A SOLUTION TO MENEGUETTE'S POLYNOMIAL PROBLEM

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ABSTRACT. In 1979, Jeltsch [Numer. Math. 32 (1979), 167–181] conjectured that every zero-stable Brown method is stiffly stable. For Brown (K, L) methods, Meneguette subsequently reduced the relevant A_0-stability question to a zero-location property for a family of perturbed characteristic polynomials. A closely related perturbation question already appeared in his 1987 Oxford doctoral thesis, and he later posed a broader purely polynomial version of the problem in 1994 [SIAM Review 36 (1994), 656–657]. We solve Meneguette's polynomial problem in its original form. More precisely, we prove the stronger statement that, even when all the zeros of the original polynomial are allowed to lie in the closed unit disc, every positive perturbation of its leading coefficient yields a polynomial all of whose zeros lie in the open unit disc. Our argument is entirely polynomial. Combined with Meneguette's coefficient and strong-stability results, the theorem establishes Jeltsch's conjecture for the corresponding subclass of zero-stable Brown methods.

1. Introduction and main result

Questions concerning the location of polynomial zeros are often elementary to formulate but unexpectedly delicate. The question considered here arose in the stability theory of Brown's multistep multiderivative methods.

In 1979, Jeltsch conjectured that every zero-stable Brown method is stiffly stable [3]. Meneguette later reduced the relevant A_0-stability question for Brown (K, L) methods to the location of the zeros of a family of perturbed characteristic polynomials [4, Theorem 2.13 and Remark 2.16, pp. 81–82]; see also [4, pp. 105–107]. He also established the coefficient monotonicity required in that setting [4, Theorem 2.9, pp. 66–67]; see also [2, Section 5].

Within this framework, the question has its roots in Meneguette's 1987 Oxford doctoral thesis [4, pp. 105–106]. He returned to it in 1994, recasting it as the broader, purely polynomial problem stated below [5, Problem 94-19]. Let

$$P(z) = a_0 + a_1 z + \cdots + a_{n-1} z^{n-1} + a_n z^n,$$

where

$$0 < a_0 \leq a_1 \leq \cdots \leq a_{n-1}, \quad a_n > 0.$$

Date: July 22, 2026.

2020 *Mathematics Subject Classification.* 30C15, 26C10, 65L20.

Key words and phrases. Polynomial zeros, coefficient perturbation, unit disc, Brown methods, zero stability.

Suppose that all the zeros of P lie in the unit disc and that

$$na_n > (n-1)a_{n-1}.$$

Must all the zeros of

$$P_d(z) := P(z) + dz^n$$

also lie in the unit disc for every $d > 0$?

The original statement does not specify whether "the unit disc" is meant in the open or closed sense. The theorem below removes this ambiguity: the original polynomial may have zeros on the unit circle, whereas every positive perturbation of its leading coefficient yields a polynomial whose zeros all lie strictly inside the unit circle.

In 2012, Botta, Meneguette, Cuminato and McKee restated the assertion as Meneguette's conjecture under the additional condition $a_n < a_{n-1}$ [2]. They established the desired conclusion for polynomials whose zeros lie in the closed unit disc and whose coefficients satisfy the stronger assumptions

$$0 < a_0 < a_1 < \cdots < a_{n-1}, \quad 0 < a_n < a_{n-1},$$

together with

$$a_j < ra_{j+1}, \quad j = 0, 1, \ldots, n-2,$$

and

$$a_n > ra_{n-1}$$

for some $r \in (0, 1)$. Under these hypotheses, they showed that, for every $\gamma > 0$, the polynomial

$$P(z) + \gamma z^n$$

has all its zeros in the open unit disc. They also applied this result to establish the corresponding case of Jeltsch's conjecture for a subclass of Brown methods.

In 2014, a complementary result was obtained by Botta, Marques and Meneguette [1] for the palindromic family

$$R(z) = 1 + \lambda\big(z + z^2 + \cdots + z^{n-1}\big) + z^n.$$

They completely characterised the values of λ for which the zeros of R lie on the unit circle and determined the exact perturbation threshold ensuring that the resulting polynomial has all its zeros in the closed unit disc. In the parameter range in which this palindromic family satisfies the hypotheses of Problem 94-19, their result applies to every positive perturbation of the leading coefficient.

The theorem below settles Problem 94-19 in the stronger form described above. Its proof is purely polynomial and does not revisit Meneguette's numerical-analysis results. A concluding remark explains how the theorem, combined with his coefficient estimates and strong-stability result, establishes Jeltsch's conjecture for zero-stable Brown (K, L) methods with $L \geq 10$ and $K \leq K_L$.

Throughout, we write

$$\mathbb{D} := \{z \in \mathbb{C} : |z| < 1\}, \quad \overline{\mathbb{D}} := \{z \in \mathbb{C} : |z| \le 1\}.$$

Theorem 1. *Let n be a positive integer and let*

$$P(z) = a_0 + a_1 z + \cdots + a_{n-1} z^{n-1} + a_n z^n,$$

where

$$0 < a_0 \le a_1 \le \cdots \le a_{n-1}, \quad a_n > 0.$$

Assume that all the zeros of P lie in $\overline{\mathbb{D}}$ and that

$$n a_n > (n-1) a_{n-1}.$$

Then, for every $d > 0$, all the zeros of

$$P_d(z) := P(z) + d z^n$$

lie in $\mathbb{D}$.

2. Proof of the main result

If $n = 1$, then the unique zero of P is $-a_0/a_1$. Since this zero lies in $\overline{\mathbb{D}}$, we have $a_0 \le a_1$. Therefore, the unique zero of

$$P_d(z) = a_0 + (a_1 + d) z$$

has modulus

$$\frac{a_0}{a_1 + d} < \frac{a_0}{a_1} \le 1,$$

and hence lies in $\mathbb{D}$. We may therefore assume that $n > 1$.

For $t \ge a_n$, consider the one-parameter family

$$P_t(z) := a_0 + a_1 z + \cdots + a_{n-1} z^{n-1} + t z^n.$$

Then

$$P_{a_n} = P, \quad P_{a_n + d} = P_d.$$

It is convenient to introduce the reciprocal polynomial

$$Q_t(w) := w^n P_t\left(\frac{1}{w}\right).$$

Since $a_0 > 0$, the polynomial Q_t has degree n and is given by

$$Q_t(w) = t + a_{n-1} w + a_{n-2} w^2 + \cdots + a_0 w^n.$$

For $j = 1, \ldots, n$, set

$$b_j := a_{n-j}.$$

Thus

$$Q_t(w) = t + \sum_{j=1}^{n} b_j w^j. \tag{1}$$

The monotonicity of the coefficients of P yields

$$b_1 \ge b_2 \ge \cdots \ge b_n > 0.$$

In particular, $b_1 = a_{n-1}$.

Since $a_0 > 0$ and $t > 0$, neither P_t nor Q_t has a zero at the origin. By the definition of Q_t, a non-zero complex number w is a zero of Q_t if and only if $1/w$ is a zero of P_t. Consequently, for every $t > a_n$, all the zeros of P_t lie in $\mathbb{D}$ if and only if every zero w of Q_t satisfies $|w| > 1$.

We shall use the following auxiliary lemma, whose proof is deferred to Section 3.

Lemma 2. *Let*

$$b_1 \geq b_2 \geq \cdots \geq b_n \geq 0,$$

and let $\theta \in (0, 2\pi)$. *If*

$$\sum_{j=1}^{n} b_j \sin(j\theta) = 0, \tag{2}$$

then

$$(n-1)b_1 + \sum_{j=1}^{n} (n-j) b_j \cos(j\theta) \geq 0. \tag{3}$$

Suppose that, for some $t \geq a_n$, the polynomial Q_t has a zero w on the unit circle. Thus

$$Q_t(w) = 0, \quad |w| = 1.$$

Write $w = e^{i\theta}$. The value $w = 1$ cannot occur, since

$$Q_t(1) = t + \sum_{j=1}^{n} b_j > 0.$$

We may therefore choose θ so that $0 < \theta < 2\pi$. Taking real and imaginary parts in the identity

$$t + \sum_{j=1}^{n} b_j e^{ij\theta} = 0,$$

we obtain

$$\sum_{j=1}^{n} b_j \sin(j\theta) = 0 \tag{4}$$

and

$$t + \sum_{j=1}^{n} b_j \cos(j\theta) = 0. \tag{5}$$

Set

$$S := \sum_{j=1}^{n} j b_j \cos(j\theta).$$

Applying Lemma 2 to (4), we find that

$$(n-1)b_1 + \sum_{j=1}^{n} (n-j) b_j \cos(j\theta) \geq 0. \tag{6}$$

On the other hand, by (5),

$$\sum_{j=1}^{n}(n-j)b_j\cos(j\theta) = -nt - S.$$

Consequently, (6) gives

$$S \le (n-1)b_1 - nt. \tag{7}$$

Since $b_1 = a_{n-1}$ and $t \ge a_n$, the hypothesis of the theorem yields

$$nt \ge na_n > (n-1)a_{n-1} = (n-1)b_1.$$

It follows from (7) that $S < 0$.

Differentiating (1) with respect to w, we obtain

$$Q_t'(w) = \sum_{j=1}^{n} jb_j w^{j-1}.$$

Since $w = e^{i\theta}$, multiplication by w gives

$$wQ_t'(w) = \sum_{j=1}^{n} jb_j e^{ij\theta}.$$

Taking real parts and recalling the definition of S, we find that

$$\operatorname{Re}\big(wQ_t'(w)\big) = \sum_{j=1}^{n} jb_j\cos(j\theta) = S.$$

Hence

$$\operatorname{Re}\big(wQ_t'(w)\big) < 0.$$

In particular, $wQ_t'(w) \ne 0$. Since $|w| = 1$, we have $w \ne 0$, and hence

$$Q_t'(w) \ne 0. \tag{8}$$

Thus every zero of Q_t on the unit circle is simple.

Let $t_0 \ge a_n$, and suppose that

$$Q_{t_0}(w_0) = 0, \quad |w_0| = 1.$$

Although the family was introduced for $t \ge a_n$, the same formula defines Q_t for every real t. By (8), the zero w_0 is simple. Applying the real implicit function theorem to the real and imaginary parts of Q_t, we obtain a unique local C^1 branch $w(t)$, defined for t in a neighbourhood of t_0, such that

$$w(t_0) = w_0, \quad Q_t(w(t)) = 0.$$

Differentiating $Q_t(w(t)) = 0$ with respect to t, we obtain

$$w'(t) = -\frac{1}{Q_t'(w(t))}. \tag{9}$$

Set $\rho(t) := |w(t)|^2$. Then

$$\rho'(t) = 2\operatorname{Re}\big(\overline{w(t)}\,w'(t)\big).$$

Since $|w_0| = 1$, using (9) we obtain

$$\rho'(t_0) = -2\,\frac{\operatorname{Re}\big(w_0 Q'_{t_0}(w_0)\big)}{\big|w_0 Q'_{t_0}(w_0)\big|^2}.$$

By the strict inequality obtained above, the numerator is strictly negative. Hence $\rho'(t_0) > 0$. Since $\rho(t_0) = 1$, there exists $\varepsilon > 0$ such that

$$\rho(t) > 1, \quad t_0 < t < t_0 + \varepsilon,$$

and

$$\rho(t) < 1, \quad t_0 - \varepsilon < t < t_0.$$

Thus, if a local zero branch meets the unit circle at $t = t_0$, then, as t increases through t_0, it crosses transversally from $|w| < 1$ to $|w| > 1$. Consequently, no zero branch can cross the unit circle in the opposite direction.

We next use this local crossing property to obtain the global conclusion. Since all the zeros of $P = P_{a_n}$ lie in $\overline{\mathbb{D}}$, reciprocity implies that every zero of Q_{a_n} satisfies $|w| \geq 1$. Some of these zeros may lie on the unit circle. Each such zero is simple, however, and the corresponding local branch satisfies

$$\frac{d}{dt}|w(t)|^2\bigg|_{t=a_n} > 0.$$

Let $\xi_1, \ldots, \xi_m$ be the distinct zeros of Q_{a_n} on the unit circle, with $m = 0$ if there are no such zeros. Each ξ_j is simple and hence lies on a unique local C^1 branch $\xi_j(t)$ satisfying

$$\xi_j(a_n) = \xi_j, \quad Q_t(\xi_j(t)) = 0,$$

and

$$\frac{d}{dt}|\xi_j(t)|^2\bigg|_{t=a_n} > 0.$$

Since there are only finitely many such zeros, there exists $\varepsilon_0 > 0$ such that

$$|\xi_j(t)| > 1, \quad a_n < t \leq a_n + \varepsilon_0, \quad j = 1, \ldots, m. \tag{10}$$

Choose pairwise disjoint closed discs $U_1, \ldots, U_m$, with U_j centred at ξ_j, such that each U_j contains no zero of Q_{a_n} other than ξ_j and the boundary of U_j contains no zero of Q_{a_n}. By decreasing ε_0 if necessary, we may assume that

$$\xi_j(t) \in \operatorname{int} U_j, \quad a_n \leq t \leq a_n + \varepsilon_0, \quad j = 1, \ldots, m.$$

Since

$$Q_t(w) - Q_{a_n}(w) = t - a_n,$$

we have

$$|Q_t(w) - Q_{a_n}(w)| = |t - a_n|$$

on ∂U_j. For each j, if t is sufficiently close to a_n, this quantity is strictly smaller than $|Q_{a_n}(w)|$ on ∂U_j. Rouché's theorem therefore shows that Q_t and Q_{a_n} have the same number of zeros, counted with multiplicity, inside U_j. Since ξ_j is a simple zero of Q_{a_n}, the polynomial Q_t has exactly one zero in

U_j. This zero is necessarily the branch $\xi_j(t)$ and, by (10), lies strictly outside the unit circle when $t > a_n$.

We now consider the zeros of Q_{a_n} which already lie strictly outside the unit circle. Let $\eta_1, \ldots, \eta_r$ be the distinct such zeros, with $r = 0$ if there are no zeros of Q_{a_n} strictly outside the unit circle, and let μ_k denote the multiplicity of η_k.

Choose pairwise disjoint closed discs $D_1, \ldots, D_r$, each centred at the corresponding zero η_k, such that the discs are disjoint from $U_1, \ldots, U_m$, lie entirely in the region $|w| > 1$, and have boundaries containing no zero of Q_{a_n}. Figure 1 depicts a schematic configuration of the two families.

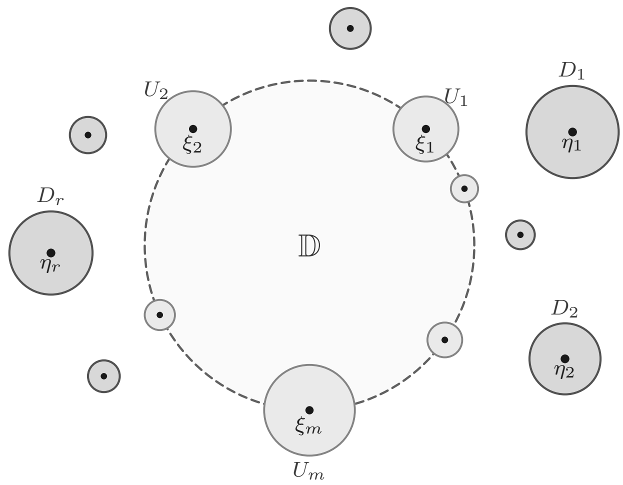

FIGURE 1. Schematic representation of the chosen closed neighbourhoods of the zeros of Q_{a_n}. The closed discs U_j are centred at the simple zeros $\xi_j \in \partial\mathbb{D}$. The closed discs D_k are centred at the distinct exterior zeros η_k, lie entirely in $\mathbb{C} \setminus \overline{\mathbb{D}}$, and are pairwise disjoint and disjoint from all the U_j. Their boundaries contain no zero of Q_{a_n}. The smaller unlabelled discs indicate possible further members of the two finite families. The radii are chosen independently, and the figure is not drawn to scale.

Applying the same Rouché argument on each ∂D_k, we find that, for t sufficiently close to a_n, the polynomial Q_t has exactly μ_k zeros, counted with multiplicity, in D_k. Since $D_k \subset \{w : |w| > 1\}$, all these zeros lie strictly outside the unit circle.

The zeros $\xi_1, \ldots, \xi_m, \eta_1, \ldots, \eta_r$ are all the distinct zeros of Q_{a_n}. Since every ξ_j is simple and η_k has multiplicity μ_k, we have

$$m + \sum_{k=1}^{r} \mu_k = n.$$

Because the two families are finite, the branch estimates (10) and the Rouché inequalities on the boundaries ∂U_j and ∂D_k may be made simultaneous. Consequently, there exists

$$\varepsilon \in (0, \varepsilon_0]$$

such that, for every t satisfying

$$a_n < t \leq a_n + \varepsilon,$$

the polynomial Q_t has exactly one zero in each U_j and exactly μ_k zeros in each D_k, all lying strictly outside the unit circle. These account for

$$m + \sum_{k=1}^{r} \mu_k = n$$

zeros, counted with multiplicity. Since Q_t has degree n, these are all its zeros. It follows that every zero of Q_t satisfies $|w| > 1$ whenever

$$a_n < t \leq a_n + \varepsilon.$$

Set

$$t_1 := a_n + \varepsilon.$$

Suppose, to the contrary, that Q_t has a zero in $\overline{\mathbb{D}}$ for some $t > t_1$. Consider the set

$$\mathcal{B} := \left\{ t \geq t_1 : Q_t(w) = 0 \text{ for some } w \in \overline{\mathbb{D}} \right\}.$$

By assumption, $\mathcal{B}$ is non-empty.

We first show that $\mathcal{B}$ is closed. Let (t_m) be a sequence in $\mathcal{B}$ such that

$$t_m \longrightarrow t.$$

For each m, choose $w_m \in \overline{\mathbb{D}}$ satisfying

$$Q_{t_m}(w_m) = 0.$$

Since $\overline{\mathbb{D}}$ is compact, there exist a subsequence (w_{m_ℓ}) and a point $w \in \overline{\mathbb{D}}$ such that

$$w_{m_\ell} \longrightarrow w.$$

The corresponding subsequence (t_{m_ℓ}) also converges to t. Therefore, by the continuity of the map

$$(s, z) \longmapsto Q_s(z),$$

we obtain

$$Q_t(w) = \lim_{\ell \to \infty} Q_{t_{m_\ell}}(w_{m_\ell}) = 0.$$

Since $t_m \geq t_1$ for every m, we also have $t \geq t_1$. Consequently, $t \in \mathcal{B}$, and hence $\mathcal{B}$ is closed.

Since $\mathcal{B}$ is non-empty and bounded below, let $T := \inf \mathcal{B}$. The closedness of $\mathcal{B}$ implies that $T \in \mathcal{B}$, and therefore

$$T = \min \mathcal{B}.$$

By the choice of t_1, all the zeros of Q_{t_1} satisfy $|w| > 1$. Thus $t_1 \notin \mathcal{B}$. Since $T \in \mathcal{B}$ and $T \geq t_1$, it follows that

$$T > t_1.$$

Moreover, since $T \in \mathcal{B}$, there exists a zero w_T of Q_T such that

$$|w_T| \leq 1.$$

We claim that

$$|w_T| = 1. \tag{11}$$

Suppose, on the contrary, that $|w_T| < 1$. Choose a closed disc centred at w_T, contained entirely in $\mathbb{D}$, whose boundary contains no zero of Q_T. Since

$$Q_t(w) - Q_T(w) = t - T,$$

Rouché's theorem implies that, for $t < T$ sufficiently close to T, the polynomials Q_t and Q_T have the same number of zeros, counted with multiplicity, inside the disc. This number is positive, since w_T is a zero of Q_T. Choosing such a t with

$$t_1 < t < T,$$

we obtain $t \in \mathcal{B}$, contradicting the minimality of T. This proves (11).

By the preceding local analysis, the simple zero w_T lies on a unique local C^1 branch $w(t)$ such that

$$w(T) = w_T, \quad Q_t(w(t)) = 0,$$

and

$$\frac{d}{dt}|w(t)|^2\bigg|_{t=T} > 0.$$

Hence, as $t \longrightarrow T$,

$$|w(t)|^2 = 1 + \frac{d}{dt}|w(t)|^2\bigg|_{t=T}(t-T) + o(|t-T|).$$

It follows that

$$|w(t)| < 1$$

for $t < T$ sufficiently close to T. Since $T > t_1$, such a value of t may be chosen with

$$t_1 < t < T.$$

We then have $t \in \mathcal{B}$, contradicting the minimality of T.

This contradiction shows that $\mathcal{B}$ is empty. Together with the local conclusion obtained for $a_n < t \leq t_1$, it follows that every zero w of Q_t satisfies

$$|w| > 1, \quad t > a_n.$$

By reciprocity, every zero of P_t lies in $\mathbb{D}$ whenever $t > a_n$. Taking

$$t = a_n + d$$

gives $P_t = P_d$ and completes the proof.

Remark 3 (Connection with Jeltsch's stability conjecture)**.** *We briefly indicate the consequence for Brown's methods. In this remark, we use Meneguette's notation for the parameters K, L and K_L, and for the characteristic coefficients α_j.*

Meneguette's root-locus analysis shows that the required A_0-stability follows if, for every $\gamma > 0$, all the zeros of

$$p_\gamma(z) = (\alpha_K + \gamma)z^K + \alpha_{K-1}z^{K-1} + \cdots + \alpha_0$$

lie in $\mathbb{D}$ [4, Theorem 2.13 and Remark 2.16, pp. 81–82]. *The same family is considered explicitly in connection with the Eneström–Kakeya theorem in* [4, pp. 105–107].

Set

$$\widetilde{p}_\gamma(z) := (-1)^K p_\gamma(-z).$$

This transformation replaces every zero by its negative and therefore preserves its modulus. For $j = 0, \ldots, K-1$, set

$$c_j := (-1)^{K+j}\alpha_j, \quad c_K := \alpha_K.$$

By the sign formula for the characteristic coefficients and the positive-coefficient formulation used by Meneguette [4, (2.16), p. 65; p. 105]*, we have*

$$c_j > 0, \quad j = 0, \ldots, K.$$

Moreover,

$$\widetilde{p}_\gamma(z) = c_0 + c_1 z + \cdots + c_{K-1}z^{K-1} + (c_K + \gamma)z^K.$$

Meneguette's coefficient results give

$$0 < c_0 \leq c_1 \leq \cdots \leq c_{K-1}$$

for $K \leq K_L$ [4, Theorem 2.9, pp. 66–67]. *In the same positive-coefficient formulation, he further verifies that*

$$Kc_K > (K-1)c_{K-1}$$

for $K \leq K_L$ when $L \geq 10$ [4, pp. 106–107].

If the Brown method is zero-stable, then p_0 satisfies the root condition [4, Definition 2.4, p. 47]. *Hence all the zeros of $\widetilde{p}_0$ lie in $\overline{\mathbb{D}}$. It follows from Theorem 1 that, for every $\gamma > 0$, all the zeros of $\widetilde{p}_\gamma$, and consequently all the zeros of p_γ, lie in $\mathbb{D}$. By Meneguette's root-locus reduction, the method is therefore A_0-stable.*

Meneguette also proves that the Brown (K, L) method is strongly stable for $K \leq K_L$ [4, Theorem 2.12, pp. 70–72]. *Finally, stiff stability is equivalent to the conjunction of A_0-stability and strong stability* [4, Theorem 2.16, pp. 96–97]. *We therefore obtain Jeltsch's conjecture for every zero-stable Brown method in the corresponding subclass, namely, in Meneguette's notation, for*

$$L \geq 10, \quad K \leq K_L.$$

3. Proof of Lemma 2

Set

$$b_{n+1} := 0, \quad \delta_k := b_k - b_{k+1}, \quad k = 1, \ldots, n.$$

By the monotonicity of the sequence (b_j),

$$\delta_k \geq 0, \quad k = 1, \ldots, n.$$

Moreover, telescoping gives

$$b_j = \sum_{k=j}^{n} \delta_k, \quad j = 1, \ldots, n, \tag{12}$$

and, in particular,

$$b_1 = \sum_{k=1}^{n} \delta_k. \tag{13}$$

Using (12) and interchanging two finite sums, the hypothesis (2) becomes

$$\sum_{k=1}^{n} \delta_k \sum_{j=1}^{k} \sin(j\theta) = 0. \tag{14}$$

Let

$$L := (n-1)b_1 + \sum_{j=1}^{n} (n-j) b_j \cos(j\theta).$$

It follows from (12) and (13) that

$$L = \sum_{k=1}^{n} \delta_k \left[(n-1) + \sum_{j=1}^{k} (n-j) \cos(j\theta) \right]. \tag{15}$$

Put

$$x := \frac{\theta}{2}.$$

Since $0 < \theta < 2\pi$, we have

$$0 < x < \pi, \quad \sin x > 0.$$

In particular, $\cot x$ is well defined. Recalling that $\theta = 2x$, we may use (14) to add zero to (15). We thus obtain

$$L = \sum_{k=1}^{n} \delta_k E_{n,k}(x), \tag{16}$$

where

$$\begin{aligned} E_{n,k}(x) &:= (n-1) + \sum_{j=1}^{k} (n-j) \cos(2jx) \\ &\quad + (n-1) \cot x \sum_{j=1}^{k} \sin(2jx). \end{aligned} \tag{17}$$

Since $\delta_k \geq 0$, it is enough to prove that

$$E_{n,k}(x) \geq 0, \quad k = 1, \dots, n.$$

For $r = 0, 1, \dots$, define

$$U_r(x) := \frac{\sin((r+1)x)}{\sin x}.$$

Thus $U_r(x)$ is the value at $\cos x$ of the Chebyshev polynomial of the second kind of degree r.

We first observe that

$$\cos(2jx) + \cot x \, \sin(2jx) = \frac{\sin((2j+1)x)}{\sin x}.$$

Together with the finite trigonometric sum

$$\sum_{j=0}^{k} \sin((2j+1)x) = \frac{\sin^2((k+1)x)}{\sin x},$$

this gives

$$1 + \sum_{j=1}^{k} [\cos(2jx) + \cot x \, \sin(2jx)] = U_k(x)^2. \tag{18}$$

Writing $n - j = (n-1) - (j-1)$ in (17) and using (18), we obtain

$$E_{n,k}(x) = (n-1)U_k(x)^2 - \sum_{j=1}^{k} (j-1)\cos(2jx).$$

Set

$$D_k(x) := (k-1)U_k(x)^2 - \sum_{j=1}^{k} (j-1)\cos(2jx). \tag{19}$$

Then

$$E_{n,k}(x) = (n-k)U_k(x)^2 + D_k(x). \tag{20}$$

It remains to show that

$$D_k(x) \geq 0, \quad k = 1, \dots, n.$$

For $r = 0, 1, \dots$, we have the finite Fourier expansion

$$U_r(x)^2 = (r+1) + 2\sum_{j=1}^{r} (r+1-j)\cos(2jx). \tag{21}$$

Indeed,

$$U_r(x) = \sum_{\ell=0}^{r} e^{i(r-2\ell)x}.$$

Since this sum is real, we have

$$U_r(x)^2 = \left|\sum_{\ell=0}^{r} e^{i(r-2\ell)x}\right|^2 = \sum_{\ell,m=0}^{r} e^{2i(m-\ell)x}$$

$$= (r+1) + 2\sum_{j=1}^{r}(r+1-j)\cos(2jx),$$

which proves (21).

Applying (21) with $r = k$ and $r = k-1$, a direct calculation gives

$$2D_k(x) = kU_{k-1}(x)^2 + (k-1)U_k(x)^2 - 1. \tag{22}$$

For $k = 1$, definition (19) immediately gives

$$D_1(x) = 0.$$

For $k = 2$, we have

$$U_1(x) = 2\cos x, \quad U_2(x) = 4\cos^2 x - 1.$$

Hence (22) yields

$$2D_2(x) = 2U_1(x)^2 + U_2(x)^2 - 1 = 16\cos^4 x.$$

Therefore

$$D_2(x) = 8\cos^4 x \geq 0.$$

Suppose now that $k \geq 3$. We use the elementary identity

$$U_k(x)^2 + U_{k-1}(x)^2 - 2\cos x\, U_k(x)U_{k-1}(x) = 1, \tag{23}$$

which follows directly from the sine addition formula. Combining (23) with (22), we obtain

$$2D_k(x) = (k-2)U_k(x)^2 + (k-1)U_{k-1}(x)^2 + 2\cos x\, U_k(x)U_{k-1}(x).$$

Since $|\cos x| \leq 1$ and $2|uv| \leq u^2 + v^2$, we have

$$2\cos x\, U_k(x)U_{k-1}(x) \geq -U_k(x)^2 - U_{k-1}(x)^2.$$

Consequently,

$$2D_k(x) \geq (k-3)U_k(x)^2 + (k-2)U_{k-1}(x)^2 \geq 0.$$

Thus $D_k(x) \geq 0$ also for $k \geq 3$. Together with the cases $k = 1$ and $k = 2$, this proves that

$$D_k(x) \geq 0, \quad k = 1, \dots, n.$$

Since $n - k \geq 0$ for $k = 1, \dots, n$, equation (20) yields

$$E_{n,k}(x) \geq 0, \quad k = 1, \dots, n.$$

Finally, (16) and $\delta_k \geq 0$ give

$$L = \sum_{k=1}^{n} \delta_k E_{n,k}(x) \geq 0.$$

This proves (3) and hence the lemma.

Acknowledgements

The author is deeply grateful to his colleague V. Botta for drawing his attention to this problem, for generously sharing the relevant documentation, including a copy of M. Meneguette's doctoral thesis, and for many stimulating and insightful discussions. He is also especially grateful to M. Meneguette for kindly agreeing to read the manuscript and comment on the proof. The author acknowledges financial support from the Centre for Mathematics of the University of Coimbra (CMUC), funded by the Portuguese Foundation for Science and Technology (FCT), under the projects UID/00324/2025 (`https://doi.org/10.54499/UID/00324/2025`) and UID/PRR/00324/2025. The author also acknowledges financial support from the FCT under the grant `https://doi.org/10.54499/2022.00143.CEECIND/CP1714/CT0002`.

References

[1] V. Botta, L. F. Marques and M. Meneguette, Palindromic and perturbed polynomials: zeros location, Acta Math. Hungar. 143 (2014) 81–87.

[2] V. Botta, M. Meneguette, J. A. Cuminato and S. McKee, On the zeros of polynomials: An extension of the Eneström–Kakeya theorem, J. Math. Anal. Appl. 385 (2012) 1151–1161.

[3] R. Jeltsch, A_0-stability and stiff stability of Brown's multistep multiderivative methods, Numer. Math. 32 (1979) 167–181.

[4] M. Meneguette, *Multistep Multiderivative Methods and Related Topics*, Ph.D. thesis, Linacre College, University of Oxford, 1987.

[5] M. Meneguette, Zeros in the unit disk, SIAM Review 36 (1994) 656–657, Problem 94-19.

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