}_g$, which are elements of the form $\ell_g c$.

As we mentioned already in Remark 3.39, we could take such auxiliary elements in $\widehat{V}_G^{\mathrm{cat}}$. Given a cleavage Σ, the equation

$$\Sigma_g v = -r_g h_g v + H_g v$$

reveals a new description of the fat groupoid in terms of the associated split ruth. This will be explained in detail in Section 4.3.

We can understand this isomorphism from Proposition 3.40 more conceptually as follows.

Proposition 3.42. *The smooth map*

$$\widehat{V}_G^* \times_G \widehat{V}_G \to \mathrm{H}(V_M, C_M) \qquad (\omega_g, H_g) \mapsto \omega_g H_g$$

is non-degenerate in the sense that, given $H_g \in \widehat{V}_G$ (resp. $\omega_g \in \widehat{V}_G^*$) and $h_{\mathbf{s}g} \in \mathrm{H}(V_M, C_M)$, there is a unique element $\omega_g \in \widehat{V}_G^*$ (resp. $H_g \in \widehat{V}_G$) such that $\omega_g H_g = h_{\mathbf{s}g}$.

Proof. Let $H_g \in \widehat{V}_G^*$ and $h_{\mathbf{s}g} \in \mathrm{H}(V_M, C_M)$. If $h_{\mathbf{s}g} = 0$, we saw in Proposition 3.40 that

$$\omega_g^{H_g} v = r_g(H_g^{-1}\mathbf{t}_g v - v^{-1})$$

is unique with the property that

$$\omega_g^{H_g} H_g = 0.$$

Now, if $h_{\mathbf{s}g} \neq 0$, consider

$$\omega_g v = \omega_g^{H_g} v + h_{\mathbf{s}g}(\mathbf{t}_g H_g)^{-1}\mathbf{t}_g v.$$

Then $\omega_g \in \widehat{V}_G^*$. Indeed, the inverse of precomposition with r_g is given by

$$c \mapsto r_{g^{-1}}(\ell_g c + H_g \partial c) + (h_{\mathbf{s}g}\partial c)^{-1}.$$

The element ω_g is unique with the property that $\omega_g H_g = h_{\mathbf{s}g}$, so we proved the statement. □

Remark 3.43. Given $V_{G,1}, \ldots, V_{G,n}$ VB-groupoids over G, we can see

$$\widehat{V}_{G,1} \times_G \cdots \times_G \widehat{V}_{G,n}$$

as a groupoid over M (with structure defined componentwise; see Example 3.34). However, with this groupoid structure, the above map $\widehat{V}_G^* \times_G \widehat{V}_G \to \mathrm{H}(V_M, C_M)$ is not a groupoid map (see the discussion after this remark). Rather, this map should be interpreted as the identity map of fat extensions, and we can interpret the map as a multiplicative tensor if we view $\widehat{V}_G^* \times_G \widehat{V}_G$ as a groupoid. That it maps into $\mathrm{H}(V_M, C_M)$ is a feature of the identity map being an isomorphism. We will be more precise in Section 7, but a more general treatment of viewing multiplicative tensors this way is a topic of our upcoming work [MOV].

Using $\widehat{V}_G^* \cong \widehat{V}_G$, the fat pairing is simply the map

$$\widehat{V}_G \times_G \widehat{V}_G \to \mathrm{H}(V_M, C_M) \qquad (H_g, H_g') \mapsto -1 + H_g' \cdot H_g^{-1}$$

that describes the inverse to the (principal) action map

$$\mathrm{H}(V_M, C_M) \ltimes \widehat{V}_G \to \widehat{V}_G \times_G \widehat{V}_G.$$

Remark 3.44. Proposition 3.40 reveals that a map of fat extensions $F_{G,1} \to F_{G,2}$ (over the same base groupoid G) is not simply a commutative diagram

$$\begin{tikzcd}
1 \arrow[r] & \mathrm{H}(V_{M,1}, C_{M,1}) \arrow[r] \arrow[d] & F_{G,1} \arrow[r] \arrow[d] & G \arrow[r] \arrow[d, "="] & 1 \\
1 \arrow[r] & \mathrm{H}(V_{M,2}, C_{M,2}) \arrow[r] & F_{G,2} \arrow[r] & G \arrow[r] & 1
\end{tikzcd}$$

Indeed, $\widehat{V}_G$ and $\widehat{V}_G^*$ are fat extensions, and there is a commutative diagram

$$\begin{tikzcd}
1 \arrow[r] & \mathrm{H}(V_M, C_M) \arrow[r] \arrow[d, "\cong"] & \widehat{V}_G \arrow[r] \arrow[d, "\cong"] & G \arrow[r] \arrow[d, "="] & 1 \\
1 \arrow[r] & \mathrm{H}(C_M^*, V_M^*) \arrow[r] & \widehat{V}_G^* \arrow[r] & G \arrow[r] & 1.
\end{tikzcd}$$

In fact, we should not even expect that a map of fat extensions comes with a groupoid map $F_{G,1} \to F_{G,2}$ (see Section 7). Rather, maps of fat extensions are "comparison maps" like the ones we encountered in Section 3.6.

What Proposition 3.40 shows is that the dual of a fat extension should be seen as having the same underlying extension, but considering the dual cochain complex representation:

Definition 3.45. Let F_G be a fat extension of G over $C_M \to V_M$. The dual of F_G is the extension

$$1 \longrightarrow \mathrm{H}(V_M, C_M) \longrightarrow F_G \longrightarrow G \longrightarrow 1$$

together with the dual cochain complex representation $V_M^* \to C_M^*$.

4. Fat extensions and equivalent formulations

Having encountered the structure of *fat extension* in the previous section, we proceed to show how to go back and forth between fat extensions, VB-groupoids and 2-term ruths. We dedicate Section 5 to the correspondence with (general linear) PB-groupoids after this section.

We call the 2-term ruth associated to a fat extension an *invariant complex*. This type of complex is a *relative complex* as in [Sal24]. This new point of view leads to interesting connections between the "natural candidates" for tensor product of VB-groupoids and of 2-term ruths. As already mentioned a few times, this is the topic of the upcoming work [MOV]. Defining the correct notion of morphism for the category of fat extensions (that yields the claimed equivalence of categories between fat extensions and VB-groupoids or 2-term ruths) is a little more subtle, but similar to how one defines morphisms of (split) 2-term ruths. We delay the notion of morphism of fat extensions to Section 7, where we also go into the functorial aspects of the equivalence we start describing in this section. The reason for the postponement is that we believe the discussion of this section should help understand the functorial aspects better.

4.1. From VB-groupoids to fat extensions.

In Section 3, we developed the notion of fat extension through the language of VB-groupoids. For referential purposes and convenience, we repeat the definition here.

Definition 4.1. Let $G \rightrightarrows M$ be a Lie groupoid, and consider a 2-term complex

$$C_M \xrightarrow{\partial} V_M.$$

A *fat extension* of G, with underlying complex $C_M \to V_M$, consists of a Lie groupoid

$$F_G \rightrightarrows M,$$

that comes with a cochain complex representation on $C_M \to V_M$, and a short exact sequence of Lie groupoids (over M)

$$1 \longrightarrow \mathrm{H}(V_M, C_M) \longrightarrow F_G \longrightarrow G \longrightarrow 1.$$

Moreover, the cochain complex representation of F_G on $C_M \to V_M$ restricts to the canonical cochain complex representation of $\mathrm{H}(V_M, C_M)$, and the two natural conjugation actions of F_G on $\mathrm{H}(V_M, C_M)$ agree.

We will simply write F_G for a fat extension, and the underlying groupoid F_G is called the fat groupoid of the fat extension. Now, in Sections 3.1, 3.2 and 3.3, we proved:

Proposition 4.2. *Let V_G be a VB-groupoid over G. Then, the fat groupoid $\widehat{V}_G$ of V_G, together with its cochain complex representation* (3.1)*, is a fat extension of G over $C_M \to V_M$.*

Similarly, we have:

Proposition 4.3. *Let V_G be a VB-groupoid over G. Then, the fat category $\widehat{V}_G^{\mathrm{cat}}$ of V_G, together with its cochain complex representation on $C_M \to V_M$ and the (comparison) map*

$$h : F_G^{\mathrm{cat}} \times_G F_G^{\mathrm{cat}} \to \mathrm{Hom}(\mathbf{t}^* V_M, \mathbf{s}^* C_M)$$

is a fat category extension of G over $C_M \to V_M$.

4.2. **From fat extensions to VB-groupoids.** In this section we prove how to recover the VB-groupoid from a fat extension. Let F_G be a fat extension and consider the VB-groupoid associated to the cochain complex representation $C_M \to V_M$. We denote this VB-groupoid by

$$F_G \ltimes (C_M \to V_M) = C_M \times_M F_G \times_M V_M \rightrightarrows V_M.$$

The source and target maps are given by $\mathbf{s}(c, H_g, v) = v$ and $\mathbf{t}(c, H_g, v) = \partial c + H_g \cdot v$, and

$$(c_1, H_g, \partial c_2 + H_h \cdot v) \cdot (c_2, H_h, v) = (c_1 + H_g \cdot c_2, H_g \cdot H_h, v).$$

Now, to recover a VB-groupoid V_G from its fat extension $F_G = \widehat{V}_G$, observe that an element $H_g \in \widehat{V}_G$ defines a linear isomorphism

$$r_g + H_g : (C_M)_{\mathbf{t}g} \oplus (V_M)_{\mathbf{s}g} \to (V_G)_g \qquad (c, v) \mapsto c \cdot 0_g + H_g v.$$

The inverse is given by $((\omega_g r_g)^{-1}\omega_g, \mathbf{s}_g)$, where $\omega_g = \omega_g^{H_g} \in \widehat{V}_G^*$ is unique with the property that $\omega_g H_g = 0$ (see Proposition 3.40). This observation leads to the following statement.

Proposition 4.4. *The VB-groupoid map*

$$\widehat{V}_G \ltimes (C_M \to V_M) \to V_G \qquad (c, H_g, v) \mapsto r_g c + H_g v$$

is surjective, and the associated exact sequence (of VB-groupoids) is given by

$$1 \longrightarrow \mathrm{H}(V_M, C_M) \ltimes V_M \longrightarrow \widehat{V}_G \ltimes (C_M \to V_M) \longrightarrow V_G \longrightarrow 1,$$

where $\mathrm{H}(V_M, C_M)$ acts trivially on V_M, and the inclusion is given by $(h_x, v) \mapsto (-h_x v, 1 + h_x, v)$.

Now, given again an abstract fat extension

$$1 \longrightarrow \mathrm{H}(V_M, C_M) \longrightarrow F_G \longrightarrow G \longrightarrow 1$$

we obtain an injective map of VB-groupoids[37]

$$\mathrm{H}(V_M, C_M) \ltimes V_M \to F_G \ltimes (C_M \to V_M) \qquad (h_x, v) \mapsto (-h_x v, h_x, v).$$

The quotient naturally inherits the structure of a VB-groupoid over G that we denote by $V(F_G)$. The resulting short exact sequence

$$1 \longrightarrow \mathrm{H}(V_M, C_M) \ltimes V_M \longrightarrow F_G \ltimes (C_M \to V_M) \longrightarrow V(F_G) \longrightarrow 1$$

is a short exact sequence of VB-groupoids over the extension

$$1 \longrightarrow \mathrm{H}(V_M, C_M) \longrightarrow F_G \longrightarrow G \longrightarrow 1.$$

In particular, the underlying cochain complex of $V(F_G)$ is given by $C_M \to V_M$. We will now show that VB-groupoids and fat extensions are in a one-to-one correspondence with each other. As we delayed the discussion on morphisms of fat extensions to Section 7, we won't go into the functorial aspects here. However, the notion of isomorphism of fat extensions necessarily appears.

Proposition 4.5. *The assignment*

$$\{\text{VB-groupoids}\} \to \{\text{Fat extensions}\} \qquad V_G \mapsto \widehat{V}_G$$

sets a one-to-one correspondence, and the assignment

$$\{\text{Fat extensions}\} \to \{\text{VB-groupoids}\} \qquad F_G \mapsto V(F_G)$$

is inverse to it (up to canonical isomorphisms).

Proof. That, given a VB-groupoid V_G, there is a canonical isomorphism

$$V(\widehat{V}_G) \xrightarrow{\sim} V_G$$

is a straightforward consequence of Proposition 4.4.

Given a fat extension F_G, observe that the map

$$F_G \to \widehat{V}(F_G) \qquad H_g \mapsto (v \mapsto [0, H_g, v])$$

is a Lie groupoid map that intertwines the representations on $C_M \to V_M$. In the remainder of the proof, we show that this map is invertible.[38] Notice for this that elements of $\widehat{V}(F_G)$ can be represented by maps

$$(V_M)_{\mathbf{s}g} \to (C_M)_{\mathbf{t}g} \times (F_G)_g \times (V_M)_{\mathbf{s}g} \qquad v \mapsto (hv, H_g^v, v)$$

such that h is linear, and

$$(V_M)_{\mathbf{s}g} \to (V_M)_{\mathbf{t}g} \qquad v \mapsto \partial h(v) + H_g^v \cdot v$$

is a well-defined linear isomorphism. We claim that, for all elements of $\widehat{V}(F_G)$, we can take $h = 0$ and $H_g^v = H_g^0$. Indeed, given a representative as above, write, for all $v \in (V_M)_{\mathbf{s}g}$,

$$h_{\mathbf{t}g}^v w = h((H_g^v)^{-1} \cdot w) \in \mathrm{H}(V_M, C_M).$$

Then, for all $v \in (V_M)_{\mathbf{s}g}$,

$$(h_{\mathbf{t}g}^v, \partial h(v) + H_g^v \cdot v) \cdot (h(v), H_g^v, v) = (0, h_{\mathbf{t}g}^v \cdot H_g^v, v),$$

[37] As above, $\mathrm{H}(V_M, C_M)$ acts trivially on V_M. Notice that now we identified $h_x \in \mathrm{H}(V_M, C_M)$ with its image in F_G; if $F_G = \widehat{V}_G$, this identifies h_x with its image $1 + h_x$.

[38] Therefore, we obtain an isomorphism of fat extensions; see Section 7 for more details.

so $v \mapsto (0, h^v_{\mathbf{t}g} \cdot H^v_g, v)$ represents the same element as $v \mapsto (h(v), H^v_g, v)$ in $\widehat{V}(F_G)$. Now, for all $v \in (V_M)_{\mathbf{s}g}$, we can take the unique element $h^v_{\mathbf{s}g} \in \mathrm{H}(V_M, C_M)$ such that

$$h^v_{\mathbf{t}g} \cdot H^v_g \cdot h^v_{\mathbf{s}g} = H^0_g.$$

Then

$$(0, h^v_{\mathbf{t}g} \cdot H^v_g, v) \cdot (h^v_{\mathbf{s}g}, v) = (0, H^0_g, v),$$

which shows that, $v \mapsto (0, H^0_g, v)$ represents the same element as $v \mapsto (h(v), H^v_g, v)$ in $\widehat{V}(F_G)$. This proves the claim, so the map $F_G \to \widehat{V}(F_G)$ is indeed invertible. □

This result can be seen as the essential surjectivity of functors that is described in Section 7.

4.3. **From split 2-term ruths to fat extensions.** Given a VB-groupoid V_G, we observed in Remark 3.41 that we can construct an inverse to the canonical map $\widehat{V}_G \to \widehat{V}^*_G$ using a cleavage Σ. The key ingredient we used was the map

$$h_g : (V_M)_{\mathbf{s}g} \to (C_M)_{\mathbf{t}g} \qquad h_g = r_{g^{-1}}(H_g - \Sigma_g)$$

that satisfies

$$\Sigma_g + r_g h_g = H_g.$$

This map h_g is a homotopy, showing that the quasi-action $R_1(g)$ and the action by H_g define the same map in cohomology: for all $v \in (V_M)_{\mathbf{s}g}$, we have

$$R_1(g)v + [\partial, h_g]v = H_g \cdot v.$$

Defining H_g out of h_g and Σ using the above formula, we obtain another description of $\widehat{V}_G$ in terms of the split 2-term ruth associated to V_G. It should be seen as the fat groupoid associated to a split 2-term ruth:

Definition 4.6. The fat groupoid $\widehat{\mathcal{E}}_G$ of a split 2-term ruth $\mathcal{E}_G$ is the Lie groupoid

$$\widehat{\mathcal{E}}_G = \{h_g : (V_M)_{\mathbf{s}g} \to (C_M)_{\mathbf{t}g} \mid \varphi_g = R_1(g) + \partial h_g \text{ is a linear isomorphism}\}$$

over M.

Notice that, given $h_g \in \widehat{\mathcal{E}}_G$, the map

$$\varphi_g = R_1(g) + [\partial, h_g]$$

is a cochain map that is an isomorphism: it defines the cochain complex representation.[39] To clarify, the structure maps are defined again as composition of homotopies (see Lemma 3.7): if h_g and $h_{g'}$ are composable (i.e. g and g' are composable), then the homotopy $h_g \cdot h_{g'}$ is such that

$$\varphi_g \varphi_{g'} = R_1(gg') + [\partial, h_g \cdot h_{g'}].$$

That is,

$$\begin{aligned} h_g \cdot h_{g'} &:= R_2(g, g') + h_g R_1(g') + R_1(g) h_{g'} + h_g \partial h_{g'} \\ h_g^{-1} &:= -(R_2(g^{-1}, g) + R_1(g^{-1}) h_g)\varphi_g^{-1} = -\varphi_g^{-1}(R_2(g, g^{-1}) + h_g R_1(g^{-1})). \end{aligned}$$

[39]That $R_1(g) + h_g\partial$ is an isomorphism as well follows from the five-lemma, for φ_g is homotopic to $R_1(g)$. Indeed, φ_g is therefore a quasi-isomorphism, and since $R_1(g) + \partial h_g$ is an isomorphism, $R_1(g) + h_g\partial$ must be as well.

Alternatively, we may think of $\widehat{\mathcal{E}}_G$ as the open set of invertible elements of the Lie category

$$\widehat{\mathcal{E}}_G^{\mathrm{cat}} = \mathrm{Hom}(\mathbf{s}^*V_M, \mathbf{t}^*C_M) \rightrightarrows M$$

with product defined as above. We have an isomorphism of Lie categories

$$h_\Sigma : \widehat{V}_G^{\mathrm{cat}} \to \widehat{\mathcal{E}}_G^{\mathrm{cat}} \qquad H_g \mapsto r_{g^{-1}}(H_g - \Sigma_g)$$

and this isomorphism restricts to an isomorphism $\widehat{V}_G \to \widehat{\mathcal{E}}_G$ (see Remark 4.8 and Section 6.2).

Remark 4.7. Thinking of $R = (\partial = R_0, R_1, R_2)$ as a Maurer-Cartan element (see Remark 2.18), we can perhaps think of the above product on $\mathrm{Hom}(\mathbf{s}^*V_M, \mathbf{t}^*C_M)$ as a "twisting" of the trivial product

$$h_g \cdot h_{g'} = 0$$

by R. The product

$$h_g \cdot h_{g'} := h_g \partial h_{g'}$$

we used in Section 3.5 can also be seen this way by taking $R = (\partial, 0, 0)$, but notice that this is not a unital ruth.

Remark 4.8. The map $h_\Sigma : \widehat{V}_G \to \mathrm{Hom}(\mathbf{s}^*V_M, \mathbf{t}^*C_M)$ we constructed above is "invariant": given $H_g, H'_g \in \widehat{V}_G$, we have

$$h_\Sigma(H_g) - h_\Sigma(H'_g) = -h(H_g, H'_g)$$

where

$$h : \widehat{V}_G \times \widehat{V}_G \to \mathrm{Hom}(\mathbf{s}^*V_M, \mathbf{t}^*C_M) \qquad h(H_g, H'_g) := r_{g^{-1}}(H'_g - H_g).$$

Conversely, Σ can be recovered as

$$\Sigma_g = -r_g h_\Sigma(H_g) + H_g$$

where $H_g \in \widehat{V}_G$ is arbitrary. That Σ is unital translates into $h_\Sigma(1 + h_x) = h_x$ for all $h_x \in \mathrm{H}(V_M, C_M)$. We can therefore think abstractly of splittings of fat extensions as being invariant maps as above. In Section 6.2 we put some more details about these *fat splittings*.

4.4. **From fat extensions to 2-term ruths.** Let F_G be a fat extension of G over $C_M \to V_M$. Recall that we can construct an exact sequence

$$1 \longrightarrow \mathrm{H}(V_M, C_M) \ltimes V_M \longrightarrow F_G \ltimes (C_M \to V_M) \longrightarrow V(F_G) \longrightarrow 1$$

of VB-groupoids, where we view V_M as a representation of $\mathrm{H}(V_M, C_M)$ trivially.

Definition 4.9. Given a short exact sequence of groupoids

$$1 \longrightarrow K \longrightarrow G \longrightarrow H \longrightarrow 1$$

over M,[40] the subcomplex[41]

$$C^\bullet_{K\text{-rel}}G = \{f \in C^\bullet G \mid f(k_1 g_1 k_2^{-1}, \ldots, k_\bullet g_\bullet k_{\bullet+1}^{-1}) = f(g_1, \ldots, g_\bullet)\}$$

of CG is called the *K-relative complex*.

The relevance of this complex, that was introduced in general in [Sal24], becomes clear from the following characterisation.

[40] All three groupoids are groupoids over M and the maps in the exact sequence are over the identity maps.

[41] The condition should be satisfied for all $(g_1, \ldots, g_\bullet) \in G^{(\bullet)}$ and $k_1, \ldots, k_{\bullet+1} \in K$ with $\mathbf{s}g_{\ell-1} = \mathbf{s}k_\ell = \mathbf{t}g_\ell$.

Proposition 4.10. *Given a short exact sequence of groupoids*

$$1 \longrightarrow K \longrightarrow G \xrightarrow{\Phi} H \longrightarrow 1$$

over M, the induced pullback map

$$\Phi^* : CH \to CG$$

is an isomorphism onto $C_{K\text{-rel}}G$.

Proof. To see that Φ^* maps into $C_{K\text{-rel}}G$ is readily verified by direct computation. The inverse is given by choosing a set-theoretic section σ of $G \to H$ (over the identity map) and setting

$$C^\bullet_{K\text{-rel}}G \to C^\bullet H \qquad f \mapsto ((h_1, \ldots, h_\bullet) \mapsto f(\sigma h_1, \ldots, \sigma h_\bullet)).$$

The map $(h_1, \ldots, h_\bullet) \mapsto f(\sigma h_1, \ldots, \sigma h_\bullet)$ is indeed smooth, for locally (in $H^{(\bullet)}$) we can replace σ in the ℓ-th component with a smooth section σ_ℓ. □

Returning to the setup where F_G is a fat extension, we write

$$C_{\mathrm{inv}}(F_G; V_M^* \to C_M^*) = C_{(\mathrm{H}(V_M, C_M) \ltimes V_M)\text{-rel}}(F_G; V_M^* \to C_M^*).$$

(with respect to the short exact sequence of VB-groupoids above) and we call its elements *invariant*. The reason we passed to the dual cochain complex representation is because that corresponds to the VB-complex of the VB-groupoid $F_G \ltimes (C_M \to V_M)$. Similarly, we write

$$C_{\mathrm{inv}}(F_G; C_M \to V_M) = C_{(\mathrm{H}(V_M, C_M) \ltimes C_M^*)\text{-rel}}(F_G; C_M \to V_M).$$

As a corollary of Proposition 4.10, we obtain:

Corollary 4.11. *Let F_G be a fat extension over $C_M \to V_M$. Then the pullback induces an isomorphism of differential graded CG-modules:*

$$C_{\mathrm{VB}}V(F_G)^* \xrightarrow{\sim} C_{\mathrm{inv}}(F_G; C_M \to V_M).$$

To make the invariance condition more concrete, thereby also explaining the terminology, notice that the left action of $\mathrm{H}(V_M, C_M) \ltimes V_M$ on $F_G \ltimes (C_M \to V_M)$ is given by

$$(h_{\mathbf{t}g}, \partial c + H_g \cdot v) \cdot (c, H_g, v) = (h_{\mathbf{t}g}(H_g \cdot v) + c, h_{\mathbf{t}g} \cdot H_g, v)$$

and the right action by

$$(c, H_g, v) \cdot (h_{\mathbf{s}g}, v) = (c - H_g \cdot (h_{\mathbf{s}g} v), H_g \cdot h_{\mathbf{s}g}, v). \tag{4.1}$$

Working out the condition of being invariant using these explicit expressions, we obtain:

Proposition 4.12. *The invariant cochains in $C^\bullet(F_G; C_M \to V_M)$ are those pairs*

$$(f_0, f_1) \in C^\bullet(F_G; C_M) \oplus C^{\bullet-1}(F_G; V_M)$$

for which

$$f_1(H_{g_2}, \ldots, H_{g_\bullet}) = f_1(H'_{g_2}, \ldots, H'_{g_\bullet}) \tag{4.2}$$

and

$$f_0(H_{g_1}, \ldots, H_{g_\bullet}) - f_0(H'_{g_1}, \ldots, H'_{g_\bullet}) = h_{\mathbf{t}g}(H_{g,1} \cdot f_1(H_{g_2}, \ldots, H_{g_\bullet})), \tag{4.3}$$

where $h_{\mathbf{t}g}$ is the unique element in $\mathrm{H}(V_M, C_M)$ such that $h_{\mathbf{t}g} \cdot H_{g,1} = H'_{g,1}$ in F_G.

Proof. Explicitly, the identification

$$C(F_G; C_M \to V_M) \xrightarrow{\sim} C_{\mathrm{VB}}(F_G \ltimes (V_M^* \to C_M^*)) \qquad (f_0, f_1) \mapsto f$$

is given by

$$f((f_{\mathbf{t}g_1}^{V_M}, H_{g_1}, f_{\mathbf{s}g_1}^{C_M}), g_2, \ldots, g_\bullet) = f_{\mathbf{s}g_1}^{C_M}(H_{g_1}^{-1} \cdot f_0(H_{g_1}, \ldots, H_{g_n})) - f_{\mathbf{t}g_1}^{V_M}(H_{g_1} \cdot f_1(H_{g_2}, \ldots, H_{g_\bullet})).$$

Now, the elements $f \in C_{\mathrm{VB}}(F_G \ltimes (V_M^* \to C_M^*))$ that are inside the relative complex with respect to the sequence

$$1 \longrightarrow \mathrm{H}(V_M, C_M) \ltimes C_M^* \longrightarrow F_G \ltimes (V_M^* \to C_M^*) \longrightarrow V(F_G) \longrightarrow 1$$

are those for which

$$f((f_{\mathbf{t}g_1}^{V_M}, H_{g_1}, f_{\mathbf{s}g_1}^{C_M}), g_2, \ldots, g_\bullet) = f((f_{\mathbf{s}g_1}^{C_M}(H_{g_1}^{-1} \cdot h_{\mathbf{t}g_1}) + f_{\mathbf{t}g_1}^{V_M}, h_{\mathbf{t}g_1} \cdot H_{g_1}, f_{\mathbf{s}g_1}^{C_M}), g_2, \ldots, g_\bullet).$$

Therefore, f_1 is independent of its entries:

$$f_1(H_{g_2}, \ldots, H_{g_\bullet}) = f_1(H'_{g_2}, \ldots, H'_{g_\bullet}) = f(g_2, \ldots, g_\bullet)$$

and

$$\begin{aligned} f_0(H_{g_1}, \ldots, H_{g_\bullet}) &= f_0(H_{g_1}, g_2, \ldots, g_\bullet) \\ &= f_0(h_{\mathbf{t}g_1} \cdot H_{g_1}, g_2, \ldots, g_\bullet) - h_{\mathbf{t}g_1}(H_{g_1} \cdot f_1(g_2, \ldots, g_\bullet)). \end{aligned}$$

Taking $h_{\mathbf{t}g_1}$ such that $h_{\mathbf{t}g_1} \cdot H_{g_1} = H'_{g_1}$, we obtain the desired result. □

Notice that the dual result is that

$$C_{\mathrm{VB}} V(F_G) \xrightarrow{\sim} C_{\mathrm{inv}}(F_G; V_M^* \to C_M^*),$$

but this isomorphism can be obtained as the dual of the above isomorphism.

Remark 4.13. Given (f_0, f_1) satisfying the conditions from Proposition 4.12, we often write

$$f_0(H_{g_1}, g_2, \ldots, g_\bullet) = f_0(H_{g_1}, \ldots, H_{g_\bullet}) \qquad f_1(g_2, \ldots, g_\bullet) = f_1(H_{g_2}, \ldots, H_{g_\bullet})$$

as above. We can write the invariance condition also in the following way: for all $h_{\mathbf{t}g_1} \in \mathrm{H}(V_M, C_M)$,

$$f_0(h_{\mathbf{t}g_1} \cdot H_{g_1}, g_2, \ldots, g_\bullet) = f_0(H_{g_1}, g_2, \ldots, g_\bullet) + h_{\mathbf{t}g_1}(H_{g_1} \cdot f_1(g_2, \ldots, g_\bullet)).$$

Yet another way is that, for all $h_{\mathbf{s}g_1} \in \mathrm{H}(V_M, C_M)$,

$$f_0(H_{g_1} \cdot h_{\mathbf{s}g_1}, g_2, \ldots, g_\bullet) = f_0(H_{g_1}, g_2, \ldots, g_\bullet) + H_{g_1} \cdot (h_{\mathbf{s}g_1} f_1(g_2, \ldots, g_\bullet)).$$

If $C_M \neq 0$, then invariant cochains (f_0, f_1) are determined by the element $f_0 \in C(F_G; C_M)$. For this reason, we usually write $C^\bullet_{\mathrm{inv}}(F_G; C_M \to V_M)$ as the subcomplex

$$C^\bullet_{\mathrm{inv}}(F_G; C_M) = \{f_0 \in C^\bullet(F_G; C_M) \mid \exists f_1 \in C^{\bullet-1}(F_G; V_M) \text{ such that (4.2) and (4.3) hold}\}$$

of $C(F_G; C_M)$. This is done mostly for notational convenience, so usually $C^\bullet_{\mathrm{inv}}(F_G; C_M \to V_M)$ is meant when $C^\bullet_{\mathrm{inv}}(F_G; C_M)$ is written.

For clarity, we arrived at the following definition for abstract fat extensions:

Definition 4.14. Let F_G be a fat extension of G over $C_M \to V_M$. The *invariant complex*

$$C_{\mathrm{inv}}(F_G; C_M) \subset C(F_G; C_M \to V_M)$$

consists of pairs $(f_0, f_1) \in C(F_G; C_M \to V_M)$ such that, for all $h_{\mathbf{t}g} \in \mathrm{H}(V_M, C_M)$,[42]

$$f_0(h_{\mathbf{t}g} \cdot H_{g_1}, g_2, \ldots, g_\bullet) = f_0(H_{g_1}, g_2, \ldots, g_\bullet) + h_{\mathbf{t}g_1}(H_{g_1} \cdot f_1(g_2, \ldots, g_\bullet)).$$

Remark 4.15. If $F_G = \widehat{V}_G$ for a VB-groupoid V_G, and $H_g, H'_g \in \widehat{V}_G$,

$$h_{\mathbf{t}g}(H_g \cdot v) = h(H_g, H'_g) = r_{g^{-1}}(H'_g v - H_g v)$$

where $h_{\mathbf{t}g} \in \mathrm{H}(V_M, C_M)$ is the map

$$h_{\mathbf{t}g} = -1 + H'_g \cdot H_g^{-1}.$$

This can be verified by direct computation.

Remark 4.16. The invariant complex of a fat category extension is defined in exactly the same way as above.

For convenience of the reader, we write the VB-complex of a VB-groupoid V_G in terms of the fat groupoid $\widehat{V}_G$ (as the invariant complex) in the next section explicitly, and we prove this fact directly.

4.5. **From VB-groupoids to 2-term ruths using the fat groupoid.** From the previous section, Section 4.4, we see that there is a model for $C_{\mathrm{VB}} V_G$ in terms of the fat groupoid $\widehat{V}_G$. As mentioned there, it is a subcomplex

$$C_{\mathrm{inv}}(\widehat{V}_G; V_M^* \to C_M^*) \subset C(\widehat{V}_G; V_M^* \to C_M^*)$$

of invariant cochains, but one can often view it as a subcomplex

$$C_{\mathrm{inv}}(\widehat{V}_G; V_M^*) \subset C(\widehat{V}_G; V_M^*)$$

(if V_M is not the zero bundle) which we sometimes use for notational convenience. We write here an explicit isomorphism between $C_{\mathrm{VB}} V_G$ and $C_{\mathrm{inv}}(\widehat{V}_G; V_M^* \to C_M^*)$.

Consider an element f of $C^\bullet_{\mathrm{VB}} V_G$. Then we can construct elements

$$f_0 \in C^\bullet(\widehat{V}_G; V_M^*) \text{ and } f_1 \in C^{\bullet-1}(\widehat{V}_G; C_M^*)$$

by setting

$$\begin{aligned} f_0(H_{g_1}, \ldots, H_{g_\bullet})v &= f(H_{g_1}(\mathbf{t}H_{g_1})^{-1}v, g_2, \ldots, g_\bullet) \\ f_1(H_{g_2}, \ldots, H_{g_\bullet})c &= f(-c^{-1}, g_2, \ldots, g_\bullet). \end{aligned}$$

Notice the resemblance with the equations that appeared in the alternative proof of Proposition 2.19 in Section 2.

Proposition 4.17. *The map*

$$C_{\mathrm{VB}} V_G \to C(\widehat{V}_G, V_M^* \to C_M^*); \quad f \mapsto (f_0, f_1)$$

is a cochain map. Moreover, it maps bijectively onto $C_{\mathrm{inv}}(\widehat{V}_G, V_M^* \to C_M^*)$, i.e. the pairs (f_0, f_1) such that

$$f_1(H_{g_2}, \ldots, H_{g_\bullet}) = f_1(H'_{g_2}, \ldots, H'_{g_\bullet}) \tag{4.4}$$

[42]Implicit from the notation is that f_0 does not depend on the last $\bullet - 1$ entries and f_1 is independent of its entries.

and

$$f_0(H_{g_1},\dots,H_{g_\bullet})v - f_0(H'_{g_1},\dots,H'_{g_\bullet})v = f_1(H_{g_2},\dots,H_{g_\bullet})h(H_{g_1}^{-1},(H'_{g_1})^{-1})v \tag{4.5}$$

where

$$h(H_{g_1^{-1}},H'_{g_1^{-1}}) : (V_M)_{\mathbf{t}g_1} \to (C_M)_{\mathbf{t}g_2}$$

is given by

$$h(H_{g_1^{-1}},H'_{g_1^{-1}})v = r_{g_1}(H'_{g_1^{-1}}v - H_{g_1^{-1}}v).$$

Proof. The map is a cochain map, for it can be seen as the pullback of the VB-groupoid map

$$(C_M \to V_M) \rtimes \widehat{V}_G \to V_G \qquad (v,H_g,c) \mapsto \ell_g c + H_g(\mathbf{t}H_g)^{-1}v.$$

Notice that we view here $(C_M \to V_M) \rtimes \widehat{V}_G$ as a VB-groupoid with $\mathbf{s}(v,H_g,c) = \partial c + H_g^{-1}\cdot v$, $\mathbf{t}(v,H_g,c) = v$, and

$$(v,H_g,c_1)\cdot(\partial c_1 + H_g^{-1}\cdot v,H_h,c_2) = (v,H_g\cdot H_h,H_h^{-1}\cdot c_1 + c_2).$$

We can recover f from f_0 and f_1 as follows. Given $(v_1,\dots,v_\bullet)\in V_G^{(\bullet)}$ over $(g_1,\dots,g_\bullet)$, we take any $(H_{g_1},\cdots,H_{g_\bullet})$ and then we set

$$f(v_1,\dots,v_\bullet) = f_1(H_{g_2},\dots,H_{g_\bullet})\omega_{g_1}^{H_{g_1}}v_1 + f_0(H_{g_1},\dots,H_{g_\bullet})\mathbf{t}v_1. \tag{4.6}$$

The map

$$\omega_{g_1}^{H_{g_1}}v = r_g(H_{g_1}^{-1}\mathbf{t}v - v^{-1})$$

is as in Proposition 3.40: the unique map $(V_G)_g \to (C_M)_{\mathbf{s}g}$ such that $\omega_{g_1}^{H_{g_1}}H_{g_1} = 0$. The right hand side of (4.6) is independent of $H_{g_1},\dots,H_{g_\bullet}$. So, if we are given $f_0 \in C^\bullet(\widehat{V}_G;V_M^*)$ and $f_1 \in C^{\bullet-1}(\widehat{V}_G;C_M^*)$ such that the sum (as in the right hand side of the equation (4.6) above) is independent of $H_{g_1},\dots,H_{g_\bullet}$, then $f\in C_{\mathrm{VB}}V_G$ can be defined by the equation (4.6) above. This gives rise to a one-to-one correspondence, and this last condition is exactly the condition as written in the statement. □

As a corollary, we obtain:

Corollary 4.18. *The map*

$$C_{\mathrm{lin}}V_G \to C(\widehat{V}_G,V_M^*); \quad f\mapsto f_0$$

is a cochain map. If $V_M\neq 0$, then the restriction to $C^\bullet_{\mathrm{VB}}V_G$ maps bijectively onto $C_{\mathrm{inv}}(\widehat{V}_G;V_M^*)$, so those elements $f_0\in C^\bullet(\widehat{V}_G,V_M^*)$ for which there exists $f_1\in C^{\bullet-1}(\widehat{V}_G,C_M^*)$ such that (4.4) *and* (4.5) *hold.*[43]

Remark 4.19. The VB-complex $C_{\mathrm{VB}}V_G^*$, in terms of $\widehat{V}_G$, is given by $C_{\mathrm{inv}}(\widehat{V}_G;C_M\to V_M)$. The latter complex consist of those pairs (f_0,f_1) such that

$$f_1(H_{g_2},\dots,H_{g_\bullet}) = f_1(H'_{g_2},\dots,H'_{g_\bullet})$$

[43]If $V_M = 0$, then the map

$$C^\bullet_{\mathrm{lin}}V_G \to C^{\bullet-1}(\widehat{V}_G,C_M^*); \quad f\mapsto f_1$$

is a cochain map, and the restriction to $C^\bullet_{\mathrm{VB}}V_G$ maps bijectively onto the elements $f_1\in C^{\bullet-1}(\widehat{V}_G,C_M^*)$ that are independent of its arguments.

and

$$f_0(H_{g_1},\dots,H_{g_\bullet}) - f_0(H'_{g_1},\dots,H'_{g_\bullet}) = h(H_{g_1},H'_{g_1})f_1(H_{g_2},\dots,H_{g_\bullet})$$

where again

$$h(H_{g_1},H'_{g_1}) : (V_M)_{\mathbf{s}g_1} \to (C_M)_{\mathbf{t}g_1}$$

is given by

$$h(H_{g_1},H'_{g_1})v = r_{g_1^{-1}}(H'_{g_1}v - H_{g_1}v).$$

We complement the last remark by observing that the isomorphism

$$C^\bullet_{\mathrm{inv}}(\widehat{V}_G; C_M \to V_M) \to C_{\mathrm{VB}}V_G^*$$

is particularly simple to write when using the projectable cochains:

$$C_{\mathrm{VB}}V_G^* \cong C_{\mathrm{proj}}V_G$$

(see Section 1 for the definition).

Proposition 4.20. *The complex $C_{\mathrm{inv}}(\widehat{V}_G; C_M)$ is isomorphic to $C_{\mathrm{proj}}V_G$ via the map*

$$C_{\mathrm{inv}}(\widehat{V}_G; C_M) \to C_{\mathrm{proj}}V_G$$

given by

$$c(g_1,\dots,g_\bullet) = r_{g_1}f_0(H_{g_1},g_2,\dots,g_\bullet) + H_{g_1}f_1(g_2,\dots,g_\bullet).$$

Proof. The map is CG-linear and a bijection, for we can read the equation the other way around (using $f_1 = \mathbf{s}c$). That the map is a cochain map, is a straightforward computation on generators. □

Remark 4.21. In this remark we will draw some parallels with the discussion presented here and the proofs of Proposition 2.19 and Proposition 4.10. In the discussion of this section and the previous one (Section 4.4), we employed exactly the strategy presented in the proof of Proposition 4.10. Namely, in the proof of the latter proposition, we constructed an isomorphism $CH \xrightarrow{\sim} C_{K\text{-rel}}G$ that is given by the pullback of a groupoid map $G \to H$ (over the same base) which is a surjective submersion. In the above setup we described, we took one of the surjective VB-groupoid maps

$$\widehat{V}_G \ltimes (C_M \to V_M) \to V_G \qquad (C_M \to V_M) \rtimes \widehat{V}_G \to V_G$$

In general, the inverse of $CH \xrightarrow{\sim} C_{K\text{-rel}}G$ is given explicitly by using local sections of $G \to H$; the choice of local sections is irrelevant. In our case, the local sections are determined by local sections of $\widehat{V}_G \to G$, and this explains the appearance of them in the alternative proof of Proposition 2.19. The proof of Proposition 2.19 using the cleavage is almost like the usage of a set-theoretic section in the proof of Proposition 4.10 in that it is a global version of the argument. Of course, for the proof of Proposition 2.19 to work, we need a smooth section, and therefore we have to pass to $\widehat{V}_G^{\mathrm{cat}} \to G$. That is, we drop the invertibility assumption on the values of the section and we need to work "up to homotopy". See also Section 6.2, where we further clarify the step from fat extensions to the associated split 2-term ruth.

5. Fat extensions and general linear PB-groupoids

We dedicate this section to the correspondence between fat extension and general linear PB-groupoids. In [CG26], where the notion of PB-groupoid is introduced, it is already observed that the fat groupoid of a VB-groupoid is useful to construct the general linear PB-groupoid of a VB-groupoid. Following the previous section, Section 4, it becomes clear that this should be interpreted as a composition of functors

$$\{\text{VB-groupoids}\} \to \{\text{Fat extensions}\} \to \{\text{General linear PB-groupoids}\}.$$

The functorial aspects are not discussed in [CG26], but we will do so in Section 7.

Although the correspondence works as in [CG26], we will go a slightly different route. The difference lies in that the general linear PB-groupoids we consider have a different structural strict Lie 2-groupoid than in [CG26]. We use the one associated to two vector bundles C_M and V_M instead of the one associated to two *trivial* vector bundles. Moreover, the structure maps of the general linear strict Lie 2-groupoid are slightly different than in [CG26]. These changes are done to improve the exposition.

In Appendix A we will discuss a related correspondence between a class of double groupoids that we call *core-transitive*, and structures that we call *core extensions*. The discussion presented in the appendix generalises [BM92], but can be seen as a clarification of the remarks made at the end of that work. General linear PB-groupoids come with a gauge double groupoid when the underlying complex is regular, and such double groupoids are core-transitive. The associated core extension is exactly the fat extension. At the end of this section we mention how to think of fat extensions as double groupoids, but the details are left for the appendix.

5.1. **PB-groupoids.**

In analogy with VB-groupoids, a PB-groupoid is a diagram

$$\begin{tikzcd}
P_G \arrow[r, shift left] \arrow[r, shift right] \arrow[d] & P_M \arrow[d] \\
G \arrow[r, shift left] \arrow[r, shift right] & M
\end{tikzcd}$$

where P_G and P_M are principal groupoid bundles over G and M, respectively. The compatibility between the principal bundle structures and the groupoid structures involved can be phrased as follows. The groupoids acting on P_G and on P_M are both groupoids over the same base:

$$\begin{tikzcd}
 & H_G \arrow[rr, shift left] \arrow[rr, shift right] \arrow[dd, shift left] \arrow[dd, shift right] & & H_M \arrow[dd, shift left] \arrow[dd, shift right] \\
P_G \arrow[ur, phantom, "\circlearrowright"] \arrow[rr, shift left] \arrow[rr, shift right] \arrow[dr] \arrow[dd] & & P_M \arrow[ur, phantom, "\circlearrowright"] \arrow[dr] \arrow[dd] & \\
 & N \arrow[rr, shift left] \arrow[rr, shift right] & & N \\
G \arrow[rr, shift left] \arrow[rr, shift right] & & M &
\end{tikzcd}$$

and these groupoids fit into a double groupoid over the unit groupoid 1_N:

$$\begin{tikzcd} H_G \arrow[r, shift left] \arrow[r, shift right] \arrow[d, shift left] \arrow[d, shift right] & H_M \arrow[d, shift left] \arrow[d, shift right] \\ N \arrow[r, shift left] \arrow[r, shift right] & N \end{tikzcd}$$

That is, it is a strict Lie 2-groupoid: after appropriate identifications, the structure maps of $H_G \rightrightarrows H_M$ are groupoid maps over the identity (see also Remark 5.3 and Appendix A).

Now, the compatibility between the actions in a PB-groupoid can be phrased as the actions defining a (principal) 2-action of the structural strict Lie 2-groupoid.

Definition 5.1. Let $P_G \rightrightarrows P_M$ be a Lie groupoid and let $H_G \rightrightarrows H_M \rightrightarrows N$ be a strict Lie 2-groupoid. A 2-*action* of H_G on P_G consists of two groupoid actions

$$P_G \circlearrowleft H_G \qquad P_M \circlearrowleft H_M$$

whose moment maps form a groupoid map

$$P_G \to 1_N$$

and whose action maps form a groupoid map

$$\begin{tikzcd} P_G \times_N H_G \arrow[r] \arrow[d, shift left] \arrow[d, shift right] & P_G \arrow[d, shift left] \arrow[d, shift right] \\ P_M \times_N H_M \arrow[r] & P_M \end{tikzcd}$$

where the groupoid structure of $P_G \times_N H_G \rightrightarrows P_M \times_N H_M$ is defined componentwise. A 2-action $P_G \circlearrowleft H_G$ whose actions are principal is called a *PB-groupoid*, and we say it is over $G \rightrightarrows M$ if $P_G/H_G \rightrightarrows P_M/H_M$ is $G \rightrightarrows M$ (up to isomorphism).

Example 5.2. An interesting class of examples is when $H_G = H_M$ is a Lie group G. Such PB-groupoids appear in [BGG17] and are called G-*groupoids* there.

To end this small section, we conclude with some comments on strict Lie 2-groupoids.

Remark 5.3. Given a strict Lie 2-groupoid $H_2 \rightrightarrows H_1 \rightrightarrows N$, we can think of the elements $h \in H_2$ as abstract homotopy data between the source and target isomorphisms in H_1. In this analogy, H_2 is a groupoid in two compatible ways: there is one composition of homotopies which should be compared to the fact that being homotopic is a transitive relation, and the other composition of homotopies should be compared to Lemma 3.7. In the main example of interest to us, the general linear strict Lie 2-groupoid, the analogy can be taken literally (see also Remark 6.10).

A more geometric perspective is that $H_2 \rightrightarrows N$ is a Lie groupoid whose source and target fibers have a compatible internal groupoid structure. The foliation by source fibers form a *family of Lie groupoids*,[44] and so does the foliation by target fibers. If two elements are composable in $H_2 \rightrightarrows H_1$, then they are both elements of a single source fiber (of $H_2 \rightrightarrows N$), and also of a single target fiber. Their product in $H_2 \rightrightarrows H_1$, in turn, also lies in these fibers. Now, take two pairs of composable elements in $H_2 \rightrightarrows H_1$, say, one pair in a source fiber over $x \in N$ and the other pair in a target fiber over x. Either, we multiply first in $H_2 \rightrightarrows H_1$ and then in $H_2 \rightrightarrows N$, or we multiply first in $H_2 \rightrightarrows N$ and then in $H_2 \rightrightarrows H_1$. The results are the same, and this compatibility between the two groupoid structures $H_2 \rightrightarrows N$ and $H_2 \rightrightarrows H_1$ is called the *interchange law*.

[44] A family of Lie groupoids is a Lie groupoid together with a groupoid map that is a surjective submersion onto some unit groupoid.

5.2. **The general linear strict Lie 2-groupoid.** General linear PB-groupoids are PB-groupoids with structural strict Lie 2-groupoid the general linear strict Lie 2-groupoid. We introduce the general linear strict Lie 2-groupoid of two vector bundles C_M and V_M next, but we do so in a slightly different way than in [CG26]. We will comment on this in Remark 5.6. Notice that, below, the groupoid

$$\mathrm{GL}(C_M,V_M)_M = \mathrm{Hom}(C_M,V_M) \rtimes \mathrm{GL}(C_M,V_M)$$

from Remark 2.25 is used.

Example 5.4. Consider the bundle of Lie groups

$$\mathrm{Pert}(C_M,V_M) \rightrightarrows \mathrm{Hom}(C_M,V_M)$$

from Remark 3.10; its multiplication is given by

$$(h_x,\partial_x)\cdot(h'_x,\partial_x) := (h_x+h'_x+h_x\partial_x h'_x,\partial_x).$$

We then set

$$\mathrm{GL}(C_M,V_M)_G := \mathrm{Pert}(C_M,V_M) \ltimes \mathrm{GL}(C_M,V_M)_M$$

where $\mathrm{Pert}(C_M,V_M)$ acts (on the left) on $\mathrm{GL}(C_M,V_M)_M$ via

$$(\partial_y,h_y)\cdot(\partial_y,\Psi_{y,x}) = (\partial_y,(1+h_y\partial_y)\Psi^{C_M}_{y,x},(1+\partial_y h_y)\Psi^{V_M}_{y,x}).$$

Following Remark 5.3, the elements $(h_y,\partial_y,\Psi_{y,x})$ of $\mathrm{GL}(C_M,V_M)_G$ can be interpreted as pointwise homotopy data: we can interpret

$$\mathbf{s}(h_y,\partial_y,\Psi_{y,x}) = (\partial_y,\Psi^{C_M}_{y,x},\Psi^{V_M}_{y,x}) \qquad \mathbf{t}(h_y,\partial_y,\Psi_{y,x}) = (\partial_y,(1+h_y\partial_y)\Psi^{C_M}_{y,x},(1+\partial_y h_y)\Psi^{V_M}_{y,x})$$

as isomorphisms of complexes with respect to the differentials

$$(\Psi^{V_M}_{y,x})^{-1}\partial_y\Psi^{C_M}_{y,x} \text{ and } \partial_y$$

and they are homotopic via $h_y\Psi^{V_M}_{y,x}$. The two multiplications of the two groupoid structures that $\mathrm{GL}(C_M,V_M)_G$ carries can now both be seen as composition of homotopies (see Lemma 3.7). The groupoid structure of

$$\mathrm{GL}(C_M,V_M)_G \rightrightarrows \mathrm{GL}(C_M,V_M)_M$$

has the above source and target maps, and

$$(h_y,\partial_y,\Psi_{y,x})\cdot(h'_y,\partial_y,\Psi'_{y,x}) = (h_y+h'_y+h_y\partial_y h'_y,\partial_y,\Psi'_{y,x}).$$

The groupoid multiplication of

$$\mathrm{GL}(C_M,V_M)_G \rightrightarrows \mathrm{Hom}(C_M,V_M)$$

is again a type of composition of homotopies: we set $\mathbf{s}(h_y,\partial_y,\Psi_{y,x}) = (\Psi^{V_M}_{y,x})^{-1}\partial_y\Psi^{C_M}_{y,x}$ and $\mathbf{t}(h_y,\partial_y,\Psi_{y,x}) = \partial_y$, and

$$(h_z,\partial_z,\Psi_{z,y})\cdot(h_y,\partial_y,\Psi_{y,x}) := (h_z+\Psi^{C_M}_{z,y}h_y(\Psi^{V_M}_{z,y})^{-1}+h_z\partial_z\Psi^{C_M}_{z,y}h_y(\Psi^{V_M}_{z,y})^{-1},\partial_z,\Psi_{z,y}\Psi_{y,x}).$$

Precomposing the homotopy on the right hand side with $\Psi^{V_M}_{z,y}\Psi^{V_M}_{y,x}$ is indeed composition of homotopies of $h_z\Psi^{V_M}_{z,y}$ and $h_y\Psi^{V_M}_{y,x}$. Another way of interpreting this multiplication is that it can be seen as being induced by the product of $\mathrm{Pert}(C_M,V_M)$:

$$(h_z,\partial_z,\Psi_{z,y})\cdot(h_y,\partial_y,\Psi_{y,x}) = ((h_z,\partial_z)\cdot(\Psi^{C_M}_{z,y}h_y(\Psi^{V_M}_{z,y})^{-1},\partial_z),\Psi_{z,y}\Psi_{y,x}).$$

Taking $C_M = M\times\mathbb{R}^k$ and $V_M = M\times\mathbb{R}^\ell$, we write $\mathrm{GL}(k,\ell)_G \rightrightarrows \mathrm{GL}(k,\ell)_M \rightrightarrows \mathrm{Hom}(\mathbb{R}^k,\mathbb{R}^\ell)$ for the resulting strict Lie 2-groupoid.

Remark 5.5. In essence, the above object is a strict Lie 2-groupoid defined from a *crossed module of Lie groupoids* as in [Mac89, And05, LGW08]. Crossed modules and strict Lie 2-groupoids correspond to each other. In the above case, the structure of crossed module is encoded in the Lie groupoid map from $\mathrm{Pert}(C_M, V_M)$ to $\mathrm{GL}(C_M, V_M)_M$, and the canonical action of $\mathrm{GL}(C_M, V_M)_M$ on $\mathrm{Pert}(C_M, V_M)$ (which is a *bundle of Lie groups representation* of $\mathrm{GL}(C_M, V_M)_M$ on $\mathrm{Pert}(C_M, V_M)$). We refer to Section 5.9 and Appendix A for more details.

To lighten the notation, we often write $(h_y, \Psi_{y,x})$, or even just (h, Ψ), for elements

$$(h_y, \partial_y, \Psi_{y,x}) \in \mathrm{GL}(C_M, V_M)_G.$$

Similarly, we often write $\Psi = \Psi_{y,x} = (\partial_y, \Psi_{y,x}) \in \mathrm{GL}(C_M, V_M)_M$.

Remark 5.6. A useful way to depict elements $(h_y, \Psi_{y,x}) \in \mathrm{GL}(C_M, V_M)_G$ is using blockmatrices. Recall for this the observation we made in Remark 3.12. There, we observed that an element $(\partial_x, h_x) \in \mathrm{Pert}(V_M, C_M)$ can be represented by a matrix

$$\begin{pmatrix} 1 + h_x\partial_x & h_x \\ 0 & 1 \end{pmatrix} \in \mathrm{GL}(C_M \oplus V_M).$$

Notice that we keep track of the differential ∂_x separately, but we suppress it from the notation. Then, elements of $\mathrm{GL}(C_M, V_M)_G$ can be represented by pairs of matrices

$$\left(\begin{pmatrix} 1 + h_y\partial_y & h_y \\ 0 & 1 \end{pmatrix}, \begin{pmatrix} \Psi_{y,x}^{C_M} & 0 \\ 0 & \Psi_{y,x}^{V_M} \end{pmatrix} \right).$$

This way, the groupoid structures of $\mathrm{GL}(C_M, V_M)_G$ are turned into matrix multiplications; for example, the multiplication of $\mathrm{GL}(C_M, V_M)_G \rightrightarrows \mathrm{Hom}(C_M, V_M)$ is given by

$$\begin{aligned} &\left(\begin{pmatrix} 1 + h_z\partial_z & h_z \\ 0 & 1 \end{pmatrix}, \begin{pmatrix} \Psi_{z,y}^{C_M} & 0 \\ 0 & \Psi_{z,y}^{V_M} \end{pmatrix} \right) \cdot \left(\begin{pmatrix} 1 + h_y\partial_y & h_y \\ 0 & 1 \end{pmatrix}, \begin{pmatrix} \Psi_{y,x}^{C_M} & 0 \\ 0 & \Psi_{y,x}^{V_M} \end{pmatrix} \right) \\ &= \left(\begin{pmatrix} 1 + h_z\partial_z & h_z \\ 0 & 1 \end{pmatrix} \begin{pmatrix} \Psi_{z,y}^{C_M} & 0 \\ 0 & \Psi_{z,y}^{V_M} \end{pmatrix} \begin{pmatrix} 1 + h_y\partial_y & h_y \\ 0 & 1 \end{pmatrix} \begin{pmatrix} \Psi_{z,y}^{C_M} & 0 \\ 0 & \Psi_{z,y}^{V_M} \end{pmatrix}^{-1}, \begin{pmatrix} \Psi_{z,y}^{C_M} & 0 \\ 0 & \Psi_{z,y}^{V_M} \end{pmatrix} \begin{pmatrix} \Psi_{y,x}^{C_M} & 0 \\ 0 & \Psi_{y,x}^{V_M} \end{pmatrix} \right). \end{aligned}$$

In [CG26], the correct matrix to consider does not have the "$1 + h_x\partial_x$-factor" in the upper left corner (of the matrices representing the elements of $\mathrm{Pert}(C_M, V_M)$). This is a consequence of the fact that the groupoid structure of $\mathrm{GL}(C_M, V_M)_G \rightrightarrows \mathrm{Hom}(C_M, V_M)$ we use is slightly different. We find the convention presented above more suitable for our discussion, especially with an eye on the correspondence we will set with fat extensions.

Remark 5.7. The previous remark already touched on it, but recall the following from [CG26]: the strict Lie 2-groupoid $\mathrm{GL}(C_M, V_M)_G$ comes with a "2-representation" (in [CG26] called "anchored 2-representation") on the 2-term complex

$$\partial \times 1 : C_M \times_M \mathrm{Hom}(C_M, V_M) \to V_M \times_M \mathrm{Hom}(C_M, V_M).$$

One way to put this is that there is a map of strict Lie 2-groupoids

$$\mathrm{GL}(C_M, V_M)_G \to \mathrm{GL}(C_M \times_M \mathrm{Hom}(C_M, V_M), V_M \times_M \mathrm{Hom}(C_M, V_M))_G.$$

It is an embedding determined by the base map, which is the diagonal

$$\mathrm{Hom}(C_M, V_M) \hookrightarrow \mathrm{Hom}(C_M, V_M) \times_M \mathrm{Hom}(C_M, V_M).$$

Following Proposition 3.10 of [CG26], it can also be seen as a linear 2-action of $\mathrm{GL}(C_M, V_M)_G$ on the above 2-term complex seen as a VB-groupoid over $1_{\mathrm{Hom}(C_M,V_M)}$:

$$(C_M \times_M \mathrm{Hom}(C_M, V_M)) \ltimes (V_M \times_M \mathrm{Hom}(C_M, V_M)) \rightrightarrows V_M \times_M \mathrm{Hom}(C_M, V_M)$$

(see Example 3.26). That $\mathrm{GL}(C_M, V_M)_G$ acts "linearly" on this VB-groupoid becomes clear when using the matrix notation: the (left) action is given by applying the linear transformation

$$\begin{pmatrix} 1 + h_y\partial_y & h_y \\ 0 & 1 \end{pmatrix} \begin{pmatrix} \Psi^{C_M}_{y,x} & 0 \\ 0 & \Psi^{V_M}_{y,x} \end{pmatrix} = \begin{pmatrix} (1 + h_y\partial_y)\Psi^{C_M}_{y,x} & h_y\Psi^{V_M}_{y,x} \\ 0 & \Psi^{V_M}_{y,x} \end{pmatrix} \in \mathrm{GL}(C_M \oplus V_M)$$

(and conjugating the differential). Moreover, $\mathrm{GL}(C_M, V_M)_M$ acts on $V_M \times_M \mathrm{Hom}(C_M, V_M)$ via

$$\Psi_{y,x} \cdot v_x = \Psi^{V_M}_{y,x} v_x$$

(and conjugating the differential). Recall from Definition 5.1 that the key feature of these two actions defining a 2-action is that the action map is a groupoid map.[45] Here, the groupoid

$$\begin{array}{c} \mathrm{GL}(C_M, V_M)_G \times_{\mathrm{Hom}(C_M,V_M)} ((C_M \times_M \mathrm{Hom}(C_M, V_M)) \ltimes (V_M \times_M \mathrm{Hom}(C_M, V_M))) \\ \downarrow\downarrow \\ \mathrm{GL}(C_M, V_M)_M \times_{\mathrm{Hom}(C_M,V_M)} (V_M \times_M \mathrm{Hom}(C_M, V_M)) \end{array}$$

is used, whose structure is defined componentwise. Notice that this groupoid is a VB-groupoid over $\mathrm{GL}(C_M, V_M)_G \rightrightarrows \mathrm{GL}(C_M, V_M)_M$, and with respect to this structure the action is linear. These facts are readily verified using the matrix notation from above.

Using the differential ∂ as moment map, we see that there is also a 2-representation of $\mathrm{GL}(C_M, V_M)_G$ on the 2-term complex $C_M \to V_M$. The linear 2-action is defined in an analogous way as above, and it corresponds to the identity map $\mathrm{GL}(C_M, V_M)_G \xrightarrow{=} \mathrm{GL}(C_M, V_M)_G$.

5.3. **General linear PB-groupoids.** In this section, we define what we mean with a general linear PB-groupoid. We also discuss some important properties that we will use.

Definition 5.8. Let C_M and V_M be vector bundles. An abstract general linear PB-groupoid with underlying vector bundles C_M and V_M is a PB-groupoid $P_G \rightrightarrows P_M$ with structural strict Lie 2-groupoid $\mathrm{GL}(C_M, V_M)_G \rightrightarrows \mathrm{GL}(C_M, V_M)_M \rightrightarrows \mathrm{Hom}(C_M, V_M)$. It is called trivial if

$$P_M = \mathrm{GL}(C_M, V_M)$$

is the canonical principal $\mathrm{GL}(C_M, V_M)_M$-bundle. A trivial abstract general linear PB-groupoid is called a *general linear PB-groupoid.*

To be clear, the (right) principal $\mathrm{GL}(C_M, V_M)_M$-bundle structure on $\mathrm{GL}(C_M, V_M)$ is given by

$$\Psi_{z,y} \cdot \Psi_{y,x}(= \Psi_{z,y} \cdot ((\Psi^{V_M}_{z,y})^{-1}\partial_z\Psi^{C_M}_{z,y}, \Psi_{y,x})) := \Psi_{z,y}\Psi_{y,x}.$$

To explain the definition, we want to emphasise two features of general linear PB-groupoids (Proposition 5.10 and Remark 5.17).

Remark 5.9. We first observe that the moment map of P_M in an abstract general linear PB-groupoid $P_G \rightrightarrows P_M$ is always given by conjugation in the following sense. Let $p \in P_M$, and let

$$\Psi_{y,x} = (\partial_y, \Psi_{y,x}) \in \mathrm{GL}(C_M, V_M)_M$$

[45] Implicitly, we use that the moment maps of the two actions are compatible.

be composable with p. Then $1_y = (\partial_y, 1_y)$ is composable with p, and therefore also any other $\Psi'_{y,x} = (\partial_y, \Psi'_{y,x}) \in \mathrm{GL}(C_M, V_M)_M$ is. Since all $p' \in P_M$ with the same image as p under the projection map $P_M \to M$ are of the form

$$p' = p \cdot \Psi'_{y,x},$$

the moment map is given on the fibers of the projection $P_M \to M$ by conjugation of the differential ∂_y by an element of $\mathrm{GL}(C_M, V_M)$.

Now, to justify the terminology in the above definition, observe the following.

Proposition 5.10. *Suppose $P_G \rightrightarrows P_M$ is an abstract general linear PB-groupoid. If $P_M \to M$ has a global section, then P_G is isomorphic to a (trivial abstract) general linear PB-groupoid.*

Proof. If the projection map $P_M \to M$ has a global section σ, then we can send $p \in P_M$ to the unique $\Psi(p) = (\partial_y, \Psi(p)) \in \mathrm{GL}(C_M, V_M)_M$ such that

$$1_p \cdot (0_y, \Psi(p)^{-1}) = 1_{\sigma\pi p} \in P_G.$$

This turns the image of the section in P_M into the unit section of $\mathrm{GL}(C_M, V_M) \to M$. In turn, the moment map of $\mathrm{GL}(C_M, V_M)$ becomes the conjugation of a differential, and the action transforms into the canonical action of $\mathrm{GL}(C_M, V_M)_M$ on $\mathrm{GL}(C_M, V_M)$. This proves the statement. □

To be clear, from now on we always assume general linear PB-groupoids are trivial abstract general linear PB-groupoids. As expected, a fundamental example of a general linear PB-groupoid is given by a 2-term complex $C_M \to V_M$.

Example 5.11. Given a 2-term complex $C_M \to V_M$, observe that $\mathrm{H}(C_M, V_M)$ (see Section 3.2) acts on $\mathrm{GL}(C_M, V_M)$ via its canonical representation. The Lie groupoid

$$\mathrm{H}(V_M, C_M) \ltimes \mathrm{GL}(C_M, V_M) \rightrightarrows \mathrm{GL}(C_M, V_M)$$

then has a structure of general linear PB-groupoid given by

$$(h_z, \Psi_{z,y}) \cdot (h_y, \Psi_{y,x}) = (h_z \cdot (\Psi^{C_M}_{z,y} h_y (\Psi^{V_M}_{z,y})^{-1}), \Psi_{z,y}\Psi_{y,x}).$$

The moment map is given by conjugating the differential of $C_M \to V_M$. Observe that, as is implicit from the notation, $\Psi^{C_M}_{z,y} h_y (\Psi^{V_M}_{z,y})^{-1}$ is an element of $\mathrm{H}(V_M, C_M)$. Indeed,

$$1 + \partial \Psi^{C_M}_{z,y} h_y (\Psi^{V_M}_{z,y})^{-1} = \Psi^{V_M}_{z,y} (1 + \partial_y h_y)(\Psi^{V_M}_{z,y})^{-1}$$

where ∂_y is the (suppressed) differential in $(h_y, \Psi_{y,x}) = (h_y, \partial_y, \Psi_{y,x})$, which satisfies

$$\partial_y = (\Psi^{V_M}_{z,y})^{-1} \partial \Psi^{C_M}_{z,y}$$

by assumption.

Remark 5.12. In the previous example, $\mathrm{H}(V_M, C_M) \ltimes \mathrm{GL}(C_M, V_M)$ simply inherits the above structure from the differential

$$\partial : M \to \mathrm{Hom}(C_M, V_M)$$

through the embedding of strict Lie 2-groupoids (a 2-representation)

$$(\mathrm{H}(C_M, V_M) \ltimes \mathrm{GL}(C_M, V_M) \rightrightarrows \mathrm{GL}(C_M, V_M) \rightrightarrows M) \hookrightarrow \mathrm{GL}(C_M, V_M)_G.$$

As $\mathrm{H}(C_M, V_M)$, together with its representations, carries little information about the differential (see Proposition 3.11), the strict Lie 2-groupoid $\mathrm{H}(C_M, V_M) \ltimes \mathrm{GL}(C_M, V_M)$ itself does not carry information about the differential.

The next fact points towards the relation between general linear PB-groupoids, fat extensions, VB-groupoids and 2-term ruths. Namely, every general linear PB-groupoid over $C_M \to V_M$ contains the above example:

Proposition 5.13. *Let $P_G \rightrightarrows \mathrm{GL}(C_M, V_M)$ be a general linear PB-groupoid with underlying complex $C_M \to V_M$. Then*[46]

$$\mathrm{H}(V_M, C_M) \ltimes \mathrm{GL}(C_M, V_M) \hookrightarrow P_G \qquad (h_y, \Psi_{y,x}) \mapsto 1_{1_y} \cdot (h_y, \partial, \Psi_{y,x})$$

is an embedding of general linear PB-groupoids. Moreover, this map fits into a short exact sequence

$$1 \longrightarrow \mathrm{H}(V_M, C_M) \ltimes \mathrm{GL}(C_M, V_M) \longrightarrow P_G \longrightarrow G \longrightarrow 1$$

of Lie groupoids, where $P_G \to G$ is the projection map.

Proof. That the map is an embedding of general linear PB-groupoids is readily verified and follows from the compatibility assumptions between the Lie groupoid structure of P_G and its principal $\mathrm{GL}(C_M, V_M)_G$-structure. The last claim is a consequence of all elements in the kernel of $P_G \to G$ being related to $1_{1_y} \in P_G$ by a unique element $(h_y, \partial_y, \Psi_{y,x}) \in \mathrm{GL}(C_M, V_M)_G$. Since then $\partial_y = \partial$ is the differential $C_M \to V_M$, $h_y \in \mathrm{H}(V_M, C_M)$ and the result follows. □

Remark 5.14. Notice that the short exact sequence of Lie groupoids

$$1 \longrightarrow \mathrm{H}(V_M, C_M) \ltimes \mathrm{GL}(C_M, V_M) \longrightarrow P_G \longrightarrow G \longrightarrow 1.$$

is not over the identity, but over the sequence

$$\mathrm{GL}(C_M, V_M) \xrightarrow{=} \mathrm{GL}(C_M, V_M) \xrightarrow{\mathbf{t}} M.$$

A general linear PB-groupoid P_G comes with a cochain complex representation on

$$\partial \times 1 : C_M \times_M \mathrm{GL}(C_M, V_M) \to V_M \times_M \mathrm{GL}(C_M, V_M).$$

This cochain complex representation of P_G can be described as follows:

Definition 5.15. Let P_G be a general linear PB-groupoid. The cochain complex representation associated to $P_G \rightrightarrows \mathrm{GL}(C_M, V_M)$ on

$$\partial \times 1 : C_M \times_M \mathrm{GL}(C_M, V_M) \to V_M \times_M \mathrm{GL}(C_M, V_M)$$

is defined as follows. Given $p_g \in P_G$, $(c_{\mathbf{s}g}, \Psi_{\mathbf{s}g,x}) \in C_M \times_M \mathrm{GL}(C_M, V_M)$ and $(v_{\mathbf{s}g}, \Psi_{\mathbf{s}g,x}) \in V_M \times_M \mathrm{GL}(C_M, V_M)$, we set

$$p_g \cdot (c_{\mathbf{s}g}, \Psi_{\mathbf{s}g,x}) := (\mathbf{t}(p_g)\Psi^{-1}_{\mathbf{s}g,x} c_{\mathbf{s}g}, \mathbf{t}p_g) \qquad p_g \cdot (c_{\mathbf{s}g}, \Psi_{\mathbf{s}g,x}) := (\mathbf{t}(p_g)\Psi^{-1}_{\mathbf{s}g,x} c_{\mathbf{s}g}, \mathbf{t}p_g).$$

Remark 5.16. The cochain complex representation is "$\mathrm{GL}(C_M, V_M)_M$-equivariant". More precisely, $\mathrm{GL}(C_M, V_M)_M$ acts on the complex (on the right): if $(c_{\mathbf{s}g}, \Psi_{\mathbf{s}g,y}) \in C_M \times_M \mathrm{GL}(C_M, V_M)$, $(v_{\mathbf{s}g}, \Psi_{\mathbf{s}g,y}) \in V_M \times_M \mathrm{GL}(C_M, V_M)$ and $\Psi_{y,x} \in \mathrm{GL}(C_M, V_M)_M$, then

$$(c_{\mathbf{s}g}, \Psi_{\mathbf{s}g,y}) \cdot \Psi_{y,x} = (c_{\mathbf{s}g}, \Psi_{\mathbf{s}g,y}\Psi_{y,x}) \qquad (v_{\mathbf{s}g}, \Psi_{\mathbf{s}g,y}) \cdot \Psi_{y,x} = (v_{\mathbf{s}g}, \Psi_{\mathbf{s}g,y}\Psi_{y,x}).$$

Given $p_g \in P_G$, $(c_{\mathbf{s}g}, \Psi_{\mathbf{s}g,y}) \in C_M \times_M \mathrm{GL}(C_M, V_M)$ and $\Psi_{y,x} \in \mathrm{GL}(C_M, V_M)_M$, we therefore have

$$(p_g \cdot \Psi_{y,x}) \cdot ((c_{\mathbf{s}g}, \Psi_{\mathbf{s}g,y}) \cdot \Psi_{y,x}) = (p_g \cdot (c_{\mathbf{s}g}, \Psi_{\mathbf{s}g,y})) \cdot \Psi_{y,x},$$

[46]We used 1 for both unit embeddings $\mathrm{GL}(C_M, V_M) \hookrightarrow P_G$ and $M \hookrightarrow \mathrm{GL}(C_M, V_M)$.

and similarly for $(v_{\mathbf{s}g}, \Psi_{\mathbf{s}g,y}) \in V_M \times_M \mathrm{GL}(C_M, V_M)$. Because of this fact, the cochain complex representation is determined by the induced cochain complex representation of the subgroupoid

$$F_G \coloneqq \{p_g \in P_G \mid \mathbf{s}p_g = 1_{\mathbf{s}g}\} \subset P_G.$$

The suggestive notation is of course to emphasise that this subgroupoid, together with the above cochain complex representation, reveals the fat extension associated to a general linear PB-groupoid.

Before we move on to give the correspondences, we make another remark which relates our setup of general linear PB-groupoids back to [CG26].

Remark 5.17. The general linear groupoid $\mathrm{GL}(C_M, V_M)$, seen as a groupoid over M, contains the gauge groupoid of the principal $\mathrm{GL}(k,\ell)_M$-bundle

$$\mathrm{Fr}(C_M, V_M) = \mathrm{Fr}\, C_M \times_M \mathrm{Fr}\, V_M = \{\phi_x^{C_M} \times \phi_x^{V_M} : \mathbb{R}^k \oplus \mathbb{R}^\ell \xrightarrow{\sim} (C_M)_x \oplus (V_M)_x \mid x \in M\},$$

if it exists, that is. The frame bundle $\mathrm{Fr}(C_M, V_M)$ has moment map given by conjugating the differential ∂ using the frames, and the action is the canonical one. This is completely analogous to how we defined the action of $\mathrm{GL}(C_M, V_M)_M$ on $\mathrm{GL}(C_M, V_M)$. In fact, the gauge groupoids of $\mathrm{GL}(C_M, V_M)$ and $\mathrm{Fr}(C_M, V_M)$ are equal and given by the cochain isomorphisms $\mathrm{Aut}(C_M \to V_M)$ from Remark 2.24

$$\begin{tikzcd} (C_M)_x \arrow[r, "\partial"] \arrow[d, "\Phi^{C_M}_{y,x}"] & (V_M)_x \arrow[d, "\Phi^{V_M}_{y,x}"] \\ (C_M)_x \arrow[r, "\partial"] & (V_M)_x \end{tikzcd}$$

(which is smooth only when ∂ has constant rank). Now, as follows from [CG26], a 2-term complex of vector bundles $C_M \to V_M$ is actually equivalently described by a principal $\mathrm{GL}(k,\ell)_M$-bundle (using the associated vector bundle construction in the other direction). Instead of making this point again, we simply encode the 2-term complex as a map on the gauge groupoid $\mathrm{GL}(C_M, V_M)$. Or rather, we reinterpret this data again as a type of general linear principal groupoid bundle:

$$\mathrm{GL}(C_M, V_M) \circlearrowright \mathrm{GL}(C_M, V_M)_M.$$

Then, the structure of general linear PB-groupoid over it will, as for PB-groupoids with structural strict Lie 2-groupoid $\mathrm{GL}(k,\ell)_G \rightrightarrows \mathrm{GL}(k,\ell)_M \rightrightarrows \mathrm{Hom}(C_M, V_M)$, exactly describe the data of a fat extension. Moreover, whether we use $\mathrm{GL}(k,\ell)_G$ or $\mathrm{GL}(C_M, V_M)_G$, the constructions that appear in the equivalence will be defined in analogous ways.

If the gauge groupoids of $\mathrm{GL}(C_M, V_M)$ (or $\mathrm{Fr}(C_M, V_M)$) is smooth, we can also take a smooth gauge groupoid of the total space P_G of a general linear PB-groupoid. The result is a double groupoid, and this gives rise to another way to encode the structure of a fat extension that will be explained in Section 5.9.

We will first describe the correspondence between fat extensions and general linear PB-groupoids.

5.4. **From fat extensions to general linear PB-groupoids.** Given a fat extension F_G, the associated general linear PB-groupoid is given as follows. We define

$$P_G \coloneqq F_G \ltimes \mathrm{GL}(C_M, V_M)$$

where we used the canonical (left) action of F_G on $\mathrm{GL}(C_M, V_M)$ by using the actions of F_G on C_M and V_M. We proceed to define the right $\mathrm{GL}(C_M, V_M)_G$-action on P_G as another canonical action:

$$(H_g, \Psi_{\mathbf{s}g,x}) \cdot (h_x, (\Psi_{\mathbf{s}g,x})^{-1}\partial\Psi_{\mathbf{s}g,x}, \Psi'_{x,y}) := (H_g \cdot (\Psi^{C_M}_{\mathbf{s}g,x} h_x (\Psi^{V_M}_{\mathbf{s}g,x})^{-1}), \Psi_{\mathbf{s}g,x}\Psi'_{x,y}).$$

Using the explicit descriptions of all the structure involved, we obtain:

Proposition 5.18. *Let F_G be a fat extension. Then*

$$F_G \ltimes \mathrm{GL}(C_M, V_M) \rightrightarrows \mathrm{GL}(C_M, V_M)$$

is a general linear PB-groupoid.

Remark 5.19. Notice the similarities with the PB-groupoid associated to the fat groupoid as explained in [CG26]. Instead of defining the PB-groupoid

$$F_G \ltimes \mathrm{GL}(C_M, V_M) \rightrightarrows \mathrm{GL}(C_M, V_M)$$

we can set[47]

$$F_G \ltimes \mathrm{Fr}(C_M, V_M) \rightrightarrows \mathrm{Fr}(C_M, V_M).$$

In exactly the same way as above, this defines a PB-groupoid with structural strict Lie 2-groupoid $\mathrm{GL}(k,\ell)_G \rightrightarrows \mathrm{GL}(k,\ell)_M \rightrightarrows \mathrm{Hom}(\mathbb{R}^k, \mathbb{R}^\ell)$. Roughly speaking, the shift in perspective is as the difference between GL E and Fr E for a vector bundle E. Instead of thinking of usual frames of E, namely frames modelled on Euclidean space, we think of frames modelled on its own fibers. We therefore need to consider a $\mathrm{GL}(E)$-symmetry instead of a $\mathrm{GL}(k)$-symmetry.

5.5. **From general linear PB-groupoids to fat extensions.** The construction we did in the previous section, Section 5.4, gives a clear hint on how to recover the fat extension from its general linear PB-groupoid $P_G \rightrightarrows \mathrm{GL}(C_M, V_M)$. Namely, the subgroupoid we saw in Remark 5.16

$$\{p_g \in P_G \mid \mathbf{s}p_g = 1_{\mathbf{s}g}\} \subset P_G \tag{5.1}$$

turns out to form a unique representative set for the Lie groupoid

$$F_G := P_G/\mathrm{GL}(C_M, V_M)_M \rightrightarrows M.$$

We used here that $\mathrm{GL}(C_M, V_M)_M$ acts freely and properly, via the unit embedding, on P_G. As observed in Remark 5.16, the description of F_G using the representatives above makes it clear that the cochain complex representation descends to F_G: given $H_g = [p_g] \in F_G$, where $\mathbf{s}p_g = 1_{\mathbf{s}g} \in \mathrm{GL}(C_M, V_M)$, then the cochain complex representation is given by

$$F_G \to \mathrm{GL}(C_M, V_M)_M \qquad H_g \mapsto \mathbf{t}p_g.$$

The surjective submersion $P_G \to G$ is a groupoid map, and the induced groupoid map $F_G \to G$ is still a surjective submersion. We therefore obtain a short exact sequence of groupoids

$$1 \longrightarrow \mathrm{H} \longrightarrow F_G \longrightarrow G \longrightarrow 1.$$

The cochain complex representation of F_G induces a cochain complex representation of H, and then we conclude with the following corollary of Proposition 5.13:

[47] Recall that $\mathrm{Fr}(C_M, V_M) = \mathrm{Fr}\, C_M \times_M \mathrm{Fr}\, V_M$.

Corollary 5.20. *The embedding* $\mathrm{H}(V_M, C_M) \hookrightarrow P_G$ *induced by Proposition 5.13 is a Lie groupoid isomorphism onto* H. *Moreover, this map is equivariant with respect to the cochain complex representations.*[48]

Proof. Its inverse is given as follows. Given $h_x = [p_x] \in \mathrm{H}$ (with $\mathbf{s}p_x = 1_x$), denote by Φh_x the unique element in $\mathrm{H}(V_M, C_M)$ such that

$$h_x \cdot (\Phi h_x, \partial, 1_x)^{-1} = 1_{1_x} \in P_G.$$

Then, the map

$$\Phi : \mathrm{H} \to \mathrm{H}(V_M, C_M)$$

is the desired inverse. □

It is readily verified that we set a true one-to-one correspondence between fat extensions and general linear PB-groupoids:

Proposition 5.21. *The assignment*

$$\{\text{Fat extensions}\} \to \{\text{General linear PB-groupoids}\} \qquad F_G \mapsto F_G \ltimes \mathrm{GL}(C_M, V_M)$$

sets a one-to-one correspondence, and the assignment

$$\{\text{General linear PB-groupoids}\} \to \{\text{Fat extensions}\} \qquad P_G \mapsto P_G/\mathrm{GL}(C_M, V_M)_M.$$

is inverse to it (up to isomorphisms).

We find it worthwhile to take a moment to also describe the correspondences with VB-groupoids and 2-term ruths. This is what we discuss next, starting with VB-groupoids.

5.6. **From VB-groupoids to general linear PB-groupoids.** As we use a different strict structural Lie 2-groupoid $\mathrm{GL}(C_M, V_M)_G \rightrightarrows \mathrm{GL}(C_M, V_M)_M \rightrightarrows \mathrm{Hom}(C_M, V_M)$ for general linear PB-groupoids, the correspondence between VB-groupoids and general linear PB-groupoids looks slightly different than in [CG26]. But the shift in perspective is analogous to the one from the previous correspondence between fat extensions and general linear PB-groupoids. Instead of looking at (adapted) frames of a VB-groupoid V_G modelled on $\mathbb{R}^k \oplus \mathbb{R}^\ell$, we will consider "frames" modelled on fibers of $C_M \oplus V_M$:

$$\phi_g : (C_M)_x \oplus (V_M)_x \xrightarrow{\sim} (V_G)_g.$$

In fact, from the previous section, Section 5.5, we can infer what maps we should really be considering: a composition of an isomorphism defined by an element $H_g \in \widehat{V}_G$:

$$(C_M)_{\mathbf{t}g} \oplus (V_M)_{\mathbf{s}g} \xrightarrow{\sim} (V_G)_g \qquad (c, v) \mapsto r_g c + H_g v$$

with two linear isomorphisms

$$(C_M)_x \xrightarrow{\sim} (C_M)_{\mathbf{t}g} \qquad (V_M)_x \xrightarrow{\sim} (V_M)_{\mathbf{s}g}.$$

The collection of such elements form the general linear PB-groupoid

$$\mathrm{GL}_{\mathbf{s}} V_G \rightrightarrows \mathrm{GL}(C_M, V_M)$$

[48]We consider here the cochain complex representation on $\mathrm{H} \subset F_G$ that we constructed above, and the canonical cochain complex representation of $\mathrm{H}(V_M, C_M)$.

of interest. This is analogous to how one defines the (adapted) frame bundle of a VB-groupoid in [CG26]. More precisely, we define this general linear PB-groupoid, without using the fat groupoid, as follows.[49] We set

$$\mathrm{GL}_{\mathbf{s}}V_G = \{\phi_g : (C_M)_x \oplus (V_M)_x \xrightarrow{\sim} (V_G)_g \mid \mathbf{s}_g\phi_g|_{(C_M)_x} = 0,\ \mathbf{t}_g\phi_g|_{(V_M)_x},\ \mathbf{s}_g\phi_g|_{(V_M)_x} \in \mathrm{GL}(V_M)\}$$

which will be the total groupoid P_G of the general linear PB-groupoid associated to V_G. To define the structure maps of this groupoid, we use the following diagram:

$$\begin{tikzcd}
(C_M)_x \arrow[d, hook] & (C_M)_{\mathbf{s}g} \arrow[dr, "\ell_g"] & & (C_M)_{\mathbf{t}g} \arrow[dl, "r_g"'] \arrow[dd, "\partial"] \\
(C_M)_x \oplus (V_M)_x \arrow[rr, "\phi_g"] & & (V_G)_g \arrow[dl, "\mathbf{s}"] \arrow[dr, "\mathbf{t}"] & \\
(V_M)_x \arrow[u, hook] & (V_M)_{\mathbf{s}g} & & (V_M)_{\mathbf{t}g}
\end{tikzcd}$$

Now, we construct four maps that will describe the source and target map of $\mathrm{GL}_{\mathbf{s}}V_G \rightrightarrows \mathrm{GL}(C_M, V_M)$. Namely, given $\phi_g \in \mathrm{GL}_{\mathbf{s}}V_G$, we write

$$\phi^{C_M}_{\mathbf{t}g} := r_{g^{-1}}\phi_g|_{(C_M)_x} \qquad \phi^{V_M}_{\mathbf{t}g} := \mathbf{t}_g\phi_g|_{(V_M)_x} \qquad \phi^{V_M}_{\mathbf{s}g} := \mathbf{s}_g\phi_g|_{(V_M)_x}$$

and[50]

$$\phi^{C_M}_{\mathbf{s}g} := \ell_g^{-1}\phi_g(1, -(\phi^{V_M}_{\mathbf{t}g})^{-1}\partial\phi^{C_M}_{\mathbf{t}g}).$$

We then set $\mathbf{s}\phi_g = (\phi^{C_M}_{\mathbf{s}g}, \phi^{V_M}_{\mathbf{s}g})$, $\mathbf{t}\phi_g = (\phi^{C_M}_{\mathbf{t}g}, \phi^{V_M}_{\mathbf{t}g})$, and

$$\phi_g \cdot \phi_h = r_h\phi_g|_{(C_M)_x} + \phi_g|_{(V_M)_x} \cdot \phi_h|_{(V_M)_x} \qquad \phi_g^{-1} = -(\phi_g(1, -1 - (\phi^{V_M}_{\mathbf{t}g})^{-1}\partial\phi^{C_M}_{\mathbf{t}g}))^{-1}.$$

Realising elements $(h_y, \Psi_{y,x})$ of $\mathrm{GL}(C_M, V_M)_G$ as block matrices

$$\begin{pmatrix} (1 + h_y\partial_y)\Psi^{C_M}_{y,x} & h_y\Psi^{V_M}_{y,x} \\ 0 & \Psi^{V_M}_{y,x} \end{pmatrix} \in \mathrm{GL}(C_M \oplus V_M)$$

the (right) action of $\mathrm{GL}(C_M, V_M)_G \rightrightarrows \mathrm{Hom}(C_M, V_M)$ on $\mathrm{GL}_{\mathbf{s}}V_G$ is given by matrix application:

$$\begin{pmatrix} \phi|_{(C_M)_y} & \phi|_{(V_M)_y} \end{pmatrix} \begin{pmatrix} (1 + h_y\partial_y)\Psi^{C_M}_{y,x} & h_y\Psi^{V_M}_{y,x} \\ 0 & \Psi^{V_M}_{y,x} \end{pmatrix} = \begin{pmatrix} \phi(1 + h_y\partial_y)\Psi^{C_M}_{y,x} & \phi h\Psi^{V_M}_{y,x} + \phi\Psi^{V_M}_{y,x} \end{pmatrix}.$$

This matrix showed up in Remark 5.7 where we discussed the canonical 2-representations of $\mathrm{GL}(C_M, V_M)_G$. For similar reasons, the action defined above defines a 2-action, and this is the desired principal 2-action. We therefore obtain:

Proposition 5.22. *Let V_G be a VB-groupoid. Then*

$$\mathrm{GL}_{\mathbf{s}}V_G \rightrightarrows \mathrm{GL}(C_M, V_M)$$

is a general linear PB-groupoid.

---

[49] See for this the older versions of the paper [CG26] on the arXiv.

[50] Notice that

$$\ell_g^{-1}v = -r_g(v^{-1})$$

where $v \in \ker \mathbf{t}$.

Remark 5.23. Using the fat groupoid to describe this general linear PB-groupoid, i.e. seeing it as an action groupoid

$$\widehat{V}_G \ltimes \mathrm{GL}(C_M, V_M) \rightrightarrows \mathrm{GL}(C_M, V_M)$$

describes the composition of functors

$$\{\text{VB-groupoids}\} \longrightarrow \{\text{Fat extensions}\} \longrightarrow \{\text{General linear PB-groupoids}\}.$$

As we already mentioned, this realises elements $\phi \in \mathrm{GL}_{\mathbf{s}} V_G$ as compositions of maps

$$r_g \phi_{\mathbf{t}g}^{C_M} + H_g \phi_{\mathbf{s}g}^{V_M} : (C_M)_x \oplus (V_M)_x \xrightarrow{\sim} (C_M)_{\mathbf{t}g} \oplus (V_M)_{\mathbf{s}g} \xrightarrow{\sim} (V_G)_g.$$

Working with $\mathrm{GL}(k,\ell)_G$ instead of $\mathrm{GL}(C_M, V_M)_G$ (so with $\mathrm{Fr}(C_M, V_M)$ instead of $\mathrm{GL}(C_M, V_M)$), this action groupoid was used in [CG26].

5.7. **From general linear PB-groupoids to VB-groupoids.** Following [CG26], it should not come as a surprise now that we can recover a VB-groupoid from its associated general linear PB-groupoid using a type of associated bundle construction. However, since we passed from $\mathrm{Fr}(C_M, V_M)$ to $\mathrm{GL}(C_M, V_M)$, the construction will especially be reminiscent to the association of a VB-groupoid from a fat extension.

Now, given a general linear PB-groupoid P_G, recall from Definition 5.15 that we can view

$$\partial \times 1 : C_M \times_M \mathrm{GL}(C_M, V_M) \to V_M \times_M \mathrm{GL}(C_M, V_M)$$

as a cochain complex representation of P_G. Therefore, it comes with an associated VB-groupoid that we simply denote by

$$P_G \ltimes (C_M \to V_M) = C_M \times_M P_G \times_M V_M \rightrightarrows V_M \times_M \mathrm{GL}(C_M, V_M).$$

Explicitly, the structure maps are as follows: the source and target maps are given, respectively, by $\mathbf{s}(c_{\mathbf{t}g}, p_g, v_{\mathbf{s}g}) = (v_{\mathbf{s}g}, \mathbf{s}p_g)$, $\mathbf{t}(c_{\mathbf{t}g}, p_g, v_{\mathbf{s}g}) = (\partial c_{\mathbf{t}g} + p_g \cdot v_{\mathbf{s}g}, \mathbf{t}p_g)$ and

$$(c_{\mathbf{t}g}, p_g, \partial c_{\mathbf{s}g} + p_{g'} \cdot v_{\mathbf{s}g'}) \cdot (c_{\mathbf{s}g}, p_{g'}, v_{\mathbf{s}g'}) = (c_{\mathbf{t}g} + p_g \cdot c_{\mathbf{s}g}, p_g \cdot p_{g'}, v_{\mathbf{s}g'}).$$

The VB-groupoid inherits a natural free and proper (right 2-) action of $\mathrm{GL}(C_M, V_M)_G$:

$$(c_{\mathbf{t}g}, p_g, v_{\mathbf{s}g}) \cdot (h_y, \Psi_{y,x}) = (c_{\mathbf{t}g} - p_g \cdot (\mathbf{s}(p_g) h_y \mathbf{s}(p_g)^{-1} v_{\mathbf{s}g}), p_g \cdot (h_y, \Psi_{y,x}), v_{\mathbf{s}g}).$$

Since the action indeed preserves the linear structure, the result $V(P_G)$ is a VB-groupoid over V_M.

Proposition 5.24. *The assignment*

$$\{\text{VB-groupoids}\} \longrightarrow \{\text{General linear PB-groupoids}\} \qquad V_G \mapsto \mathrm{GL}_{\mathbf{s}} V_G$$

sets a one-to-one correspondence, and the assignment

$$\{\text{General linear PB-groupoids}\} \longrightarrow \{\text{VB-groupoids}\} \qquad P_G \mapsto V(P_G)$$

is inverse to it (up to canonical isomorphisms).

Remark 5.25. The above quotient can be taken in steps. Indeed, we can take a quotient of $P_G \ltimes (C_M \to V_M)$ by $\mathrm{GL}(C_M, V_M)_M$, and then by $\mathrm{H}(C_M, V_M) \ltimes V_M$ (see Section 4.2 and (4.1) for the explicit formula of this action). This way, we can interpret the first quotient as recovering $F_G \ltimes (C_M \to V_M)$, and then the second quotient is as in Section 4.2. Doing this, we described the composition of functors

$$\{\text{General linear PB-groupoids}\} \longrightarrow \{\text{Fat extensions}\} \longrightarrow \{\text{VB-groupoids}\}.$$

5.8. **The abstract ruth of a general linear PB-groupoid.** We conclude by setting the correspondence between general linear PB-groupoids and 2-term ruths. Consider the abstract ruth associated to $P_G \ltimes (V_M^* \to C_M^*)$, so

$$C^\bullet(P_G; C_M \to V_M) = C^\bullet(P_G; C_M) \oplus C^{\bullet-1}(P_G; V_M).$$

Note the abuse of notation; it would be more correct to write the complex as

$$C^\bullet(P_G; C_M \times_M \mathrm{GL}(C_M, V_M) \to V_M \times_M \mathrm{GL}(C_M, V_M)).$$

But if (f_0, f_1) is a cochain in this complex, then the $\mathrm{GL}(C_M, V_M)$ components of $f_0(p_1, \dots, p_\bullet)$ and $f_0(p_2, \dots, p_\bullet)$ are equal to $\mathbf{t}p_1$ and $\mathbf{t}p_2$, respectively, so we simplify the notation.

If we argue through the language of fat extensions or VB-groupoids, for example, using Proposition 4.10, we see that the following subcomplex should be considered:

Definition 5.26. The GL-*equivariant cochains*

$$(f_0, f_1) \in C(P_G; C_M \to V_M) = C^\bullet(P_G; C_M) \oplus C^{\bullet-1}(P_G; V_M)$$

are those satisfying, for all $(p_{g_1}, \dots, p_{g_\bullet}) \in P^{(\bullet)}$,

$$f_1(p_{g_2}, \dots, p_{g_\bullet}) = f_1(p'_{g_2}, \dots, p'_{g_\bullet})$$

and, for all $(h_y, \Psi_{y,x}) \in \mathrm{GL}(C_M, V_M)_G$,

$$f_0(p_{g_1} \cdot (h_y, \Psi_{y,x}), \dots, p_{g_\bullet} \cdot (h_y, \Psi_{y,x})) = f_0(p_{g_1}, \dots, p_{g_\bullet}) + p_{g_1} \cdot (\mathbf{s}p_{g_1} h_y (\mathbf{s}p_{g_1})^{-1} f_1(p_{g_2}, \dots, p_{g_\bullet})).$$

The resulting subcomplex is denoted by $C_{\mathrm{GL}}(P_G; C_M \to V_M)$ (or $C_{\mathrm{GL}}(P_G; C_M)$).

Remark 5.27. Given $(f_0, f_1) \in C_{\mathrm{GL}}(P_G; C_M)$, we write

$$f_0(p_{g_1}, g_2, \dots, g_\bullet) = f_0(p_{g_1}, p_{g_2}, \dots, p_{g_\bullet}) \qquad f_1(\mathbf{t}(p_{g_2}), g_2, \dots, g_\bullet) = f_1(p_{g_2}, \dots, p_{g_\bullet}).$$

The above condition then reads

$$f_0(p_{g_1} \cdot (h_y, \Psi_{y,x}), g_2, \dots, g_\bullet) = f_0(p_{g_1}, g_2, \dots, g_\bullet) + p_{g_1} \cdot (\mathbf{s}p_{g_1} h_y (\mathbf{s}p_{g_1})^{-1} f_1(\mathbf{s}(p_{g_1}), g_2, \dots, g_\bullet)).$$

From our previous discussions, we can infer that the above abstract ruth is isomorphic to $C_{\mathrm{VB}} V(P_G)^*$ and to $C_{\mathrm{inv}}(F_G; C_M)$. For example: the PB-groupoid associated to a fat extension F_G is given by

$$P_G = F_G \ltimes \mathrm{GL}(C_M, V_M).$$

The representation on P_G is given by the pullback along the canonical groupoid map

$$P_G \to F_G.$$

This pullback then sets the desired isomorphism.

Proposition 5.28. *Let F_G be a fat extension of G over $C_M \to V_M$. The pullback*

$$C^\bullet(F_G; C_M \to V_M) \to C^\bullet(F_G \ltimes \mathrm{GL}(C_M, V_M); C_M \to V_M)$$

restricts to an isomorphism

$$C^\bullet_{\mathrm{inv}}(F_G; C_M) \xrightarrow{\sim} C^\bullet_{\mathrm{GL}}(F_G \ltimes \mathrm{GL}(C_M, V_M); C_M).$$

Remark 5.29. To explain the terminology "GL-equivariance", notice that we can conjugate elements $(f_0, f_1) \in C_{\mathrm{GL}}(P_G; C_M)$ to

$$\widetilde{f_0}(p_{g_1}, g_2, \ldots, g_\bullet) := \mathbf{t}(p_{g_1})^{-1} f_0(p_{g_1}, g_2, \ldots, g_\bullet)$$
$$\widetilde{f_1}(\Psi_{\mathbf{t}g_2,x}, g_2, \ldots, g_\bullet) := \Psi^{-1}_{\mathbf{t}g_2,x} f_1(\Psi_{\mathbf{t}g_2,x}, g_2, \ldots, g_\bullet).$$

We can think of $(\widetilde{f_0}, \widetilde{f_1})$ as the result of postcomposing (f_0, f_1) with the evaluation/action map

$$\begin{tikzcd} C_M \times_M \mathrm{GL}(C_M, V_M) \arrow[r] \arrow[d] & V_M \times_M \mathrm{GL}(C_M, V_M) \arrow[d] \\ C_M \arrow[r] & V_M \end{tikzcd}$$

The conditions of Definition 5.26 in terms of $(\widetilde{f_0}, \widetilde{f_1})$ are as follows:

$$\begin{pmatrix} \widetilde{f_0}(p_{g_1} \cdot (h_y, \Psi_{y,x}), g) \\ \widetilde{f_1}(\mathbf{s}(p_{g_1} \cdot (h_y, \Psi_{y,x})), g) \end{pmatrix} = \begin{pmatrix} (\Psi^{C_M}_{y,x})^{-1}(1 + h_y\partial_y)^{-1} & -(\Psi^{V_M}_{y,x})^{-1} h_y^{-1} \\ 0 & (\Psi^{V_M}_{y,x})^{-1} \end{pmatrix} \begin{pmatrix} \widetilde{f_0}(p_{g_1}, g) \\ \widetilde{f_1}(\mathbf{s}p_{g_1}, g) \end{pmatrix}.$$

The result is therefore a complex $\widetilde{C}_{\mathrm{GL}}(P_G; C_M)$ consisting of "GL-equivariant" cochains $(\widetilde{f_0}, \widetilde{f_1})$ with respect to the 2-actions of $\mathrm{GL}(C_M, V_M)_G$ on P_G and on $C_M \ltimes V_M \rightrightarrows V_M$ (see Remark 5.7). In terms of this complex, the differential is given by

$$\delta\widetilde{f_0}(p_{g_1}, g_2, \ldots, g_{\bullet+1}) = (-1)^\bullet(\widetilde{f_0}(p_{g_2}, g_3, \ldots, g_{\bullet+1}) - \widetilde{f_0}(p_{g_1} \cdot p_{g_2}, g_3, \ldots, g_{\bullet+1}))$$
$$+ (-1)^\bullet \textstyle\sum_{k=2}^{\bullet+1} (-1)^k d_k^* \widetilde{f_0}(p_{g_1}, g_2, \ldots, g_{\bullet+1})$$

where p_{g_2} is arbitrary and composable with p_{g_1} (and similarly for $\delta\widetilde{f_1}$).

5.9. **From general linear PB-groupoids to general linear double groupoids.** There is yet another way to encode the data of a fat extension. This becomes clear from the fact that general linear PB-groupoids sometimes come with a *gauge double groupoid*.

Remark 5.30. Recall from Remark 5.17 that the groupoid of cochain isomorphisms

$$\mathrm{Aut}(C_M \to V_M) \rightrightarrows M$$

can be seen as the gauge groupoid of the principal groupoid bundle $\mathrm{GL}(C_M, V_M)$ (or $\mathrm{Fr}(C_M, V_M)$). Given a general linear PB-groupoid P_G, what if we also take the gauge groupoid of the principal $\mathrm{GL}(C_M, V_M)_G$-bundle P_G?

In general, given a PB-groupoid $P_G \rightrightarrows P_M$ with structural strict Lie 2-groupoid $H_G \rightrightarrows H_M \rightrightarrows N$, then P_G comes with a (not necessarily smooth) *gauge double groupoid*

$$\begin{tikzcd} P_G \times_{H_G} P_G \arrow[r, shift left] \arrow[r, shift right] \arrow[d, shift left] \arrow[d, shift right] & P_M \times_{H_M} P_M \arrow[d, shift left] \arrow[d, shift right] \\ G \arrow[r, shift left] \arrow[r, shift right] & M \end{tikzcd}$$

Doing this for a general linear PB-groupoid (also if we take a general linear PB-groupoid as in [CG26]), we obtain a double groupoid whose vertical base groupoid is given by $\mathrm{Aut}(C_M \to V_M)$. A natural condition to put on the PB-groupoid so that the above gauge groupoids are smooth is to assume the moment maps to be surjective submersions onto an embedded submanifold of N.[51] This happens for general linear PB-groupoids precisely when the differential ∂ is of constant rank.

[51]By the properties of the moment maps, one moment map has this property if the other does.

Now, to be more precise, we introduce a special class of double groupoids. First of all, we use the following notation for double groupoids:

$$\begin{tikzcd} G \arrow[r, shift left] \arrow[r, shift right] \arrow[d, shift left] \arrow[d, shift right] & G_\downarrow \arrow[d, shift left] \arrow[d, shift right] \\ G_\rightarrow \arrow[r, shift left] \arrow[r, shift right] & M \end{tikzcd}$$

Here, $G_\rightarrow$ and $G_\downarrow$ are Lie groupoids, and we denote their structure maps using the same arrow-index. Moreover, G is a Lie groupoid in two ways, over $G_\rightarrow$ and over $G_\downarrow$, and we denote the groupoid structure maps of $G \rightrightarrows G_\downarrow$ with indices $\rightarrow$ and $G \rightrightarrows G_\rightarrow$ with indices $\downarrow$.

Definition 5.31. A *double groupoid* is a diagram of groupoids

$$\begin{tikzcd} G \arrow[r, shift left] \arrow[r, shift right] \arrow[d, shift left] \arrow[d, shift right] & G_\downarrow \arrow[d, shift left] \arrow[d, shift right] \\ G_\rightarrow \arrow[r, shift left] \arrow[r, shift right] & M \end{tikzcd}$$

such that the structure maps of $G \rightrightarrows G_\rightarrow$ are groupoid maps over the respective structure maps of $G_\downarrow \rightrightarrows M$.[52] We also assume that the double source map

$$(\mathbf{s}_\rightarrow, \mathbf{s}_\downarrow) : G \to G_\downarrow \times_M G_\rightarrow$$

is a (not necessarily surjective) submersion.

In the compatibility assumption made in this definition, we see, for example, the composable arrows of $G \rightrightarrows G_\rightarrow$ as a groupoid

$$G \times_{G_\rightarrow} G \rightrightarrows G_\downarrow \times_M G_\downarrow$$

with structure defined componentwise. The compatibility of the two products is called the *interchange law* and takes the form

$$(g \cdot_\rightarrow h) \cdot_\downarrow (g' \cdot_\rightarrow h') = (g \cdot_\downarrow g') \cdot_\rightarrow (h \cdot_\downarrow h').$$

Remark 5.32. As in [Ste07, MT11], we do not require the double source map $(\mathbf{s}_\rightarrow, \mathbf{s}_\downarrow)$ to be a *surjective* submersion. However, gauge double groupoids of PB-groupoids (that come with a gauge double groupoid) have this property. Following [MT11], we call double groupoids with a double source map that is a surjective submersion *full*.

In general, since the double source map $(\mathbf{s}_\rightarrow, \mathbf{s}_\downarrow)$ is a submersion, we obtain a *core groupoid* that we suggestively denote by $F \rightrightarrows M$:

Definition 5.33. The *core groupoid* of a double groupoid is given by

$$F := \ker \mathbf{s}_\downarrow \cap \ker \mathbf{s}_\rightarrow \rightrightarrows M.$$

Explicitly, its structure maps are $\mathbf{s}(g) = \mathbf{s}_\rightarrow \mathbf{s}_\downarrow g = \mathbf{s}_\downarrow \mathbf{s}_\rightarrow g$, $\mathbf{t}(g) = \mathbf{t}_\rightarrow \mathbf{t}_\downarrow g = \mathbf{t}_\downarrow \mathbf{t}_\rightarrow g$, and[53]

$$g \cdot h = (g \cdot_\downarrow 1_\rightarrow(\mathbf{t}_\rightarrow h)) \cdot_\rightarrow h = (g \cdot_\rightarrow 1_\downarrow(\mathbf{t}_\downarrow h)) \cdot_\downarrow h$$
$$g^{-1} = (g)^{-1}_\rightarrow \cdot_\downarrow 1_\rightarrow(\mathbf{t}_\rightarrow g)^{-1} = (g)^{-1}_\downarrow \cdot_\rightarrow 1_\downarrow(\mathbf{t}_\downarrow g)^{-1}.$$

[52]Equivalently, the structure maps of $G \rightrightarrows G_\downarrow$ are groupoid maps over the respective structure maps of $G_\rightarrow \rightrightarrows M$

[53]That is, in the definition of $g \cdot h$, we right-translate g to be composable with h.

Moreover, F comes with two groupoid maps

$$G_{\downarrow} \xleftarrow{\mathbf{t}_{\to}} F \xrightarrow{\mathbf{t}_{\downarrow}} G_{\to}$$

which are just the restriction to the target maps. We now concentrate on the following very particular case of double groupoids:

Definition 5.34. A double groupoid as above is called *(vertically) core-transitive* if the map

$$\mathbf{t}_{\downarrow} : F \to G_{\to}$$

is a surjective submersion.[54]

Core-transitive double groupoids turn out to be determined by, what we will call, *core extensions*.

Definition 5.35. A *core extension* is a diagram[55]

$$\begin{tikzcd} 1 \arrow[r] & \mathrm{H} \arrow[r] \arrow[d, phantom, "\circlearrowright"] & F \arrow[r] \arrow[dl] & G_{\to} \arrow[r] & 1 \\ & G_{\downarrow} & & & \end{tikzcd}$$

such that the restriction of the map $F \to G_{\downarrow}$ to H is equivariant (with respect to conjugation in $G_{\downarrow}$) and the Peiffer identity[56] holds: the conjugation action of F on H, viewing H as a normal subgroupoid of F, equals the action by F on H induced by the representation $G_{\downarrow} \circlearrowright \mathrm{H}$ and $F \to G_{\downarrow}$.

This is directly related to [BM92], a result that is generalised in Appendix A. In the appendix we present a proof, and we explain the already apparent connection with fat extensions. In the rest of this section we focus on the implication that regular fat extensions (fat extensions over regular cochain complexes) are equivalently described by *general linear double groupoids*.

The general linear double groupoids that encode regular 2-term complexes are strict Lie 2-groupoids. Such a general linear double groupoid sits naturally inside the strict Lie 2-groupoid of Remark 5.12.

Example 5.36. Given a regular 2-term complex $C_M \to V_M$, we denote again by $\mathrm{Aut}(C_M \to V_M)$ its Lie groupoid of cochain isomorphisms. Consider the double groupoid

$$\begin{tikzcd} \mathrm{H} \times_M \mathrm{Aut}(C_M \to V_M) \times_M \mathrm{H} \arrow[r, shift left] \arrow[r, shift right] \arrow[d, shift left] \arrow[d, shift right] & \mathrm{Aut}(C_M \to V_M) \arrow[d, shift left] \arrow[d, shift right] \\ \mathrm{H} \arrow[r, shift left] \arrow[r, shift right] & M \end{tikzcd}$$

The groupoid structures are given as follows. The vertical groupoid structure has source and target map given, respectively, by $\mathbf{s}_{\downarrow}(h_y, \Psi_{y,x}, h_x) = h_x$, $\mathbf{t}_{\downarrow}(h_y, \Psi_{y,x}, h_x) = h_y$ and

$$(h_z, \Psi_{z,y}, h_y) \cdot_{\downarrow} (h_y, \Psi_{y,x}, h_x) = (h_z, \Psi_{z,y}\Psi_{y,x}, h_x).$$

[54] Of course, there is an analogous definition of horizontally core-transitive if we interchange the vertical structures with the horizontal ones.

[55] The action $G_{\downarrow} \circlearrowright \mathrm{H}$ is a representation; that is, it acts by Lie group automorphisms (See Definition A.1).

[56] The *Peiffer identity* is a name given to a condition that a *crossed module* of Lie groupoids satisfies, of which the Peiffer identity in our case is a generalisation.

The horizontal groupoid structure is given by $\mathbf{s}_{\to}(h_y, \Psi_{y,x}, h_x) = \Psi_{y,x}$,

$$\mathbf{t}_{\to}(h_y, \Psi_{y,x}, h_x) = ((1 + h_y\partial)\Psi^{C_M}_{y,x}(1 + h_x\partial)^{-1}, (1 + \partial h_y)\Psi^{V_M}_{y,x}(1 + \partial h_x)^{-1}),$$

and

$$(h_y, \Psi_{y,x}, h_x) \cdot_{\to} (h'_y, \Psi'_{y,x}, h'_x) = (h_y h'_y, \Psi'_{y,x}, h_x h'_x).$$

The vertical groupoid structure has a normal subgroupoid of the form

$$\mathrm{Aut}(C_M \to V_M) \ltimes \mathrm{H} \rightrightarrows \mathrm{H}$$

through the embedding

$$\mathrm{Aut}(C_M \to V_M) \ltimes \mathrm{H} \to \mathrm{H} \times_M \mathrm{Aut}(C_M \to V_M) \times_M \mathrm{H} \qquad (\Psi_{y,x}, h_x) \mapsto (\Psi_{y,x} h_x \Psi^{-1}_{y,x}, \Psi_{y,x}, h_x).$$

We can then take the quotient with respect to this normal subgroupoid to obtain a double groupoid (a strict Lie 2-groupoid) of the form

$$\begin{tikzcd} \mathrm{H}_{\mathrm{Aut}} \arrow[r, shift left] \arrow[r, shift right] \arrow[d, shift left] \arrow[d, shift right] & \mathrm{Aut}(C_M \to V_M) \arrow[d, shift left] \arrow[d, shift right] \\ M \arrow[r, shift left] \arrow[r, shift right] & M \end{tikzcd}$$

This is the general linear double groupoid associated to a 2-term complex. It sits naturally inside of $\mathrm{H}(V_M, C_M) \ltimes \mathrm{GL}(C_M, V_M)$ (seen as a strict Lie 2-groupoid; see Remark 5.12), for it is also this strict Lie 2-groupoid restricted to $\mathrm{Aut}(C_M \to V_M)$:

$$\mathrm{H} \ltimes \mathrm{Aut}(C_M \to V_M) \xrightarrow{\sim} \mathrm{H}_{\mathrm{Aut}} \qquad (h_y, \Psi_{y,x}) \mapsto [h_y, \Psi_{y,x}, 0_x].$$

The reason for writing the double groupoid as above is that this description generalises to produce the general linear double groupoid of a fat extension (see Appendix A).

We now define:

Definition 5.37. Let G be a Lie groupoid, and let $C_M \to V_M$ be a regular 2-term complex. A double groupoid

$$\begin{tikzcd} G_{\mathrm{Aut}} \arrow[r, shift left] \arrow[r, shift right] \arrow[d, shift left] \arrow[d, shift right] & \mathrm{Aut}(C_M \to V_M) \arrow[d, shift left] \arrow[d, shift right] \\ G \arrow[r, shift left] \arrow[r, shift right] & M \end{tikzcd}$$

that is vertically core-transitive, and whose restriction to the vertical units is the strict Lie 2-groupoid

$$\begin{tikzcd} \mathrm{H}_{\mathrm{Aut}} \arrow[r, shift left] \arrow[r, shift right] \arrow[d, shift left] \arrow[d, shift right] & \mathrm{Aut}(C_M \to V_M) \arrow[d, shift left] \arrow[d, shift right] \\ M \arrow[r, shift left] \arrow[r, shift right] & M \end{tikzcd}$$

is called a *general linear double groupoid* of G over $C_M \to V_M$.

Our observation in Remark 5.30, together with Proposition 5.13, reveals:

Proposition 5.38. *The gauge double groupoid of a regular general linear PB-groupoid P_G*

$$\begin{tikzcd} P_G \times_{\mathrm{GL}} P_G \arrow[r, shift left] \arrow[r, shift right] \arrow[d, shift left] \arrow[d, shift right] & \mathrm{Aut}(C_M \to V_M) \arrow[d, shift left] \arrow[d, shift right] \\ G \arrow[r, shift left] \arrow[r, shift right] & M \end{tikzcd}$$

is a general linear double groupoid.

To go from general linear double groupoids to regular general linear PB-groupoids requires a slight generalisation (and clarification) of [BM92].[57] If we consider a regular general linear PB-groupoid, then the core extension of its gauge double groupoid recovers precisely the fat extension. So, with this correspondence in mind, general fat extensions can be thought of as "core extensions" of general linear PB-groupoids. We delay the discussion now to the appendix, and proceed with making some further comments on the correspondences.

6. Fat splittings and comments on the correspondences

Now that we went through the correspondences, we make some final comments.

6.1. Cohomological vanishing for proper groupoids.

We mention here that the cohomological vanishing theorem for 2-term ruths of proper groupoids is straightforward using fat extensions:

Proposition 6.1. *Let F_G be a fat extension over a proper groupoid G. Moreover, let $\int_x = \int_{\mathbf{t}^{-1}x} d\mu(g)$ be a proper normalised Haar system of G. Then the maps*

$$\eta : C^{\bullet}_{\text{inv}}(F_G; C_M) \to C^{\bullet-1}_{\text{inv}}(F_G; C_M)$$

given by

$$\eta f(H_{g_1}, g_2, \dots, g_{\bullet-1}) = (-1)^{\bullet} \int_{\mathbf{s}g_{\bullet-1}} f(H_{g_1}, g_2, \dots, g_{\bullet-1}, g)$$

define a contraction homotopy $[\delta, \eta] = 1$ in degrees $\bullet \geq 2$.

Proof. The proof is a straightforward computation, entirely analogous to the proof that groupoid cohomology with trivial coefficients of G vanishes. □

6.2. Splittings of fat extensions and trivialisations.

In this section we clarify some aspects of "split" fat extensions. Naturally, this discussion comes with several clarifications of the correspondences between 2-term ruths, VB-groupoids and fat extensions. Following Section 4.3, we define fat splittings as follows.

Definition 6.2. Let F_G be a fat extension of G over $C_M \to V_M$. A *fat splitting* of F_G is a map

$$h : F_G \to \mathrm{Hom}(\mathbf{s}^* V_M, \mathbf{t}^* C_M)$$

that is invariant: for all $H_g, H'_g \in F_G$, [58]

$$h(H_g) - h(H'_g) = -h(H_g, H'_g) = -H'_g \cdot H_g^{-1}$$

and unital: for all $h_x \in \mathrm{H}(V_M, C_M)$,

$$h(h_x) = h_x.$$

It is called multiplicative if

$$h(H_{g_1} \cdot H_{g_2})v = h(H_{g_1}) H_{g_2} \cdot v - h(H_{g_1}) \partial h(H_{g_2}) v + H_{g_1} \cdot h(H_{g_2}) v.$$

A disclaimer is that, although the similarity will become apparent, the multiplicativity condition is different than the multiplicativity condition for tensors (that appears in Section 7).

[57] To emphasise: most essential points are explained in [BM92], especially in the last remarks made in that work.

[58] Recall that $h(H_g, H'_g)$

Proposition 6.3. *Let V_G be a VB-groupoid. The association*

$$\{\text{Cleavages on } V_G\} \to \{\text{Fat splittings on } \widehat{V}_G\} \qquad \Sigma \mapsto h_\Sigma(H_g) := r_{g^{-1}}(H_g - \Sigma_g)$$

sets a one-to-one correspondence between (unital) cleavages of V_G and (unital) fat splittings. Moreover, for all $H_{g_1}, H_{g_2} \in \widehat{V}_G$ composable,

$$h_\Sigma(H_{g_1} \cdot H_{g_2})v = R_2(g_1, g_2)v + H_{g_1} \cdot h_\Sigma(H_{g_2})v - h_\Sigma(H_{g_1})\partial h_\Sigma(H_{g_2})v + h_\Sigma(H_{g_1})H_{g_2} \cdot v,$$

so under this correspondence, flat cleavages correspond to fat multiplicative splittings.

Proof. As mentioned before, an invariant map h defines a cleavage Σ by setting

$$\Sigma_g v = -r_g h(H_g)v + H_g v$$

where H_g is arbitrary. It is readily verified that the two constructions are mutually inverse. The last statement follows by a direct computation:

$$\begin{aligned} r_{g_1g_2}(H_{g_1} \cdot h_\Sigma(H_{g_2})v + h_\Sigma(H_{g_1})H_{g_2} \cdot v) &= H_{g_1}(H_{g_2} \cdot v - R_1(g_2)v) \cdot (H_{g_2}v - \Sigma_{g_2}v) \\ &\quad + (H_{g_1} - \Sigma_{g_1})(H_{g_2} \cdot v) \\ &= (H_{g_1} \cdot H_{g_2})v + r_{g_2}(H_{g_1} - \Sigma_{g_1})(H_{g_2} \cdot v - R_1(g_2)v) - \Sigma_{g_1} \cdot \Sigma_{g_2}v \\ &= r_{g_1g_2}(h_\Sigma(H_{g_1} \cdot H_{g_2}) + h_\Sigma(H_{g_1})\partial h_\Sigma(H_{g_2}) - R_2(g_1, g_2)) \end{aligned}$$

where we added $\pm\Sigma_{g_1} \cdot \Sigma_{g_2}v$ in the second equality and $\pm\Sigma_{g_1g_2}$ in the last. □

Remark 6.4. Notice that, given h_Σ as above, then, for all $H_{g_1}, H_{g_2} \in \widehat{V}_G$ composable,

$$\begin{aligned} h(H_{g_1})\partial h(H_{g_2})v &= h_\Sigma(H_{g_1})H_{g_2} \cdot v - h_\Sigma(H_{g_1})R_1(g_2)v \\ &= H_{g_1} \cdot h_\Sigma(H_{g_2})v - R_1(g_1)h_\Sigma(H_{g_2})v. \end{aligned}$$

Therefore, $h_\Sigma(H_{g_1} \cdot H_{g_2})$ equals $h_\Sigma(H_{g_1}) \cdot h_\Sigma(H_{g_2})$, where the product is defined as in Section 4.3.

Remark 6.5. Another way to arrive at the correct notion of fat splitting is Proposition 3.17 of Section 3.5. Namely, since

$$\widehat{V}_G^{\text{cat}} \cong (\text{Hom}(\mathbf{s}^*V_M, \mathbf{t}^*C_M) \times_G \widehat{V}_G)/H(V_M, C_M) \qquad H_g \mapsto (-h(H_g, H'_g), H'_g),$$

we see that a cleavage Σ, i.e. a map $\Sigma : G \to \widehat{V}_G^{\text{cat}}$, can also be seen as a fat splitting h. Indeed, h defines the map

$$(h, 1) : \widehat{V}_G \to \text{Hom}(\mathbf{s}^*V_M, \mathbf{t}^*C_M) \times_G \widehat{V}_G$$

and, by invariance, this last map descends to a map $G \to \widehat{V}_G^{\text{cat}}$, which is Σ if $h = h_\Sigma$. The failure for h to be multiplicative measures the failure of this last map being a Lie category map with respect to the structure of Lie category on $\text{Hom}(\mathbf{s}^*V_M, \mathbf{t}^*C_M)$ as defined in Section 3.5.

Remark 6.6. We can translate a fat splitting h to a map

$$\widetilde{h} : \widehat{V}_G \to \mathbf{t}^*\text{Hom}(V_M, C_M) = \text{Hom}(\mathbf{t}^*V_M, \mathbf{t}^*C_M)$$

by using the representation:

$$\widetilde{h}(H_g)v := h(H_g)(H_g^{-1} \cdot v)$$

Then, $\widetilde{h}$ satisfies the invariance condition

$$\widetilde{h}(H_g)H_g \cdot v - \widetilde{h}(H'_g)H'_g \cdot v = -h(H_g, H'_g)v$$

(such that $\widetilde{h}(h_x) = 0$ for all $h_x \in \mathrm{H}(V_M, C_M)$). We can rewrite the condition for h to be multiplicative in terms of $\widetilde{h}$ as

$$\widetilde{h}(H_{g_1} \cdot H_{g_2})v = H_{g_1} \cdot \widetilde{h}(H_{g_2})H_{g_1}^{-1} \cdot v - \widetilde{h}(H_{g_1})\partial(H_{g_1} \cdot \widetilde{h}(H_{g_2})H_{g_1}^{-1} \cdot v) + \widetilde{h}(H_{g_1})v.$$

Using the following terminology, it is straightforward to see that a fat multiplicative splitting "trivialises" the associated fat extension.

Definition 6.7. Suppose we are given a cochain complex representation of G on $C_M \to V_M$. The trivial fat extension is the fat extension corresponding to the fat groupoid

$$\mathrm{H}(V_M, C_M) \times_M G \rightrightarrows M$$

with $\widehat{\mathbf{t}}(h_{\mathbf{t}g}, g) = \mathbf{t}g$, $\widehat{\mathbf{s}}(h_{\mathbf{t}g}, g) = \mathbf{s}g$, and multiplication and inversion are given by[59]

$$(h_{\mathbf{t}g_1}, g_1) \cdot (h_{\mathbf{t}g_2}, g_2) = (h_{\mathbf{t}g_1} \cdot (g_1 \cdot h_{\mathbf{t}g_2}), g_1 g_2) \qquad (h_{\mathbf{t}g_1}, g_1)^{-1} = (g_1^{-1} \cdot h_{\mathbf{t}g_1}^{-1}, g_1^{-1}).$$

The representation of $\mathrm{H}(V_M, C_M) \times_M G \rightrightarrows M$ on $C_M \to V_M$ is given by

$$(h_{\mathbf{t}g}, g) \cdot c = (1 + h_{\mathbf{t}g}\partial)(g \cdot c) \qquad (h_{\mathbf{t}g}, g) \cdot v = (1 + \partial h_{\mathbf{t}g})(g \cdot v).$$

A trivialisation of a fat extension is a cochain complex representation of G on $C_M \to V_M$, and an isomorphism of fat extensions

$$F_G \cong \mathrm{H}(V_M, C_M) \times_M G.$$

Proposition 6.8. *Let F_G be a fat extension. A fat multiplicative splitting h is equivalent to a trivialisation of F_G. More precisely, the correspondence is given by*

$$F_G \cong \mathrm{H}(V_M, C_M) \times_M G \qquad H_g \mapsto (v \mapsto -h(H_g)(H_g^{-1} \cdot v), g),$$

where the representation of G on $C_M \to V_M$ is given by

$$g \cdot c := H_g \cdot c - h(H_g)\partial c \qquad g \cdot v = H_g \cdot v - \partial h(H_g)v.$$

To conclude, since the invariant cochains $C_{\mathrm{inv}}(F_G; C_M)$ form an abstract ruth, a fat splitting splits the invariant cochains $C_{\mathrm{inv}}(F_G; C_M)$:

Proposition 6.9. *The complex $C_{\mathrm{inv}}(\widehat{V}_G; C_M)$ is an abstract ruth. Moreover, writing $\mathcal{E}_G = C^\bullet(G; C_M) \oplus C^{\bullet-1}(G; V_M)$, a fat splitting h defines an isomorphism*

$$\mathcal{E}_G \xrightarrow{\sim} C_{\mathrm{inv}}(F_G; C_M) \qquad (f_0^{\mathcal{E}}, f_1^{\mathcal{E}}) \mapsto (f_0, f_1)$$

by setting

$$f_0(H_{g_1}, g_2, \ldots, g_\bullet) = f_0^{\mathcal{E}}(g_1, \ldots, g_\bullet) - h(H_{g_1}) f_1^{\mathcal{E}}(g_2, \ldots, g_\bullet)$$
$$f_1(g_2, \ldots, g_\bullet) = f_1^{\mathcal{E}}(g_2, \ldots, g_\bullet).$$

The induced operators R_1 and R_2 on $\mathcal{E}_G$ are given by

$$R_1(g)c := H_g \cdot c - h(H_g)\partial c \qquad R_1(g)v = H_g \cdot v - \partial h(H_g)v$$
$$R_2(g_1, g_2)v := h(H_{g_1} \cdot H_{g_2})v - h(H_{g_1})H_{g_2} \cdot v + h(H_{g_1})\partial h(H_{g_2})v - H_{g_1} \cdot h(H_{g_2})v.$$

Proof. It is clear that the given map is a map of CG-modules. The inverse of the map is given by reading the equation the other way around (which makes sense by the properties (4.4) and (4.5)). The last statement follows from the above discussion in this section. □

[59]Here, G acts on $\mathrm{H}(V_M, C_M)$ by conjugation.

6.3. **Fat extensions as a functor to the quasi general linear Lie 2-groupoid.** In this section, we will explain how fat extensions of G relate to "pseudofunctors" from G to a type of general linear 2-groupoid as defined in [dHS19]. Here, we will refer to that general linear Lie 2-groupoid as the quasi-general linear Lie 2-groupoid. It is a Lie 2-groupoid that is not strict.

Remark 6.10. In the analogy we made in Remark 5.3, we can relax "strictness" as follows. A Lie 2-groupoid (so not necessarily strict) can similarly be described by a diagram $H_2 \rightrightarrows H_1 \rightrightarrows N$, but $H_1 \rightrightarrows N$ is only a groupoid "up to homotopy". More precisely, we relax the condition that $H_2 \rightrightarrows N$ and $H_1 \rightrightarrows N$ are Lie groupoids to them being possibly only Lie categories. Still, $g \in H_1$ is assumed to have "quasi-inverses" $g^{-1} \in H_1$. The elements of H_2, which are invertible in the groupoid $H_2 \rightrightarrows H_1$, can then again be interpreted as homotopies h that show two elements in H_1 are homotopic. In particular, given $g \in H_1$, there will exist a $g^{-1} \in H_1$ and homotopies in H_2 that show gg^{-1} and $g^{-1}g$ are homotopic to identity elements of H_1.

We explain next the definition of the quasi general linear groupoid (the general linear groupoid that appeared in [dHS19]). We will use the following object (that will also appear in Section 7.2): given two vector bundles E_1 and E_2 over M, we write

$$\mathrm{Lin}(E_1, E_2) := \{E_{1,x} \to E_{2,y} \mid x, y \in M\}$$

for the vector bundle over $M \times M$ of linear maps between fibers of E_1 and E_2.

Example 6.11. Recall the general linear strict Lie 2-groupoid from Example 5.4. Relaxing the conditions for elements of $\mathrm{GL}(C_M, V_M)_M$ to be quasi-isomorphisms (see Remark 6.10), we will show here that we obtain a well-defined quasi general linear Lie 2-groupoid (as we call it here)

$$\mathrm{QGL}(C_M, V_M)_G \rightrightarrows \mathrm{QGL}(C_M, V_M)_M \rightrightarrows M.$$

As observed in [dHS19], notice that the quasi-isomorphisms

$$\mathrm{QGL}(C_M, V_M)_M \subset \mathrm{Hom}(C_M, V_M) \times_M \mathrm{End}\, C_M \times_{M\times M} \mathrm{End}\, V_M \times_M \mathrm{Hom}(C_M, V_M)$$

form an embedded submanifold. To see this, observe first that, given a cochain map $(\partial_y, \Psi_{y,x}, \partial_x)$, then the existence of maps

$$[\partial, \Psi^{-1}] = 0 \qquad 1 - \Psi\Psi^{-1} = [\partial, h_y] \qquad 1 - \Psi^{-1}\Psi = [\partial, h_x]$$

such that additionally[60]

$$h_y \Psi^{V_M} = \Psi^{C_M} h_x \qquad (\Psi^{C_M})^{-1} h_y = h_x (\Psi^{V_M})^{-1}$$

is equivalent to the existence of an isomorphism

$$\begin{pmatrix} \Psi^{C_M}_{y,x} & -h_y \\ \partial_x & (\Psi^{V_M}_{y,x})^{-1} \end{pmatrix} : (C_M)_x \oplus (V_M)_y \xrightarrow{\sim} (C_M)_y \oplus (V_M)_x$$

(so the left column is fixed) with inverse given by

$$\begin{pmatrix} (\Psi^{C_M}_{y,x})^{-1} & h_x \\ -\partial_y & \Psi^{V_M}_{y,x} \end{pmatrix} : (C_M)_y \oplus (V_M)_x \xrightarrow{\sim} (C_M)_x \oplus (V_M)_y$$

(where the bottom row is fixed). Therefore, $(\partial_y, \Psi_{y,x}, \partial_x)$ is a quasi-isomorphism if and only if

$$\ker \Psi^{C_M}_{y,x} \cap \ker \partial_x = 0 \qquad \mathrm{im}\, \Psi^{V_M}_{y,x} + \mathrm{im}\, \partial_y = (V_M)_y. \tag{6.1}$$

[60] If Ψ is a quasi-isomorphism, it is indeed possible to find homotopy data satisfying these extra conditions.

In [dHS19] (Proposition 5.4 there) it is then shown that, on the open set U of (not-necessarily cochain maps) $(\partial_y, \Psi_{y,x}, \partial_x)$ satisfying (6.1), the map

$$U \to \mathrm{Lin}(C_M, V_M) \qquad (\partial_y, \Psi_{y,x}, \partial_x) \mapsto \partial_y \Psi_{y,x}^{C_M} - \Psi_{y,x}^{V_M} \partial_x$$

has maximal rank. We now define $\mathrm{QGL}(C_M, V_M)_M \rightrightarrows \mathrm{Hom}(C_M, V_M)$ as a Lie category with composition, and we set

$$\mathrm{GL}(C_M, V_M)_G := \mathrm{QGL}(C_M, V_M)_M \times_{M \times M} \mathrm{Lin}(V_M, C_M)$$

thought of as elements $(\partial_y, \Psi_{y,x}, \Psi'_{y,x}, \partial_x, h_{y,x})$ (or simply $(\partial_y, \Psi_{y,x}, \partial_x, h_{y,x})$) that form homotopy data between the quasi-isomorphisms $\Psi_{y,x}$ and $\Psi'_{y,x}$:[61]

$$\Psi_{y,x} - \Psi'_{y,x} = \partial_x h_{x,y} + h_{x,y} \partial_y.$$

Both products can be interpreted as homotopy composition in the following way. The Lie category structure $\mathrm{QGL}(C_M, V_M)_G \rightrightarrows \mathrm{Hom}(C_M, V_M)$ is given by[62]

$$(\partial_z, \Psi_{z,y}, \partial_y, h_{z,y}) \cdot (\partial_y, \Psi_{y,x}, \partial_x, h_{y,x}) := (\partial_z, \Psi_{z,y}\Psi_{y,x}, \partial_x, \Psi'_{z,y} h_{y,x} + h_{z,y}\Psi_{y,x}).$$

The groupoid structure $\mathrm{QGL}(C_M, V_M)_G \rightrightarrows \mathrm{QGL}(C_M, V_M)_M$ has source and target map given by $\mathbf{s}(\partial_z, \Psi_{z,y}, \partial_y, h_{z,y}) = (\partial_z, \Psi'_{z,y}, \partial_y)$ and $\mathbf{t}(\partial_z, \Psi_{z,y}, \partial_y, h_{z,y}) = (\partial_z, \Psi_{z,y}, \partial_y)$, and

$$(\partial_z, \Psi_{z,y}, \partial_y, h_{z,y}) \cdot (\partial_z, \Psi'_{z,y}, \partial_y, h'_{z,y}) := (\partial_z, \Psi_{z,y}, \Psi''_{z,y}, \partial_y, h_{z,y} + h'_{z,y}).$$

Now, an element $(\partial_y, \Psi_{y,x}, \partial_x) \in \mathrm{QGL}(C_M, V_M)_M$ is not always invertible, but, given homotopy data $(h_x, \Psi_{y,x}^{-1}, h_y)$ as above, $(\partial_x, \Psi_{y,x}^{-1}, \partial_y)$ is a "quasi-inverse" in the sense of Remark 6.10, which shows that $\mathrm{QGL}(C_M, V_M)_G \rightrightarrows \mathrm{QGL}(C_M, V_M)_M \rightrightarrows \mathrm{Hom}(C_M, V_M)$ is indeed a Lie 2-groupoid.

Remark 6.12. Notice that the we can identify $\mathrm{GL}(C_M, V_M)_G \subset \mathrm{QGL}(C_M, V_M)_G$ as the invertible elements of the Lie category $\mathrm{QGL}(C_M, V_M)_G \rightrightarrows \mathrm{Hom}(C_M, V_M)$ via

$$(h_y, \partial_y, \Psi_{y,x}) \mapsto (\partial_y, (1 + [\partial_y, h_y])\Psi_{y,x}, (\Psi_{y,x}^{V_M})^{-1} \partial_y \Psi_{y,x}^{C_M}, h_y \Psi_{y,x}^{V_M}).$$

We can interpret the step to pseudofunctors efficiently using the fat category extension. A fat category extension F_G^{cat} of G over $C_M \to V_M$ comes with a Lie category map

$$F_G^{\mathrm{cat}} \to \mathrm{QGL}(C_M, V_M)_M$$

which defines the cochain complex representation. Now, a fat category splitting is a section of the projection $F_G^{\mathrm{cat}} \to G$ (see Remark 3.23 or Section 6.2) and therefore defines a map

$$R_1 : G \to F_G^{\mathrm{cat}} \to \mathrm{QGL}(C_M, V_M)_M$$

which might fail to be a functor. Nevertheless, its image lies in $\mathrm{QGL}(C_M, V_M)_M$, so for all $g \in G$,

$$[\partial, R_1(g)] = 0.$$

The failure for this map to define a functor can now be seen as a map

$$R_2 : G^{(2)} \to \mathrm{QGL}(C_M, V_M)_G.$$

That it maps into $\mathrm{QGL}(C_M, V_M)_G$ is saying that, for all $(g_1, g_2) \in G^{(2)}$,

$$R_1(g_1)R_1(g_2) - R_1(g_1 g_2) = [\partial, R_2(g_1, g_2)].$$

[61] We sometimes leave $\Psi'_{y,x}$ implicit for it is determined by $\Psi_{y,x}$ and $h_{y,x}$. Moreover, given $\Psi_{y,x}$, any $h_{y,x}$ defines the cochain map $\Psi'_{y,x} := \Psi_{y,x} - \partial_y h_{y,x} - h_{y,x}\partial_x$ which is homotopic to $\Psi_{y,x}$.

[62] Notice that $\Psi'_{z,y} h_{y,x} + h_{z,y}\Psi_{y,x} = \Psi_{z,y} h_{y,x} + h_{z,y}\Psi'_{y,x}$ (see Lemma 3.7).

The condition that R_2 is unital, and that for all $(g_1, g_2, g_3) \in G^{(3)}$,

$$-R_2(g_1g_2, g_3) + R_2(g_1, g_2g_3) = -R_1(g_1)R_2(g_2, g_3) + R_2(g_1, g_2)R_1(g_3)$$

is then equivalent to the fact that R_1 and R_2 together define a so-called pseudofunctor.[63] The functorial aspects of this construction are a bit subtle. It is proved in [dHS19] that the category of (split) 2-terms ruths over a fixed 2-term complex $C_M \to V_M$, together with *quasi-isomorphisms* of 2-term ruths, is equivalent to the category of pseudofunctors together with lax-functors. A strict version of this correspondence is not discussed in [dHS19], but some more comments about this issue are made in that work.

As we already mentioned, a fat category splitting of a fat category extension turns it into a *weak representation* as in [Wol17], and such objects come with a natural notion of morphism that turns the correspondence into an equivalence of categories.

7. Functorial aspects

We will introduce a morphism of fat extensions as a multiplicative tensor. This will be a first instance where we view a multiplicative tensor as an object that naturally occurs in the theory of fat extensions. This should therefore also be seen as part of the bigger project together with Joaõ Nuno Mestre and Luca Vitagliano [MOV]. A morphism of general linear PB-groupoids is similar in nature, but its properties can succinctly be described as GL-*equivariance*. We introduce both types of morphisms through the language of VB-groupoids. For this reason, in introducing the correct notions of morphisms, we will come to see naturally how the respective categories are equivalent to the category of VB-groupoids.

A notational comment: throughout, we will denote a cochain map as follows:

$$\begin{tikzcd} C_{M,1} \arrow[r, "\partial"] \arrow[d, "\Phi^{C_M}"] & V_{M,1} \arrow[d, "\Phi^{V_M}"] \\ C_{M,2} \arrow[r, "\partial"] & V_{M,2} \end{tikzcd}$$

but often we simply write Φ. A VB-groupoid map is denoted by

$$\Phi = \Phi^{V_G} : V_{G,1} \to V_{G,2}.$$

We focus on maps over the identity map of G. Such a map is a groupoid map $V_{G,1} \to V_{G,2}$ that is also linear (i.e. intertwines the scalar multiplications). It comes with a cochain map by setting

$$\Phi^{C_M} := \Phi^{V_G}|_{C_M} \qquad \Phi^{V_M} := \mathbf{s}\Phi^{V_G}|_{V_M}$$

where we used the natural inclusion

$$C_M \oplus V_M = V_G|_M \hookrightarrow V_G.$$

Remark 7.1. We restrict attention to VB-groupoid maps over the identity for exposition. In order to pass to more general maps (over general groupoid maps), notice that the category of VB-groupoids

[63] The pseudofunctor conditions say that R_2 is unital and that the two homotopies

$$R_2(g_1g_2, g_3) + R_2(g_1, g_2)R_1(g_3) \qquad R_2(g_1, g_2g_3) + R_1(g_1)R_2(g_2, g_3)$$

are equal.

is naturally fibred, just as, for example, the category of vector bundles. That is, general VB-groupoid maps from V_G to V_H can be seen as VB-groupoid maps

$$V_G \to \varphi^* V_H$$

over the identity, where $\varphi : G \to H$ is a Lie groupoid map, and $\varphi^* V_H$ is the pullback VB-groupoid. The other categories we will consider (of fat extensions and of general linear PB-groupoids) are fibred analogously, and passing to more general morphisms is therefore straightforward and omitted.

7.1. **Morphisms of fat extensions.** Our aim is now to describe, given a map of VB-groupoids $\Phi = \Phi^{V_G} : V_{G,1} \to V_{G,2}$, a type of morphism $f = f_\Phi : \widehat{V}_{G,1} \to \widehat{V}_{G,2}$ that consists of exactly the same data. However, observe that there is no obvious groupoid map $\widehat{V}_{G,1} \to \widehat{V}_{G,2}$ in general. Rather, we obtain a notion of "related" elements. We therefore aid our discussion with the following intermediate definition.

Definition 7.2. Given a VB-groupoid $\Phi : V_{G,1} \to V_{G,2}$, we say $H_{g,1} \in \widehat{V}_{G,1}$ and $H_{g,2} \in \widehat{V}_{G,2}$ are Φ-related if the diagram

$$\begin{tikzcd} (V_{M,1})_{\mathbf{s}g} \arrow[r, "H_{g,1}"] \arrow[d, "\Phi^{V_M}"] & (V_{G,1})_g \arrow[d, "\Phi^{V_G}"] \\ (V_{M,2})_{\mathbf{s}g} \arrow[r, "H_{g,2}"] & (V_{G,2})_g \end{tikzcd}$$

commutes. In this case, we write $H_{g,1} \sim_\Phi H_{g,2}$.

Given $(g, g') \in G^{(2)}$, $H_{g,1} \sim_\Phi H_{g,2}$ and $H_{g',1} \sim_\Phi H_{g',2}$, then also $H_{g,1} \cdot H_{g',1} \sim_\Phi H_{g,2} \cdot H_{g',2}$. But for our purposes such a notion of "multiplicative relation" is too narrow. For example, there will often exist $H_{g,1} \in \widehat{V}_{G,1}$ not Φ-related to any $H_{g,2} \in \widehat{V}_{G,2}$.[64] In general, however, given $H_{g,1} \in \widehat{V}_{G,1}$ and $H_{g,2} \in \widehat{V}_{G,2}$, we can always produce the map

$$f_\Phi(H_{g,1}, H_{g,2}) : (V_{M,1})_{\mathbf{s}g} \to (C_{M,2})_{\mathbf{t}g} \qquad v \mapsto r_{g^{-1}}(H_{g,2}\Phi^{V_M} v - \Phi^{V_G} H_{g,1} v) \tag{7.1}$$

that is zero precisely when $H_{g,1} \sim_\Phi H_{g,2}$. The argument that the Φ-relation is multiplicative can be generalised to the following property of f_Φ:

Lemma 7.3. *Given a VB-groupoid map* $\Phi : V_{G,1} \to V_{G,2}$*,*

$$f_\Phi : \widehat{V}_{G,1} \times_G \widehat{V}_{G,2} \to \mathrm{Hom}(\mathbf{s}^* V_M, \mathbf{t}^* C_M),$$

defined as above in (7.1)*, satisfies, for all* $(H_{g,1}, H_{g',1}) \in \widehat{V}^{(2)}_{G,1}$ *and* $(H_{g,2}, H_{g',2}) \in \widehat{V}^{(2)}_{G,2}$*,*

$$f_\Phi(H_{g,1} \cdot H_{g',1}, H_{g,2} \cdot H_{g',2})v = f_\Phi(H_{g,1}, H_{g,2})H_{g',1} \cdot v + H_{g,2} \cdot f_\Phi(H_{g',1}, H_{g',2})v.$$

Proof. We compute directly:

$$\begin{aligned} r_{gg'} f_\Phi(H_{g,1} \cdot H_{g',1}, H_{g,2} \cdot H_{g',2})v &= H_{g,2}(\mathbf{t}H_{g',2}\Phi v) \cdot H_{g',2}\Phi v - \Phi(H_{g,1}(\mathbf{t}H_{g',1}v)) \cdot \Phi(H_{g',1}v) \\ &= r_{g'}(H_{g,2}(\Phi(\mathbf{t}H_{g',1}v)) - \Phi(H_{g,1}(\mathbf{t}H_{g',1}v)))+ \\ &\quad + H_{g,2}\mathbf{t}(H_{g',2}\Phi v - \Phi H_{g',1}v) \cdot (H_{g',2}\Phi v - \Phi H_{g',1}v) \\ &= r_{gg'} f_\Phi(H_{g,1}, H_{g,2})(H_{g',1} \cdot v) + r_{gg'}(H_{g,2} \cdot f_\Phi(H_{g',1}, H_{g',2})v). \end{aligned}$$

This proves the statement. □

[64]For example, if there is a $v \in \ker\Phi \subset (V_{M,1})_{\mathbf{s}g}$ and $H_{g,1}$ such that $\Phi H_{g,1} v \neq 0$.

Aside from this multiplicativity property, which generalises the multiplicativity of the Φ-relation, we also need a generalisation of the following fact. If $H_{g,1} \sim_\Phi H_{g,2}$, then the cochain map $\Phi = (\Phi^{C_M}, \Phi^{V_M})$ intertwines the actions by $H_{g,1}$ and $H_{g,2}$, respectively:

$$H_{g,2} \cdot \Phi^{C_M} c = \Phi^{C_M}(H_{g,1} \cdot c) \qquad H_{g,2} \cdot \Phi^{V_M} v = \Phi^{V_M}(H_{g,1} \cdot v).$$

That is, we observe next that the map $f_\Phi(H_{g,1}, H_{g,2})$ can actually be interpreted as the homotopy that measures the failure of Φ to intertwine the actions by $H_{g,1}$ and $H_{g,2}$:

Lemma 7.4. *Given a VB-groupoid map* $\Phi : V_{G,1} \to V_{G,2}$*,*

$$f_\Phi : \widehat{V}_{G,1} \times_G \widehat{V}_{G,2} \to \mathrm{Hom}(\mathbf{s}^* V_M, \mathbf{t}^* C_M),$$

defined as above in (7.1)*, satisfies, for all* $H_{g,1} \in \widehat{V}_{G,1}$ *and* $H_{g,2} \in \widehat{V}_{G,2}$*,*

$$f_\Phi(H_{g,1}, H_{g,2})\partial c = H_{g,2} \cdot \Phi^{C_M} c - \Phi^{C_M}(H_{g,1} \cdot c),$$
$$\partial f_\Phi(H_{g,1}, H_{g,2}) v = H_{g,2} \cdot \Phi^{V_M} v - \Phi^{V_M}(H_{g,1} \cdot v).$$

There is one more important property that relate the map f_Φ and the cochain map Φ: invariance. To introduce it, it is helpful to understand how one recovers Φ^{V_G} from f_Φ and the cochain map Φ.

Recall that an element, say, $H_{g,1} \in \widehat{V}_{G,1}$, defines a linear isomorphism

$$r_g + H_{g,1} : (C_{M,1})_{\mathbf{t}g} \oplus (V_{M,1})_{\mathbf{s}g} \xrightarrow{\sim} (V_{G,1})_g \qquad (c, v) \mapsto c \cdot 0_g + H_{g,1} v.$$

We then have (for future reference, we write $f = f_\Phi$)

$$\Phi_g^{V_G}(r_g c + H_{g,1} v) = r_g(\Phi^{C_M}(c) - f(H_{g,1}, H_{g,2})v) + H_{g,2}\Phi^{V_M}(v). \tag{7.2}$$

Interpreting f_Φ as a "correcting term" in this way points to the missing piece of information, invariance, that f_Φ and the cochain map Φ satisfy.

Lemma 7.5. *Given a VB-groupoid map* $\Phi : V_{G,1} \to V_{G,2}$*,*

$$f_\Phi : \widehat{V}_{G,1} \times_G \widehat{V}_{G,2} \to \mathrm{Hom}(\mathbf{s}^* V_M, \mathbf{t}^* C_M)$$

defined as above in (7.1) *satisfies, for all* $H_{g,1}, H'_{g,1} \in \widehat{V}_{G,1}$ *and* $H_{g,2}, H'_{g,2} \in \widehat{V}_{G,2}$*,*

$$f_\Phi(H_{g,1}, H_{g,2})v - f_\Phi(H'_{g,1}, H'_{g,2})v = h_2(H_{g,2}, H'_{g,2})\Phi^{V_M} v - \Phi^{C_M} h_1(H_{g,1}, H'_{g,1})v$$

where

$$h_1(H_{g,1}, H'_{g,1})v = r_{g^{-1}}(H'_{g,1} v - H_{g,1} v),$$

and h_2 *is defined similarly.*

Recognise that, in the above lemma, h_1 and h_2 are equal to f_Φ if we take Φ to be the respective identity maps. The similarity with the invariance condition that appeared in Section 4.4 is, of course, not coincidental. In Remark 7.8 we clarify this connection further.

We organise the properties that (f, Φ) should satisfy, in order to set the correspondence, in the following way.

Definition 7.6. Consider fat extensions $F_{G,1}$ and $F_{G,2}$ of G, over $C_{M,1} \to V_{M,1}$ and $C_{M,2} \to V_{M,2}$, respectively. Consider a pair (f, Φ), where $\Phi = (\Phi^{C_M}, \Phi^{V_M})$ are two vector bundle maps

$$C_{M,1} \xrightarrow{\Phi^{C_M}} C_{M,2} \qquad V_{M,1} \xrightarrow{\Phi^{V_M}} V_{M,2}$$

and f is a smooth map

$$f : F_{G,1} \times_G F_{G,2} \to \mathrm{Hom}(\mathbf{s}^* V_{M,1}, \mathbf{t}^* C_{M,2}).$$

Now, (f, Φ) is a *morphism* $F_{G,1} \to F_{G,2}$ *of fat extensions* if it is invariant:

$$f(H_{g,1}, H_{g,2})v - f(H'_{g,1}, H'_{g,2})v = h_2(H_{g,2}, H'_{g,2})\Phi^{V_M} v - \Phi^{C_M} h_1(H_{g,1}, H'_{g,1})v$$

and multiplicative: this means that Φ is a cochain map

$$\begin{array}{ccc} C_{M,1} & \xrightarrow{\partial} & V_{M,1} \\ \big\downarrow \Phi^{C_M} & & \big\downarrow \Phi^{V_M} \\ C_{M,2} & \xrightarrow{\partial} & V_{M,2} \end{array}$$

that

$$f(H_{g,1}, H_{g,2})\partial c = H_{g,2} \cdot \Phi^{C_M} c - \Phi^{C_M}(H_{g,1} \cdot c),$$
$$\partial f(H_{g,1}, H_{g,2})v = H_{g,2} \cdot \Phi^{V_M} v - \Phi^{V_M}(H_{g,1} \cdot v),$$

and that

$$f(H_{g,1} \cdot H_{h,1}, H_{g,2} \cdot H_{h,2})v = f(H_{g,1}, H_{g,2})(H_{h,1} \cdot v) + H_{g,2} \cdot f(H_{h,1}, H_{h,2})v.$$

Remark 7.7. We often simplify and write a map of fat extensions (f, Φ) by f, so we leave Φ implicit from the notation. Similar as happens for the invariant cochains we described in Section 4.4 (see Remark 4.13), f usually determines the cochain map Φ by invariance. But, aside from suppressing Φ from the notation sometimes, we keep track of Φ as part of the data.

Remark 7.8. The point of this remark is to clarify why we used the term "multiplicative" in the definition above. In fact, we will argue here that we can think of a morphism of fat extensions as a multiplicative tensor.

To do this, notice that we can translate a map of fat extensions f so that it takes values in

$$\mathbf{t}^*\mathrm{Hom}(V_M, C_M) = \mathrm{Hom}(\mathbf{t}^* V_M, \mathbf{t}^* C_M).$$

Indeed, simply replace f with

$$\widetilde{f}(H_{g,1}, H_{g,2}) : (V_{M,1})_{\mathbf{t}g} \to (C_{M,2})_{\mathbf{t}g} \qquad v \mapsto f(H_{g,1}, H_{g,2})(H_{g,1}^{-1} \cdot v).$$

In terms of $\widetilde{f}$, the invariance condition becomes

$$\widetilde{f}(H_{g,1}, H_{g,2})(H_{g,1} \cdot v) - \widetilde{f}(H'_{g,1}, H'_{g,2})(H'_{g,1} \cdot v) = h_2(H_{g,2}, H'_{g,2})\Phi^{V_M} v - \Phi^{C_M} h_1(H_{g,1}, H'_{g,1})v,$$

and the two multiplicativity conditions involving f translate to

$$\partial \widetilde{f}(H_{g,1}, H_{g,2})v = H_{g,2} \cdot \Phi^{V_M}(H_{g,1}^{-1} \cdot v) - \Phi^{V_M} v$$
$$\widetilde{f}(H_{g,1}, H_{g,2})\partial c = H_{g,2} \cdot \Phi^{C_M}(H_{g,1}^{-1} \cdot c) - \Phi^{C_M} c$$

and

$$\widetilde{f}(H_{g,1} \cdot H_{h,1}, H_{g,2} \cdot H_{h,2})v = \widetilde{f}(H_{g,1}, H_{g,2})v + H_{g,2} \cdot \widetilde{f}(H_{h,1}, H_{h,2})(H_{g,1}^{-1} \cdot v).$$

This way, f (or rather $(\widetilde{f}, \Phi)$) can be thought of as an "invariant" 1-cocycle in (the 3-term ruth corresponding to) the cochain complex representation

$$\mathrm{Hom}(V_{M,1}, C_{M,2}) \to \mathrm{Hom}(C_{M,1}, C_{M,2}) \oplus \mathrm{Hom}(V_{M,1}, V_{M,2}) \to \mathrm{Hom}(C_{M,1}, V_{M,2})$$

of the groupoid $F_{G,1} \times_G F_{G,2}$, whose groupoid structure is defined componentwise. Here, the cochain complex representation can be thought of as a tensor product of the cochain complexes

$$V^*_{M,1} \to C^*_{M,1} \qquad C_{M,2} \to V_{M,2}$$

with the action defined "componentwise". Studying more general tensor products this way is part of the objective of the work in progress [MOV].

Before we move on, we discuss the composition of morphisms.

Definition 7.9. Given maps of fat extensions

$$F_{G,1} \times_G F_{G,2} \xrightarrow{f_{21}} \mathrm{Hom}(\mathbf{s}^* V_{M,1}, \mathbf{t}^* C_{M,2}) \qquad F_{G,2} \times_G F_{G,3} \xrightarrow{f_{32}} \mathrm{Hom}(\mathbf{s}^* V_{M,2}, \mathbf{t}^* C_{M,3}),$$

over cochain maps Φ_{21} and Φ_{32}, respectively, the composition $f_{31} = f_{32} \circ f_{21}$ over $\Phi_{31} = \Phi_{32} \circ \Phi_{21}$ is given by

$$f_{31}(H_{g,1}, H_{g,3}) = f_{32}(H_{g,2}, H_{g,3})\Phi_{21}^{V_M} + \Phi_{32}^{C_M} f_{21}(H_{g,1}, H_{g,2}),$$

where $H_{g,2} \in F_{G,2}$ is arbitrary.

Remark 7.10. The identity map $F_G \to F_G$ is the identity cochain map between $C_M \to V_M$ and itself, together with the map

$$h : F_G \times_G F_G \to \mathrm{Hom}(\mathbf{s}^* V_M, \mathbf{t}^* C_M) \qquad h(H_g, H'_g) = h_{\mathbf{t}g}(H_g \cdot v),$$

where $h_{\mathbf{t}g}$ is unique with the property that

$$h_{\mathbf{t}g} \cdot H_g = H'_g.$$

If $F_G = \widehat{V}_G$ for a VB-groupoid V_G, we recover the map

$$h(H_g, H'_g) = r_{g^{-1}}(H'_g - H_g)$$

which we already mentioned in Remark 4.15.

The following Remark should be compared to Remark 3.33, where we discussed the "inclusion" map of a fat subextension and the "projection" map of a fat quotient.

Remark 7.11. We observe here that f^{-1} is a left/right inverse to f if and only if

$$f^{-1}(H_{g,2}, H_{g,1}) = -\Phi^{-1} f(H_{g,1}, H_{g,2})\Phi^{-1},$$

where Φ^{-1} is a left/right inverse to the cochain map Φ underlying f. In this case, the map that sends $H_{g,1} \in F_{G,1}$ to the unique $H_{g,2} \in F_{G,2}$ for which[65]

$$f(H_{g,1}, H_{g,2}) = 0$$

is a groupoid immersion/submersion $F_{G,1} \to F_{G,2}$ that we can fit into a commutative diagram

$$\begin{tikzcd}
1 \arrow[r] & \mathrm{H}(C_{M,1}, V_{M,1}) \arrow[r] \arrow[d] & F_{G,1} \arrow[r] \arrow[d] & G \arrow[r] \arrow[d, "="] & 1 \\
1 \arrow[r] & \mathrm{H}(C_{M,2}, V_{M,2}) \arrow[r] & F_{G,2} \arrow[r] & G \arrow[r] & 1
\end{tikzcd}$$

[65]Explicitly, $H_{g,2}$ is given by $H'_{g,2} \cdot h_{\mathbf{s}g}$, where $H'_{g,2} \in F_{G,2}$ is arbitrary, and

$$h_{\mathbf{s}g} := (H'_{g,2})^{-1} \cdot f(H_{g,1}, H'_{g,2})(\Phi^{V_M})^{-1}.$$

Conversely, all such commutative diagrams arise in this way; that is, those for which the vertical map on the left is given by

$$\mathrm{H}(C_{M,1}, V_{M,1}) \to \mathrm{H}(C_{M,2}, V_{M,2}) \qquad h_x \mapsto \Phi^{C_M} h_x (\Phi^{V_M})^{-1}$$

if Φ^{-1} a left inverse of Φ, and

$$\mathrm{H}(C_{M,1}, V_{M,1}) \to \mathrm{H}(C_{M,2}, V_{M,2}) \qquad h_x \mapsto (\Phi^{C_M})^{-1} h_x \Phi^{V_M}$$

if Φ^{-1} a right inverse of Φ.

There are of course many other classes of maps of interest. For example, the class of maps for which the underlying cochain map is a quasi-isomorphism. Such maps correspond to VB-Morita maps (as can be inferred from [dHO20]). This will be further clarified and explored in [MMO].

To conclude this section, we summarise:

Proposition 7.12. *Let $\Phi : V_{G,1} \to V_{G,2}$ be a VB-groupoid map. Then,*

$$f_\Phi : \widehat{V}_{G,1} \to \widehat{V}_{G,2},$$

defined as in (7.1)*, is a morphism of fat extensions. Moreover, the assignment $\Phi \mapsto f_\Phi$ respects composition.*

Proof. Only the last statement is left to be verified. Given Φ_{21} and Φ_{32} VB-groupoid maps, we compute: any $H_{g,2} \in \widehat{V}_{G,2}$ gives

$$\begin{aligned} r_g f_{31}(H_{g,1}, H_{g,3}) &= H_{g,3}(\Phi_{32}\Phi_{21} v) - \Phi_{32}\Phi_{21}(H_{g,1} v) \\ &= \Phi_{32}(H_{g,2}\Phi_{21} v - \Phi_{21}(H_{g,1} v)) + (H_{g,3}\Phi_{32} - \Phi_{32} H_{g,2})\Phi_{21} v \\ &= r_g \Phi_{32} f_{21}(H_{g,1}, H_{g,2}) v + r_g f_{32}(H_{g,2}, H_{g,3}) \Phi_{21} v. \end{aligned}$$

This proves the statement. □

Starting with a map of fat extensions f, the invariance of f makes sure the map Φ, defined as in (7.2), is well-defined. The multiplicativity of f ensures this map Φ is a VB-groupoid map.

We arrived at the main theorem of this work.

Theorem 7.13. *The functor*

$$\{\text{VB-groupoids}\} \to \{\text{Fat extensions}\} \qquad V_G \to \widehat{V}_G \quad \Phi \mapsto f_\Phi$$

sets an equivalence of categories.

Proof. In this section we established the (fully faithful) functoriality, and in Section 4 (see Section 4.2; also see Section 3) we showed the essential surjectivity. □

Remark 7.14. The proof of [dHO20] that the functor

$$\{\text{Split 2-term ruths}\} \to \{\text{VB-groupoids}\}$$

is an equivalence can now be understood as follows. We call {Split VB-groupoids} the category of VB-groupoids equipped with a cleavage. Then the above functor decomposes as

$$\{\text{Split 2-term ruths}\} \to \{\text{Split VB-groupoids}\} \to \{\text{VB-groupoids}\}$$

where the latter equivalence is given by the forgetful functor. To see how the first equivalence works, observe that, from a VB-groupoid map $\Phi : V_{G,1} \to V_{G,2}$, we can construct the map $f = f_\Phi$ as above on $\widehat{V}_G^{\mathrm{cat}} \times_G \widehat{V}_G^{\mathrm{cat}}$. So, cleavages Σ_1 and Σ_2 of $V_{G,1}$ and $V_{G,2}$, respectively, define a map

$$\mu_g := f(\Sigma_{g,1}, \Sigma_{g,2}) : (V_{M,1})_{\mathbf{s}g} \to (C_{M,2})_{\mathbf{t}g}.$$

In the first multiplicativity condition involving f (see Definition 7.6) we can replace f with μ, but the second condition is true only up to the commutator $[R_2(g,g'), \Phi]$:[66]

$$\mu_{gg'}v = \mu_g R_1(g')v + R_1(g)\mu_h v + R_2(g,g')\Phi v - \Phi R_2(g,g')v.$$

This precisely means the pair (Φ, μ) defines a ruth map, and, as above in (7.2), the equation

$$\Phi^{V_G}(r_g c + \Sigma_{g,1}v) = r_g(\Phi^{C_M}(c) - \mu_g v) + \Sigma_{g,2}\Phi^{V_M} v$$

sets a one-to-one correspondence between VB-groupoid maps and ruth maps.

The notions of morphisms for fat extensions and for fat category extensions are formally identical. Therefore, it is readily verified that we also obtain:

Theorem 7.15. *The functor*

$$\{\text{Fat category extensions}\} \to \{\text{Fat extensions}\} \qquad F_G^{\mathrm{cat}} \to F_G \quad f \mapsto f|_{F_G \times F_G}$$

sets an equivalence of categories.

Remark 7.16. Similarly, we can establish equivalences of the form

$$\{\text{Fat extensions}\} \to \{\text{VB-groupoids}\} \qquad F_G \mapsto V(F_G)$$
$$\{\text{Fat extensions}\} \to \{\text{Abstract 2-term ruths}\} \qquad F_G \mapsto C_{\mathrm{inv}}(F_G; C_M)$$
$$\{\text{Split 2-term ruths}\} \to \{\text{Split fat extensions}\} \qquad \mathcal{E}_G \mapsto \widehat{\mathcal{E}}_G$$

but we will not go any further into these details here.

7.2. **Morphisms of general linear PB-groupoids.** To understand maps between general linear PB-groupoids, we first go through the equivalence of categories between ordinary vector bundles and ordinary general linear principal bundles. Doing so, we will already come across some of the most essential ingredients to understand the more general case.

7.2.1. *Morphisms of general linear principal bundles.* Let $E_1 \to M$ be a vector bundle of rank k_1, $E_2 \to M$ be a vector bundle of rank k_2, and consider a vector bundle map $E_1 \to E_2$ over M. If we then consider two frames over the same basepoint $x \in M$, one of E_1 and one of E_2, we obtain the following diagram

$$\begin{tikzcd} \mathbb{R}^{k_1} \arrow[r, "\sim"] \arrow[d, dashed] & E_{1,x} \arrow[d] \\ \mathbb{R}^{k_2} \arrow[r, "\sim"] & E_{2,x} \end{tikzcd}$$

So, a linear map $E_1 \to E_2$ produces a map

$$f_\Phi : \mathrm{Fr}\, E_1 \times_M \mathrm{Fr}\, E_2 \to \mathrm{Hom}(\mathbb{R}^{k_1}, \mathbb{R}^{k_2}) \qquad f_\Phi(\phi_1, \phi_2) := \phi_2^{-1}\Phi\phi_1.$$

This map has an additional special property, which leads to the following definition.

[66] We simply wrote the associated split ruths of $V_{G,1}$ and $V_{G,2}$ using the same notation.

Definition 7.17. A *morphism of principal* GL*-bundles* is a map

$$f : P_1 \times_M P_2 \to \mathrm{Hom}(\mathbb{R}^{k_1}, \mathbb{R}^{k_2})$$

that is GL-equivariant: given $p_1 \in P_1$, $p_2 \in P_2$, $\Psi_1 \in \mathrm{GL}(k_1)$ and $\Psi_2 \in \mathrm{GL}(k_2)$, we have

$$f(p_1\Psi_1, p_2\Psi_2) = \Psi_2^{-1} f(p_1, p_2)\Psi_1.$$

Moreover, composition $f_{31} = f_{32} \circ f_{21} : P_1 \to P_3$ of morphisms $f_{21} : P_1 \to P_2$ and $f_{32} : P_2 \to P_3$ is given by

$$f_{31}(p_1, p_3) = f_{32}(p_2, p_3) \circ f_{21}(p_1, p_2)$$

where $p_2 \in P_2$ is arbitrary.

The term "equivariant" is justified for $\mathrm{GL}(k_1)$ and $\mathrm{GL}(k_2)$ act canonically on $\mathrm{Fr}\, E_1 \times_M \mathrm{Fr}\, E_2$ and also on $\mathrm{Hom}(\mathbb{R}^{k_1}, \mathbb{R}^{k_2})$. The GL-equivariance condition can then literally be understood as the equivariance with respect to these actions.

Remark 7.18. The identity map $P \to P$ is the map

$$P \times_M P \to \mathrm{Hom}(\mathbb{R}^k, \mathbb{R}^k)$$

which assigns to $(p_1, p_2) \in P \times_M P$ the unique $\Psi \in \mathrm{GL}(k)$ with the property that $p_1 = p_2\Psi$. In particular, f is an isomorphism if and only if f maps into $\mathrm{GL}(k)$, for then

$$f^{-1}(p_2, p_1) = f(p_1, p_2)^{-1}$$

where the inverse on the right hand side is taken in $\mathrm{GL}(k)$. In this case, the map sending $p_1 \in P_1$ to the unique $p_2 \in P_2$ for which

$$f(p_1, p_2) = 1$$

is an equivariant isomorphism $P_1 \xrightarrow{\sim} P_2$. Conversely, all such equivariant isomorphisms arise in this way.

With the correct notion of morphism of general linear principal bundles at hand, we now prove:

Proposition 7.19. *The assignment*

$$\{\text{Vector bundles}\} \to \{\text{Principal GL-bundles}\} \qquad E \mapsto \mathrm{Fr}\, E \qquad \Phi \mapsto f_\Phi$$

sets an equivalence of categories.

Proof. Suppose we are given a map

$$\mathrm{Fr}\, E_1 \times_M \mathrm{Fr}\, E_2 \to \mathrm{Hom}(\mathbb{R}^{k_1}, \mathbb{R}^{k_2})$$

that is GL-equivariant. We can then define a linear map $E_{1,x} \to E_{2,x}$ by choosing any two frames:

$$\begin{tikzcd} \mathbb{R}^{k_1} \arrow[r, "\sim"] \arrow[d] & E_{1,x} \arrow[d, dashed] \\ \mathbb{R}^{k_2} \arrow[r, "\sim"] & E_{2,x} \end{tikzcd}$$

By the equivariance, this map is independent of the chosen frames. That we set a true one-to-one correspondence now is readily verified. ◻

We now move to understanding morphisms of general linear PB-groupoids. Although a natural guess arises from the above discussion, to avoid mistakes, it is especially useful to simply go through the above one-to-one correspondence in the more general setup. We start with setting the equivalence between general linear PB-groupoids and VB-groupoid as it seems the most appropriate following the above discussion. However, as we will see, the equivalence on the level of morphisms between general linear PB-groupoids and fat extensions is set in a similar way.

7.2.2. *Morphisms of general linear PB-groupoids.* Recall that the total space of the associated PB-groupoid of a VB-groupoid V_G is the following space of frames:

$$\mathrm{GL}_{\mathbf{s}}V_G = \{\phi_g : (C_M)_x \oplus (V_M)_x \xrightarrow{\sim} (V_G)_g \mid \mathbf{s}_g\phi_g|_{(C_M)_x} = 0, \mathbf{t}_g\phi_g|_{(V_M)_x}, \mathbf{s}_g\phi_g|_{(V_M)_x} \in \mathrm{GL}(V_M)\}$$

Given a VB-groupoid map $V_{G,1} \to V_{G,2}$ over G, and picking one such frame ϕ_1 for $\mathrm{GL}_{\mathbf{s}}V_{G,1}$ and ϕ_2 for $\mathrm{GL}_{\mathbf{s}}V_{G,2}$, we obtain the following diagram

$$\begin{tikzcd} (C_{M,1})_{x_1} \oplus (V_{M,1})_{x_1} \arrow[r, "\sim"] \arrow[d, dashed] & (V_{G,1})_g \arrow[d] \\ (C_{M,2})_{x_2} \oplus (V_{M,2})_{x_2} \arrow[r, "\sim"] & (V_{G,2})_g \end{tikzcd}$$

Before we move on, observe that the resulting map

$$(C_{M,1})_{x_1} \oplus (V_{M,1})_{x_1} \to (C_{M,2})_{x_2} \oplus (V_{M,2})_{x_2}$$

is not an element of $\mathrm{Hom}(C_{M,1} \oplus V_{M,1}, C_{M,2} \oplus V_{M,2})$, but rather of

$$\mathrm{Lin}(C_{M,1} \oplus V_{M,1}, C_{M,2} \oplus V_{M,2})$$

where, as in Section 6.3, for E_1 and E_2 vector bundles over M,

$$\mathrm{Lin}(E_1, E_2) := \{E_{1,x} \to E_{2,y} \mid x, y \in M\}$$

denotes the vector bundle over $M \times M$ of linear maps between fibers of E_1 and E_2. We often denote elements of $\mathrm{Lin}(C_{M,1} \oplus V_{M,1}, C_{M,2} \oplus V_{M,2})$ by blockmatrices, and use

$$\mathrm{U}(C_{M,1} \oplus V_{M,1}, C_{M,2} \oplus V_{M,2})$$

to mean upper triangular blockmatrices. The reason for introducing this subbundle is because the maps from above are of this form, for $\mathbf{s}\phi_2 = 0$. We write the components of the above maps now (suggestively) as follows:

$$\begin{pmatrix} \Phi(\phi_1, \phi_2) & f_\Phi(\phi_1, \phi_2) \\ 0 & \Phi(\phi_1, \phi_2) \end{pmatrix} (C_{M,1})_{x_1} \oplus (V_{M,1})_{x_1} \to (C_{M,2})_{x_2} \oplus (V_{M,2})_{x_2}.$$

Up to conjugation by frames, $\Phi(\phi_1, \phi_2)$ is precisely the cochain map from $C_{M,1} \to V_{M,1}$ to $C_{M,2} \to V_{M,2}$. In analogy with Proposition 7.19, this does not come as a surprise. Rests to consider the map $f_\Phi : (V_{M,1})_{x_1} \to (C_{M,2})_{x_2}$. To understand its essential property, we compute:

Proposition 7.20. *Let $\Phi : V_{G,1} \to V_{G,2}$ be a VB-groupoid map, and consider $\Phi(\phi_1, \phi_2)$ and $f_\Phi(\phi_1, \phi_2)$ defined as above. Then, given $\Psi_1 \in \mathrm{GL}(C_{M,1}, V_{M,1})_M$ composable with $\phi_1 \in \mathrm{GL}(C_M, V_M)$ and $\Psi_2 \in \mathrm{GL}(C_{M,2}, V_{M,2})_M$ composable with $\phi_2 \in \mathrm{GL}(C_M, V_M)$, we have*

$$\Phi(\phi_1 \cdot \Psi_1, \phi_2 \cdot \Psi_2) = \Psi_2^{-1}\Phi(\phi_1, \phi_2)\Psi_1, \tag{7.3}$$

and, given $(h_1, \Psi_1) \in \mathrm{GL}(C_{M,1}, V_{M,1})_G$ *composable with* $\phi_1 \in P_G$ *and* $(h_2, \Psi_2) \in \mathrm{GL}(C_{M,2}, V_{M,2})_G$ *composable with* $\phi_2 \in P_G$*, we have*[67]

$$\begin{aligned} &f_\Phi(\phi_1 \cdot (h_1, \Psi_1), \phi_2 \cdot (h_2, \Psi_2)) = \\ &= (\Psi_2^{C_M})^{-1}(1 + h_2\partial_2)^{-1}(f_\Phi(\phi_1, \phi_2) + \Phi^{C_M}(\phi_1, \phi_2)h_1 - h_2\Phi^{V_M}(\phi_1, \phi_2))\Psi_1^{V_M}. \end{aligned} \tag{7.4}$$

Proof. We prove (7.4). Using the matrix notation we developed, the map

$$\begin{tikzcd} (C_{M,1})_{x_1} \oplus (V_{M,1})_{x_1} \arrow[r, "\sim"] \arrow[d, dashed] & (V_{G,1})_g \arrow[d] \\ (C_{M,2})_{x_2} \oplus (V_{M,2})_{x_2} \arrow[r, "\sim"] & (V_{G,2})_g \end{tikzcd}$$

that we considered above is given by

$$\begin{pmatrix} \Phi^{C_M}(\phi_1, \phi_2) & f_\Phi(\phi_1, \phi_2) \\ 0 & \Phi^{V_M}(\phi_1, \phi_2) \end{pmatrix}.$$

The left hand side of (7.4) corresponds to the diagram

$$\begin{tikzcd} (C_{M,1})_{y_1} \oplus (V_{M,1})_{y_1} \arrow[r, "\sim"] \arrow[d, dashed] & (C_{M,1})_{x_1} \oplus (V_{M,1})_{x_1} \arrow[r, "\sim"] & (V_{G,1})_g \arrow[d] \\ (C_{M,2})_{y_2} \oplus (V_{M,2})_{y_2} \arrow[r, "\sim"] & (C_{M,2})_{x_2} \oplus (V_{M,2})_{x_2} \arrow[r, "\sim"] & (V_{G,2})_g \end{tikzcd}$$

which is the map

$$\begin{pmatrix} \Phi^{C_M}(\phi_1 \cdot (h_1, \partial_1, \Psi_1), \phi_2 \cdot (h_2, \partial_2, \Psi_2)) & f_\Phi(\phi_1 \cdot (h_1, \partial_1, \Psi_1), \phi_2 \cdot (h_2, \partial_2, \Psi_2)) \\ 0 & \Phi^{V_M}(\phi_1 \cdot (h_1, \partial_1, \Psi_1), \phi_2 \cdot (h_2, \partial_2, \Psi_2)) \end{pmatrix}.$$

But, on the other hand, the diagram also represents the composition

$$\begin{pmatrix} (1 + h_2\partial_2)\Psi_2^{C_M} & h_2\Psi_2^{V_M} \\ 0 & \Psi_2^{V_M} \end{pmatrix}^{-1} \begin{pmatrix} \Phi^{C_M}(\phi_1, \phi_2) & f_\Phi(\phi_1, \phi_2) \\ 0 & \Phi^{V_M}(\phi_1, \phi_2) \end{pmatrix} \begin{pmatrix} (1 + h_1\partial_1)\Psi_1^{C_M} & h_1\Psi_1^{V_M} \\ 0 & \Psi_1^{V_M} \end{pmatrix}.$$

This proves the statement. □

Observe that we can view the last matrix multiplication in the above proof as describing two right 2-actions, one of $\mathrm{GL}(C_{M,1}, V_{M,1})_G$ and one of $\mathrm{GL}(C_{M,2}, V_{M,2})_G$ (see Remark 5.7). That is, these 2-actions of $\mathrm{GL}(C_{M,1}, V_{M,1})_G$ and of $\mathrm{GL}(C_{M,2}, V_{M,2})_G$ — the two matrix multiplications — are well-defined 2-actions on $\mathrm{U}(C_{M,1} \oplus V_{M,1}, C_{M,2} \oplus V_{M,2})$. The above property of such maps

$$\mathrm{GL}_{\mathbf{s}}V_{G,1} \times_G \mathrm{GL}_{\mathbf{s}}V_{G,2} \to \mathrm{U}(C_{M,1} \oplus V_{M,1}, C_{M,2} \oplus V_{M,2})$$

can then literally be interpreted as GL-equivariance.

Definition 7.21. A *morphism of general linear PB-groupoids* is a map

$$\begin{pmatrix} \Phi^{C_M} & f_\Phi \\ 0 & \Phi^{V_M} \end{pmatrix} : P_{G,1} \times_G P_{G,2} \to \mathrm{U}(C_{M,1} \oplus V_{M,1}, C_{M,2} \oplus V_{M,2})$$

[67]The differentials ∂_1 and ∂_2 that appear are the suppressed differentials in the elements (h_1, Ψ_1) and (h_2, Ψ_2).

that is GL-equivariant; that is, given $(h_1, \Psi_1) \in \mathrm{GL}(C_{M,1}, V_{M,1})_G$ composable with $p_1 \in P_G$ and $(h_2, \Psi_2) \in \mathrm{GL}(C_{M,2}, V_{M,2})_G$ composable with $p_2 \in P_G$, we have

$$\begin{pmatrix} \Phi^{C_M}(p_1 \cdot (h_1,\Psi_1), p_2 \cdot (h_2,\Psi_2)) & f_\Phi(p_1 \cdot (h_1,\Psi_1), p_2 \cdot (h_2,\Psi_2)) \\ 0 & \Phi^{V_M}(p_1 \cdot (h_1,\Psi_1), p_2 \cdot (h_2,\Psi_2)) \end{pmatrix} =$$

$$= \begin{pmatrix} (1+h_2\partial_2)\Psi_2^{C_M} & h_2\Psi_2^{V_M} \\ 0 & \Psi_2^{V_M} \end{pmatrix}^{-1} \begin{pmatrix} \Phi^{C_M}(p_1,p_2) & f_\Phi(p_1,p_2) \\ 0 & \Phi^{V_M}(p_1,p_2) \end{pmatrix} \begin{pmatrix} (1+h_1\partial_1)\Psi_1^{C_M} & h_1\Psi_1^{V_M} \\ 0 & \Psi_1^{V_M} \end{pmatrix}.$$

Starting with a pair (Φ, f_Φ) that is GL-equivariant, we can define $\Phi : V_{G,1} \to V_{G,2}$ by picking, for all $g \in G$, any two frames and using the diagram

$$\begin{tikzcd} (C_{M,1})_{x_1} \oplus (V_{M,1})_{x_1} \arrow[r, "\sim"] \arrow[d] & (V_{G,1})_g \arrow[d, dashed] \\ (C_{M,2})_{x_2} \oplus (V_{M,2})_{x_2} \arrow[r, "\sim"] & (V_{G,2})_g \end{tikzcd}$$

that (Φ, f_Φ) induces. The resulting map $\Phi_g : (V_{G,1})_g \to (V_{G,2})_g$ is independent of chosen frames by GL-equivariance. So, similar to Proposition 7.19, we have:

Theorem 7.22. *The assignment*

$$\{\text{VB-groupoids}\} \to \{\text{General linear PB-groupoids}\} \qquad V_G \mapsto \mathrm{GL}_{\mathbf{s}} V_G \qquad \Phi^{V_G} \mapsto \begin{pmatrix} \Phi^{C_M} & f_\Phi \\ 0 & \Phi^{V_M} \end{pmatrix}$$

sets an equivalence of categories.

Proof. In this section we established the (fully faithful) functoriality, and in Section 5 (see Section 5.6) we showed the essential surjectivity (see also [CG26]). □

Remark 7.23. Analogous to Remark 7.18, the identity map $P_G \to P_G$ of a PB-groupoid P_G over $C_M \to V_M$ is the map

$$P_G \times_G P_G \to \mathrm{U}(C_M \oplus V_M, C_M \oplus V_M)$$

which assigns to $(p_1, p_2) \in P_G \times_G P_G$ the unique (matrix representation of) $(h, \Psi) \in \mathrm{GL}(C_M, V_M)_G$ with the property that $p_1 = p_2 \cdot (h, \Psi)$. In particular, f is an isomorphism if and only if f maps into $\mathrm{GL}(C_M \oplus V_M)$, for then

$$f^{-1}(p_2, p_1) = f(p_1, p_2)^{-1}$$

where the inverse on the right hand side is taken in $\mathrm{GL}(C_M \oplus V_M)$. In this case, the map sending $p_1 \in P_{G,1}$ to the unique $p_2 \in P_{G,2}$ for which

$$f(p_1, p_2) = 1$$

is an equivariant isomorphism $P_1 \xrightarrow{\sim} P_2$. Conversely, all such equivariant isomorphisms arise in this way.

7.2.3. *Morphisms of general linear PB-groupoids using fat extensions.* Given a map of general linear PB-groupoids

$$\begin{pmatrix} \Phi_{\mathrm{PB}}^{C_M} & f_{\mathrm{PB}} \\ 0 & \Phi_{\mathrm{PB}}^{V_M} \end{pmatrix} : P_{G,1} \times_G P_{G,2} \to \mathrm{U}(C_{M,1} \oplus V_{M,1}, C_{M,2} \oplus V_{M,2})$$

the cochain map Φ from $C_{M,1} \to V_{M,1}$ to $C_{M,2} \to V_{M,2}$ over $x \in M$ can simply be recovered as

$$\Phi_x := \Phi_{\mathrm{PB}}(1_{1_x}, 1_{1_x}).$$

But we can also pick any other elements $\Psi_1 \in \mathrm{GL}(C_{M,1}, V_{M,1})$ and $\Psi_2 \in \mathrm{GL}(C_{M,2}, V_{M,2})$ over x, for then we have

$$\Phi_x = \Psi_2 \Phi_{\mathrm{PB}}(\Psi_1, \Psi_2)\Psi_1^{-1}.$$

by GL-equivariance.[68] Reading this equation the other way around, given Φ, we can generate the map Φ_{PB} from the cochain map Φ by imposing GL-equivariance.

Now, if we realise $P_{G,1}$ and $P_{G,2}$ through their associated fat extensions; that is, writing

$$P_{G,1} = F_{G,1} \ltimes \mathrm{GL}(C_{M,1}, V_{M,1}) \qquad P_{G,2} = F_{G,2} \ltimes \mathrm{GL}(C_{M,2}, V_{M,2}),$$

then we can similarly recover the map of fat extensions as

$$f_{\mathrm{fat}}(H_{g,1}, H_{g,2}) \coloneqq f_{\mathrm{PB}}(H_{g,1}, 1_{\mathbf{s}g}, H_{g,2}, 1_{\mathbf{s}g}).$$

A straightforward computation shows that f_{fat} is indeed a map of fat extensions. On the other hand, we can generate the map of general linear PB-groupoids f_{PB} from the map of fat extensions f_{fat} similar to how we generated Φ_{PB} from Φ. We therefore obtain:

Theorem 7.24. *The functor*

$$\{\text{Fat extensions}\} \to \{\text{General linear PB-groupoids}\} \qquad F_G \mapsto F_G \ltimes \mathrm{GL}(C_M, V_M) \qquad f_{\mathrm{fat}} \mapsto f_{\mathrm{PB}}$$

sets an equivalence of categories.

Proof. In this section we established the (fully faithful) functoriality, and in Section 5 (see Sections 5.4 and 5.5) we showed the essential surjectivity (see also [CG26]). ◻

Remark 7.25. It might be worthwhile to realise that, when before we used VB-groupoids, we built the map of PB-groupoids using the diagram

$$\begin{tikzcd}
(C_{M,1})_{x_1} \oplus (V_{M,1})_{x_1} \arrow[r, "\sim"] \arrow[d, dashed] & (V_{G,1})_g \arrow[d] \\
(C_{M,2})_{x_2} \oplus (V_{M,2})_{x_2} \arrow[r, "\sim"] & (V_{G,2})_g
\end{tikzcd}$$

But if we go through the composition of functors

$$\{\text{VB-groupoids}\} \to \{\text{Fat extensions}\} \to \{\text{General linear PB-groupoids}\},$$

we, instead, build the map of PB-groupoids as follows:

$$\begin{tikzcd}
(C_{M,1})_{x_1} \oplus (V_{M,1})_{x_1} \arrow[r, "\sim"] \arrow[d, dashed] & (C_{M,1})_{\mathbf{t}g} \oplus (V_{M,1})_{\mathbf{s}g} \arrow[r, "\sim"] \arrow[d] & (V_{G,1})_g \\
(C_{M,2})_{x_2} \oplus (V_{M,2})_{x_2} \arrow[r, "\sim"] & (C_{M,2})_{\mathbf{t}g} \oplus (V_{M,2})_{\mathbf{s}g} \arrow[r, "\sim"] & (V_{G,2})_g
\end{tikzcd}$$

where we can think of the middle vertical map as taking the right-most horizontal maps as input. The end result is the same, because the diagram

$$\begin{tikzcd}
(C_{M,1})_{\mathbf{t}g} \oplus (V_{M,1})_{\mathbf{s}g} \arrow[r, "\sim"] \arrow[d] & (V_{G,1})_g \arrow[d] \\
(C_{M,2})_{\mathbf{t}g} \oplus (V_{M,2})_{\mathbf{s}g} \arrow[r, "\sim"] & (V_{G,2})_g
\end{tikzcd}$$

[68]The identification we use here uses Proposition 5.13.

commutes. The following full commutative diagram is perhaps a nice illustration of how all three different maps fit together (from left to right: a map of PB-groupoids, a map of fat extensions and a map of VB-groupoids):

$$\begin{tikzcd}
(C_{M,1})_{x_1} \oplus (V_{M,1})_{x_1} \arrow[r, "\sim"] \arrow[d] & (C_{M,1})_{\mathbf{t}g} \oplus (V_{M,1})_{\mathbf{s}g} \arrow[r, "\sim"] \arrow[d] & (V_{G,1})_g \arrow[d] \\
(C_{M,2})_{x_2} \oplus (V_{M,2})_{x_2} \arrow[r, "\sim"] & (C_{M,2})_{\mathbf{t}g} \oplus (V_{M,2})_{\mathbf{s}g} \arrow[r, "\sim"] & (V_{G,2})_g
\end{tikzcd}$$

The first vertical map (the map of PB-groupoids) takes as input the two horizontal maps (in both rows), the second vertical map (the map of fat extensions) takes as input the right horizontal maps (in both rows), and the last vertical map is simply defined without extra input.

8. The infinitesimal picture

We now turn to the infinitesimal theory of what we have discussed until now. As already observed in [GSM10], there is a fat algebroid $\widehat{V}_A$ associated to a VB-algebroid V_A. It carries canonical representations on C_M and V_M, and it is shown in [GSM10] how one obtains the (split) ruth associated to V_A using these representations and a splitting of a canonical exact sequence:

$$0 \longrightarrow \mathrm{Hom}(V_M, C_M) \longrightarrow \widehat{V}_A \longrightarrow A \longrightarrow 0.$$

In this section we elaborate on these remarks, explaining the infinitesimal counterparts of the theory we have seen in the above sections. We keep the discussion shorter, and a discussion on "PB-algebroids" (a concept which is not defined in the literature yet) is not included. We put an emphasis on describing the correspondences between VB-algebroids, 2-term ruths, and the newly defined fat extensions. In the section after, Section 9, we explain the process of differentiation.

8.1. **Notation for VB-algebroids.** See, for example, [BCdH16, CD17] for more details on the theory of VB-algebroids.

We denote the structure maps of a Lie algebroid by $([\cdot,\cdot],\rho)$. Writing $\Omega^\bullet A = \Gamma \bigwedge A^*$, we write

$$d : \Omega^\bullet A \to \Omega^{\bullet+1} A$$

for the algebroid differential (the de Rham differential using $[\cdot,\cdot]$ and $\mathcal{L} = \mathcal{L}_\rho$). Then ΩA becomes a (commutative) differential graded algebra with d and the wedge product.

For $A \Rightarrow M$ a Lie algebroid, we denote a VB-algebroid over A by $V_A \Rightarrow V_M$.

Remark 8.1. We think of a VB-algebroid as a diagram

$$\begin{tikzcd}
V_A \arrow[r, Rightarrow] \arrow[d] & V_M \arrow[d] \\
A \arrow[r, Rightarrow] & M
\end{tikzcd}$$

so that we think of the algebroid structures as being "horizontal", while the other vector bundle structures (on V_A and V_M) are "vertical". In line with this convention, we write the projection maps $V_A \to V_M$ and $A \to M$ by $\mathrm{pr}_\to$, and for the projection maps $V_A \to A$ and $V_M \to M$ we write $\mathrm{pr}_\downarrow$. Moreover, we write

$$\Gamma_\to V_A = \Gamma(V_M, V_A) \text{ and } \Gamma_\downarrow V_A = \Gamma(A, V_A)$$

for the spaces of sections, and similarly $\Omega_\rightarrow V_A$ and $\Omega_\downarrow V_A$ for the respective forms, and we write the scalar multiplications and additions of $V_A \to A$ and $V_M \to M$ by $h^\lambda_\downarrow$ and $+_\downarrow$, respectively, and the scalar multiplications and additions of $V_A \to V_M$ and $A \to M$ by $h^\lambda_\rightarrow$ and $+_\rightarrow$. However, we often simplify to

$$h^\lambda_\downarrow v = \lambda_\downarrow v \qquad h^\lambda_\rightarrow v = \lambda_\rightarrow v$$

In particular, $\lambda_\downarrow$ is infinitesimally multiplicative:

$$\lambda^*_\downarrow [v, w] = [\lambda^*_\downarrow v, \lambda^*_\downarrow w]$$

(with $\lambda^*_\downarrow v = \lambda^{-1}_\downarrow \circ v \circ \lambda_\downarrow$). That is, the scalar multiplication $\lambda_\downarrow$ by $\lambda \in \mathbb{R}_{>0}$ is an algebroid automorphism $V_A \to V_A$ [BCdH16]. On the other hand, the scalar multiplication $\lambda_\rightarrow$ by $\lambda \in \mathbb{R}_{>0}$ is a vector bundle isomorphism $V_A \to V_A$ with respect to the vertical vector bundle structures. Recall that on the core

$$C_M = \ker \mathrm{pr}_\rightarrow \cap \ker \mathrm{pr}_\downarrow \to M$$

of V_A the two vector bundle structures coincide, i.e. for all $c \in C_M$, $\lambda_\downarrow c = \lambda_\rightarrow c$.

Since $\lambda_\downarrow$ is an algebroid map, the 1-homogeneous cochains (with respect to $\lambda_\downarrow$)

$$\Omega^\bullet_{\mathrm{lin}} V_A = \{\omega \in \Omega^\bullet_\rightarrow V_A \mid \lambda^*_\downarrow \omega = \lambda \cdot \omega\} \subset \Omega^\bullet_\rightarrow V_A \tag{8.1}$$

form a differential graded submodule of ΩV_A called the linear (sub)complex. More generally, a form $\omega \in \Omega_\rightarrow V_A$ is called k-homogeneous if

$$\lambda^*_\downarrow \omega = \lambda^k \omega$$

and the k-homogeneous forms form a differential graded submodule of ΩV_A.

Similarly, given a section $\alpha \in \Gamma_\rightarrow V_A$, it is k-homogeneous if[69]

$$\lambda^*_\downarrow \alpha (= \lambda^{-1}_\downarrow \circ \alpha \circ \lambda_\downarrow) = \lambda^k_\rightarrow \alpha$$

and we write $\Gamma_{\mathrm{lin}} V_A$ for the linear (i.e. 0-homogeneous) sections.

Remark 8.2. A word on terminology and notation is perhaps on its place. Given a function $f \in C^\infty V_A$, we have two notions of f being k-homogeneous, for it can be homogeneous "horizontally" or "vertically":

$$\lambda^*_\rightarrow f = \lambda^k f \qquad \lambda^*_\downarrow f = \lambda^k f.$$

We therefore write

$$C^\infty_{\rightarrow\mathrm{lin}} V_A \qquad C^\infty_{\downarrow\mathrm{lin}} V_A$$

for horizontally linear (i.e. 1-homogeneous) functions and vertically linear functions respectively.

Notice now that, since[70]

$$\Omega^1_\rightarrow V_A = \Gamma V_A^{*V_M} = C^\infty_{\rightarrow\mathrm{lin}} V_A$$

we can think of elements ω of $\Omega^1_\rightarrow V_A$ as linear functions on V_A with respect to the horizontal vector bundle structure. If $\omega \in \Omega^1_{\mathrm{lin}} V_A$, then the induced function of V_A is also linear with respect to the

[69] Notice that the appearance of $\lambda^k_\rightarrow$ on the right hand side is similar to the appearance of λ^k on the right hand side of the equation of homogeneity of functions on V_A. More precisely, $V_A \times \mathbb{R} \Rightarrow V_A$ can be seen as a VB-algebroid over 0_{V_M} or 0_A, and k-homogeneous sections of these VB-algebroids are exactly the $k+1$-homogeneous functions.

[70] The notation $V_A^{*V_M}$ is used to indicate that we dualise V_A with respect to the vector bundle structure $V_A \to V_M$.

vertical vector bundle structure. This means that ω defines an element of $\Omega^1_\downarrow V_A$ as well. For general $\omega \in \Omega^\bullet_{\mathrm{lin}} V_A$, we have multilinearity "horizontally" and linearity "vertically" in the sense that

$$\omega(\lambda_\downarrow v_1 +_\downarrow \mu_\downarrow w_1, \ldots, \lambda_\downarrow v_\bullet +_\downarrow \mu_\downarrow w_\bullet) = \lambda\omega(v_1, \ldots, v_\bullet) + \mu\omega(w_1, \ldots, w_\bullet).$$

That is, "linearity" refers here to the vertical vector bundle structure, and it would, in line with our convention, be sensible to write $\Omega_{\mathrm{lin}} V_A$ as $\Omega_{\to,\downarrow\mathrm{lin}} V_A$. We avoid this cumbersome notation as it is clear from the context that, in this case, we are interested in forms on the algebroid $V_A \Rightarrow V_M$ that are linear with respect to the compatible vector bundle structures on $V_A \to A$ and $V_M \to M$.

In local coordinates it becomes clear that $\Gamma_\to V_A$ is generated, as a $C^\infty V_M$-module, by the (-1)-homogeneous sections, called *core sections*, and the 0-homogeneous sections, called *linear sections*.

Remark 8.3. It is convenient to think about linear (or more generally homogeneous) forms as follows. The space of homogeneous sections in $\Gamma_\to V_A$ are naturally graded. We can think of a form $\omega \in \Omega_{\mathrm{lin}} V_A$ therefore as a map

$$\omega : \Gamma_\to V_A \otimes_{C^\infty V_M} \cdots \otimes_{C^\infty V_M} \Gamma_\to V_A \to C^\infty V_M$$

that additionally restricts to a map of degree $+1$ between homogeneous elements (considering the total degree on the left hand side). Indeed,

$$\omega(\alpha_\ell) \circ \lambda_\downarrow = (\lambda^*_\downarrow \omega)(\lambda^*_\downarrow \alpha_\ell)$$

so the functions $\omega(\alpha_\ell)$, for k_ℓ-homogeneous $\alpha_\ell \in \Gamma_\to V_A$, are $k + \sum_\ell k_\ell$-homogeneous if and only if ω is k-homogeneous.

8.2. **From VB-algebroids to fat extensions.** We begin with a continuation of the previous section.

8.2.1. *The appearance of the fat algebroid.* From Remark 8.3, we can conclude that $\omega \in \Omega_{\mathrm{lin}} V_A$ is completely determined by a map of the form

$$\omega_0 : \Gamma_{\mathrm{lin}} V_A \times \cdots \times \Gamma_{\mathrm{lin}} V_A \to C^\infty_{\mathrm{lin}} V_M.$$

Now, $\Gamma_{\mathrm{lin}} V_A$ is the space of sections of a Lie algebroid $\widehat{V}_A$ over M, called the fat algebroid (see [GSM10]). This algebroid fits into a short exact sequence (of Lie algebroids)

$$0 \longrightarrow \mathrm{Hom}(V_M, C_M) \longrightarrow \widehat{V}_A \longrightarrow A \longrightarrow 0$$

where the projection $\widehat{V}_A \to A$ is induced by the projection $V_A \to A$ (every linear section is projectable to a section of A), and the inclusion is given by

$$\mathrm{Hom}(V_M, C_M) \to \widehat{V}_A \qquad h_x \mapsto (v \mapsto h_x(v) +_\downarrow 0_\to(v)).$$

As a vector bundle, the fat algebroid is given by

$$\widehat{V}_A = \{H_a : (V_M)_{\mathrm{pr}_\to a} \to (V_A)_a \mid \mathrm{pr}_\to H_a = 1\}.$$

The anchor map is induced by the projection:

$$\widehat{\rho} H_a = \rho a$$

and the Lie bracket is induced by the bracket of V_A (the bracket of linear sections is linear).

Remark 8.4. In [GSM10] the fat algebroid is written as $\hat{A}$. However, we prefer to keep track of the dependence on V_A.

The Lie bracket of the bundle of Lie algebras $\mathrm{Hom}(V_M, C_M)$ is given by

$$[h, h'] := h'\partial h - h\partial h'.$$

As is clear from its definition, it is defined for every 2-term complex $C_M \to V_M$.

Definition 8.5. Let

$$C_M \xrightarrow{\partial} V_M$$

be a 2-term complex. We call the bundle of Lie algebras $\mathrm{Hom}(V_M, C_M)$ the *bundle of infinitesimal homotopies* of the 2-term complex $C_M \to V_M$.

Remark 8.6. The bundle of infinitesimal homotopies $\mathrm{Hom}(V_M, C_M)$ comes canonically with a cochain complex representation on $C_M \to V_M$: given $\gamma \in \Gamma C_M$ and $v \in \Gamma V_M$, we set

$$\nabla_h \gamma := -h\partial\gamma \qquad \nabla_h v := -\partial h v.$$

We see that, if $\omega \in \Omega^\bullet_{\mathrm{lin}} V_A$, then

$$\omega_0 : \Gamma\widehat{V}_A \times \overset{\bullet}{\cdots} \times \Gamma\widehat{V}_A \to \Gamma V_M^* \qquad \omega_0(\alpha_1, \ldots, \alpha_\bullet) := \iota_{\alpha_\bullet} \cdots \iota_{\alpha_1}\omega$$

and

$$\omega_1 : \Gamma\widehat{V}_A \times \overset{\bullet-1}{\cdots} \times \Gamma\widehat{V}_A \to \Gamma C_M^* \qquad \omega_1(\alpha_1, \cdots, \alpha_{\bullet-1})\gamma := \iota_\gamma\iota_{\alpha_{\bullet-1}} \cdots \iota_{\alpha_1}\omega$$

together satisfy the following property: for all $h \in \mathrm{Hom}(V_M, C_M)$ we have

$$\iota_h\omega_0 = \omega_1 h.$$

Notice that, in the definition $\omega_1 h$, a sign appears:

$$(\omega_1 h)(\alpha_2, \ldots, \alpha_\bullet)v := (-1)^{\bullet-1}\omega_1(\alpha_2, \ldots, \alpha_\bullet)hv$$

We call (ω_0, ω_1) "invariant" forms, and in the next sections we will explain how such invariant forms constitute a 2-term ruth $\Omega_{\mathrm{inv}}(\widehat{V}_A; V_M^*)$ that is isomorphic to $\Omega_{\mathrm{lin}} V_A$.

8.2.2. *The fat algebroid and its canonical representations.* The fat algebroid $\widehat{V}_A$ has canonical representations on C_M and V_M^* [71]

$$\nabla_\alpha := [\alpha, \cdot] : \Gamma C_M \to \Gamma C_M \qquad \nabla_\alpha := \rho_{V_A}\alpha : \Gamma V_M^* \to \Gamma V_M^* \tag{8.2}$$

and we can put them together to form a cochain complex representation on

$$\partial : C_M \to V_M.$$

Dually, we also have a cochain complex representation on

$$V_M^* \xrightarrow{\partial^*} C_M^*.$$

We think of V_M^* as sitting in degree 0 and C_M^* as sitting in degree 1. We denote the resulting differential graded ΩA-module by

$$\Omega(\widehat{V}_A; V_M^* \to C_M^*)^\bullet = \Omega^\bullet(\widehat{V}_A; V_M^*) \oplus \Omega^{\bullet-1}(\widehat{V}_A; C_M^*).$$

In analogy with Proposition 3.15, we have:

[71] We use here that $[\cdot,\cdot]$ is a degree +1 operator on homogeneous sections and, on homogeneous functions, $\rho_{V_A}(\alpha)$ has degree equal to the degree of α.

Proposition 8.7. *The cochain complex representation of $\widehat{V}_A$ restricts to the canonical cochain complex representation of* $\mathrm{Hom}(V_M, C_M)$*. Moreover, the action of $\widehat{V}_A$ on* $\mathrm{Hom}(V_M, C_M)$*, viewing* $\mathrm{Hom}(V_M, C_M)$ *as an ideal of $\widehat{V}_A$, equals the representation of $\widehat{V}_A$ on* $\mathrm{Hom}(V_M, C_M)$*.*

Proof. We prove the second statement, and we do a direct computation: locally, $\mathrm{Hom}(V_M, C_M)$ is generated by elements $f \otimes \gamma$, where $f \in \Gamma V_M^*$ and $\gamma \in \Gamma C_M$. Given $\alpha \in \Gamma\widehat{V}_A$, we then have

$$[\alpha, f \otimes \gamma] = f \otimes \nabla_\alpha \gamma + \nabla_\alpha f \otimes \gamma.$$

This proves the statement. □

Remark 8.8. Given a vector bundle E_M, algebroid representations of A on E_M correspond to algebroid maps

$$A \to \mathfrak{gl}(E_M),$$

where $\mathfrak{gl}(E_M)$ is the algebroid of derivations

$$\mathfrak{gl}(E_M) := \{D = (D, X) \mid D(fs) = fD(s) + X(f)s\} \subset \mathrm{End}\, E_M \oplus TM.$$

Given two vector bundles C_M and V_M, we set

$$\mathfrak{gl}(C_M, V_M) := \mathfrak{gl}(C_M) \times_{TM} \mathfrak{gl}(V_M).$$

Using the action algebroid

$$\mathfrak{gl}(C_M, V_M)_M := \mathrm{Hom}(C_M, V_M) \rtimes \mathfrak{gl}(C_M, V_M)$$

associated to the representation of $\mathfrak{gl}(C_M, V_M)$ on $\mathrm{Hom}(C_M, V_M)$ given by

$$\nabla_D \partial := D^{V_M}\partial - \partial D^{C_M},$$

we see that cochain complex representations of A on $C_M \to V_M$ correspond to algebroid maps

$$A \to \mathfrak{gl}(C_M, V_M)_M$$

(see also Remark 8.28).

8.2.3. *Fat extensions.* We discussed now the structure behind the fat algebroid of a VB-algebroid that makes it into a *fat extension*. Here is the abstract definition:

Definition 8.9. Let $A \Rightarrow M$ be a Lie algebroid, and consider a 2-term complex

$$C_M \xrightarrow{\partial} V_M.$$

A *fat extension* of A, with underlying complex $C_M \to V_M$, consists of a Lie algebroid

$$F_A \rightrightarrows M,$$

that comes with a cochain complex representation on $C_M \to V_M$, and a short exact sequence of Lie algebroids (over M)

$$0 \longrightarrow \mathrm{Hom}(V_M, C_M) \longrightarrow F_A \longrightarrow A \longrightarrow 0.$$

Moreover, the cochain complex representation of F_A on $C_M \to V_M$ restricts to the canonical cochain complex representation of $\mathrm{Hom}(V_M, C_M)$, and the two natural actions of F_A on $\mathrm{Hom}(V_M, C_M)$ — viewing $\mathrm{Hom}(V_M, C_M)$ as an ideal or as a representation — agree.

We will simply write F_A for a fat extension, and the underlying algebroid F_A is called the fat algebroid of the fat extension. From the above discussion, we obtain:

Proposition 8.10. *Let V_A be a VB-algebroid over A. Then, the fat algebroid $\widehat{V}_A$ of V_A, together with its cochain complex representation* (8.2)*, is a fat extension of A over $C_M \to V_M$.*

Remark 8.11. We observe here that, given a VB-algebroid V_A, the fat extension $\widehat{V}_A$ with its dual cochain complex representation should be seen as the fat extension of the dual VB-algebroid $V_A^* \Rightarrow C_M^*$. First of all, the fat groupoid $\widehat{V}_A^*$ fits into a short exact sequence

$$0 \longrightarrow \mathrm{Hom}(C_M^*, V_M^*) \longrightarrow \widehat{V}_A^* \longrightarrow A \longrightarrow 0.$$

Then, $\widehat{V}_A$ and $\widehat{V}_A^*$ are isomorphic as Lie algebroids via[72]

$$\widehat{V}_A \to \widehat{V}_A^*; \quad \alpha \mapsto \xi_\alpha(f)v = f(v -_\downarrow \alpha \mathrm{pr}_\to v).$$

Using the isomorphism

$$\mathrm{Hom}(V_M, C_M) \to \mathrm{Hom}(C_M^*, V_M^*); \quad h \mapsto -h^*,$$

the short exact sequences of $\widehat{V}_A$ and $\widehat{V}_A^*$ correspond to each other.

Notice that the structure of $V_A^* \Rightarrow C_M^*$ is revealed from the fat extension $\widehat{V}_A^*$ as follows: given $\alpha, \beta \in \Gamma\widehat{V}_A$ (and writing $\xi_\alpha, \xi_\beta \in \Gamma\widehat{V}_A^*$ as above), $\gamma \in \Gamma C_M$ and $f \in \Gamma V_M^*$,

$$\rho_{V_A^*}(\xi_\alpha)\gamma = [\alpha, \gamma] \qquad \rho_{V_A^*}(f)\gamma = \partial^*(f)\gamma \qquad [\xi_\alpha, \xi_\beta] = \xi_{[\alpha,\beta]} \qquad [\xi_\alpha, f] = \rho_{V_A}(\alpha)f.$$

Starting with V_A, these equations define the algebroid structure on V_A^*.

8.2.4. *Extended Cartan calculus.* Before we continue, we develop a type of "extended" Cartan calculus on $\Omega_{\mathrm{lin}}V_A$ and on $\Omega(\widehat{V}_A; V_M^* \to C_M^*)$. We find it particularly useful to see how $\Omega_{\mathrm{lin}}V_A$ can be described in terms of the fat algebroid $\widehat{V}_A$ using invariant cochains.

We start with the ordinary Cartan calculus on $\Omega_\to V_A$. We have three types of operators: a differential d, and then for each section $\alpha \in \Gamma V_A$ a Lie derivative $\mathcal{L}_\alpha$ and a contraction ι_α. The operators are derivations and of degree $+1$, 0 and -1, respectively. Together, they satisfy the Cartan calculus identities:

$$[d, \iota_\alpha] = \mathcal{L}_\alpha, \quad [d, \mathcal{L}_\alpha] = 0 \quad [\iota_\alpha, \iota_\beta] = 0 \quad [\mathcal{L}_\alpha, \mathcal{L}_\beta] = \mathcal{L}_{[\alpha,\beta]} \quad [\mathcal{L}_\alpha, \iota_\beta] = \iota_{[\alpha,\beta]}. \tag{8.3}$$

Similarly, on $\Omega_{\mathrm{lin}}V_A$ we have a Cartan calculus if we restrict attention to linear sections (see Remark 8.3), and see the operators as satisfying the Leibniz rule.

However, from the inclusion $\Omega_{\mathrm{lin}}V_A \subset \Omega V_A$, we can also apply contractions and Lie derivatives of (not necessarily linear) sections to linear forms ω. In particular, we can do this for core sections $\gamma \in \Gamma C_M$. Instead of thinking of $\iota_\gamma\omega$ as another form of V_A, we know from Remark 8.3 that we can see it as a form of $\widehat{V}_A$ (or even A). That is, we can define a contraction of the following form:

Definition 8.12. Let V_A be a VB-algebroid. Given $\gamma \in \Gamma C_M$, we set

$$\iota_\gamma : \Omega^\bullet_{\mathrm{lin}}V_A \to \Omega^{\bullet-1}A \subset \Omega^{\bullet-1}\widehat{V}_A \qquad \iota_\gamma\omega(\alpha_2, \ldots, \alpha_\bullet) := \omega(\gamma, \alpha_2, \ldots, \alpha_\bullet).$$

We think of this contraction as having degree -1. Notice that it is ΩA-linear. Taking this point of view, we see that there is another type of contraction:

[72] Here, say, $\alpha \in \Gamma_{\mathrm{lin}}V_A$ is related to $\alpha \in \Gamma A$ via the projection, $\xi_\alpha \in \Gamma_{\mathrm{lin}}V_A^*$ is also related to $\alpha \in \Gamma A$ via the projection, $f \in \Gamma C_M^* = C^\infty_{\mathrm{lin}}C_M$ and $v \in \Gamma_\downarrow V_A$.

Definition 8.13. Let V_A be a VB-algebroid. Given $v \in \Gamma V_M$, we set

$$\iota_v : \Omega^\bullet_{\mathrm{lin}} V_A \to \Omega^\bullet \widehat{V}_A \qquad \iota_v\omega(\alpha_1, \dots, \alpha_\bullet) := \omega(\alpha_1, \dots, \alpha_\bullet)v.$$

We think of this contraction as having degree 0. Also this map is ΩA-linear. In turn, the contractions define Lie derivatives:

Definition 8.14. Let V_A be a VB-algebroid. Given $\gamma \in C_M$ and $v \in \Gamma V_M$, we set

$$\mathcal{L}_\gamma : \Omega^\bullet_{\mathrm{lin}} V_A \to \Omega^\bullet \widehat{V}_A \qquad \mathcal{L}_v : \Omega^\bullet_{\mathrm{lin}} V_A \to \Omega^{\bullet+1} \widehat{V}_A,$$

(of degree 0 and +1, respectively) given by

$$\begin{aligned} \mathcal{L}_\gamma\omega(\alpha_1, \dots, \alpha_\bullet) &= \omega(\alpha_1, \dots, \alpha_\bullet)\partial\gamma - \textstyle\sum_k (-1)^k \omega([\alpha_k, \gamma], \alpha_1, \dots, \widehat{\alpha}_k, \dots, \alpha_\bullet) \\ \mathcal{L}_v\omega(\alpha_0, \dots, \alpha_\bullet) &= - \textstyle\sum_k (-1)^k (\rho_{V_A}(\alpha_k)(\omega(\alpha_1, \dots, \widehat{\alpha}_k, \dots, \alpha_\bullet))v \\ &\quad - \rho_A(\alpha_k)(\omega(\alpha_1, \dots, \widehat{\alpha}_k, \dots, \alpha_\bullet)v)). \end{aligned}$$

Notice that both these maps satisfy the Leibniz rule. We "extended" successfully now the Cartan calculus on $\Omega_{\mathrm{lin}} V_A$: the usual Cartan calculus identities (8.3) hold, but we should interpret equalities like $[d, \iota_\gamma] = \mathcal{L}_\gamma$ as equality of maps $\Omega_{\mathrm{lin}} V_A \to \Omega \widehat{V}_A$.[73]

Similarly, on $\Omega(\widehat{V}_A; V_M^* \to C_M^*)$ we have a Cartan calculus, for $\widehat{V}_A$ is a Lie algebroid, and V_M^* and C_M^* are representations: for $\alpha \in \Gamma\widehat{V}_A$ we have ι_α as usual and $\mathcal{L}_\alpha = \nabla_\alpha$. Notice that again it makes sense to apply contractions and Lie derivatives of sections of C_M and V_M in the following sense:

Definition 8.15. Let V_A be a VB-algebroid. Given $\gamma \in \Gamma C_M$ and $v \in \Gamma V_M$, we set

$$\begin{aligned} \iota_\gamma &: \Omega(\widehat{V}_A; V_M^* \to C_M^*)^\bullet \to \Omega^{\bullet-1}\widehat{V}_A \qquad \iota_\gamma(\omega_0, \omega_1) = \iota_\gamma\omega_1 \\ \iota_v &: \Omega(\widehat{V}_A; V_M^* \to C_M^*)^\bullet \to \Omega^{\bullet}\widehat{V}_A \qquad \iota_v(\omega_0, \omega_1) = \iota_v\omega_0. \end{aligned}$$

where

$$\begin{aligned} \iota_\gamma\omega_1(\alpha_1, \dots, \alpha_{\bullet-1}) &= (-1)^{\bullet-1}\omega_1(\alpha_1, \dots, \alpha_{\bullet-1})\gamma \\ \iota_v\omega_0(\alpha_1, \dots, \alpha_\bullet) &= \omega_0(\alpha_1, \dots, \alpha_\bullet)v. \end{aligned}$$

Moreover,

$$\mathcal{L}_\gamma : \Omega(\widehat{V}_A; V_M^* \to C_M^*)^\bullet \to \Omega^\bullet\widehat{V}_A \qquad \mathcal{L}_v : \Omega(\widehat{V}_A; V_M^* \to C_M^*)^\bullet \to \Omega^{\bullet+1}\widehat{V}_A,$$

are given, respectively, by

$$\begin{aligned} \mathcal{L}_\gamma(\omega_0, \omega_1)(\alpha_1, \dots, \alpha_\bullet) &= \omega_0(\alpha_1, \dots, \alpha_\bullet)\partial\gamma + \textstyle\sum_k (-1)^{k+\bullet}\omega_1(\alpha_1, \dots, \widehat{\alpha}_k, \dots, \alpha_\bullet)[\alpha_k, \gamma] \\ \mathcal{L}_v(\omega_0, \omega_1)(\alpha_0, \dots, \alpha_\bullet) &= - \textstyle\sum_k (-1)^k (\rho_{V_A}(\alpha_k)(\omega_0(\alpha_1, \dots, \widehat{\alpha}_k, \dots, \alpha_\bullet))v \\ &\quad - \rho(\alpha_k)(\omega_0(\alpha_1, \dots, \widehat{\alpha}_k, \dots, \alpha_\bullet)v)). \end{aligned}$$

[73]Note that ι_v is a degree 0 operator, and $\mathcal{L}_v$ is a degree +1 operator, so for example $\mathcal{L}_v = [d, \iota_v] = d\iota_v - \iota_v d$.

8.3. **From fat extensions to 2-term ruths.** We will now show that, as expected, $\Omega_{\text{lin}}V_A$ defines a subcomplex of $\Omega(\widehat{V}_A; V_M^* \to C_M^*)$. We will denote this subcomplex by $\Omega_{\text{inv}}(\widehat{V}_A; V_M^*)$ and call its elements "invariant". This way, we will introduce the abstract 2-term ruth $\Omega_{\text{inv}}(F_A; C_M)$ of an abstract fat extension F_A. As for Lie groupoids, we can interpret the invariant complex as a type of relative complex from [Sal24]. Notice that for $F_A = J^1A$, this complex appeared in [She12].

Consider an element ω of $\Omega^\bullet_{\text{lin}}V_A$. From Remark 8.3, we know that linear forms are determined by the induced map[74]

$$\omega_0 : \Gamma\widehat{V}_A \times \cdots \times \Gamma\widehat{V}_A \to \Gamma V_M^* \qquad \omega_0(\alpha_1, \ldots, \alpha_\bullet) := \iota_{\alpha_\bullet} \cdots \iota_{\alpha_1}\omega.$$

This map is $C^\infty M$-multilinear and can therefore be thought of as an element of $\Omega^\bullet(\widehat{V}_A; V_M^*)$. In fact, this defines a cochain map

$$\Omega_{\text{lin}}V_A \to \Omega(\widehat{V}_A; V_M^*) \qquad \omega \mapsto \omega_0$$

using the representation on V_M^* discussed above. Since

$$\iota_{\alpha_\bullet} \cdots \iota_{\alpha_2}\iota_h\omega(v) = \iota_{\alpha_\bullet} \cdots \iota_{\alpha_2}\iota_{hv}\omega,$$

the elements ω_0 have the property that, for all $h \in \text{Hom}(V_M, C_M)$,

$$\iota_h\omega_0 = \omega_1 h,$$

where $\omega_1 \in \Omega^{\bullet-1}(A; C_M^*)$ is given by[75]

$$\omega_1(\alpha_2, \cdots, \alpha_\bullet)\gamma := \iota_\gamma\iota_{\alpha_\bullet} \cdots \iota_{\alpha_2}\omega$$

and[76]

$$(\omega_1 h)(\alpha_2, \ldots, \alpha_\bullet)v = (-1)^{\bullet-1}\omega_1(\alpha_2, \ldots, \alpha_\bullet)h(v).$$

We can see (ω_0, ω_1) as an element of the cochain complex representation $\Omega(\widehat{V}_A, V_M^* \to C_M^*)^\bullet$. Using the Cartan calculus we developed in the previous section, Section 8.2.4, a way to write the above condition is now that, for all $h \in \text{Hom}(V_M, C_M)$ and $v \in \Gamma V_M$,

$$\iota_v\iota_h = \iota_{hv}.$$

The map $\omega \mapsto (\omega_0, \omega_1)$ is ΩA-linear, and it intertwines the contraction operators and Lie derivatives. Therefore:

Proposition 8.16. *The map*

$$\Omega_{\text{lin}}V_A \to \Omega(\widehat{V}_A; V_M^* \to C_M^*)$$

is a map differential graded ΩA-modules, and it maps bijectively onto those pairs (ω_0, ω_1) such that $\omega_1 \in \Omega^{\bullet-1}(\widehat{V}_A; C_M^*)$ is independent of its entries[77] *and* (8.3) *holds.*

---

[74] We assume for simplicity that V_M is not the zero bundle, otherwise it is determined by a form on A with values in C_M.

[75] Here, $\gamma \in \Gamma C_M$, and we use that $\iota_\gamma\iota_{\alpha_{\bullet-1}} \cdots \iota_{\alpha_1}\omega \in C^\infty M$ by Remark 8.3.

[76] To explain the sign, observe that we can think of h^* as a graded endomorphism of degree -1 of the graded vector bundle $E_M = V_M^* \oplus C_M^*[-1]$. The expression ωh can then be interpreted as $h^* \wedge \omega$, where $\wedge$ is induced by the canonical pairing $\text{End}\, E_M \otimes E_M \to E_M$ (see, for example, [AAC12]).

[77] By independence we mean that $\omega_1(\alpha_2, \ldots, \alpha_\bullet)$ only depends on the base sections underlying $\alpha_2, \ldots, \alpha_\bullet$.

Proof. The proof is a straightforward computation using the Cartan calculus we developed:

$$\begin{aligned}(d\omega)_0(\alpha_0,\dots,\alpha_\bullet) &= \textstyle\sum_k(-1)^k\mathcal{L}_{\alpha_k}\iota_{\alpha_\bullet}\cdots\widehat{\iota_{\alpha_k}}\cdots\iota_{\alpha_0}\omega\\ &\quad+\textstyle\sum_{k<\ell}(-1)^{k+\ell}\iota_{\alpha_\bullet}\cdots\widehat{\iota_{\alpha_\ell}}\cdots\widehat{\iota_{\alpha_k}}\cdots\iota_{\alpha_0}\iota_{[\alpha_k,\alpha_\ell]}\omega = d\omega_0(\alpha_0,\dots,\alpha_\bullet)\end{aligned}$$

and

$$\begin{aligned}(d\omega)_1(\alpha_1,\dots,\alpha_\bullet)\gamma &= (-1)^\bullet\omega_0(\alpha_1,\dots,\alpha_\bullet)\partial\gamma\\ &\quad-\textstyle\sum_k(-1)^k(\mathcal{L}_{\alpha_k}\iota_\gamma-\iota_{[\alpha_k,\gamma]})\iota_{\alpha_\bullet}\cdots\widehat{\iota_{\alpha_k}}\cdots\iota_{\alpha_1}\omega\\ &\quad+\textstyle\sum_{k<\ell}(-1)^{k+\ell}\iota_\gamma\iota_{\alpha_\bullet}\cdots\widehat{\iota_{\alpha_\ell}}\cdots\widehat{\iota_{\alpha_k}}\cdots\iota_{\alpha_1}\iota_{[\alpha_k,\alpha_\ell]}\omega\\ &= \partial^*(\omega_0)(\alpha_1,\dots,\alpha_\bullet)\gamma + d\omega_1(\alpha_0,\dots,\alpha_\bullet)\gamma.\end{aligned}$$

This proves that the map $\omega\mapsto(\omega_0,\omega_1)$ is a cochain map. That it maps onto those pairs (ω_0,ω_1) satisfying (8.3) is a consequence of Remark 8.3. □

We can immediately conclude the following.

Corollary 8.17. *The map*

$$\Omega_{\mathrm{lin}}V_A \to \Omega(\widehat{V}_A;V_M^*)$$

is a map of differential graded ΩA-modules. If $V_M\neq 0$, it maps bijectively onto $\Omega_{\mathrm{inv}}(\widehat{V}_A;V_M^*)$, so those forms $\omega_0\in\Omega^\bullet(\widehat{V}_A,V_M^*)$ for which there exists $\omega_1\in\Omega^{\bullet-1}(\widehat{V}_A,C_M^*)$, independent of its entries, such that (8.3) *holds.*[78]

Remark 8.18. Although we will usually write $\Omega_{\mathrm{inv}}(\widehat{V}_A;V_M^*)$, we do this for notational convenience, so $\Omega_{\mathrm{inv}}(\widehat{V}_A;V_M^*\to C_M^*)$ is meant when writing $\Omega_{\mathrm{inv}}(\widehat{V}_A;V_M^*)$.

To conclude, we formulate the abstract definition of invariant cochains in terms of fat extensions.

Definition 8.19. Let F_A be a fat extension of A over $C_M\to V_M$. The *invariant complex*

$$\Omega_{\mathrm{inv}}(F_G;C_M)\subset\Omega(F_G;C_M\to V_M)$$

consists of pairs $(\omega_0,\omega_1)\in\Omega(F_G;C_M\to V_M)$ such that ω_1 is independent of its entries, and, for all $h\in\mathrm{Hom}(V_M,C_M)$,

$$\iota_h\omega_0 = h\omega_1.$$

Remark 8.20. Notice that the above condition can be written as

$$\iota_f\iota_h = \iota_{fh}$$

for all $f\in\Gamma C_M^*$ and $h\in\mathrm{Hom}(V_C,C_M)$.

[78] If $V_M=0$, then the map

$$C^\bullet_{\mathrm{lin}}V_A\to C^{\bullet-1}(A,C_M^*);\quad f\mapsto f_1$$

is a bijective cochain map.

8.4. **Fat extensions and split 2-term ruths.** In [GSM10] it is shown how a VB-algebroid induces a split 2-term ruth on A. This goes as follows:

Remark 8.21. The choice of a splitting $\sigma : A \to \widehat{V}_A$ defines A-connections $\nabla_\alpha = \nabla_{\sigma\alpha}$ on V_M^* and C_M^*. These connections are usually not flat, but they are up to the homotopy $R_2 \in \Omega^2(A; \mathrm{Hom}(C_M^*, V_M^*) = \mathrm{Hom}(V_M, C_M))$ given by

$$R_2(\alpha, \beta) = \sigma[\alpha, \beta] - [\sigma\alpha, \sigma\beta].$$

That is, $d = \partial^* + d_\nabla + R_2$[79] squares to zero:

$$[\partial^*, d_\nabla] = 0 \qquad d_\nabla^2 + [\partial^*, R_2] = 0 \qquad [d_\nabla, R_2] = 0.$$

Although it is well-known that $\Omega_{\mathrm{lin}} V_A$ is isomorphic (using the splitting σ) to $\Omega(A; V_M^* \to C_M^*)$, we observe that this is readily verified using the canonical model of invariant cochains from the previous section, Section 8.3. The pullback of the splitting defines a map

$$\sigma^* : \Omega(\widehat{V}_A; V_M^* \to C_M^*) \to \Omega(A; V_M^* \to C_M^*).$$

This is usually not a cochain map, but it restricts to one on $\Omega_{\mathrm{inv}}(\widehat{V}_A; V_M^*)$ (i.e. those pairs (ω_0, ω_1) for which (8.3) holds).

Proposition 8.22. *Let $\sigma : A \to \widehat{V}_A$ be a splitting and consider the split ruth $\Omega(A; V_M^* \to C_M^*)$. Then the map*

$$\sigma^* : \Omega_{\mathrm{inv}}(\widehat{V}_A; V_M^*) \to \Omega(A; V_M^* \to C_M^*)$$

is an isomorphism of ruths.

Proof. Indeed, for all $\omega = (\omega_0, \omega_1) \in \Omega(\widehat{V}_A; V_M^* \to C_M^*)$ we have

$$\begin{aligned}([d, \sigma^*]\omega)_0(\alpha_0, \dots, \alpha_\bullet) &= \textstyle\sum_{k<\ell} (-1)^{k+\ell+\bullet} \omega_1(\sigma\alpha_0, \dots, \widehat{\sigma\alpha_k}, \dots, \widehat{\sigma\alpha_\ell}, \dots, \sigma\alpha_\bullet) R_2(\alpha_k, \alpha_\ell) \\ &\quad + \textstyle\sum_{k<\ell} (-1)^{k+\ell} \omega_0(R_2(\alpha_k, \alpha_\ell), \sigma\alpha_0, \dots, \widehat{\sigma\alpha_k}, \dots, \widehat{\sigma\alpha_\ell}, \dots, \sigma\alpha_\bullet) \\ ([d, \sigma^*]\omega)_1(\alpha_0, \dots, \alpha_{\bullet-1}) &= \textstyle\sum_{k<\ell} (-1)^{k+\ell} \omega_1(R_2(\alpha_k, \alpha_\ell), \sigma\alpha_0, \dots, \widehat{\sigma\alpha_k}, \dots, \widehat{\sigma\alpha_\ell}, \dots, \sigma\alpha_{\bullet-1})\end{aligned}$$

which are zero on forms satisfying (8.3). On the other hand, using the projection $\widehat{V}_A \to A$, its inverse is given by

$$\Omega(A; V_M^* \to C_M^*) \to \Omega(\widehat{V}_A; V_M^* \to C_M^*) \qquad (\omega_0, \omega_1) \mapsto (\omega_0 - h(\omega_1), \omega_1)$$

where we wrote

$$h : \Omega^{\bullet-1}(A; C_M^*) \to \Omega^\bullet(\widehat{V}_A; V_M^*) \qquad h(\omega_1)(\alpha_1, \dots, \alpha_\bullet) := (-1)^{\bullet-1} \omega_1(\alpha_2, \dots, \alpha_\bullet)(\alpha_1 - \sigma\alpha_1).$$

It is readily verified that the maps are ΩA-linear, so this proves the statement. □

Remark 8.23. In each step of passing to a different model

$$\Omega_{\mathrm{lin}} V_A \to \Omega_{\mathrm{inv}}(\widehat{V}_A; V_M^*) \to \Omega(A; V_M^* \to C_M^*)$$

we are realizing that $\omega \in \Omega_{\mathrm{lin}} V_A$ is determined by "fewer" sections, at the cost of increasing the combinatorial complexity. In the first step, we can canonically reduce from thinking of forms on V_A to forms on $\widehat{V}_A$. In the second step, we choose a splitting $\sigma : A \to \widehat{V}_A$, which puts us in the position to think of forms on A.

[79] Here, d_∇ is given by the De Rham type formulas on $\Omega(A; V_M^*)$ and on $\Omega(A; C_M^*)$ using the A-connections.

Remark 8.24. As was already observed in [AAC12], a splitting $\sigma : A \to \widehat{V}_A$ splits the groupoid $\widehat{V}_A$ as follows. As a vector bundle, it takes the form

$$\widehat{V}_A \cong \mathrm{Hom}(V_M, C_M) \oplus A,$$

and its Lie bracket transforms into

$$[(h,\alpha),(h',\alpha')] := (R_2(\alpha,\alpha') + \nabla_h \alpha' - \nabla_{h'}\alpha + [h,h'], [\alpha,\alpha']).$$

This should be seen as the fat extension $\widehat{\mathcal{E}}_A$ associated to a split ruth $\mathcal{E}_A$. In [AAC12], after a sequence of observations about what we now recognise as the fat algebroid of a split ruth (Proposition 3.9 — Proposition 3.12), it is said that "*...there doesn't seem to be any construction which associates to a Lie algebroid extension of A a representation up to homotopy, so that, applying $J^1(A)$ one recovers the adjoint representation.*". While that might be true, we now know that the ruth can be recovered from J^1A and its extension when we think of J^1A as a fat extension. That is, we can recover the ruth if we additionally keep track of the cochain complex representation $A \to TM$ of J^1A and its compatibility with the extension.

8.5. **From fat extensions to VB-algebroids.** The step we have not discussed yet is how to build the VB-algebroid out of an abstract fat extension. So, let F_A be an abstract fat extension of A over $C_M \to V_M$. Consider the VB-algebroid

$$F_A \ltimes (C_M \to V_M) \Rightarrow V_M$$

associated to the cochain complex representation of F_A on $C_M \to V_M$. That is, we consider

$$F_A \ltimes (C_M \to V_M) := C_M \times_M F_A \times_M V_M.$$

as a vector bundle, and we define

$$[(\gamma,\alpha),(\gamma',\alpha')] := (\nabla_\alpha \gamma' - \nabla_{\alpha'}\gamma, [\alpha,\alpha'])$$

on sections $(\gamma,\alpha) = (\gamma,\alpha,1) : V_M \to F_A \ltimes (C_M \to V_M)$ that are constant along the fibers of V_M.

Remark 8.25. When $F_G = \widehat{V}_A$ for a VB-algebroid V_A, then $\widehat{V}_A \ltimes (C_M \to V_M)$ projects onto V_A via

$$\widehat{V}_A \ltimes (C_M \to V_M) \to V_A \qquad (\gamma,\alpha,v) \mapsto (\gamma +_\to 0_\downarrow \alpha) +_\downarrow \alpha v = (\gamma +_\downarrow 0_\to v) +_\to \alpha v.$$

As expected, the kernel can be identified with $\mathrm{Hom}(V_M, C_M) \ltimes V_M$, where V_M is seen as a representation of $\mathrm{Hom}(V_M, C_M)$ trivially.

Following the above remark, we observe next that $F_A \ltimes (C_M \to V_M)$ contains the ideal

$$\mathrm{Hom}(V_M, C_M) \ltimes V_M \hookrightarrow F_A \ltimes (C_M \to V_M) \qquad (h,v) \mapsto (-hv, h, v)$$

and then the quotient is the required VB-algebroid $V(F_G)$:

Proposition 8.26. *The assignment*

$$\{\text{VB-algebroids}\} \to \{\text{Fat extensions}\} \qquad V_A \mapsto \widehat{V}_A$$

sets a one-to-one correspondence, and the assignment

$$\{\text{Fat extensions}\} \to \{\text{VB-algebroids}\} \qquad F_A \mapsto V(F_A)$$

is inverse to it (up to canonical isomorphisms).

Notice that the above result generalises [CLS14].

8.6. **Comments on PB-algebroids and core-transitive double algebroids.** The notion of PB-algebroid has not been developed in the literature yet. But the discussion in Section 5 gives a good hint on what the "PB-algebroid" of a fat extension should be. Also for algebroids we can formulate a category of (vertically/horizontally) *core-transitive* double algebroids, and such double algebroids correspond to *core extensions*. This is a slight generalisation of the result in [JLM21] (similar to the generalisation for groupoids presented in Appendix A). The definition is as follows.

Definition 8.27. A *core extension* is a diagram

$$\begin{tikzcd} 0 \arrow[r] & \mathfrak{h} \arrow[r] \arrow[d, phantom, "\circlearrowright"] & F \arrow[r] \arrow[dl] & A_{\to} \arrow[r] & 0 \\ & A_{\downarrow} & & & \end{tikzcd}$$

which is a short exact sequence of Lie algebroids, and the action $A_{\downarrow} \circlearrowright \mathfrak{h}$ is a *bundle of Lie algebras representation*, i.e. a Lie algebroid action ∇ of $A_{\downarrow}$ on $\mathfrak{h}$ that is a derivation with respect to the bracket of $\mathfrak{h}$:

$$\nabla_{\alpha}[h_1, h_2] = [\nabla_{\alpha} h_1, h_2] + [h_1, \nabla_{\alpha} h_2].$$

Moreover, the map $F \to A_{\downarrow}$ restricted to $\mathfrak{h}$ intertwines ∇_{α} with $[\alpha, \cdot]$, and the *Peiffer identity* holds: the action of F on $\mathfrak{h}$, viewing $\mathfrak{h} \subset F$ as an ideal, agrees with the action of F on $\mathfrak{h}$ induced by $A_{\downarrow} \circlearrowright \mathfrak{h}$ and the map $F \to A_{\downarrow}$.

Remark 8.28. Although we do not discuss it in full detail, we mention here what the main objects are that define the "general linear PB-algebroid" associated to a fat extension of algebroids. Since we do not discuss "2-actions" (and in what way they are "principal"), we mainly hint to some analogies with our discussion in Section 5.

If we are given a fat extension F_A of A over $C_M \to V_M$, we can consider

$$F_A \ltimes \mathfrak{gl}(C_M, V_M) \Rightarrow \mathfrak{gl}(C_M, V_M)$$

associated to the representation of F_A on $C_M \to V_M$ (see Remark 8.8). This Lie algebroid carries a "2-action" of the strict Lie 2-algebroid

$$\mathfrak{gl}(C_M, V_M)_A \Rightarrow \mathfrak{gl}(C_M, V_M)_M \Rightarrow \mathrm{Hom}(C_M, V_M).$$

Following the above discussion, this strict Lie 2-algebroid is actually determined by a *crossed module of Lie algebroids*, so a core extension as above for which $\mathfrak{h} = F$. The bundle of Lie algebras $\mathfrak{h}$ in this case is

$$\mathfrak{pert}(C_M, V_M) \coloneqq \mathrm{Hom}(C_M, V_M) \oplus \mathrm{Hom}(V_M, C_M) \Rightarrow \mathrm{Hom}(C_M, V_M)$$

whose bracket is given by (we write h_{∂} for the value of the section $h \in \Gamma\mathfrak{pert}(C_M, V_M)$ at ∂):

$$[h, h']_{\partial} \coloneqq h'_{\partial} \partial h_{\partial} - h_{\partial} \partial h'_{\partial}.$$

The Lie algebroid $\mathfrak{gl}(C_M, V_M)_M$ (see Remark 8.8) acts on $\mathfrak{pert}(C_M, V_M)$ as a bundle of Lie algebras representation via

$$(\nabla_D h)_{\partial} \coloneqq (D^{C_M} h - h D^{V_M})_{\partial}.$$

The Lie algebroid map that is part of the core extension is given by

$$\mathfrak{pert}(C_M, V_M) \to \mathfrak{gl}(C_M, V_M)_M \qquad h \mapsto (D^{C_M}_{\partial} \coloneqq -h_{\partial}\partial,\ D^{V_M}_{\partial} \coloneqq -\partial h_{\partial}).$$

As in Example 5.4, we can think of the resulting strict Lie 2-algebroid as being of the form

$$\mathfrak{gl}(C_M, V_M)_A := \mathfrak{pert}(C_M, V_M) \ltimes \mathfrak{gl}(C_M, V_M)_M.$$

8.7. **Morphisms of fat extensions.** We briefly introduce here the notion of morphisms of fat extensions of Lie algebroids. We do this by introducing it directly as an infinitesimally multiplicative tensor. Given two fat extensions $F_{A,1}$ and $F_{A,2}$ of A, we have a Lie algebroid

$$F_{A,1} \times_A F_{A,2} \Rightarrow M.$$

This Lie algebroid carries representations that are the tensor product of a representation of $F_{A,1}$ with a representation of $F_{A,2}$.

Definition 8.29. Consider fat extensions $F_{A,1}$ and $F_{A,2}$ of A, over $C_{M,1} \to V_{M,1}$ and $C_{M,2} \to V_{M,2}$, respectively. Consider linear maps

$$\begin{array}{ccc} C_{M,1} & & V_{M,1} \\ \big\downarrow{\scriptstyle \Phi^{C_M}} & & \big\downarrow{\scriptstyle \Phi^{V_M}} \\ C_{M,2} & & V_{M,2} \end{array}$$

and a smooth map (that is linear over M)

$$f : F_{A,1} \times_A F_{A,2} \to \mathrm{Hom}(V_{M,1}, C_{M,2}).$$

Then (f, Φ) is a *morphism of fat extensions* if it is invariant: for all $v \in \Gamma V_M$ and $(h_1, h_2) \in \mathrm{Hom}(V_{M,1}, C_{M,1}) \oplus \mathrm{Hom}(V_{M,2}, C_{M,2})$,

$$f(h_1, h_2) = h_2 \Phi^{V_M} - \Phi^{C_M} h_1,$$

and infinitesimally multiplicative: $d(f, \Phi) = 0$, where d is the differential associated to the 3-term cochain complex representation

$$\mathrm{Hom}(V_{M,1}, C_{M,2}) \to \mathrm{Hom}(C_{M,1}, C_{M,2}) \oplus \mathrm{Hom}(V_{M,1}, V_{M,2}) \to \mathrm{Hom}(C_{M,1}, V_{M,2})$$

of $F_{A,1} \to F_{A,2}$.

Remark 8.30. The multiplicativity conditions for a morphism (f, Φ) are given more concretely by the following three conditions. The first one is that the map Φ is a cochain map with respect to the differential ∂. The second condition realises $f(\alpha_1, \alpha_2)$, for all $(\alpha_1, \alpha_2) \in F_{A,1} \times_A F_{A,2}$, as a homotopy measuring the failure of Φ to intertwine ∇_{α_1} with ∇_{α_2}:

$$f(\alpha_1, \alpha_2)\partial = \nabla_{\alpha_1} \Phi^{C_M} - \Phi^{C_M} \nabla_{\alpha_2}$$
$$\partial f(\alpha_1, \alpha_2) = \nabla_{\alpha_1} \Phi^{V_M} - \Phi^{V_M} \nabla_{\alpha_2}.$$

The last multiplicativity condition can be written as follows: given sections $\alpha, \beta \in \Gamma(F_{A,1} \times_A F_{A,2})$,

$$f([\alpha, \beta])v = \nabla_{\alpha_2}(f(\beta)v) - f(\beta)(\nabla_{\alpha_1} v) + f(\alpha)(\nabla_{\beta_1} v) - \nabla_{\beta_2}(f(\alpha)v),$$

where we wrote $\alpha = (\alpha_1, \alpha_2)$ and $\beta = (\beta_1, \beta_2)$.

Composition is defined in the same way as in Definition 7.9.

Definition 8.31. Given maps of fat extensions

$$F_{A,1} \times_A F_{A,2} \xrightarrow{f_{21}} \mathrm{Hom}(V_{M,1}, C_{M,2}) \qquad F_{A,2} \times_A F_{A,3} \xrightarrow{f_{32}} \mathrm{Hom}(V_{M,2}, C_{M,3}),$$

over cochain maps Φ_{21} and Φ_{32}, respectively, the composition $f_{31} = f_{32} \circ f_{21}$ over $\Phi_{31} = \Phi_{32} \circ \Phi_{21}$ is given by

$$f_{31}(\alpha_1, \alpha_3) = f_{32}(\alpha_2, \alpha_3)\Phi_{21}^{V_M} + \Phi_{32}^{C_M} f_{21}(\alpha_1, \alpha_2),$$

where $\alpha_2 \in F_{G,2}$ is arbitrary.

We can arrive at the above definition of morphism by starting with a VB-algebroid map

$$\Phi^{V_A} : V_{A,1} \to V_{A,2}$$

and defining a comparison map

$$f_\Phi : \widehat{V}_{A,1} \to \widehat{V}_{A,2} \qquad f_\Phi(\alpha_1, \alpha_2)v := (\alpha_2(\Phi^{V_M} v) -_{\downarrow} \Phi^{V_A}(\alpha_1 v)) -_{\to} 0_{\downarrow}\alpha.$$

The map Φ^{V_A} is recovered from (f_Φ, Φ) by realising that

$$\Phi^{V_A}((\gamma +_{\to} 0_{\downarrow}\alpha) +_{\downarrow} \alpha_1 v) = ((\Phi^{C_M}(\gamma) - f_\Phi(\alpha_1, \alpha_2)v) +_{\to} 0_{\downarrow}\alpha) +_{\downarrow} \alpha_2 \Phi^{V_M}(v).$$

We then have:

Theorem 8.32. *The functor*

$$\{\text{VB-algebroids}\} \to \{\text{Fat extensions}\} \qquad V_A \to \widehat{V}_A \quad \Phi \mapsto f_\Phi$$

sets an equivalence of categories.

Remark 8.33. Similarly, we can establish equivalences of the form

$$\{\text{Fat extensions}\} \to \{\text{VB-algebroids}\} \qquad F_A \mapsto V(F_A)$$
$$\{\text{Fat extensions}\} \to \{\text{Abstract 2-term ruths}\} \qquad F_A \mapsto C_{\text{inv}}(F_A; C_M)$$
$$\{\text{Split 2-term ruths}\} \to \{\text{Split fat extensions}\} \qquad \mathcal{E}_A \mapsto \widehat{\mathcal{E}}_A$$

but we will not go any further into these details here.

In the next section, Section 9, we will give a detailed description of the differentiation procedure in "Fat Lie theory". Apart from differentiating fat extensions, we will work out the details of the differentiation of the fat groupoid of a VB-groupoid to the fat algebroid of a VB-algebroid. The differentiation of the associated abstract 2-term ruths is, as expected, a type of Van Est map.

9. Fat Lie theory

In this section we explain the differentiation procedure of fat extensions. As we will see next, the differentiation procedure for fat extensions is relatively straightforward. Afterwards, we go through the details of differentiating the fat groupoid of a VB-groupoid to the fat algebroid of its VB-algebroid. To do that, some particular identifications have to be made. This step is also discussed in [DE19] (Proposition 2.7), but we elaborate on some aspects. Afterwards, we comment on the Van Est map in this setup.

9.1. **The Fat Lie functor.** To obtain the fat extension of a the VB-algebroid of a VB-groupoid, we can simply apply the Lie functor multiple times in several ways:

Proposition 9.1. *Let F_G be a fat extension of G over $C_M \to V_M$. Set*

$$F_A := \mathrm{Lie}\, F_G.$$

Moreover, consider the cochain complex representation $C_M \to V_M$ of A obtained by differentiation, and the exact sequence

$$0 \longrightarrow \mathrm{Hom}(V_M, C_M) \longrightarrow F_A \longrightarrow A \longrightarrow 0,$$

which is obtained from the fat extension of G by differentiation. Then, together with this structure, F_A is a fat extension of A over $C_M \to V_M$.

The assignment is functorial, for morphisms can be differentiated as well. Following Remark 7.8 and Remark 8.30, this should be seen as an application of a Van Est map. A general form of such Van Est maps will be discussed in [MOV]. Here is the special case of interest:

Proposition 9.2. *Let $f_G : F_{G,1} \to F_{G,2}$ be a morphism of fat extensions over G. Then*

$$f_A(\alpha_1, \alpha_2) := \partial_{\epsilon=0} f(\exp(\epsilon\alpha_1), \exp(\epsilon\alpha_2))$$

defines a morphism of fat extensions $f_A : F_{A,1} \to F_{A,2}$ over A. This defines a functor

$$\mathrm{Lie} : \{\text{fat extensions of } G\} \to \{\text{fat extensions of } A\} \qquad F_G \mapsto F_A \quad f_G \mapsto f_A.$$

In the next section 9.2 we show that, given a VB-groupoid V_G with VB-algebroid V_A, the Lie algebroid of $\widehat{V}_G$ can be seen as $\widehat{V}_A$.

9.2. **The tangent bundle of the fat groupoid of a VB-groupoid.** We often take the kinematic approach to tangent vectors in this section. Given $M \ni x$, we write tangent vectors as

$$\partial_{\epsilon=0} x_\epsilon = \tfrac{\partial}{\partial\epsilon}|_{\epsilon=0} x_\epsilon \in T_x M$$

where x_ϵ is a curve in M with $x_0 = x$. We will show a precise relation relating $T\widehat{V_G}$ and $\widehat{TV_G}$, where we view TV_G as a VB-groupoid over TG (a similar statement holds for VB-algebroids). We start with the following observation about vector bundles.

If $E \to M$ and $F \to M$ are vector bundles, then $T\mathrm{Hom}(E, F)$ is a vector bundle both over $\mathrm{Hom}(E, F)$ and over TM.[80] We consider the vector bundle structure over TM, and we will show that it embeds into $\mathrm{Hom}_{TM}(TE, TF)$.

Let $\partial_{\epsilon=0}\varphi_\epsilon \in T_\varphi \mathrm{Hom}(E, F)$ be a tangent vector of $\mathrm{Hom}(E, F)$, say, with

$$\varphi_\epsilon : E_{x_\epsilon} \to F_{x_\epsilon},$$

$x = x_0$ and $\varphi = \varphi_0 : E_x \to F_x$. Write $v = \partial_{\epsilon=0} x_\epsilon \in T_x M$ for the projection. Then, we define

$$(TE)_v \to (TF)_v \qquad \partial_{\epsilon=0} e_\epsilon \mapsto \partial_{\epsilon=0} \varphi_\epsilon e_\epsilon.$$

This map is injective and linear, so we obtained the following:

Proposition 9.3. *Let E and F be vector bundles over M. Then, the map*

$$T\mathrm{Hom}(E, F) \hookrightarrow \mathrm{Hom}_{TM}(TE, TF) \qquad \partial_{\epsilon=0}\varphi_\epsilon \mapsto (\partial_{\epsilon=0} e_\epsilon \mapsto \partial_{\epsilon=0}\varphi_\epsilon e_\epsilon)$$

is a vector bundle embedding over TM.

[80]Given a vector bundle $E \to M$, we write $T_e E$ for the fiber over $e \in E$ and $(TE)_v$ for the fiber over $v \in TM$.

This is probably easiest to see by writing the map in (standard) local coordinates. The above map over the zero section of $TM \to M$ takes the following form.

Remark 9.4. Here, we write the above embedding for $v = 0_x \in T_xM$. We decompose[81]

$$(TE)_{0_x} = E_x \oplus E_x^{\text{vert}}$$

canonically. Then, $\mathrm{Hom}((TE)_{0_x}, (TF)_{0_x})$ splits canonically as

$$\mathrm{Hom}(E_x, F_x) \oplus \mathrm{Hom}(E_x^{\text{vert}}, F_x) \oplus \mathrm{Hom}(E_x, F_x^{\text{vert}}) \oplus \mathrm{Hom}(E_x^{\text{vert}}, F_x^{\text{vert}}).$$

The image of the fiber $(T\mathrm{Hom}(E,F))_{0_x}$, under the embedding, can be identified with

$$\mathrm{Hom}(E_x, F_x) \oplus \mathrm{Hom}(E_x^{\text{vert}}, F_x^{\text{vert}}).$$

Indeed, let $\partial_{\epsilon=0}\varphi_\epsilon \in T_\varphi \mathrm{Hom}(E,F)$ such that its projection to TM is 0_x. Plugging in

$$\partial_{\epsilon=0} e_\epsilon \mapsto \partial_{\epsilon=0}\varphi_\epsilon e_\epsilon$$

an element of E_x gives $\varphi : E_x \to F_x$. If we plug in a vertical vector, then, using the notation from above, we can take $x_\epsilon = x$, and so $\partial_{\epsilon=0}\varphi_\epsilon e_\epsilon$ is again vertical. The map $(TE)_{0_x} \to (TF)_{0_x}$ therefore identifies with $\varphi : E_x \to F_x$ and this map $E_x^{\text{vert}} \to F_x^{\text{vert}}$.

Now, let $V_G \rightrightarrows V_M$ be a VB-groupoid, and consider the VB-groupoids

$$\begin{tikzcd} T\widehat{V}_G \arrow[r, shift left] \arrow[r, shift right] \arrow[d] & TM \arrow[d] \\ \widehat{V}_G \arrow[r, shift left] \arrow[r, shift right] & M \end{tikzcd} \qquad \begin{tikzcd} TV_G \arrow[r, shift left] \arrow[r, shift right] \arrow[d] & TV_M \arrow[d] \\ TG \arrow[r, shift left] \arrow[r, shift right] & TM \end{tikzcd}$$

Proposition 9.5. *Let $V_G \rightrightarrows V_M$ be a VB-groupoid. Then the map*

$$T\widehat{V}_G \to \widehat{TV_G} \qquad \partial_{\epsilon=0}H_{g_\epsilon} \mapsto (\partial_{\epsilon=0}v_\epsilon \mapsto \partial_{\epsilon=0}H_{g_\epsilon}v_\epsilon)$$

is an embedding of Lie groupoids. In fact, using the canonical embedding of Proposition 9.3, there is a commutative diagram (of short exact sequences over TM)

$$\begin{tikzcd} 1 \arrow[r] & T\mathrm{H}(V_M, C_M) \arrow[r] \arrow[d, hook] & T\widehat{V}_G \arrow[r] \arrow[d, hook] & TG \arrow[r] \arrow[d, "="] & 1 \\ 1 \arrow[r] & \mathrm{H}(TV_M, TC_M) \arrow[r] & \widehat{TV_G} \arrow[r] & TG \arrow[r] & 1 \end{tikzcd}$$

Proof. To see that the above map is a groupoid map, observe that we can write the multiplication of $TG \rightrightarrows TM$ as follows (see [Mac05]). Given $v \in T_gG$ and $w \in T_hG$ composable, we can find g_ϵ and h_ϵ curves in G, such that $v = \partial_{\epsilon=0}g_\epsilon$ and $w = \partial_{\epsilon=0}h_\epsilon$, while $\mathbf{s}g_\epsilon = \mathbf{t}h_\epsilon$ for all (small) ϵ. Then

$$v \cdot w = \partial_{\epsilon=0}(g_\epsilon \cdot h_\epsilon).$$

Using this formula, the statement is readily verified. □

[81]Here, we wrote E_x^{vert} for the space of vertical vectors, which can be identified with E_x.

9.3. **The fat algebroid of the fat groupoid of a VB-groupoid.** In this section, we prove that there is a canonical identification

$$\operatorname{Lie}\widehat{V}_G = \widehat{V}_A$$

as vector bundles. Recall that $\widehat{V}_G$ is an open subset of $\widehat{V}_G^{\mathrm{cat}}$, and $\widehat{V}_G^{\mathrm{cat}}$ is the preimage of 1 (the identity section in $\operatorname{End} V_M$) under the surjective linear map (over $\mathbf{s} : G \to M$)

$$\operatorname{Hom}(\mathbf{s}^* V_M, V_G) \to \operatorname{End} V_M \qquad H_g \mapsto \mathbf{s}_g H_g.$$

The tangent space $T_{H_g}\widehat{V}_G$ can therefore be thought of as elements

$$\partial_{\epsilon=0} H_{g_\epsilon} \in T_{H_g}\operatorname{Hom}(\mathbf{s}^* V_M, V_G)$$

where H_{g_ϵ} are maps $(V_M)_{\mathbf{s}g_\epsilon} \to (V_G)_{g_\epsilon}$, such that

$$\partial_{\epsilon=0}\mathbf{s}_g H_{g_\epsilon} = 0.$$

Using Proposition 9.3, this equation can be interpreted inside of $\operatorname{End}(TV_M)_{d\mathbf{s}X}$, where

$$X = \partial_{\epsilon=0} g_\epsilon \in T_g G.$$

Proposition 9.6. *We have*

$$\operatorname{Lie}\widehat{V}_G = \{H_a : (V_M)_{\mathrm{pr}_\to a} \to (V_A)_a \mid \mathrm{pr}_\to H_a = 1\} = \widehat{V}_A$$

as vector bundles.

Proof. Since

$$(d\widehat{\mathbf{s}})_{H_g} : T_{H_g}\widehat{V}_G \to T_{\mathbf{s}g} M \qquad \partial_{\epsilon=0} H_{g_\epsilon} \mapsto \partial_{\epsilon=0}\mathbf{s}g_\epsilon,$$

the vector $X = \partial_{\epsilon=0} g_\epsilon$ is tangent to the source fiber if $\partial_{\epsilon=0} H_{g_\epsilon} \in \ker(d\widehat{\mathbf{s}})_{H_g}$. In this situation, we can take $g_\epsilon \in \mathbf{s}^{-1}\mathbf{s}g$. An element

$$(X, \partial_{\epsilon=0} v_\epsilon) \in d\mathbf{s}^* TV_M$$

(here, $v_\epsilon \in (V_M)_{\mathbf{s}g_\epsilon} = (V_M)_{\mathbf{s}g}$) has then the property that $\partial_{\epsilon=0} v_\epsilon$ is vertical. This element $\partial_{\epsilon=0} v_\epsilon$ can now itself be thought of as an element of $(V_M)_{\mathbf{s}g}$. With this identification, the embedding of Proposition 9.5 produces a linear map

$$H_X : (V_M)_{\mathbf{s}g} \to (H_g^* \ker(d\mathbf{s})|_{(V_G)_g})_X \qquad \partial_{\epsilon=0} v_\epsilon \mapsto \partial_{\epsilon=0} H_{g_\epsilon} v_\epsilon.$$

We can see these maps as (horizontal) linear sections of the double vector bundle[82]

$$\begin{tikzcd}
H_g^* \ker(d\mathbf{s})|_{(V_G)_g} \arrow[r] \arrow[d] & (V_M)_{\mathbf{s}g} \arrow[d] \\
\ker(d\mathbf{s})_g \arrow[r] & \{g\}
\end{tikzcd}$$

(the base map of the section H_X is given by $g \mapsto X \in \ker(d\mathbf{s})_g$). The linear map we constructed

$$\ker(d\widehat{\mathbf{s}})_{H_g} \to \{H_X : (V_M)_{\mathbf{s}g} \to (H_g^* \ker d\mathbf{s}|_{(V_G)_g})_X \mid \mathrm{pr}_\to H_X = 1\} \tag{9.1}$$

is injective, so by a dimension count a bijection. The restriction to the units is the desired isomorphism of vector bundles. This proves the statement. □

Remark 9.7. The double vector bundle

[82]Notice that this makes sense as H_g has full rank.

$$\begin{tikzcd} H_g^* \ker(d\mathbf{s})|_{(V_G)_g} \arrow[r] \arrow[d] & (V_M)_{\mathbf{s}g} \arrow[d] \\ \ker(d\mathbf{s})_g \arrow[r] & \{g\} \end{tikzcd}$$

can be seen as the pullback along $H_g : (V_M)_{\mathbf{s}g} \to (V_G)_g$ of

$$\begin{tikzcd} \ker(d\mathbf{s})|_{(V_G)_g} \arrow[r] \arrow[d] & (V_G)_g \arrow[d] \\ \ker(d\mathbf{s})_g \arrow[r] & \{g\} \end{tikzcd}$$

which is a sub double vector bundle of

$$\begin{tikzcd} TV_G \arrow[r] \arrow[d] & V_G \arrow[d] \\ TG \arrow[r] & G. \end{tikzcd}$$

That is, we view $\ker(d\mathbf{s})|_{(V_G)_g} \to \ker(d\mathbf{s})_g$ as a vector bundle over $(V_G)_g \to \{g\}$, and then we change the base to $(V_M)_{\mathbf{s}g} \to \{g\}$ using H_g.

9.4. **Right invariant vector fields of the fat groupoid.** Under the correspondence

$$\mathfrak{X}_{\text{R-inv}} V_G \cong \Gamma_{\to} V_A,$$

$\mathfrak{X}_{\text{R-inv,lin}} V_G$ corresponds to $\Gamma_{\text{lin}} V_A = \Gamma \widehat{V}_A$. Linear vector fields on V_G are (horizontal) linear sections of the double vector bundle

$$\begin{tikzcd} TV_G \arrow[r] \arrow[d] & V_G \arrow[d] \\ TG \arrow[r] & G \end{tikzcd}$$

so $\Gamma \widehat{V}_A$, with the induced bracket from V_A, is isomorphic to $\mathfrak{X}_{\text{R-inv,lin}} V_G$ as Lie algebras.

On the other hand, by the discussion from the last section, Section 9.3, $\Gamma \widehat{V}_A \cong \mathfrak{X}_{\text{R-inv}} \widehat{V}_G$. Our goal is now to show that this isomorphism is an isomorphism of Lie algebras, i.e. that the induced bracket on Lie $\widehat{V}_G$ is the correct one (the one induced from V_A). That is, we want to show that

$$\mathfrak{X}_{\text{R-inv,lin}} V_G \cong \mathfrak{X}_{\text{R-inv}} \widehat{V}_G$$

is an isomorphism of Lie algebras. For the following result, see also [DE19].

Proposition 9.8. *The canonical identification of vector bundles*

$$\text{Lie } \widehat{V}_G \cong \widehat{V}_A$$

of Proposition 9.6 is an isomorphism of Lie algebroids.

Proof. This is straightforward using bisections. Recall that the bisections of $\widehat{V}_G$ can be seen as linear bisections of V_G (Remark 3.2). The exponential map (taking right-invariant flow)[83]

$$\exp : \mathfrak{X}_{\text{R-inv}} V_G \to \text{Bis } V_G$$

restricts on linear sections to a map $\mathfrak{X}_{\text{R-inv,lin}} V_G \to \text{Bis } \widehat{V}_G$, and then the diagram

[83] As usual, this should not be interpreted literally due to completeness issues.

$$\begin{tikzcd} \mathfrak{X}_{\text{R-inv,lin}} V_G \arrow[drr, "\exp"] & & \\ \Gamma \widehat{V}_A \arrow[u, "\sim"] \arrow[d, "\sim"'] \arrow[rr, dotted] & & \operatorname{Bis} \widehat{V}_G \\ \mathfrak{X}_{\text{R-inv}} \widehat{V}_G \arrow[urr, "\exp"'] & & \end{tikzcd}$$

commutes. Using the formula for the Lie bracket in terms of the exponential map, we see that the brackets on $\mathfrak{X}_{\text{R-inv,lin}} V_G$ and on $\mathfrak{X}_{\text{R-inv}} \widehat{V}_G$ coincide. □

Remark 9.9. In view of the next section, Section 9.5, we add some more details. A section $\alpha \in \Gamma \widehat{V}_A$ defines right-invariant vector fields

$$X \in \mathfrak{X}_{\text{R-inv,lin}} V_G \qquad \widehat{X} \in \mathfrak{X}_{\text{R-inv}} \widehat{V}_G,$$

both over $X \in \mathfrak{X}_{\text{R-inv}} G$. The right-invariant flow of the linear vector field X is determined by a curve of maps

$$\varphi^X_\epsilon : V_M \to V_G$$

that are linear over a map $\varphi^X_\epsilon : M \to G$. Linearity means that φ^X_ϵ restricts to linear maps[84]

$$(\varphi^X_\epsilon)_x : (V_M)_x \to (V_G)_{\varphi^X_\epsilon(x)}.$$

From the previous section, Section 9.4, recall (9.1), i.e. that $\ker(d\widehat{\mathbf{s}})_{H_g}$ can be identified with (horizontal) linear sections of the double vector bundle

$$\begin{tikzcd} H_g^* \ker(d\mathbf{s})|_{(V_G)_g} \arrow[r] \arrow[d] & (V_M)_{\mathbf{s}g} \arrow[d] \\ \ker(d\mathbf{s})_g \arrow[r] & \{g\}. \end{tikzcd}$$

For this reason, we can see the flow of some section $\widehat{X}$ of $\ker d\widehat{\mathbf{s}}$ (not necessarily right-invariant) as a curve of linear maps

$$\varphi^{\widehat{X}}_\epsilon : \widehat{V}_G \to \widehat{V}_G \qquad \varphi^{\widehat{X}}_\epsilon(H_g) : (V_M)_{\mathbf{s}g} \to (V_G)_{\varphi^X_\epsilon(g)}$$

Now, if $\widehat{X}$ is right-invariant, i.e. such that

$$\widehat{X}(H_g \cdot H_h) = (dR_{H_h})_{H_g} \widehat{X}(H_g),$$

then $\widehat{X}$ is determined on the identity elements of $\widehat{V}_G$, which defines the section $\alpha \in \Gamma \widehat{V}_A$ we started with. The flow $\varphi^{\widehat{X}}$ is therefore right-invariant and determined by its values on the identity elements; these form a family of linear maps

$$\varphi^{\widehat{X}}_\epsilon : (V_M)_x \to (V_G)_{\varphi^X_\epsilon(x)}.$$

The two maps $\varphi^{\widehat{X}}_\epsilon$ and φ^X_ϵ coincide.

Next, we explain how the differentiation procedure leads to a Van Est map on invariant cochains.

[84] Notice that, given $v \in (V_M)_x$, the curve $(\varphi^X_\epsilon)_x v$ stays within the source fiber $\mathbf{s}^{-1}\mathbf{s}v$.

9.5. **The Van Est map.** Before we begin, we briefly mention how the differentiation procedure of abstract ruths looks like.

9.5.1. *Differentiation of abstract ruths.* To emphasise, this is only a short discussion. In [MMO], more details will be included.

As is recently shown in [CdH26], the differentiation of a Lie groupoid G to a Lie algebroid A can be thought of algebraically as follows. The differential graded algebra CG contains the *differentiating ideal* $J_{\mathrm{N}} = J + \delta J$, where

$$J = \bigoplus_{\bullet \geq 0} J^{\bullet} \qquad J^{\bullet} := \{ f \in C^{\bullet}G \mid f \text{ vanishes to order} \geq 1 \text{ at } \operatorname{im} u_k \text{ for some } k \}.$$

The differential graded algebra ΩA is canonically isomorphic to $(CG)_{\mathrm{N}}/J_{\mathrm{N}}$. The quotient map

$$\mathrm{VE} : (CG)_{\mathrm{N}} \to \Omega A$$

is the ordinary *Van Est map* of [Cra03] (see [CdH26] for more details).

Now, given a ruth $\mathcal{E}_G$ of G, then we can quotient $(\mathcal{E}_G)_{\mathrm{N}}$ by $(\mathcal{E}_G)_{\mathrm{N}} \cdot J_{\mathrm{N}}$. The result

$$\mathcal{E}_A := (\mathcal{E}_G)_{\mathrm{N}}/(\mathcal{E}_G)_{\mathrm{N}} \cdot J_{\mathrm{N}} \cong (\mathcal{E}_G)_{\mathrm{N}} \otimes_{(CG)_{\mathrm{N}}} \Omega A$$

is a well-defined ruth of A, and accordingly there is a Van Est map

$$(\mathcal{E}_G)_{\mathrm{N}} \to \mathcal{E}_A.$$

There is a *Van Est theorem* (for split ruths, see [AAS11]) which extends the ordinary Van Est theorem of [Cra03]. This fact and other further details will be discussed in [MMO]. We now move to giving an explicit description of the Van Est map/theorem in terms of invariant cochains.

9.5.2. *The Van Est map of invariant cochains.* Let F_G be a fat extension of a Lie groupoid G over $C_M \to V_M$, and consider the associated fat extension F_A of A over $C_M \to V_M$. The Van Est map

$$\mathrm{VE} : C_{\mathrm{inv}}(F_G; C_M)_{\mathrm{N}} \to \Omega_{\mathrm{inv}}(F_A; C_M)$$

is simply the restriction of the ordinary Van Est map

$$\mathrm{VE} : C(F_G; C_M \to V_M)_{\mathrm{N}} \to \Omega(F_A; C_M \to V_M)$$

defined componentwise from the ordinary Van Est maps associated to the representations on C_M and V_M. Following [AAS11, CD17], there is a Van Est theorem:

Theorem 9.10. *The Van Est map*

$$\mathrm{VE} : C^{\bullet}(F_G; C_M \to V_M)_{\mathrm{N}} \to \Omega^{\bullet}(F_A; C_M \to V_M)$$

restricts to a map of differential graded modules

$$\mathrm{VE} : C^{\bullet}_{\mathrm{inv}}(F_G; C_M)_{\mathrm{N}} \to \Omega^{\bullet}_{\mathrm{inv}}(F_A; C_M).$$

Moreover, if G has (homologically) $\leq n$-connected fibers, then VE *induces an isomorphism in cohomology in degrees $\leq n$, and it induces an injective map in cohomology in degree $n+1$.*

Instead of proving the statement directly, we explain how the Van Est map looks like in terms of $F_G = \widehat{V}_G$. Following the discussion of the previous sections, we can give a clear description of the formula in this situation.

As explained in the Section 9.4, $\alpha \in \Gamma \widehat{V}_A$ exponentiates to a curve of linear maps[85]

$$\exp_x(\epsilon\alpha) = \exp_{\exp(\epsilon\alpha)x}(\epsilon\alpha) : (V_M)_x \to (V_G)_{\exp(\epsilon\alpha)x}.$$

[85] Here, $\epsilon > 0$ is small enough so that $\epsilon\alpha$ is close to the zero section.

For $\bullet \geq 0$, consider the maps

$$\partial_\alpha : C^{\bullet+1}(\widehat{V}_G; V_M^*) \to C^\bullet(\widehat{V}_G; V_M^*) \qquad \partial_\alpha : C^{\bullet+1}(\widehat{V}_G; C_M^*) \to C^\bullet(\widehat{V}_G; C_M^*)$$

given both by

$$(\partial_\alpha f)(H_{g_1}, \dots, H_{g_\bullet})e = \partial_{\epsilon=0} f(\exp_{1_{\mathrm{t}g_1}}(\epsilon\alpha), H_{g_1}, \dots, H_{g_\bullet}) \exp_{1_{\mathrm{t}g_1}}(\epsilon\alpha) \cdot e.$$

Then the Van Est map

$$\mathrm{VE} : C^\bullet(\widehat{V}_G; V_M^* \to C_M^*)_{\mathrm{N}} \to \Omega^\bullet(\widehat{V}_A; V_M^* \to C_M^*)$$

is given componentwise by[86]

$$\mathrm{VE} f(\alpha_1, \dots, \alpha_\bullet) = \textstyle\sum_{\sigma \in \mathrm{S}_\bullet} (-1)^\sigma \partial_{\alpha_{\sigma\bullet}} \cdots \partial_{\alpha_{\sigma 1}} f|_M.$$

Proposition 9.11. *The Van Est map*

$$\mathrm{VE} : C^\bullet(\widehat{V}_G; V_M^* \to C_M^*)_{\mathrm{N}} \to \Omega^\bullet(\widehat{V}_A; V_M^* \to C_M^*)$$

restricts to a cochain map

$$\mathrm{VE} : C^\bullet_{\mathrm{inv}}(\widehat{V}_G; V_M^*)_{\mathrm{N}} \to \Omega^\bullet_{\mathrm{inv}}(\widehat{V}_A; V_M^*).$$

Moreover, if G has (homologically) $\leq n$-connected fibers, then VE *induces an isomorphism in cohomology in degrees $\leq n$, and it induces an injective map in cohomology in degree $n+1$.*

Proof. Given $(f_0, f_1) \in C^\bullet_{\mathrm{inv}}(\widehat{V}_G; V_M^*)$, we want to show that

$$\iota_v \iota_h \mathrm{VE} f_0 = \iota_{hv} \mathrm{VE} f_1 \tag{9.2}$$

for all $h \in \Gamma\mathrm{Hom}(V_M, C_M)$ and $v \in \Gamma V_M$. First observe that

$$\exp_x(\epsilon h) = 1 + \epsilon h : (V_M)_x \to (V_G)_{1_x}.$$

In particular, as this is a bisection over the unit bisection $1 : M \to G$, we have, for all $\alpha \in \Gamma\widehat{V}_A$, that $\partial_h \partial_\alpha = 0$. Therefore, the only surviving terms $\partial_{\beta_\bullet} \cdots \partial_{\beta_1} f_0$ in

$$\mathrm{VE} f_0(\alpha_1, \dots, \alpha_\bullet) = \textstyle\sum_{\sigma \in \mathrm{S}_\bullet} (-1)^\sigma \partial_{\alpha_{\sigma\bullet}} \cdots \partial_{\alpha_{\sigma 1}} f_0|_M,$$

say, for $\alpha_1 = h$, are those for which $\beta_1 = h$. The result follows using the explicit formula of [Cra03]:

$$(\partial_{\beta_\bullet} \cdots \partial_{\beta_1} f)v = \partial_{\epsilon_\bullet=0} \cdots \partial_{\epsilon_1=0} f(\exp(\epsilon_1\beta_1), \dots, \exp(\epsilon_\bullet\beta_\bullet)) \exp(\epsilon_1\beta_1) \cdots \exp(\epsilon_\bullet\beta_\bullet)v.$$

The last statement is a consequence of, for example, the result of [CD17] and our discussion on flows, as it follows from that discussion that we have a commutative diagram

$$\begin{tikzcd}
(C_{\mathrm{VB}} V_G)_{\mathrm{N}} \arrow[r, "\sim"] \arrow[d, "\mathrm{VE}"] & C_{\mathrm{inv}}(\widehat{V}_G; V_M^*)_{\mathrm{N}} \arrow[d, "\mathrm{VE}"] \\
\Omega_{\mathrm{lin}} V_A \arrow[r, "\sim"] & \Omega_{\mathrm{inv}}(\widehat{V}_A; V_M^*)
\end{tikzcd}$$

One could also establish such a commutative diagram using instead the result of [AAS11]. □

One can prove the Van Est theorem also directly using similar techniques as in [Cra03]; this will appear in [MMO].

[86] We used $\mathrm{S}_\bullet$ for the group of permutations on $\bullet$ elements.

10. Deformation theory for Lie groupoids using the jet groupoid

We focus now on the fat extension of the jet groupoid J^1G. Recall that

$$\begin{aligned} J^1G &= \{j^1\sigma \mid \sigma \in \mathrm{Bis}_{\mathrm{loc}}G\} \\ &= \{H_g : T_{\mathbf{s}g}M \to T_gG \mid d\mathbf{s}_gH_g = 1 \text{ and } d\mathbf{t}_gH_g \text{ is a linear isomorphism}\} \\ &= \{\omega_g : T_gG \to A_{\mathbf{s}g} \mid \omega_g\ell_g = 1 \text{ and } \omega_g r_g \text{ is a linear isomorphism}\}. \end{aligned}$$

The deformation complex of G is $C_{\mathrm{proj}}TG$, which we now see is isomorphic to $C_{\mathrm{inv}}(J^1G; A)$. However, as we will see, it is also natural, perhaps even more natural, to work with the *jet category* whose elements are 1-jets of sections of the source map.

We already reformulated the vanishing of proper groupoids (see Proposition 6.1) and the Van Est theorem in terms of $C_{\mathrm{inv}}(J^1G; A)$ (see Proposition 9.11). We will discuss two more reformulations, namely that of describing the deformation class of a deformation of groupoids, and for the deformation cohomology in the regular case. There are many other reformulations possible of other parts of [CMS20], but we won't discuss further applications here. We think the small discussion presented here gives a good starting point for the interested reader to understand deformation theory of Lie groupoids in this setup.

Remark 10.1. Before we continue, notice that in [She12] the deformation complex of Lie algebroids is discussed in terms of the jet algebroid J^1A. That is, $\Omega_{\mathrm{inv}}(J^1A; A)$ is the deformation complex of A (see also [CM08]). The dgLa structure is given as follows: given $\omega_0 \in \Omega^{\bullet+1}_{\mathrm{inv}}(J^1A; A)$ and $\omega_0' \in \Omega^{\bullet'+1}_{\mathrm{inv}}(J^1A; A)$, we first define

$$(\omega_0 \circ \omega_0')(\alpha_1, \dots, \alpha_{\bullet+\bullet'+1}) = \textstyle\sum_\sigma (-1)^\sigma \omega_0(\omega_0'(\alpha_{\sigma 1}, \dots, \alpha_{\sigma(\bullet'+1)}), \alpha_{\sigma(1+\bullet'+1)}, \dots, \alpha_{\sigma(\bullet+\bullet'+1)})$$

and then put

$$[\omega, \omega'] = (-1)^{\bullet\bullet'}\omega \circ \omega' - \omega' \circ \omega.$$

Notice that we can see elements ω_0 in degree 2 satisfying the Maurer Cartan equation $[\omega_0, \omega_0] = 0$ as Lie brackets by setting

$$[\alpha, \beta]_\omega := \omega_0(d\alpha, d\beta).$$

The Leibniz rule follows by the existence of the form ω_1, which can be seen as the anchor map. In fact, for any form $\omega \in \Omega^\bullet_{\mathrm{inv}}(J^1A; A)$ we can define a multiderivation by

$$D(\alpha_1, \dots, \alpha_\bullet) = \omega_0(d\alpha_1, \dots, d\alpha_\bullet)$$

whose symbol is given by ω_1. This way, we obtain an isomorphism of differential graded Lie algebras between $\Omega_{\mathrm{def}}A$ of [CM08] and $\Omega_{\mathrm{inv}}(J^1A; A)$ of [She12].

10.1. Deformations of Lie groupoids.

We define a deformation of Lie groupoids as in [CMS20].

Definition 10.2. A deformation of a Lie groupoid G is a Lie groupoid $G_\mathbb{R} \rightrightarrows M_\mathbb{R}$ together with a Lie groupoid submersion

$$G_\mathbb{R} \to 1_\mathbb{R}$$

such that $G_0 = G$ (the preimage of zero).

Example 10.3. Let $G \rightrightarrows M$ be a Lie groupoid, and let $H \rightrightarrows N$ be a Lie subgroupoid of G. Consider the deformation to the normal cone (see for instance [CR08, Mei17, DS21, Obs21])[87]

$$\mathcal{D}(G,H) \rightrightarrows \mathcal{D}(M,N).$$

This is a deformation, in the above sense, of $\mathcal{N}(G,H) \rightrightarrows \mathcal{N}(M,N)$, and it is central to the linearisation problem for Lie groupoids.

Deformations can be seen as families of Lie groupoids $G_\epsilon \rightrightarrows M_\epsilon$. For the deformation theory of G, both $C_{\mathrm{def}}G$ and $C_{\mathrm{def}}G_{\mathbb{R}}$ are relevant. We can see $TG_{\mathbb{R}}$ as a deformation of TG, and elements of $J^1G_{\mathbb{R}}$ take the form

$$H_g : T_{\mathbf{s}g}M_{\mathbb{R}} \to T_gG_{\mathbb{R}}$$

that we should see as a family of maps $T_{\mathbf{s}g_\epsilon}M_\epsilon \to T_{g_\epsilon}G_\epsilon$.[88]

10.2. **The deformation class of a deformation.** Given a deformation $G_{\mathbb{R}}$ of G, we can lift the vector field $\frac{\partial}{\partial t} \in \mathfrak{X}\mathbb{R}$ to a vector field $X_{\mathbb{R}}$ of $M_{\mathbb{R}}$. Then, take a fat splitting h (see Section 6.2) and define the map

$$f_0 : J^1G_{\mathbb{R}} \to \mathbf{t}^*A_{\mathbb{R}} \qquad f_0(H_g) = -h(H_g)X_{\mathbb{R}}(\mathbf{s}g).$$

This defines an element of $C^1_{\mathrm{inv}}(J^1G_{\mathbb{R}}; A_{\mathbb{R}})$ with $f_1 = X_{\mathbb{R}} \in \mathfrak{X}M_{\mathbb{R}}$:

$$f_0(H_g) - f_0(H'_g) = h(H_g, H'_g)f_1(\mathbf{s}g).$$

Remark 10.4. It is good to realise that, using the model $C_{\mathrm{proj}}TG_{\mathbb{R}}$, and using a cleavage Σ of $TG_{\mathbb{R}}$, we can lift $X_{\mathbb{R}}$ to an $\mathbf{s}$-projectable vector field $\Sigma X_{\mathbb{R}}$ of $G_{\mathbb{R}}$ (i.e. $g \mapsto \Sigma_g X_{\mathbb{R}}(\mathbf{s}g)$) and it can directly be seen as a 1-cochain, for it is $\mathbf{s}$-projectable. On the one hand, the 1-cochain we write here makes it clear that, when we pass to the adjoint (split) ruth using Σ, the 1-cochain is nothing but $X_{\mathbb{R}} \in C^0(G_{\mathbb{R}}; TM_{\mathbb{R}})$. On the other hand, the usage of the cleavage shows that the map f above is more naturally defined on the fat category.

Taking δf_0 defines a cocycle that we can restrict to a cocycle in $C^2_{\mathrm{inv}}(J^1G; A)$. Indeed, since

$$\delta f_0(H_{g_1}, g_2) = h(H_{g_1})X_{\mathbb{R}}(\mathbf{s}g_1) - (h(H_{g_1} \cdot H_{g_2}) - H_{g_1} \cdot h(H_{g_2}))X_{\mathbb{R}}(\mathbf{s}g_2),$$

where H_{g_2} is arbitrary, and $X_{\mathbb{R}}$ projects to $\frac{\partial}{\partial t}$, the resulting element in $(A_{\mathbb{R}})_{\mathbf{t}g}$ is vertical with respect to the projection to $\mathbb{R}$. Therefore, the same equation defines a cocycle in $C^2_{\mathrm{inv}}(J^1G; A)$.

The resulting cohomology class is independent of the choice of vector field $X_{\mathbb{R}}$ or the fat splitting h. The reason is that $f_0 - f'_0$ (where f'_0 is defined using a different choice $X'_{\mathbb{R}}$ and h') restricts to $C^1_{\mathrm{inv}}(J^1G; A)$ (the same reasoning as before applies). Since

$$\delta f_0 = \delta f'_0 + \delta(f_0 - f'_0),$$

the cohomology classes defined by f_0 and by f'_0 are the same.

Remark 10.5. As can be inferred from [BBLM20, CMS20], we can linearise a proper Lie groupoid G along an invariant submanifold $N \subset M$ using $\mathcal{D}(G, G_N)$ and the homotopy operator η from Proposition 6.1. The projectable complex from [CMS20] is natural for such an application, for it is easy to see that its 1-cocycles comprise the multiplicative vector fields. The above model, especially when put in terms of the jet category, seems to also give a natural point of view.

[87] As sets, $\mathcal{D}(M,N) = M \times (\mathbb{R}\backslash\{0\}) \sqcup \mathcal{N}(M,N)$. The smooth structure is canonically defined, and it is unique with the property that $M \times (\mathbb{R}\backslash\{0\})$ is an embedded submanifold.

[88] Note that if $X \in T_{\mathbf{s}g}M$ projects to $\lambda\frac{\partial}{\partial t}$, then H_gX also projects to $\lambda\frac{\partial}{\partial t}$.

10.3. **Deformation cohomology in the regular case.** Given a regular Lie groupoid G, TG is also regular. A VB-groupoid V_G over G whose underlying complex $C_M \to V_M$ is regular, comes with a short exact sequence of complexes

$$\begin{tikzcd} \ker\partial \arrow[r] \arrow[d] & C_M \arrow[r] \arrow[d, "\partial"] & 0 \arrow[d] \\ 0 \arrow[r] & V_M \arrow[r] & \operatorname{coker}\partial \end{tikzcd}$$

all three of which we can see as cochain complex representations of $\widehat{V}_G$. Notice that the invariant complex of the outer two representations are canonically isomorphic to the corresponding representations of G. We therefore obtain a long exact sequence

$$\cdots \longrightarrow H^\bullet(G;\ker\partial) \longrightarrow H^\bullet_{\mathrm{inv}}(\widehat{V}_G;C_M) \longrightarrow H^{\bullet-1}(G;\operatorname{coker}\partial) \longrightarrow \cdots.$$

If $G = G(P)$ is the Gauge groupoid of a principal bundle P, we recover the result that

$$H^\bullet_{\mathrm{def}}G(P) \cong H^\bullet(G(P);P\times_G\mathfrak{g}),$$

where $P\times_G\mathfrak{g}$ is the adjoint bundle. If $G = G(\mathcal{F})$ is a foliation groupoid of a foliation $\mathcal{F}$, we recover the result that

$$H^\bullet_{\mathrm{def}}G(\mathcal{F}) \cong H^{\bullet-1}(G(\mathcal{F});\mathcal{N}),$$

where $\mathcal{N}$ is the normal bundle of the foliation. Similar arguments work to prove analogous results for VB-algebroids.

Appendix A. Double groupoids and core extensions

Recall the following notation we use for double groupoids:

$$\begin{tikzcd} G \arrow[r, shift left] \arrow[r, shift right] \arrow[d, shift left] \arrow[d, shift right] & G_\downarrow \arrow[d, shift left] \arrow[d, shift right] \\ G_\rightarrow \arrow[r, shift left] \arrow[r, shift right] & M \end{tikzcd}$$

We denote the structure maps of $G_\rightarrow$ and $G_\downarrow$ using the same arrow-index. We denote the groupoid structure maps of $G \rightrightarrows G_\downarrow$ with indices $\rightarrow$ and $G \rightrightarrows G_\rightarrow$ with indices $\downarrow$.

The core groupoid is given by

$$F := \ker\mathbf{s}_\downarrow \cap \ker\mathbf{s}_\rightarrow \rightrightarrows M,$$

whose structure maps are given by $\mathbf{s}(g) = \mathbf{s}_\rightarrow\mathbf{s}_\downarrow g = \mathbf{s}_\downarrow\mathbf{s}_\rightarrow g$, $\mathbf{t}(g) = \mathbf{t}_\rightarrow\mathbf{t}_\downarrow g = \mathbf{t}_\downarrow\mathbf{t}_\rightarrow g$, and[89]

$$g\cdot h = (g\cdot_\downarrow 1_\rightarrow(\mathbf{t}_\rightarrow h))\cdot_\rightarrow h = (g\cdot_\rightarrow 1_\downarrow(\mathbf{t}_\downarrow h))\cdot_\downarrow h$$
$$g^{-1} = (g)^{-1}_\rightarrow \cdot_\downarrow 1_\rightarrow(\mathbf{t}_\rightarrow g)^{-1} = (g)^{-1}_\downarrow \cdot_\rightarrow 1_\downarrow(\mathbf{t}_\downarrow g)^{-1}.$$

It comes with two groupoid maps

$$G_\downarrow \xleftarrow{\mathbf{t}_\rightarrow} F \xrightarrow{\mathbf{t}_\downarrow} G_\rightarrow.$$

Then, (vertically) core-transitive double groupoids are those for which the map

$$\mathbf{t}_\downarrow : F \to G_\rightarrow$$

is a surjective submersion.

[89]That is, we right-translate g to be composable with h.

Following [BM92] (or rather generalising their result) we will show that this type of double groupoid can be described in a much more efficient way using, what we call here, a *core extension*.

A.1. **From core-transitive double groupoids to core extensions.** Let G be a vertically core-transitive double groupoid. Then G comes with a bundle of Lie groups suggestively denoted by H, which is the kernel of the map $F \to G_{\to}$. Notice that there is a natural conjugation action of $G_{\downarrow}$ on H. Namely, for $g \in G_{\downarrow}$ and $h_x \in \mathrm{H}$, such that $\mathbf{s}_{\downarrow} g = x$, we can set

$$g \cdot h_x := 1_{\to}(g) \cdot_{\downarrow} h_x \cdot_{\downarrow} 1_{\to}(g^{-1}).$$

Notice that this action defines a "bundle of Lie groups representation" on the bundle of Lie groups H in the following sense.

Definition A.1. Let $G \rightrightarrows M$ be a Lie groupoid, and consider a bundle of Lie groups $\mathrm{H} \rightrightarrows M$. An action of G on H is called a *bundle of Lie groups representation* if the action is by Lie group automorphisms. That is, given $g \in G$ and $h_x, h'_x \in \mathrm{H}$ with $\mathbf{s}g = x$,

$$g \cdot (h_x \cdot h'_x) = (g \cdot h_x) \cdot (g \cdot h'_x).$$

We actually defined now three structures from a vertically core-transitive double groupoid G — a short exact sequence of Lie groupoids

$$1 \longrightarrow \mathrm{H} \longrightarrow F \longrightarrow G_{\to} \longrightarrow 1,$$

a Lie groupoid map $F \to G_{\downarrow}$, and a bundle of Lie groups representation $G_{\downarrow} \circlearrowright \mathrm{H}$. We summarise this data into a diagram as follows

$$\begin{tikzcd} 1 \arrow[r] & \mathrm{H} \arrow[r] \arrow[d, phantom, "\circlearrowleft"] & F \arrow[r] \arrow[dl] & G_{\to} \arrow[r] & 1. \\ & G_{\downarrow} & & & \end{tikzcd}$$

There are two compatibility assumptions between the three structures described above that are reminiscent of the axioms of a so-called *crossed module* of Lie groupoids. Firstly, the map $F \to G_{\downarrow}$ restricted to H is *equivariant*. By this we mean that the map turns the action of $G_{\downarrow}$ on H into the usual conjugation of $G_{\downarrow}$:[90]

$$\mathbf{t}_{\to}(g \cdot h_x) = g \cdot (\mathbf{t}_{\to} h_x) \cdot g^{-1}.$$

Secondly, the map $F \to G_{\downarrow}$ defines an action of F on H. The following lemma shows that this action equals the conjugation action of F on H (viewing $\mathrm{H} \subset F$ as a normal subgroupoid).

Lemma A.2. *Given a vertically core-transitive double groupoid G, we have for all $g \in F$ and $h_x \in \mathrm{H}$ with $\mathbf{s}g = x$,*[91]

$$1_{\to}(\mathbf{t}_{\to} g) \cdot_{\downarrow} h_x \cdot_{\downarrow} 1_{\to}(\mathbf{t}_{\to} g)^{-1} = g \cdot h \cdot g^{-1}.$$

We call this the Peiffer identity.

[90]Notice that H maps into the isotropy of $G_{\downarrow}$.

[91]On the right hand side the product of F is used.

Proof. This follows by the interchange law. We do a direct computation:

$$\begin{aligned} g \cdot h_x \cdot g^{-1} &= (g \cdot_\downarrow h_x \cdot_\downarrow 1_\to(\mathbf{t}_\to g)^{-1}) \cdot_\to ((g)^{-1}_\to \cdot_\downarrow 1_\to(\mathbf{t}_\to g)^{-1}) \\ &= (g \cdot_\to (g)^{-1}_\to) \cdot_\downarrow h_x \cdot_\downarrow 1_\to(\mathbf{t}_\to g)^{-1} = 1_\to(\mathbf{t}_\to g) \cdot_\downarrow h_x \cdot_\downarrow 1_\to(\mathbf{t}_\to g)^{-1}. \end{aligned}$$

This proves the statement. □

The terminology is chosen to reflect that the identity closely resembles the Peiffer identity for crossed modules of Lie groupoids. That is, the structure we arrived at is both a generalisation of that of core diagrams in [BM92] and that of crossed modules (see, for example, [Mac89, And05, LGW08]). We call it a *core extension* in analogy with our definition of fat extension.

Definition A.3. A *core extension* is a diagram

$$\begin{tikzcd} 1 \arrow[r] & \mathrm{H} \arrow[r] \arrow[d, phantom, "\circlearrowleft"] & F \arrow[r] \arrow[dl] & G_\to \arrow[r] & 1 \\ & G_\downarrow & & & \end{tikzcd}$$

such that the restriction of the map $F \to G_\downarrow$ to H is equivariant (with respect to conjugation in $G_\downarrow$) and the Peiffer identity holds: the conjugation action of F on H, viewing H as a normal subgroupoid of F, equals the action by F on H induced by the bundle of Lie groups representation $G_\downarrow \circlearrowright \mathrm{H}$ and the map $F \to G_\downarrow$.

We usually simply write F for a core extension. Notice that, in a core extension F, the orbits of F are the same as the orbits of $G_\to$.

Since a double groupoid might be both horizontally and vertically core-transitive, we sometimes call the associated core extension of a vertically core-transitive double groupoid the *vertical core extension*.

Remark A.4. It is worthwhile to observe that a strict Lie 2-groupoid $H_G \rightrightarrows H_M \rightrightarrows N$ is always vertically core-transitive. Writing F_H for its core, the above exact sequence for H_G is given by

$$1 \longrightarrow F_H \longrightarrow F_H \longrightarrow N \longrightarrow 1.$$

So, the vertical core extension of a strict Lie 2-groupoid is completely determined by a bundle of Lie groups $F_H \rightrightarrows N$ and a Lie groupoid $H_M \rightrightarrows N$, together with a Lie groupoid map $F_H \to H_M$, and a bundle of Lie groups representation $H_M \circlearrowright F_H$. This recovers the concept of crossed module of Lie groupoids. This definition is used in, for example, [Mac89, And05, LGW08]. As we will see in the next section, Section A.2, the one-to-one correspondence we set between vertically core-transitive double groupoids and core extensions reduces to a one-to-one correspondence between strict Lie 2-groupoids and crossed modules of Lie groupoids (F_H, H_M) (where F_H is a bundle of Lie groups). See also Remark A.10.

A.2. **From core extensions to core-transitive double groupoids.** In this section we show that we can recover a vertically core-transitive double groupoid from its vertical core extension. As we already mentioned before, this sets a one-to-one correspondence. For a large and essential part, this was proved already in [BM92] (see especially the last comments in that work). There, the focus lies on transitive double groupoids.[92] These are double groupoids such that all four groupoids defining the double groupoid are assumed to be transitive. Additionally, they are assumed to be horizontally

[92] In [BM92], transitive groupoids are called locally trivial groupoids.

core-transitive and vertically core-transitive. However, it is observed at the end of the work [BM92] that generalisations should be possible for vertically core-transitive double groupoids that satisfy another assumption. Writing G for the double groupoid as before, the assumption is written in [BM92] as follows: all elements of G can be written as

$$g_1 \cdot_\downarrow 1_\rightarrow(g_2) \cdot_\downarrow (g_3)_\downarrow^{-1}$$

where $g_1, g_3 \in F$ and $g_2 \in G_\downarrow$. However, essentially, this assumption is equivalent to the vertically core-transitive assumption:[93]

Lemma A.5. *Let G be a double groupoid. Then G is vertically core-transitive if and only if*

$$F \times_M G_\downarrow \times_M F \to G \qquad (g_1, g_2, g_3) \mapsto g_1 \cdot_\downarrow 1_\rightarrow(g_2) \cdot_\downarrow (g_3)_\downarrow^{-1}$$

is a surjective submersion.

Proof. A local section σ of $\mathbf{t}_\downarrow : F \to G_\rightarrow$ defines a local section of the above map by setting

$$g \mapsto (g \cdot_\downarrow \sigma(\mathbf{s}_\downarrow g) \cdot_\downarrow 1_\rightarrow(\mathbf{s}_\rightarrow g)^{-1}, \mathbf{s}_\rightarrow g, \sigma(\mathbf{s}_\downarrow g)).$$

Given $(g_1, g_2, g_3) \in F \times_M G_\downarrow \times_M F$, a local section of $F \to G_\rightarrow$ through g_3 defines (around the element $g_1 \cdot_\downarrow 1_\rightarrow(g_2) \cdot_\downarrow (g_3)_\downarrow^{-1} \in G$) a local section through (g_1, g_2, g_3). A local section σ of the above map defines a local section of $F \to G_\rightarrow$ by setting

$$g \mapsto \mathrm{pr}_1 \sigma 1_\downarrow(g) = \mathrm{pr}_3 \sigma 1_\downarrow(g).$$

This proves the statement. □

That is, in Remark 4.3 of [BM92], assumptions (4.3.1) and (4.3.2) are equivalent. The relevance of the above lemma becomes clear after the following observation. The pullback groupoid of $G_\downarrow$ along the source map of F defines a double groupoid

$$\begin{tikzcd} F \times_M G_\downarrow \times_M F \arrow[r, shift left] \arrow[r, shift right] \arrow[d, shift left] \arrow[d, shift right] & G_\downarrow \arrow[d, shift left] \arrow[d, shift right] \\ F \arrow[r, shift left] \arrow[r, shift right] & M \end{tikzcd}$$

The vertical groupoid structure takes the form $\mathbf{t}_\downarrow(g_1, g_2, g_3) = g_1$, $\mathbf{s}_\downarrow(g_1, g_2, g_3) = g_3$, and

$$(g_1, g_2, g_3) \cdot_\downarrow (g_3, g_4, g_5) = (g_1, g_2 g_4, g_5).$$

The horizontal groupoid structure $F \times_M G_\downarrow \times_M F \rightrightarrows G_\downarrow$ is given by $\mathbf{t}_\rightarrow(g_1, g_2, g_3) = (\mathbf{t}_\rightarrow g_1) g_2 (\mathbf{t}_\rightarrow g_3)^{-1}$, $\mathbf{s}_\rightarrow(g_1, g_2, g_3) = g_2$, and

$$(g_1, g_2, g_3) \cdot_\rightarrow (h_1, h_2, h_3) = (g_1 h_1, (\mathbf{t}_\rightarrow h_1)^{-1} g_2 (\mathbf{t}_\rightarrow h_3) = h_2, g_3 h_3).$$

A direct computation shows:

Lemma A.6. *Let G be a vertically core-transitive double groupoid. Then the map*

$$F \times_M G_\downarrow \times_M F \to G \qquad (g_1, g_2, g_3) \mapsto g_1 \cdot_\downarrow 1_\rightarrow(g_2) \cdot_\downarrow (g_3)_\downarrow^{-1}$$

is a map of double groupoids that is a surjective submersion.

[93] A similar remark was already made in [JLM21].

Proof. That the map intertwines the vertical products is readily verified. We do a direct computation to verify that the map intertwines the horizontal products. Suppose (g_1, g_2, g_3) and (h_1, h_2, h_3) in $F \times_M G_\downarrow \times_M F$ are horizontally composable. Then, first of all,

$$\begin{aligned}
&(g_1 \cdot_\downarrow 1_\to(g_2) \cdot_\downarrow (g_3)_\downarrow^{-1}) \cdot_\to (h_1 \cdot_\downarrow 1_\to(h_2) \cdot_\downarrow (h_3)_\downarrow^{-1}) \\
&\qquad = ((g_1 \cdot_\downarrow 1_\to(\mathbf{t}_\to h_1)) \cdot_\to h_1) \cdot_\downarrow ((1_\to((\mathbf{t}_\to h_1)^{-1} g_2) \cdot_\downarrow (g_3)_\downarrow^{-1}) \cdot_\to (1_\to(h_2) \cdot_\downarrow (h_3)_\downarrow^{-1})) \\
&\qquad = (g_1 h_1) \cdot_\downarrow ((1_\to((\mathbf{t}_\to h_1)^{-1} g_2) \cdot_\downarrow (g_3)_\downarrow^{-1}) \cdot_\to (1_\to(h_2) \cdot_\downarrow (h_3)_\downarrow^{-1})),
\end{aligned}$$

where in the first equality we replaced $g_1 \cdot_\downarrow 1_\to(g_2)$ with

$$g_1 \cdot_\downarrow 1_\to(\mathbf{t}_\to h_1) \cdot_\downarrow 1_\to((\mathbf{t}_\to h_1)^{-1} g_2)$$

and used the interchange law. We now use that $(\mathbf{t}_\to h_1)^{-1} g_2 = h_2 (\mathbf{t}_\to h_3)^{-1}$ and conclude:

$$\begin{aligned}
&(g_1 \cdot_\downarrow 1_\to(g_2) \cdot_\downarrow (g_3)_\downarrow^{-1}) \cdot_\to (h_1 \cdot_\downarrow 1_\to(h_2) \cdot_\downarrow (h_3)_\downarrow^{-1}) \\
&\qquad = (g_1 h_1) \cdot_\downarrow ((1_\to(h_2) \cdot_\downarrow 1_\to(\mathbf{t}_\to h_3)^{-1} \cdot_\downarrow (g_3)_\downarrow^{-1}) \cdot_\to (1_\to(h_2) \cdot_\downarrow (h_3)_\downarrow^{-1})) \\
&\qquad = (g_1 h_1) \cdot_\downarrow 1_\to(h_2) \cdot_\downarrow ((g_3 \cdot_\downarrow 1_\to(\mathbf{t}_\to h_3))_\downarrow^{-1} \cdot_\to (h_3)_\downarrow^{-1}) \\
&\qquad = (g_1 h_1) \cdot_\downarrow 1_\to(h_2) \cdot_\downarrow (g_3 h_3)_\downarrow^{-1}.
\end{aligned}$$

That the map is a surjective submersion was already proved in Lemma A.5. □

Now, we can realise G as a quotient of $F \times_M G_\downarrow \times_M F$ by the horizontal kernel of the above map.

Remark A.7. Given a map of double groupoids $\Phi : G \to G'$ that is a surjective submersion, notice that its horizontal kernel $K := \ker_\to \Phi$ can be seen as a double subgroupoid of G. Here,

$$\ker_\to \Phi := \{g \in G \mid \Phi(g) = \operatorname{im} 1_\to\}.$$

This is a consequence of the interchange law. We can then see the resulting exact sequence

$$1 \longrightarrow K \longrightarrow G \xrightarrow{\Phi} G' \longrightarrow 1$$

as a short "horizontally" exact sequence of double groupoids. More precisely, all maps in the exact sequence are maps of double groupoids, but the sequence is only exact with respect to the horizontal products.

The map $F \times_M G_\downarrow \times_M F \to G$ from Lemma A.5, seen as a groupoid map of the horizontal products, has kernel isomorphic to the action groupoid

$$G_\downarrow \ltimes \mathrm{H} \rightrightarrows \mathrm{H},$$

where the action is given by

$$g \cdot h_x = 1_\to(g) \cdot_\downarrow h_x \cdot_\downarrow 1_\to(g)^{-1}.$$

Its structure of double groupoid is

$$\begin{array}{ccc}
G_\downarrow \ltimes \mathrm{H} & \rightrightarrows & G_\downarrow \\
\downarrow\downarrow & & \downarrow\downarrow \\
\mathrm{H} & \rightrightarrows & M
\end{array}$$

where $G_\downarrow \times_M \mathrm{H} \rightrightarrows G_\downarrow$ is seen as a bundle of Lie groups.

Proposition A.8. *Consider the map $F \times_M G_\downarrow \times_M F \to G$ from Lemma A.6. This map, together with the map*

$$G_\downarrow \ltimes \mathrm{H} \to F \times_M G_\downarrow \times_M F \qquad (g, h_x) \mapsto (1_\to(g) \cdot_\downarrow h_x \cdot_\downarrow 1_\to(g)^{-1}, g, h_x)$$

fit into a short horizontally exact sequence of double Lie groupoids (see Remark A.7)

$$1 \longrightarrow G_\downarrow \ltimes \mathrm{H} \longrightarrow F \times_M G_\downarrow \times_M F \longrightarrow G \longrightarrow 1.$$

Proof. The horizontal kernel consists of elements (g_1, g_2, g_3) satisfying

$$g_1 \cdot_\downarrow 1_\to(g_2) \cdot_\downarrow (g_3)_\downarrow^{-1} = 1_\to(g_2),$$

so $g_1, g_3 \in H$, and

$$g_1 = 1_\to(g_2) \cdot_\downarrow g_3 \cdot_\downarrow 1_\to(g_2)^{-1}.$$

This proves that the horizontal kernel can indeed be identified with $G_\downarrow \ltimes \mathrm{H}$ as above. The map of Lemma A.6 induces an isomorphism of double groupoids

$$G(F) \to G \qquad [g_1, g_2, g_3] \mapsto g_1 \cdot_\downarrow 1_\to(g_2) \cdot_\downarrow (g_3)_\downarrow^{-1},$$

where $G(F)$ is the double groupoid

$$G(F) \coloneqq (F \times_M G_\downarrow \times_M F)/(G_\downarrow \ltimes \mathrm{H}) \rightrightarrows F/\mathrm{H} = G_\to$$

over $G_\downarrow \rightrightarrows M$. This proves the statement. □

Now, the structure being used to define

$$G(F) = (F \times_M G_\downarrow \times_M F)/(G_\downarrow \ltimes \mathrm{H})$$

uses precisely the data underlying the vertical core extension of G. To ensure $G(F)$ is defined for any core extension F, the compatibility assumptions are used in the following way. Firstly, the action groupoid $G_\downarrow \ltimes \mathrm{H} \rightrightarrows \mathrm{H}$ is a double groupoid over $G_\downarrow$ precisely because the action defines a bundle of Lie groups representation (see Definition A.1). The equivariance condition makes sure that the embedding

$$G_\downarrow \ltimes \mathrm{H} \to F \times_M G_\downarrow \times_M F$$

is a map of double groupoids. The Peiffer identity ensures this groupoid is actually normal inside of $F \times_M G_\downarrow \times_M F$. We now conclude:

Theorem A.9. *The association*

$$\{\text{Vertically core-transitive double Lie groupoids}\} \xrightarrow{\sim} \{\text{Vertical core extensions}\} \qquad G \mapsto F$$

sets a one-to-one correspondence, and the assignment

$$\{\text{Vertical core extensions}\} \xrightarrow{\sim} \{\text{Vertically core-transitive double Lie groupoids}\} \qquad F \mapsto G(F)$$

is inverse to it (up to canonical isomorphisms).

Proof. We already showed that, given G a vertically core-transitive double groupoid, G and $G(F)$ are isomorphic, where F is the vertical core extension of G. So, it remains to observe that, given a core extension F, the core groupoid of $G(F)$ can be identified with $F \ni g$ as the classes of elements represented by $[g, 1_{\mathbf{s}g}, 1_{\mathbf{s}g}]$. That we recover the core extension is now readily verified. □

Remark A.10. Restricting attention to strict Lie 2-groupoids yields an interesting consequence: they correspond to crossed modules of Lie groupoids (F_H, H_M) for which F_H is a bundle of Lie groups (see Remark A.4).

The whole discussion also applies to horizontally core-transitive double Lie groupoids (by interchanging horizontal and vertical structures everywhere). From our discussion it follows that *double core-transitive groupoids* (so double groupoids that are both horizontally and vertically core-transitive) are isomorphic to either of the two double groupoids that its core extensions define. However, starting with two core extensions, it is not immediate that the two resulting double groupoids one constructs are isomorphic. Therefore, an extra compatibility assumption between the two core extensions appears. This leads to the following definition:

Definition A.11. A *double core extension*, or core diagram, is a diagram

$$\begin{tikzcd}
1 \arrow[r] & \mathrm{H}_{\downarrow} \arrow[dr] & & \mathrm{H}_{\rightarrow} \arrow[dl] & 1 \arrow[l] \\
& & F \arrow[dl] \arrow[dr] & & \\
1 & G_{\downarrow} \arrow[l] & & G_{\rightarrow} \arrow[r] & 1
\end{tikzcd}$$

of exact sequences such that $\mathrm{H}_{\downarrow}$ and $\mathrm{H}_{\rightarrow}$ commute elementwise in F: for all $h_{x,\downarrow} \in \mathrm{H}_{\downarrow}$ and $h_{x,\rightarrow} \in \mathrm{H}_{\rightarrow}$ we have that

$$h_{x,\downarrow} h_{x,\rightarrow} = h_{x,\rightarrow} h_{x,\downarrow}$$

as elements of F.

Remark A.12. The above compatibility condition (of "commuting elementwise") is written in [BM92]. We did not specify the actions in the above definition, for specifying the actions in a double core extension is unnecessary (this should be compared to Lemma 2.6 of [BM92]). Indeed, given two core extensions

$$\begin{tikzcd}
1 \arrow[r] & \mathrm{H}_{\downarrow} \arrow[dr] \arrow[d, phantom, "\circlearrowleft"] & & \mathrm{H}_{\rightarrow} \arrow[dl] \arrow[d, phantom, "\circlearrowright"] & 1 \arrow[l] \\
& {} & F \arrow[dl] \arrow[dr] & {} & \\
1 & G_{\downarrow} \arrow[l] & & G_{\rightarrow} \arrow[r] & 1
\end{tikzcd}$$

satisfying that elements of $\mathrm{H}_{\downarrow}$ and $\mathrm{H}_{\rightarrow}$ commute elementwise in F, then, for example, the action of $G_{\rightarrow}$ on $H_{\rightarrow}$ is given by

$$g \cdot h_x = \widetilde{g} \cdot h_x \cdot \widetilde{g}^{-1}$$

where $\widetilde{g} \in F$ is arbitrary with the property that $\widetilde{g}$ maps to g under the map $F \to G_{\rightarrow}$. That this action is well-defined uses the compatibility between the two core extensions. The newly defined actions in terms of lifts therefore determine the actions that are part of the data of the given core extensions. As equivariance and the Peiffer identity are immediate from this new description of the action, a double core extension can be defined as in Definition A.11.

From our discussion we can conclude (this result was already proved in [JLM21]):

Corollary A.13. *There is a one-to-one correspondence*

$$\{\text{Double core-transitive groupoids}\} \xrightarrow{\sim} \{\text{Double core extensions}\} \qquad G \mapsto F.$$

Remark A.14. In [BM92], the focus lies on "double transitive groupoids", so horizontally and vertically transitive (and core-transitive) double groupoids. To recover the main result of [BM92] on double transitive groupoids, recall that, in a core extension

$$\begin{tikzcd} 1 \arrow[r] & \mathrm{H} \arrow[r] \arrow[loop below, phantom, "\circlearrowleft"] & F \arrow[r] \arrow[dl] & G_{\to} \arrow[r] & 1 \\ & G_{\downarrow} & & & \end{tikzcd}$$

the orbits of F are equal to the orbits of $G_{\to}$. Therefore, F is transitive if and only if $G_{\to}$ is, and we call core extensions for which this holds transitive. The resulting double groupoid is vertically transitive, and this sets an equivalence between transitive vertical core extensions and vertically transitive (that are also vertically core-transitive) double groupoids. Using Corollary A.13, this reproduces the result of [BM92] on double transitive groupoids (namely that they correspond to double transitive core extensions).

Remark A.15. A class of examples of double groupoids which are usually not (core-)transitive are given by commuting groupoid actions $G \circlearrowright P \circlearrowleft H$. Given such commuting actions, we have a double groupoid

$$\begin{tikzcd} G \ltimes P \rtimes H \arrow[r, shift left] \arrow[r, shift right] \arrow[d, shift left] \arrow[d, shift right] & P \rtimes H \arrow[d, shift left] \arrow[d, shift right] \\ G \ltimes P \arrow[r, shift left] \arrow[r, shift right] & P \end{tikzcd}$$

The core is the unit groupoid 1_P, and the core data consists further of the unit embeddings

$$G \ltimes P \hookleftarrow P \hookrightarrow P \rtimes H.$$

These maps carry no information about the actions. In this case, rather than considering the core, we can view $G \ltimes P \rtimes H$ as a groupoid over P, and there are two Lie groupoid maps

$$G \leftarrow G \ltimes P \rtimes H \rightarrow H.$$

It is well-known that there is a natural way to set an equivalence between left/right principal bibundles (commutating actions, one of which is principal) and such diagrams $G \leftarrow K \rightarrow H$ where one is a Morita map (see [dH13]).

We now go back to PB-groupoids, and show that PB-groupoids (that come with a gauge double groupoid) can be efficiently described using core extensions.

A.3. **PB-groupoids and core extensions.** Principal groupoid bundles do not always come with a smooth gauge groupoid. A typical assumption that guarantees the smoothness of the gauge groupoid is that the moment map is a surjective submersion (see e.g. [dH13]). General linear PB-groupoids come with a gauge double groupoid if the differential has constant rank. We focus here on the following class of PB-groupoids:

Definition A.16. let P_G be a PB-groupoid. If the moment map(s) of P_G are surjective submersions onto a smooth embedded submanifold, then we call P_G a *PB-groupoid with gauge double groupoid.*

To be clear, PB-groupoids with gauge double groupoid really come with a gauge double groupoid:

Proposition A.17. *Let $P_G \rightrightarrows P_M$ be a PB-groupoid with gauge double groupoid. Then*

$$P_G \times_{H_G} P_G \rightrightarrows P_M \times_{H_M} P_M$$

is a Lie groupoid with $\mathbf{s}[p_g, p_h] := [\mathbf{s}p_g, \mathbf{s}p_h]$, $\mathbf{t}[p_g, p_h] = [\mathbf{t}p_g, \mathbf{t}p_h]$, and

$$[p_g, p_h] \cdot [p_{g'}, p_{h'}] := [p_g \cdot p_{g'}, p_h \cdot p_{h'}] \qquad [p_g, p_h]^{-1} = [p_g^{-1}, p_h^{-1}].$$

(here, $\mathbf{s}p_g = \mathbf{t}p_{g'}$ and $\mathbf{s}p_h = \mathbf{t}p_{h'}$). Moreover,

$$\begin{tikzcd} P_G \times_{H_G} P_G \arrow[r, shift left] \arrow[r, shift right] \arrow[d, shift left] \arrow[d, shift right] & P_M \times_{H_M} P_M \arrow[d, shift left] \arrow[d, shift right] \\ G \arrow[r, shift left] \arrow[r, shift right] & M \end{tikzcd}$$

is a full double groupoid, called the gauge double groupoid of P_G.

Proof. We prove here that the above double groupoid is full. That is, the double source map

$$(\mathbf{s}_\to, \mathbf{s}_\downarrow) : P_G \times_{H_G} P_G \to (P_M \times_{H_M} P_M) \times_M G$$

is a surjective submersion. Indeed, to construct a section through $[p_g, p_h] \in P_G \times_{H_G} P_G$, first pick a local section σ of $P_M \to M$. Then, take a local section σ of $P_G \to G$, which is projectable to the local section σ of $P_M \to M$, and goes through

$$p_h \cdot 1_{\Psi(\sigma(\mathbf{s}h), \mathbf{s}p_h)^{-1}},$$

where we used

$$\Psi : P_M \times_M P_M \xleftarrow{\sim} P_M \times_N H_M \to H_M.$$

Lastly, we pick a local section σ_μ of the moment map $\mu : P_G \to N$ through p_g (defined on an open set in the image of μ). This way, we can build the local section

$$(P_M \times_N P_M) \times_M G \to P_G \times_N P_G \qquad (p_x, p_{\mathbf{s}k}, k) \mapsto (\sigma_\mu(\mu p_x) \cdot 1_{\Psi(\mathbf{s}\sigma_\mu(\mu p_x), p_x)}, \sigma(k) \cdot 1_{\Psi(\sigma(\mathbf{s}k), p_{\mathbf{s}k})})$$

which induces the desired well-defined local section through $[p_g, p_h] \in P_G \times_{H_G} P_G$. □

We will argue next that the gauge groupoid of a PB-groupoid with gauge double groupoid is always vertically core-transitive. Basically, this follows from the fact that the structural strict Lie 2-groupoid is always vertically core-transitive (see Remark A.4).

Lemma A.18. *Let $P_G \rightrightarrows P_M$ be a PB-groupoid with gauge double groupoid, and denote by $H_G \rightrightarrows H_M \rightrightarrows N$ its structural strict Lie 2-groupoid. Then its gauge double groupoid*

$$\begin{tikzcd} P_G \times_{H_G} P_G \arrow[r, shift left] \arrow[r, shift right] \arrow[d, shift left] \arrow[d, shift right] & P_M \times_{H_M} P_M \arrow[d, shift left] \arrow[d, shift right] \\ G \arrow[r, shift left] \arrow[r, shift right] & M \end{tikzcd}$$

is vertically core-transitive.

Proof. We write F_H for the core of H_G and F for the core of the gauge double groupoid. Note that

$$F = \{[p_g, q_{1_{\mathbf{s}g}}] \in P_G \times_{H_G} P_G \mid \mathbf{s}p = \mathbf{s}q \in P_M\}.$$

Now, let σ be a (local) section of $P_G \times_N P_G \to G \times G$ that is $\mathbf{s} \times \mathbf{s}$-projectable to a section of $P_M \times_N P_M \to M \times M$, and consider a section of $\mathbf{t}_\downarrow : F_H \to N$. Define

$$a : G \times G \xrightarrow{\sigma} P_G \times_N P_G \to N \to F_H.$$

Given $[p_g, q_{1_{\mathbf{s}g}}] \in F$, we take σ defined on an open set of $(g, 1_{\mathbf{s}g})$, and such that $\sigma(g, 1_{\mathbf{s}g}) = (p_g, q_{1_{\mathbf{s}g}})$. Then the pointwise multiplication map

$$\sigma \cdot a : G \times G \to P_G \times_{H_G} P_G$$

defines through

$$G \hookrightarrow G \times 1 \xrightarrow{\sigma \cdot a} F$$

the desired section of $F \to G$ through $[p_g, q_{1_{\mathbf{s}g}}] \in F$. This proves the statement. □

In turn, a PB-groupoid with gauge double groupoid comes with a vertical core extension. We can describe it as follows.

Definition A.19. Let $P_G \rightrightarrows P_M$ be a PB-groupoid with gauge double groupoid. The core extension of P_G is the principal groupoid bundle P_M together with the core extension

$$\begin{tikzcd}
1 \arrow[r] & \mathrm{H} \arrow[r] \arrow[d, phantom, "\circlearrowright"] & F \arrow[r] \arrow[dl] & G \arrow[r] & 1. \\
& P_M \times_{H_M} P_M & & &
\end{tikzcd}$$

Remark A.20. In the above discussion, the core groupoid F is isomorphic to

$$P_G/H_M \rightrightarrows P_M/H_M = M$$

where H_M acts on P_G via the unit embedding $H_M \hookrightarrow H_G$. The isomorphism is given by

$$P_G/H_M \to F \qquad [p] \mapsto [p, 1_{\mathbf{s}p}].$$

This should be compared with Section 5.5.

If P_G is a regular general linear PB-groupoid, then the gauge groupoid associated to the principal groupoid bundle $P_M = \mathrm{GL}(C_M, V_M)$ is the Lie groupoid $\mathrm{Aut}(C_M \to V_M)$. To tie everything together, we end this section with the following result.

Proposition A.21. *Let $F_G \rightrightarrows M$ be a Lie groupoid, and let $C_M \to V_M$ be a regular 2-term complex. Consider the canonical (conjugation) action* $\mathrm{Aut}(C_M \to V_M) \circlearrowright \mathrm{H}(C_M, V_M)$. *Then F_G defines a fat extension of G over $C_M \to V_M$ if and only if*

$$\begin{tikzcd}
1 \arrow[r] & \mathrm{H}(V_M, C_M) \arrow[r] \arrow[d, phantom, "\circlearrowright"] & F_G \arrow[r] \arrow[dl] & G \arrow[r] & 1, \\
& \mathrm{Aut}(C_M \to V_M) & & &
\end{tikzcd}$$

defines a core extension.

Proof. The equivariance condition is equivalent to F_G restricting to the canonical cochain complex representation, and the Peiffer identity translates to the fact that the two natural conjugation actions of F_G on $\mathrm{H}(V_M, C_M)$ agree. This proves the statement. □

Remark A.22. That the action in a core extension

$$\begin{tikzcd}
1 \arrow[r] & \mathrm{H} \arrow[r] \arrow[d, phantom, "\circlearrowright"] & F \arrow[r] \arrow[dl] & G_{\to} \arrow[r] & 1 \\
& G_{\downarrow} & & &
\end{tikzcd}$$

defines a bundle of Lie groups representation uses, of course, the group structure of $G_\downarrow$ explicitly. But the equivariance and Peiffer identity use the groupoid structure $G_\downarrow$ in terms of the groupoid map $F \to G_\downarrow$ only.

While the regularity of the complex in the above statement is used to ensure that $\mathrm{Aut}(C_M \to V_M)$ is a smooth Lie groupoid, this is irrelevant for the description of a fat extension F_G. The reason is that we can "bypass" $\mathrm{Aut}(C_M \to V_M)$ by embedding $\mathrm{Aut}(C_M \to V_M)$ into, for example, $\mathrm{GL}(C_M, V_M)$. Although the conjugation action of $\mathrm{GL}(C_M, V_M)$ on $\mathrm{H}(V_M, C_M)$ is not well-defined, it is well-defined on the image of $F_G \to \mathrm{GL}(C_M, V_M)$. So, a fat extension is, in essence, a type of core extension, but only when interpreting the concept in such a more general form.

In general, the gauge double groupoid of a PB-groupoid with gauge double groupoid is encoded by a core extension. But the above remark suggests that at least general linear PB-groupoids are also encoded by a type of "core extension": fat extensions. Investigating this definition for PB-groupoids in more general setups (in the style of the previous remark) seems an interesting direction that will be pursued elsewhere.

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