

Efficient and well-conditioned ghost-point discretization of boundary operators on unfitted domains

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Abstract

Unfitted boundary methods are widely used to numerically solve partial differential equations (PDEs) on irregular domains, avoiding the computational burden of generating boundary-conforming grids. In the finite-difference framework, structured Cartesian grids offer advantages such as ease of implementation and efficient parallelization, while geometry is represented implicitly, for instance, through level-set functions. In this setting, ghost point methods are commonly employed to enforce boundary conditions by introducing additional relations between interior and ghost nodes. However, constructing these relations becomes challenging for high-order accurate discretizations, which often rely on wide stencils that can reduce computational efficiency and degrade performance in large-scale parallel simulations.

In this work, we investigate alternative ghost-point discretizations based on compact stencils. We introduce a formulation based on a boundary operator that locally approximates the boundary condition near each ghost node, replacing it with linear relations involving both interior and ghost points. The operator is constructed via least-squares reconstruction, allowing flexible stencil configurations while preserving the desired order of accuracy. Several strategies for selecting and adapting compact stencils are proposed, guided by conditioning criteria and iterative refinement procedures to im-

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prove global stability. Numerical experiments on various geometries and convection-diffusion regimes demonstrate the effectiveness of the proposed approach, showing that it maintains high accuracy even in the presence of boundary layers and improves stencil compactness and conditioning of the resulting linear systems.

Keywords: Unfitted boundary methods, ghost-point methods, finite-difference methods, least-squares reconstruction, high-order accuracy, compact stencil, irregular domain.

1. Introduction

Finite differences on a Cartesian grid offer several advantages, owing to its simple structure, which enables high accuracy while supporting efficient storage and parallel computing. For problems posed in irregular domains, the computational grid does not conform to the geometry of the domain (unfitted boundary methods). Instead, the domain is implicitly represented, for instance, by means of a level set function [1, 2]. Such a method enables the use of Cartesian meshes while allowing the treatment of complex geometries. Moreover, they are particularly efficient in problems involving moving boundaries, since the underlying grid does not need to be recomputed when the geometry evolves in time.

Nevertheless, the unfitted boundary method requires an additional treatment to preserve the global accuracy and stability. To this end, embedded method with ghost points is a popular technique that substitutes the boundary condition on the physical domain with an equivalent thick boundary condition over several layers of external points surrounding the domain [3–10].

On the one hand, the discretization of PDEs yields relations that combine internal points, *i.e.*, the nodes that belong to the physical domain, and ghost points. On the other hand, additional relations that combine ghost and internal points have to be found out to close the global system (whose unknowns are the values of the internal and ghost points). A common first-order approach sets the ghost value equal to that at a near boundary point. To attain higher-order accuracy, the ghost node must be related to interior nodes in a manner that the boundary condition is enforced at a chosen boundary point. The construction of these relations is the critical issue in a ghost point framework to provide a non-singular system with a small condition number, and several parameters control the design of the additional relations. The principal ingredient to design the complementary

relations is the construction of the local stencil, involving ghost and internal nodes together with boundary conditions prescribed at some point of the physical boundary.

The choice of this boundary point is fundamental and distinguishes the various ghost-point techniques. Several second-order ghost-point techniques have been proposed in the literature. In [7], the boundary condition associated with a ghost point is imposed at the intersection between the boundary and the coordinate axes. By contrast, in [11–13] it is imposed at the closest point on the boundary, and this technique is employed in several applications that model complex-shaped boundaries, such as topography in computational volcanology [14], monument conservation [15], and moving bubbles in sorption kinetics [16]. More recently, a nodal ghost-point formulation on Cartesian meshes has been introduced within the finite element framework [17, 18] and compared with the finite difference scheme in [19]. This approach provides improved theoretical properties, such as symmetry and coercivity of the discrete operator. However, its extension to high-order accuracy is not straightforward.

Very high-order approximations (third-order or higher) on unfitted domains require more complex reconstructions within the finite-difference setting. In [20], the authors extend the method of [11] to fourth-order accuracy. We also mention the extension of the extrapolation/interpolation techniques following the normal direction or the axis direction proposed in [9, 21, 22]. Sixth-order methods are presented in [23, 24], where the ghost values are extrapolated solely from nearby interior points and remain decoupled from other ghost points.

In many ghost-point formulations, the stencils adopted for very high-order discretizations are quite large, with diameters exceeding 8 to 10 times the characteristic length Δx , which raises several issues: a significant loss of accuracy and the possible inability to construct such a stencil in some situations (corner or wedge). This occurs either because the construction of the discrete operator near the boundary relies on stencils obtained from a square neighbourhood of grid points [11, 20], or because the stencil does not include other ghost points [23, 24] (i.e., the ghost equations are decoupled from one another), which leads to the inclusion of interior values that are often far from the boundary. This concern also becomes a limitation in large-scale simulations from a computational point of view, particularly in parallel computing environments, where compact stencils are desirable to reduce communication costs and improve computational efficiency. Moreover, wide stencils may fail to accurately capture boundary effects, such as boundary layers. Compact stencils are therefore desirable for resolving solutions with

steep gradients.

In this paper, we investigate coupled ghost-point strategies for constructing ghost-point discretizations using more compact stencils. Allowing neighbouring ghost points to enter the same stencil enables more compact boundary discretizations, but introduces coupling among the ghost unknowns. While this coupling does not inherently imply instability, it requires careful control of the resulting algebraic system. The proposed strategy therefore aims to achieve stencil compactness while controlling both local and global conditioning. We propose a new formulation based on a so-called boundary operator that provides a local discretization of the boundary condition in the vicinity of any ghost point. Such a linear operator mimics the boundary condition up to a given order of accuracy, and substitutes it on the frontier with linear relations over internal and ghost points. In particular, we analyse various configurations of reduced stencils and examine their effects on the resulting discrete operators.

To this end, we introduce a representation of the boundary condition of any function ψ as a linear combination of the ψ value over the points of a stencil. We say that a reconstruction is o -th order if the representation is exact for polynomials up to degree $o - 1$. We require the number of points in the stencil to be large enough to guarantee the desired order of accuracy, possibly leading to an underdetermined linear system but with the maximal rank property. We then characterize the quality of the local representation with the condition number of an intermediate non-singular matrix used in the Moore-Penrose system. An additional criterion for assessing the quality of all the representations is the condition number of the global system. Two major criteria control the quality of the reconstruction: the relative position of the points to the target ghost point that constitute the stencil; the choice of the physical point on the boundary where we impose the condition. We then propose several strategies for constructing compact stencils that yield high-quality representations.

As such, the main contributions of this work are: i) the boundary operator formulation that replaces the boundary condition with linear relations between interior and ghost nodes; ii) a least-squares reconstruction framework allowing flexible stencil configurations; iii) conditioning-based stencil selection algorithms ensuring both local and global stability; iv) a cone-based stencil construction strategy that yields compact and well-conditioned stencils.

The paper is organized as follows. In Section 2, we introduce the general discretization of the PDE and of the boundary conditions, and describe the

least-squares approach used to construct the equations for the ghost nodes and to assemble the resulting matrix. In Section 3, we present different stencil strategies for the discretization of the boundary conditions and perform a numerical study to assess their performance and their impact on the conditioning of the resulting linear system. Section 4 is devoted to numerical experiments on different geometries and for various combinations of convection and diffusion coefficients, including tests that highlight boundary layer effects. Finally, conclusions are drawn in Section 5.

2. Model and discretization

Let $\Omega \subset \mathbb{R}^2$ be a bounded domain with a piecewise smooth Lipschitz boundary. We seek the solution $\phi : \Omega \rightarrow \mathbb{R}$ of an elliptic problem defined in Ω , complemented with Robin boundary conditions on the 1D smooth closed manifold $\partial\Omega \subset \mathbb{R}^2$.

The problem then reads

$$\begin{cases} -\kappa\Delta\phi(\mathbf{x}) + U(\mathbf{x}) \cdot \nabla\phi(\mathbf{x}) = f(\mathbf{x}) & \text{for } \mathbf{x} \in \Omega \\ a_D(\mathbf{x})\phi(\mathbf{x}) + a_N(\mathbf{x})\nabla\phi(\mathbf{x}) \cdot \mathbf{n}(\mathbf{x}) = g(\mathbf{x}) & \text{for } \mathbf{x} \in \partial\Omega \end{cases}, \quad (1)$$

with $\kappa > 0$ the diffusion coefficient; $U(\mathbf{x}) \in \mathbb{R}^2$ the assigned velocity, and $f(\mathbf{x})$ the source term. Moreover, $a_D(\mathbf{x})$, $a_N(\mathbf{x})$ stand for the Dirichlet and Neumann contribution parameters with $\mathbf{n}(\mathbf{x})$ the outward normal vector at any point $\mathbf{x} \in \partial\Omega$, while $g(\mathbf{x})$ is a regular function.

2.1. Notations

Given a spacing parameter h , we define the Cartesian grid characterized by point (x_i, y_j) with $x_i = ih$ and $y_j = jh$. The set $\mathcal{M}_h$ gathers the points $\mathbf{x}_k$ of the grid that belong to Ω , with indices $k = 1, \dots, N_I = |\mathcal{M}_h|$. The set $\mathcal{G}_h$ corresponds to the ghost points of the grid $\mathbf{x}_k \notin \Omega$ constituted of one or two layers of points around Ω in function of the expected accuracy (2nd, or 4th, respectively). Layer 1 is constituted of the horizontally and vertically adjacent points to $\mathcal{M}_h$, and we repeat the procedure for the second and third layers.

We complete the identification of the ghost points with index $k = N_I + 1 \dots, N_I + N_G$, with $N_G = |\mathcal{G}_h|$. Set $\mathcal{A}_h = \mathcal{M}_h \cup \mathcal{G}_h$ is the set of *active points*, that are the points where we aim at defining an approximation $\phi_k \approx \phi(x_k)$. We store the values in $\Phi = (\Phi_I, \Phi_G)$, with Φ_I and Φ_G the vectors associated to $\mathcal{M}_h$ and $\mathcal{G}_h$ respectively. We define a mapping $k \rightarrow (i, j) = (i(k), j(k))$ such that $\mathbf{x}_k = (x_i, y_j)$, $k = 1, \dots, N_{TOT}$, with $N_{TOT} = N_I + N_G$.

2.2. Inner operator discretization

Let us consider an o -th order finite-difference discretization of the differential operators in (1). For any grid point $\mathbf{x}_k = (x_i, y_j) \in \mathcal{M}_h$, we write the discrete equation as

$$A_x^o[\Phi]_k + A_y^o[\Phi]_k = f_k = f(\mathbf{x}_k), \quad k = 1, \dots, N_I, \quad (2)$$

where $A_x^o[\Phi]_k$ and $A_y^o[\Phi]_k$ denote the contributions of the discrete operator in the two coordinate directions. Collecting all equations for the points of $\mathcal{M}_h$, we obtain

$$A_x^o[\Phi] + A_y^o[\Phi] = f_I,$$

where $f_I \in \mathbb{R}^{N_I}$ is the vector with entries f_k .

In this manuscript, we focus on a fourth-order centred scheme for the Laplace operator as well as for the gradient; as such, $A_x^4[\Phi]_k$ is obtained as

$$\begin{aligned} A_x^4[\Phi]_k = & -\kappa \frac{-\frac{1}{12}\phi_{i-2,j} + \frac{4}{3}\phi_{i-1,j} - \frac{5}{2}\phi_{i,j} + \frac{4}{3}\phi_{i+1,j} - \frac{1}{12}\phi_{i+2,j}}{h^2} \\ & + u \frac{\frac{1}{12}\phi_{i-2,j} - \frac{2}{3}\phi_{i-1,j} + \frac{2}{3}\phi_{i+1,j} - \frac{1}{12}\phi_{i+2,j}}{2h}, \end{aligned} \quad (3)$$

with $\mathbf{x}_k = (x_i, y_j)$ and analogously $A_y^4[\Phi]_k$ reads

$$\begin{aligned} A_y^4[\Phi]_k = & -\kappa \frac{-\frac{1}{12}\phi_{i,j-2} + \frac{4}{3}\phi_{i,j-1} - \frac{5}{2}\phi_{i,j} + \frac{4}{3}\phi_{i,j+1} - \frac{1}{12}\phi_{i,j+2}}{h^2} \\ & + u \frac{\frac{1}{12}\phi_{i,j-2} - \frac{2}{3}\phi_{i,j-1} + \frac{2}{3}\phi_{i,j+1} - \frac{1}{12}\phi_{i,j+2}}{2h}. \end{aligned} \quad (4)$$

Remark 1: While point $\mathbf{x}_k = (x_i, y_j) \in \mathcal{M}_h$, this is not necessarily the case for the other points $\mathbf{x}_\ell$ that are involved in the formula. Nevertheless, the ghost points' set $\mathcal{G}_h$ has to be designed to guarantee that $\mathbf{x}_\ell \in \mathcal{A}_h$.

Consequently, Eq (2) corresponds to an $N_I \times N_{TOT}$ linear system (N_I algebraic equations with N_{TOT} unknowns) involving all points of $\mathcal{M}_h$ as well as all points of $\mathcal{G}_h$, formulated under the matrix form:

$$A_{II}\Phi_I + A_{IG}\Phi_G = f_I, \quad (5)$$

where A_{II} and A_{IG} label the discrete operators acting on interior or ghost points.

2.3. Discrete boundary operator

To close the system (5), one has to provide N_G additional algebraic equations corresponding to the entries of the vector Φ_G . To this end, let $\mathbf{x}_k$ be a ghost point and $\mathbf{p}_k \in \partial\Omega$ a point close to $\mathbf{x}_k$. For any smooth function ψ , we have

$$\mathfrak{B}[\psi](\mathbf{p}_k) = a_D(\mathbf{p}_k)\psi(\mathbf{p}_k) + a_N(\mathbf{p}_k)\nabla\psi(\mathbf{p}_k) \cdot \mathbf{n}(\mathbf{p}_k).$$

Now, let $\mathcal{S}^k$ be a stencil of size $n_k = |\mathcal{S}^k|$ with $\mathbf{x}_k \in \mathcal{S}^k$ and constituted of ghost points or inner points close to $\mathbf{x}_k$. The procedure to build efficient and well-conditioned $\mathcal{S}^k$ will be the topic of the next section.

At that stage, for any regular function ψ , we introduce the functional

$$\mathbf{a} \in \mathbb{R}^{N_k} \rightarrow \mathcal{E}^k(\mathbf{a}; \mathcal{S}^k)[\psi] = \left(\sum_{\ell \in \mathcal{S}^k} a_\ell \psi(\mathbf{x}_\ell) \right) - \mathfrak{B}[\psi](\mathbf{p}_k), \quad (6)$$

where $\mathbf{a}$ collects the coefficients a_ℓ associated with the points of the stencil $\mathcal{S}^k$. On the other hand, $\mathbb{P}^{o-1}$ is the polynomial space of degree lower or equal to $o-1$, equipped with the basis of $m_o = \frac{o(o+1)}{2}$ independent polynomials

$$\psi(\mathbf{x}; \mathbf{x}_k, \boldsymbol{\alpha}) = (\mathbf{x} - \mathbf{x}_k)^\alpha, \quad |\boldsymbol{\alpha}| \leq o-1,$$

with $\boldsymbol{\alpha} = (\alpha_x, \alpha_y)$, $|\boldsymbol{\alpha}| = \alpha_x + \alpha_y$ and $(\mathbf{x} - \mathbf{x}_k)^\alpha = (x - x_k)^{\alpha_x} (y - y_k)^{\alpha_y}$. In this paper, we aim to achieve fourth-order accuracy for both the solution and its gradient, under Dirichlet as well as Neumann boundary conditions. While fourth-order accuracy of the solution, along with Dirichlet boundary conditions, would require only $o = 4$, to attain the desired accuracy for the gradient and in the presence of mixed boundary conditions, we use $o = 5$ (then $\mathbb{P}^4$ polynomial space), corresponding to $m_o = 15$.

To determine the vector $\mathbf{a}$, we state that the functional has to satisfy the following constraints:

$$\mathcal{E}^k(\mathbf{a}; \mathcal{S}^k)[\psi(\mathbf{x}; \mathbf{x}_k, \boldsymbol{\alpha})] = 0, \quad |\boldsymbol{\alpha}| \leq o-1.$$

We then have N_k unknown coefficients with N_o constraints and reads $B^k \mathbf{a} = \mathbf{g}^k$ with

$$B_{\ell, \boldsymbol{\alpha}}^k = (\mathbf{x}_\ell - \mathbf{x}_k)^\alpha, \quad \mathbf{g}_{\boldsymbol{\alpha}}^k = \mathfrak{B}[(\mathbf{x} - \mathbf{x}_k)^\alpha](\mathbf{p}_k), \quad \ell \in \mathcal{S}^k, \quad |\boldsymbol{\alpha}| \leq o-1. \quad (7)$$

Remark 2: At the implementation level, we need an indexation $\boldsymbol{\alpha} \rightarrow m(\boldsymbol{\alpha})$ with

m ranging from 1 to N_o to manipulate the matrix and carry out the computations.

► If $n_k < m_o$, we do not have enough degrees of freedom to guarantee the desired order of accuracy, which disqualifies such a situation. One has to choose a stencil large enough such that $N_k \geq N_o$.

► If $n_k = m_o$ and assuming that the square matrix B^k is non-singular, we then obtain a unique vector $\mathbf{a}^k$ to represent the boundary condition operator at point $\mathbf{p}_k$.

► If $n_k > m_o$, we have an underdetermined system regarding the coefficients, and we use the least squares method under constraints. More precisely, we seek a vector $\mathbf{a}^k$ such that:

$$\mathbf{a}^k = \arg \min_{\mathbf{a} \in \mathbb{R}^{N_k}} \frac{1}{2} \mathbf{a}^t \mathbf{a}$$

under the linear constraints $\mathcal{E}^k(\mathbf{a}; \mathcal{S}^k)[\psi(\mathbf{x}; \mathbf{x}_k, \boldsymbol{\alpha})] = 0$, $|\boldsymbol{\alpha}| \leq o - 1$.

Remark 3: It is important to observe that $\mathbf{a}^k = \mathbf{a}^k(\mathbf{p}_k)$ since the determination of $\mathbf{a}^k$ as a minimizer of (6) depends on the choice of point $\mathbf{p}_k \in \partial\Omega$. Thus, we have an additional degree of freedom from the choice of $\mathbf{p}_k$ that will be exploited to provide the "best" boundary operator.

As a consequence, given $\mathbf{p}_k$ and the stencil $\mathcal{S}_k$, we determine the vector $\mathbf{a}^k$, and the discrete boundary operator at the ghost point $\mathbf{x}_k \in \mathcal{G}_h$ reads:

$$\mathfrak{B}_k(\Phi) = \sum_{\mathbf{x}_\ell \in \mathcal{S}_k} a_\ell^k \phi_\ell. \quad (8)$$

At last, the new equation associated to the ghost point $\mathbf{x}_k$ gives

$$\mathfrak{B}_k(\Phi) = g(\mathbf{p}_k), \quad k = N_I + 1, \dots, N_I + N_G. \quad (9)$$

In that way, we define the N_G equations from the relations (9), that we can rewrite under the compact form:

$$A_{GI}\Phi_I + A_{GG}\Phi_G = F_G. \quad (10)$$

with F_G the vector of entries $g(\mathbf{p}_k)$.

Assembling the two systems, Eqs (5) and (10), provides the global discrete $N_{TOT} \times N_{TOT}$ problem

$$\begin{bmatrix} A_{II} & A_{IG} \\ A_{GI} & A_{GG} \end{bmatrix} \begin{bmatrix} \Phi_I \\ \Phi_G \end{bmatrix} = \begin{bmatrix} F_I \\ F_G \end{bmatrix}, \quad (11)$$

where the matrices A_{II} and A_{IG} come from the interior operator discretization, whereas the matrices A_{GI} and A_{GG} are derived from the boundary operator discretization.

Remark 4: The linear system (10) has to be of maximal rank $N_G \times N_{TOT}$ to provide an eligible global system. In fact, we need to prescribe more constraints about the stencil design and the choice of $\mathbf{p}_k$ to control the condition number, both locally (matrix B_k) and globally (the global linear system).

Consistency error of the discrete boundary operator. We briefly discuss the consistency of the discrete boundary operator introduced above and its role in the accuracy of the complete ghost-point discretization. Let $\mathbf{x}_k \in G_h$ be a ghost point, let $\mathbf{p}_k \in \partial\Omega$ be the associated boundary point, and let S^k be the corresponding stencil. We assume that the stencil is compact, in the sense that its diameter is bounded by Ch , with a constant C independent of h . Let ψ be a sufficiently smooth function in a neighbourhood of $S^k \cup \{\mathbf{p}_k\}$. By construction, the coefficients a^k satisfy

$$E^k(a^k; S^k)[q] = 0, \quad \forall q \in \mathbb{P}_{o-1}.$$

Equivalently,

$$\sum_{x_\ell \in S^k} a_\ell^k q(x_\ell) = B[q](p_k), \quad \forall q \in \mathbb{P}_{o-1}.$$

Let $T_{o-1}\psi$ be the Taylor polynomial of ψ of degree $o-1$ around x_k . Since $T_{o-1}\psi \in \mathbb{P}_{o-1}$, the polynomial part is reproduced exactly, and therefore

$$E^k(a^k; S^k)[\psi] = E^k(a^k; S^k)[\psi - T_{o-1}\psi].$$

Because $|x_\ell - x_k| = O(h)$ for all $x_\ell \in S^k$ and $|p_k - x_k| = O(h)$, the Taylor remainder implies that

$$|E^k(a^k; S^k)[\psi]| \leq C_k (|a_D(p_k)|h^o + |a_N(p_k)|h^{o-1}) |\psi|_{C^o},$$

where the constant C_k depends on the scaled geometry of the stencil and on the norm of the reconstruction coefficients a^k . Hence, the discrete boundary operator is locally o -th order accurate for pure Dirichlet conditions and locally $(o-1)$ -th order accurate for Neumann or Robin conditions. This loss of one power of h is due to the normal derivative appearing in $B[\psi]$. Therefore, the choice $o = 5$, used in this work, yields a fourth-order boundary residual also for Neumann and mixed boundary conditions.

2.4. The level-set function and the boundary projection

Let the domain Ω be represented by a level-set function $\varphi(\mathbf{x})$, which is negative inside Ω , positive outside, and vanishes on the boundary:

$$\Omega = \{ \mathbf{x} \in \mathbb{R}^2 : \varphi(\mathbf{x}) < 0 \}, \quad \partial\Omega = \{ \mathbf{x} \in \mathbb{R}^2 : \varphi(\mathbf{x}) = 0 \}.$$

The outward unit normal to the boundary is given by $\mathbf{n} = \nabla\varphi / |\nabla\varphi|$, where φ is assumed to be explicitly available. Given a ghost point $\mathbf{x}_k$, its orthogonal projection onto the boundary is computed as an iterative procedure as given in [20].

3. The choice of the stencils

3.1. Stencil quality and condition number

Stability and accuracy of the approximation derive from two key stages. On a local scale, the stencil selection results in the conditioning of the local matrix, which connects the ghost point with the boundary data and interior points. At a large scale, the global matrix, which encompasses all the interior and ghost points, is generated from the local matrices. The global condition number is essential to obtain accurate approximations for general boundary configurations. Consequently, it is imperative to address the two issues for any non-polygonal frontier and boundary conditions and develop a robust and efficient algorithm to select stencils.

We first introduce the local criterion to assess the quality of the stencil and the associated discrete boundary operator, characterized by the vector $\mathbf{a}^k$.

Definition 1: Let $\mathcal{S}^k$ be a stencil associated to a ghost point $\mathbf{x}^k$. The stencil is **admissible** if the matrix B^k given by (7) satisfies the maximum rank property.

If we satisfy the maximum rank property then $B^{k\dagger} = (B^k)^\top B^k$ is non-singular and one has $\mathbf{a}^k = B^{k\dagger} \mathbf{g}^k$, with $B^{\ddagger} = (B^{k\dagger})^{-1} (B^k)^\top$ the Moore-Penrose matrix.

Definition 2: An admissible stencil is said to be **locally well conditioned** if $B^{k\dagger}$ is a well-conditioned matrix. Accordingly, we adopt the criterion $\chi(B^{k\dagger}) < \Lambda_{\text{LOC}}$, where $\chi(B^{k\dagger})$ denotes the condition number of $B^{k\dagger}$ and Λ_{LOC} is a prescribed tolerance. Specifically, we denote by $\chi(B^k) = \sigma_{\max}(B^k) / \sigma_{\min}(B^k)$ the 2-norm condition number of B^k , where $\sigma_{\max}(B^k)$ and $\sigma_{\min}(B^k)$ are the largest and smallest non-zero singular values of B^k , respectively.

On the other hand, we deal with the global matrix using a second criterion to control stability and accuracy. A heuristic argument concerns the condition number of the square matrix A_{GG} that connects the ghost points between themselves. We expect to have a well-conditioned matrix A_{GG} if each diagonal entry is not too small compared to the off-diagonal entries in the same row (in modulus). This requirement leads to the following definition.

Definition 3: *The influence of the stencil $\mathcal{S}^k$ on the global conditioning of the matrix A_{GG} is measured by the value*

$$R_k = \frac{1}{|a_k^k|} \max_{\mathbf{x}_\ell \in \mathcal{S}^k, \ell \neq k} |a_\ell^k|, \quad (12)$$

where a_ℓ^k are given in (8). An admissible stencil $\mathcal{S}^k$ is said **globally well-conditioned** with respect to the ghost point $\mathbf{x}_k$ and to the boundary point $\mathbf{p}_k$ if one has,

$$R_k < \Lambda_{\text{GLO}},$$

where Λ_{GLO} is a prescribed tolerance.

The connection between this criterion and the spectral properties of the ghost-point block can be interpreted in terms of diagonal dominance and Gershgorin's theorem. After normalizing each ghost equation by the coefficient associated with the target ghost point, the ratios controlled by the global criterion measure the strength of the coupling with neighbouring ghost unknowns. Keeping these ratios bounded prevents the diagonal coefficient from becoming arbitrarily small relative to the off-diagonal contributions and limits the radius of the corresponding Gershgorin disks. The proposed criterion should therefore be regarded as a practical proxy for controlling the spectral behaviour of the ghost-point block, rather than as a sufficient condition for the stability or conditioning of the complete discrete operator, which also depends on the interior discretization.

Remark 5: We recall from Remark 3 that $\mathbf{a}^k = \mathbf{a}^k(\mathbf{p}_k)$ for any $\mathbf{x}_k \in \mathcal{G}_h$, where $\mathbf{p}_k \in \partial\Omega$ is close to the point $\mathbf{x}_\ell$. Therefore, the quantity R_k deeply relies on the choice of the points where we prescribe the boundary condition on the physical interface. In summary, the admissibility of a stencil only depends on the relative positions of the stencil grid points, whereas the well-conditioning criterion depends on the mutual position of stencil grid points and the boundary point $\mathbf{p}_k$ where the conditions are prescribed.

Remark 6: While the non-singularity of the global matrix is a mandatory re-

quirement to provide a solution, locally and globally well-conditioned stencils are desirable that reduce computational effort and provide better accuracy.

The constrained least-squares reconstruction does not introduce an additional consistency error. Indeed, when $n_k > m_o$, the vector a^k is selected as the minimum-norm vector satisfying the exact polynomial constraints. Thus, the formal order is determined by the polynomial reproduction property, while the least-squares procedure affects the size of the constants in the error estimate. If the matrix $B^{k\dagger}$ is ill-conditioned, the coefficients a^k may amplify the Taylor remainder and round-off errors. This motivates the maximum-rank requirement and the local conditioning criterion $\chi(B^{k\dagger}) < \Lambda_{\text{LOC}}$ introduced in Definition 1 and Definition 2.

These criteria are not only algorithmic devices for selecting compact stencils; they also control the constants appearing in the boundary consistency estimate. In particular, loss of rank or large condition numbers in the local moment matrix do not change the formal polynomial exactness, but they may amplify the Taylor remainder and round-off errors sufficiently to deteriorate the observed convergence rate.

In conclusion, the main challenge is the design of algorithms that generate optimal stencil in such a way that we satisfy the two criteria given by Definitions 2 and 3. To this end, we first investigate triangular stencils, which automatically satisfy Definition 2 by construction. Within this class, we identify a configuration that also fulfills Definition 3. However, this stencil may exhibit undesirable properties, such as a large diameter, which can lead to poor spectral resolution. Consequently, an alternative stencil-generation procedure is required. To address this limitation, we propose a strategy that does not rely on triangular stencils. Specifically, we first construct a buffer of candidate points and iteratively select points until Definition 2 is satisfied. We then verify whether Definition 3 holds. If this condition is not met, we identify a critical point responsible for the issue and we replace it with a new candidate from the buffer. This process is repeated until all required conditions are satisfied. As we will demonstrate, the resulting stencil is significantly more compact.

Remark 7: Note that, if $\mathbf{x}_k$ is the unique ghost-point belonging to $\mathcal{S}^k$, the problem is explicit, giving that the approximation of ϕ_G at $\mathbf{x}_k$ is obtained through an affine boundary operator:

$$(\mathbf{x}_k \in \mathcal{S}^k, \text{ boundary condition on } \mathbf{p}_k) \rightarrow \phi_G(\mathbf{x}_k). \quad (13)$$

Such a procedure is a highly parallelizable computational framework.

3.2. The stencil construction

As anticipated at the end of the previous section, we explore different constructions to provide a suitable stencil that will be analyzed in terms of the following requirements:

- [1:] admissible stencil;
- [2:] locally well conditioned stencil;
- [3:] globally well conditioned stencil;
- [4:] compact stencil.

To formalize the different strategies, we introduce additional notation. Let $\mathbf{x}_k = (x_k, y_k)$ be a ghost-point and $\mathbf{p}_k = (p_x^k, p_y^k)$ the associated point on the physical boundary. Vector $\mathbf{d}_k = \mathbf{x}_k - \mathbf{p}_k = (d_x^k, d_y^k)$ is the displacement vector between $\mathbf{x}_k$ and $\mathbf{p}_k$, and $p > 1$ an integer that will control the stencil size $n_k = |\mathcal{S}^k|$ according to the target accuracy we intend to reach.

3.2.1. Stencil based on triangle shape

We first consider the stencil design based on points belonging to a triangular zone of the grid. We aim to produce a compact stencil that balances the number of horizontal and vertical displacements to ensure homogeneity in both directions.

Stencil S1: External Right-Triangular Stencil. It is a right triangle-shaped stencil with the ghost point positioned at the vertex of the right angle (see Figure 1.a):

$$\mathcal{S}^k = \left\{ (x_k + l \operatorname{sign}(d_x^k), y_k + m \operatorname{sign}(d_y^k)), l, m = 0, \dots, p, l + m \leq p \right\}. \quad (14)$$

Such a right-angle triangular stencil satisfies requirements 1 and 2.

Stencil S2: Internal Right-Triangular Stencil. To reduce the number of ghost points (other than $\mathbf{x}_k$) in this right-triangular stencil, the vertex of the right angle is chosen as the innermost internal point (x_R^k, y_R^k) (see Figure 1.b):

$$\begin{cases} (x_R^k, y_R^k) = (x_k + p \operatorname{sign}(d_x^k), y_k), & \text{if } |x_k - p_x^k| \geq |y_k - p_y^k|, \\ (x_R^k, y_R^k) = (x_k, y_k + p \operatorname{sign}(d_y^k)), & \text{otherwise.} \end{cases} \quad (15)$$

Then:

$$\mathcal{S}^k = \left\{ (x_R^k - l \operatorname{sign}(d_x^k), y_k + m \operatorname{sign}(d_y^k)), l, m = 0, \dots, p, l + m \leq p \right\}, \quad (16)$$

if $|x_k - p_x^k| \geq |y_k - p_y^k|$, or, otherwise,

$$\mathcal{S}^k = \left\{ (x_k + l \operatorname{sign}(d_x^k), y_k - m \operatorname{sign}(d_y^k)), l, m = 0, \dots, p, l + m \leq p \right\}. \quad (17)$$

This stencil satisfies requirements 1 and 2. However, if $|\mathbf{x}_k - \mathbf{p}_k| > h$, the magnitude of the coefficients associated with the ghost-point would result in a much smaller value than those associated with other points of the stencil. Therefore, requirement 3 is not satisfied.

Stencil S3: Internal Right-Triangular Stencil with Jump. To satisfy requirement 3 when considering an internal right-triangular stencil S2, additional internal grid points have to be included rather than ghost points. At this scope, if $|\mathbf{x}_k - \mathbf{p}_k| > h$, the diameter of the stencil is enlarged by one column and a line of ghost points is substituted with a line on internal points (see Figures 1.c). Notably, this stencil contains no ghost points other than $\mathbf{x}_k$; therefore, the problem would result as explicit (see Remark 7), and requirements 1, 2, and 3 are satisfied.

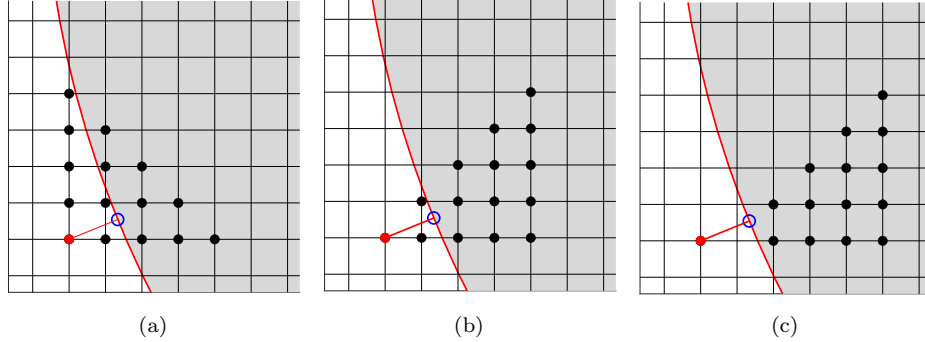


Figure 1: Stencils S1 (a), S2 (b) and S3 (c). The red point denotes the ghost node $\mathbf{x}_k$ from which the stencil is constructed, while the blue circle indicates the corresponding boundary point $\mathbf{p}_k$.

3.2.2. Stencil construction based on a cone (Stencil S4)

We consider the half-cone $\mathcal{C}(\mathbf{x}_k, \mathbf{p}_k, \vartheta)$ centred at the point $\mathbf{x}_k$ of aperture ϑ with semi-axis given by the vector $\mathbf{x}_k \mathbf{p}_k$. Then we constitute the stencil

by picking-up m_o active nodes (ghost and interior nodes) $\mathbf{y}_k \in \mathcal{C}(\mathbf{x}_k, \mathbf{p}_k, \vartheta)$, close enough to $\mathbf{p}_k$, *i.e.* the point that belongs to the physical boundary. We display in Figure 2 several situations for characteristic angles of the cone. Unlike the S1–S3 stencils, this stencil is not necessarily admissible

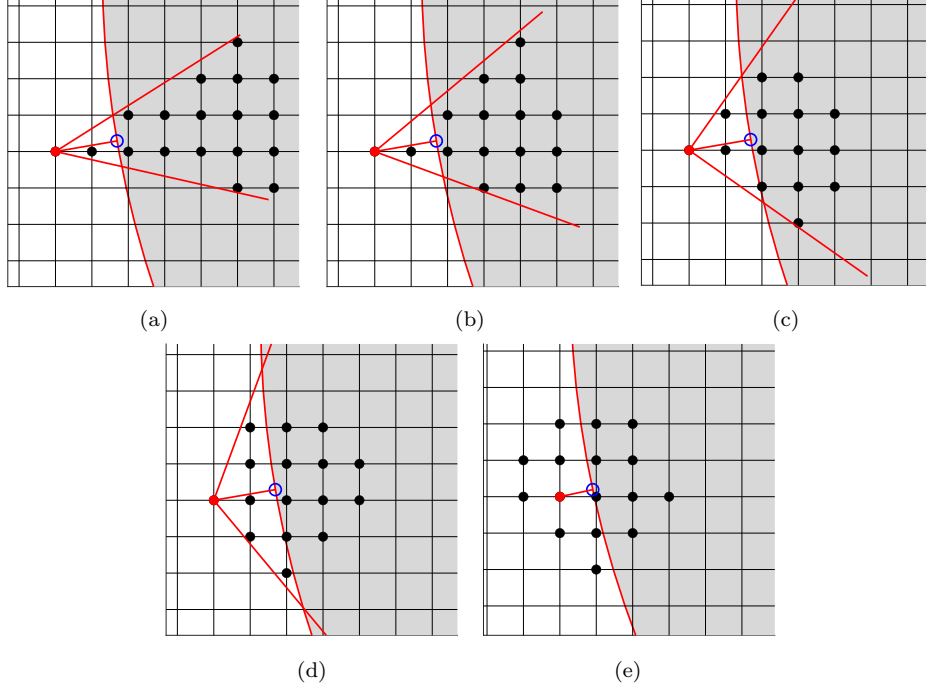


Figure 2: Stencils S4.1, with $\vartheta = 45^\circ$ (a), 60° (b), 90° (c), 120° (d), 360° (e). The red point denotes the ghost node $\mathbf{x}_k$ from which the stencil is constructed, while the blue circle indicates the corresponding boundary point $\mathbf{p}_k$.

according to Definition 1. To address such a concern, we employ three successive strategies, applied hierarchically, to generate three stencils: S4.1, S4.2 and S4.3.

S4.1: Incrementally add candidate points to the stencil until it becomes both admissible and locally well-conditioned, according to Def. 2 using a tolerance of Λ_{LOC} .

S4.2: Compute the stencil S4.1 first. If the stencil is not globally well-conditioned according to Def. 3 and with a tolerance of Λ_{GLO} , remove the point associated with the largest coefficient a_ℓ^k (see Definition 3), and replace it with the nearest available candidate point (see Figure 3,

left panel). This procedure is repeated until the stencil becomes globally well-conditioned, up to a maximum of three iterations.

S4.3: Compute the stencil S4.2 first. If the stencil remains not globally well-conditioned according to Def. 3 and with a tolerance of Λ_{GLO} , modify the boundary point $\mathbf{p}_k$ by taking the horizontal or vertical projection from the ghost point onto the boundary (choosing the closer one if both projections exist). Then, apply the strategy described above to construct the S4.2 stencil using this modified boundary point (see Figure 3, right panel).

The strategies are summarized with the Algorithm 1 in Appendix A. Unless otherwise stated, all numerical experiments use $\Lambda_{\text{LOC}} = 10^6$ and $\Lambda_{\text{GLO}} = 10$.

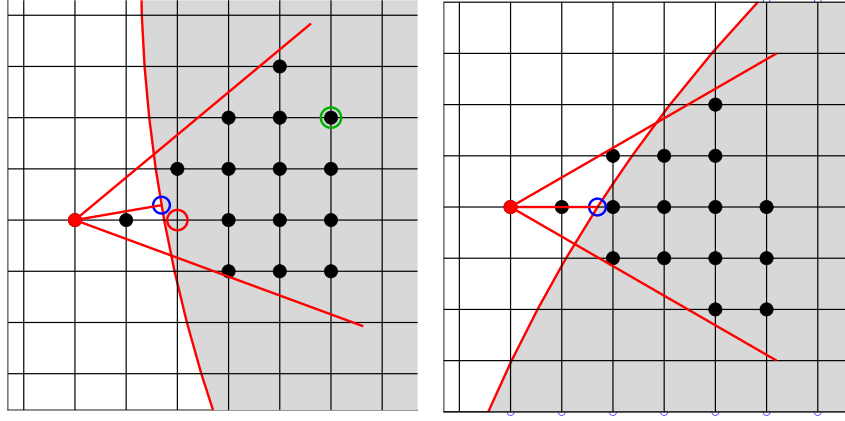


Figure 3: Stencils S4.2 (left) and S4.3 (right) with $\vartheta = 60^\circ$. The red point denotes the ghost node $\mathbf{x}_k$ from which the stencil is constructed, while the blue circle indicates the corresponding boundary point $\mathbf{p}_k$. For the Stencil S4.2 (left), the red circle denotes the node removed from the stencil, which is replaced by the node indicated by the green circle.

4. Numerical Experiments

This section presents a series of benchmarks designed to evaluate the advantages and limitations of the stencil design strategies introduced above. The domain Ω is immersed in a square $[-1, 1] \times [-1, 1] \subset \mathbb{R}^2$ equipped with a Cartesian grid of characteristic length h . The set $\mathcal{M}_h$ gathers the interior nodes, whereas $\mathcal{G}_h$ is composed by the ghost nodes of the grid. To determine

the order of convergence, we consider 13 levels of grid refinement with characteristic mesh size $h = \frac{2}{N}$, where N increases exponentially. Specifically, $N = 160, 176, 194, 213, 234, 258, 283, 312, 343, 377, 415, 456$, and 502. Numerical errors are computed using the L^1 and L^∞ norm, as detailed in the next section.

4.1. Errors and orders computation

Let us denote by $\phi^a(\mathbf{x})$ the analytical solution and by $\phi^h(\mathbf{x})$ its numerical approximation. We quantify the numerical errors in both the solution and its gradient. In particular, we consider the relative L_1 and L_∞ errors on the discrete mesh $\mathcal{M}_h$, defined as

$$\|e\|_{L_1} = \frac{\sum_{\mathbf{x}_i \in \mathcal{M}_h} |\phi^h(\mathbf{x}_i) - \phi^a(\mathbf{x}_i)|}{\sum_{\mathbf{x}_i \in \mathcal{M}_h} |\phi^a(\mathbf{x}_i)|}, \quad \|e\|_{L_\infty} = \frac{\max_{\mathbf{x}_i \in \mathcal{M}_h} |\phi^h(\mathbf{x}_i) - \phi^a(\mathbf{x}_i)|}{\max_{\mathbf{x}_i \in \mathcal{M}_h} |\phi^a(\mathbf{x}_i)|}. \quad (18)$$

The corresponding relative L_1 and L_∞ errors of the gradient are defined as

$$\|\nabla^h e\|_{L_1} = \frac{\sum_{\mathbf{x}_i \in \mathcal{M}_h} |\nabla^h \phi^h(\mathbf{x}_i) - \nabla \phi^a(\mathbf{x}_i)|}{\sum_{\mathbf{x}_i \in \mathcal{M}_h} |\nabla \phi^a(\mathbf{x}_i)|}, \quad \|\nabla^h e\|_{L_\infty} = \frac{\max_{\mathbf{x}_i \in \mathcal{M}_h} |\nabla^h \phi^h(\mathbf{x}_i) - \nabla \phi^a(\mathbf{x}_i)|}{\max_{\mathbf{x}_i \in \mathcal{M}_h} |\nabla \phi^a(\mathbf{x}_i)|}, \quad (19)$$

where the pointwise gradient error magnitude is given by

$$|\nabla^h \phi^h(\mathbf{x}_i) - \nabla \phi^a(\mathbf{x}_i)| = \left[\left(\partial_x^h \phi^h(\mathbf{x}_i) - \partial_x \phi^a(\mathbf{x}_i) \right)^2 + \left(\partial_y^h \phi^h(\mathbf{x}_i) - \partial_y \phi^a(\mathbf{x}_i) \right)^2 \right]^{1/2}.$$

Here, ∂_x^h and ∂_y^h denote fourth-order finite-difference reconstructions of the gradient computed from the numerical solution ϕ^h . For example, the discrete x -derivative is given by

$$\partial_x^h \phi^h(x, y) = \frac{-\phi^h(x + 2h, y) + 8\phi^h(x + h, y) - 8\phi^h(x - h, y) + \phi^h(x - 2h, y)}{12h},$$

with an analogous expression used for $\partial_y^h \phi^h$.

4.2. The homogeneous Poisson problem on an annulus domain

We first present a simple benchmark to assess stencil design strategies applied to Dirichlet and/or Neumann conditions. The goal is to select the algorithm that provides the best stencil in terms of accuracy and stability. To this end, we consider an annulus domain $\Omega \subset \mathbb{R}^2$ with radii $R_1 = \frac{\sqrt{5}}{5}$ and $R_2 = \frac{\sqrt{3}}{2}$, namely $\Omega = \{\mathbf{x} \in \mathbb{R}^2 \mid R_1 < |\mathbf{x}| < R_2\}$. The inner and outer

boundaries are $\partial_i\Omega = \{\mathbf{x} \in \mathbb{R}^2 \mid |\mathbf{x}| = R_1\}$ and $\partial_o\Omega = \{\mathbf{x} \in \mathbb{R}^2 \mid |\mathbf{x}| = R_2\}$, respectively. The boundary of the domain is then given by $\partial\Omega = \partial_i\Omega \cup \partial_o\Omega$.

For the sake of simplicity, problem (1) is simplified to the Laplace operator with $\kappa = 1$, $U(\mathbf{x}) = (0, 0)$, $f(\mathbf{x}) = 0$ in Ω . Moreover, for the boundary condition we use $a_D(\mathbf{x}) = 1$ and $a_N(\mathbf{x}) = 0$ on $\partial_i\Omega$, and $a_N(\mathbf{x}) = 1$ and $a_D(\mathbf{x}) = 0$ on $\partial_o\Omega$. The problem then reads

$$\begin{cases} -\Delta\phi(\mathbf{x}) = 0 & \text{for } \mathbf{x} \in \Omega, \\ \phi(\mathbf{x}) = 0 & \text{for } \mathbf{x} \in \partial_i\Omega, \\ \nabla\phi(\mathbf{x}) \cdot \mathbf{n}(\mathbf{x}) = 1 & \text{for } \mathbf{x} \in \partial_o\Omega. \end{cases} \quad (20)$$

Note that the rotational invariance of the parameters and geometry implies that the analytical solution is radial (depending only on $|\mathbf{x}|$) and reads $\phi^a(\mathbf{x}) = a + b \log(|\mathbf{x}|)$. Enforcing the boundary conditions, we have:

$$a + b \log(R_1) = 0, \quad \frac{b}{R_2} = 1. \quad (21)$$

Then, $b = R_2$ and $a = -R_2 \log(R_1)$, leading to

$$\phi^a(\mathbf{x}) = R_2 \log\left(\frac{|\mathbf{x}|}{R_1}\right). \quad (22)$$

We first analyze the impact of the S1, S2 and S3 stencil reconstructions for the simple Poisson problem. We display in Fig. 4 (left panel) the errors in norm L^1 and L^∞ by using the S1 algorithm. Computations have been carried out over the family of grids and we draw in blue the average decay from which we deduce the order method. The middle panel and the right panel correspond to the S2 and S3 algorithms, respectively. Stencil S1 leads to a scheme with low accuracy, mainly due to the treatment of ghost points located far from the boundary, where the resulting stencil is not guaranteed to be globally well-conditioned. Stencil S2 improves the error decay, although the convergence is not strictly monotonic. A more consistent decrease is achieved with stencil S3, which is globally well-conditioned; however, the L^∞ norm of the gradient of the solution does not reach the expected fourth-order accuracy.

We now consider the S4 reconstruction and examine the three variants for a fixed cone angle of $\vartheta = 60^\circ$. Fig. 5 shows the errors and convergence curves for the first (left panel), second (middle panel), and third (right panel) variants across the different grids. Despite stencil S4.1 yields reasonable results for the L^1 norms of both the solution and its gradient, the

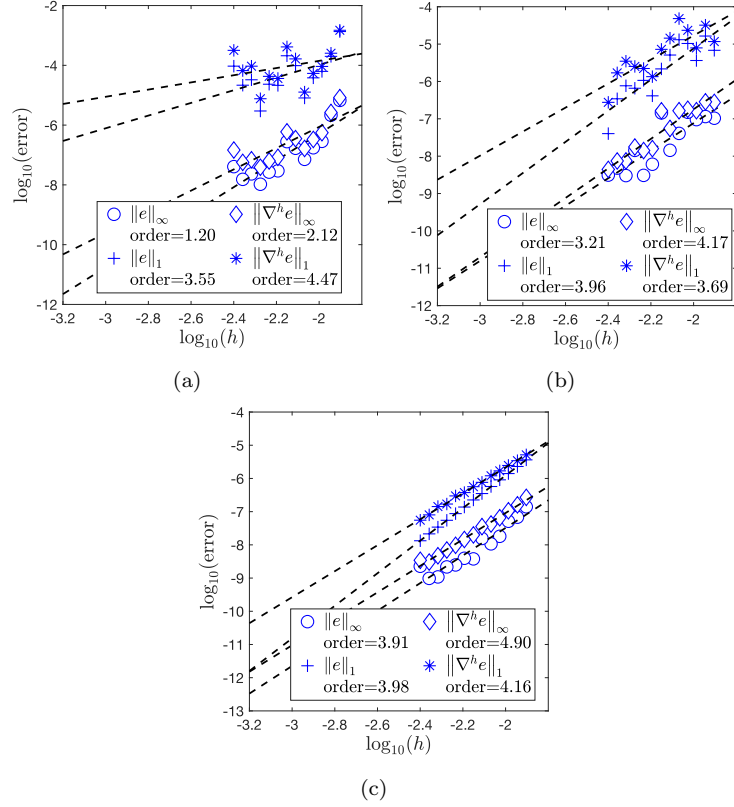


Figure 4: Numerical errors for the test of Section 4.2. We use the stencils S1 (a), S2 (b), and S3 (c). Orders (solution: L^1 , L^∞ ; gradient: L^1 , L^∞). (a) 3.55, 1.20, 4.47, 2.12. (b) 3.96, 3.21, 3.69, 4.17. (c) 3.98, 3.91, 4.16, 4.90.

L^∞ norm does not decrease monotonically and therefore we discard such a version. Stencil S4.2 shows a noticeable improvement, while the best overall performance is achieved with S4.3.

To provide a more in-depth analysis, we report in Table 1 the stencil sizes for the different variations. The first column lists the number of points in each stencil, while the remaining columns report the number of stencils for the corresponding sizes. We observe that none of the three stencils (S4.1, S4.2 and S4.3) requires more than 19 points, and the vast majority of stencils contain at most 17 points (recall that $m_o = 15$). Thus, admissibility is attained without a significant enlargement of the stencil.

Figure 6 compares the three stencils (S4.1, S4.2, and S4.3) with respect to Requirements 1-3, using $N = 502$. The left panel shows a box plot

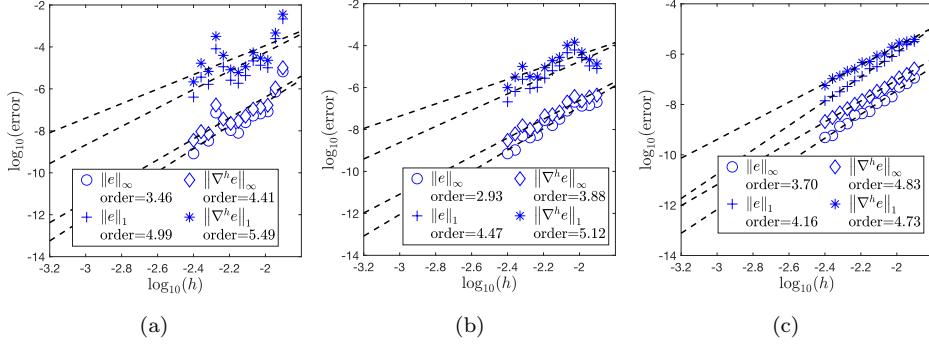


Figure 5: Numerical errors for the test of Section 4.2. We use the stencils S4.1 (a), S4.2 (b), and S4.3 (c), with a vertex angle of $\vartheta = 60^\circ$. Orders (solution: L^1 , L^∞ ; gradient: L^1 , L^∞). (a) 4.99, 3.46, 5.49, 4.41. (b) 4.47, 2.93, 5.12, 3.88. (c) 4.16, 3.70, 4.73, 4.83.

# of grid points	Stencil length					
	S4.1		S4.2		S4.3	
	Count	(%)	Count	(%)	Count	(%)
15	912	24.46	912	24.46	808	21.67
16	1348	36.16	1188	31.87	1284	34.44
17	1032	27.68	1112	29.83	1192	31.97
18	404	10.84	484	12.98	412	11.05
19	32	0.86	32	0.86	32	0.86

Table 1: Stencil size (number of points per stencil) for the three stencils S4.1, S4.2, and S4.3, with $N = 502$.

of the condition numbers on a logarithmic scale of the matrices B_k (as defined in Definition 2). Stencil S4.1 produces some highly ill-conditioned stencils, with condition numbers exceeding 3×10^5 , whereas stencils S4.2 and S4.3 substantially improve conditioning. Finally, the right panel reports the values of R_k (see Definition 3). Stencil S4.3 consistently yields stencils that satisfy the global well-conditioning criterion, performing best among the three stencils in this regard.

Sensitivity to the conditioning tolerances. The stencil construction strategy S4.3 depends on two parameters: the local conditioning tolerance Λ_{LOC} , which controls the conditioning of the local reconstruction matrix, and the global conditioning tolerance Λ_{GLO} , which limits the relative magnitude

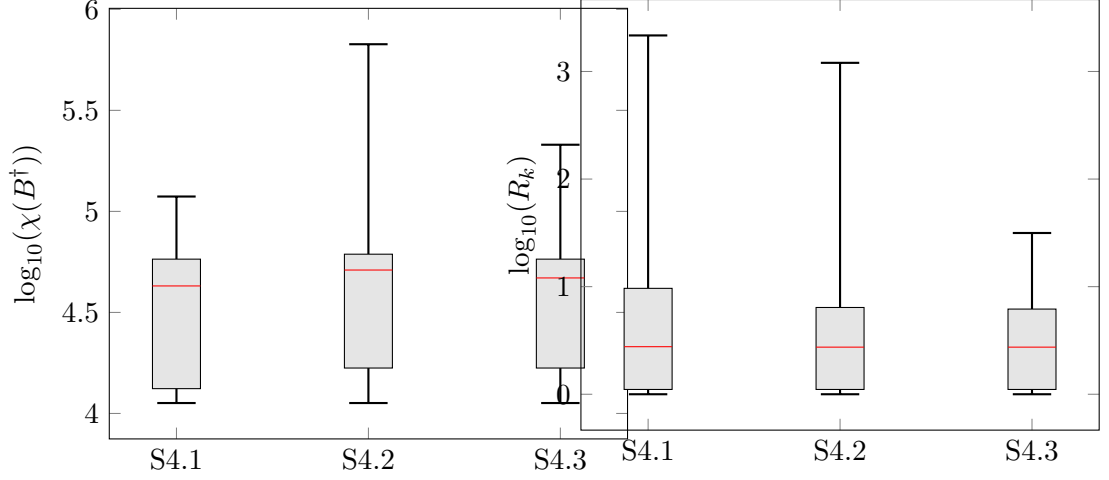


Figure 6: Boxplots comparing local and global conditioning for the three stencils S4.1, S4.2, and S4.3, with $N = 502$. Left: distribution of condition numbers of the local matrices $B^\dagger$ on a logarithmic scale. Right: distribution of the value R_k (Eq. 12) on a logarithmic scale. In each box plot, the central line denotes the median, the box spans the first and third quartiles, and the whiskers extend to the most extreme observations.

of the ghost-point coefficients. To assess the sensitivity of the proposed strategy to these parameters, we perform a series of numerical experiments in which both tolerances are varied over a representative range. Since the purpose of this study is to evaluate the robustness of the stencil construction rather than the convergence of the numerical scheme, all tests are carried out on a single representative mesh with $N = 502$. For each ghost point $\mathbf{x}^k \in \mathcal{G}_h$, let $\mathcal{S}^k$ denote its associated stencil. The normalized stencil diameter is defined as

$$\delta_k = \frac{1}{h} \max_{\mathbf{x}_i, \mathbf{x}_j \in \mathcal{S}^k} \|\mathbf{x}_i - \mathbf{x}_j\|_2, \quad (23)$$

where h is the Cartesian mesh size. We then consider the maximum stencil diameter $\delta_{\max} = \max_k \delta_k$.

Table 2 reports, for each pair $(\Lambda_{\text{LOC}}, \Lambda_{\text{GLO}})$, the discrete L^∞ relative error, the maximum normalized stencil diameter $\delta_{\max}$, and the maximum condition number among the accepted local reconstruction matrices, $\chi_{\text{loc}}^{\max} = \max_k \chi(B^{k\dagger})$. The error and condition number are reported on a logarithmic scale.

The results indicate that the proposed strategy is only mildly sensitive to the choice of the conditioning tolerances. Decreasing Λ_{LOC} makes the local admissibility criterion more restrictive, which leads to slightly larger

$\Lambda_{\text{LOC}} \backslash \Lambda_{\text{GLO}}$	1	5	10	50	100
10^2	(−4.83, 7.6, 2.02)	(−4.82, 7.4, 2.03)	(−4.82, 7.3, 2.04)	(−4.81, 7.2, 2.05)	(−4.81, 7.1, 2.06)
10^3	(−4.84, 7.1, 3.00)	(−4.84, 7.0, 3.01)	(−4.84, 6.9, 3.02)	(−4.83, 6.8, 3.03)	(−4.83, 6.7, 3.04)
10^4	(−4.84, 6.7, 4.00)	(−4.84, 6.6, 4.01)	(−4.84, 6.5, 4.02)	(−4.84, 6.4, 4.03)	(−4.83, 6.3, 4.04)
10^5	(−4.84, 6.4, 5.00)	(−4.84, 6.3, 5.01)	(−4.84, 6.2, 5.02)	(−4.84, 6.1, 5.03)	(−4.83, 6.0, 5.04)
10^6	(−4.84, 6.2, 6.00)	(−4.84, 6.1, 6.01)	(−4.84, 6.0, 6.02)	(−4.84, 5.9, 6.03)	(−4.83, 5.8, 6.04)
10^7	(−4.83, 6.1, 6.72)	(−4.83, 6.0, 6.79)	(−4.82, 5.9, 6.86)	(−4.81, 5.8, 6.93)	(−4.80, 5.7, 7.00)

Table 2: Sensitivity of strategy S4.3 to the local and global conditioning tolerances, Λ_{LOC} and Λ_{GLO} , respectively. The computations are performed on a single mesh with $N = 502$. Each entry has the form $(\log_{10} \|e\|_{\infty}, \delta_{\text{max}}, \log_{10}(\chi_{\text{loc}}^{\text{max}}))$, where $\|e\|_{\infty}$ is the L^{∞} error defined in (18), δ_{max} is the maximum normalized stencil diameter, and $\chi_{\text{loc}}^{\text{max}} = \max_k \chi(B^{k\dagger})$ is the maximum condition number of the accepted local reconstruction matrices. The parameter values of Λ_{LOC} and Λ_{GLO} adopted in the remainder of the paper are highlighted in bold.

stencil diameters while reducing the maximum local condition number. Conversely, larger values of Λ_{LOC} allow more compact stencils to be accepted at the expense of admitting less well-conditioned local reconstruction matrices. Likewise, increasing Λ_{GLO} relaxes the global admissibility criterion, reducing the need for stencil modifications and producing slightly more compact stencils. Despite these geometric changes, the numerical error remains nearly unchanged over the range of tolerances considered. This demonstrates that the proposed method is robust with respect to moderate variations of both parameters, and that the values $\Lambda_{\text{LOC}} = 10^6$ and $\Lambda_{\text{GLO}} = 10$ provide a satisfactory compromise between stencil locality and numerical robustness.

Sensitivity to the cone angle ϑ . The choice of the cone angle ϑ introduces an additional degree of freedom. We therefore analyze the stencil quality as a function of the aperture. To this end, we employ the S4.3 algorithm to construct stencils for different angles, namely $\vartheta = 30, 45, 90, 120, 360^\circ$, reproducing the error and convergence curves in Fig. 7. We clearly achieve the best results with $\vartheta = 45^\circ$ and $\vartheta = 60^\circ$.

As an intermediate conclusion, we observe that the best results are obtained with the algorithms S3 (Approach A), S4.2 with $\vartheta = 45^\circ$ (Approach B) and S4.3 with $\vartheta = 60^\circ$ (Approach C). To compare the three approaches, we now assess Requirement 4. Figure 8 illustrates the distribution of the stencil normalized diameters defined in (23). Approach C clearly provides the most compact stencils, and therefore corresponds to the best approach for Requirement 4. In conclusion, from this point onwards, we adopt Approach C (stencil S4.3 with a vertex angle of $\vartheta = 60^\circ$) for all subsequent

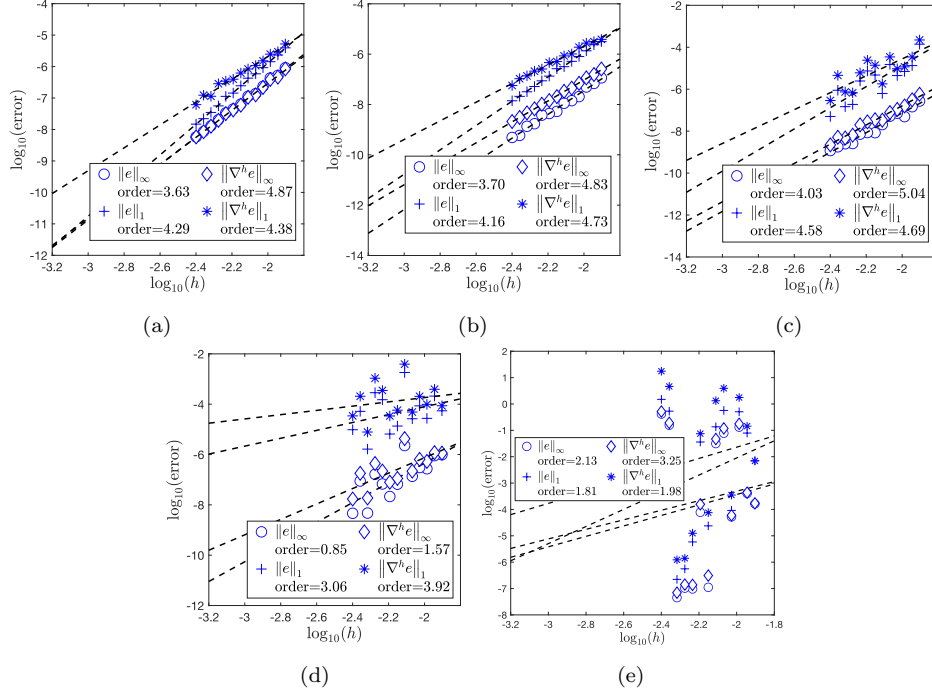


Figure 7: Numerical errors for the test of Section 4.2. We use the stencil S4.3 and a vertex angle of $\vartheta = 45^\circ$ (a), 60° (b), 90° (c), 120° (d), 360° (e). See Fig. 2 for the respective stencils. Orders (solution: L^1 , L^∞ ; gradient: L^1 , L^∞). (a) 4.29, 3.63, 4.38, 4.87. (b) 4.16, 3.70, 4.73, 4.83. (c) 4.58, 4.03, 4.69, 5.04. (d) 3.06, 0.85, 3.92, 1.57. (e) 1.81, 2.13, 1.98, 3.25.

tests.

Comparison with existing methods in the literature. We compare the proposed Approach C, corresponding to strategy S4.3 with $\vartheta = 60^\circ$, with two representative high-order ghost-point discretizations from the literature: the Coco–Russo method, originally introduced as a second-order method in [11] and subsequently extended to fourth order in [20], and a modified version of the finite-difference ROD method proposed in [23].

The original ROD method is based on a different stencil construction strategy and is designed to achieve sixth-order accuracy. In particular, the reconstruction of a ghost value may involve more than one boundary point and several neighbouring internal points. Consequently, a direct comparison would combine the effects of different approximation orders and substantially different stencil structures. To provide a more homogeneous comparison, we

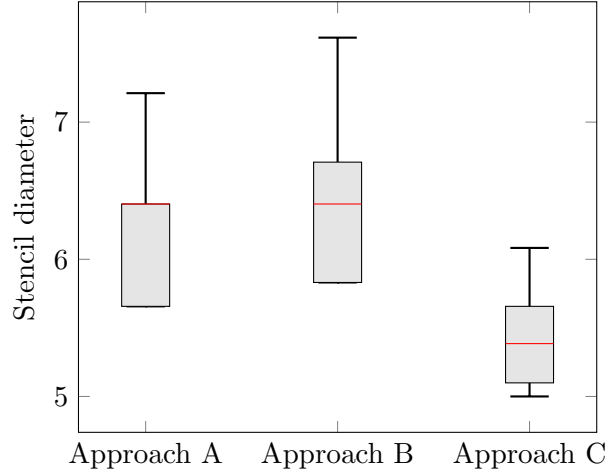


Figure 8: Boxplots of stencil diameters (Eq. (23)) for the three most accurate approaches: S3 (Approach A), and S4.3 for $\vartheta = 45^\circ$ (Approach B) and $\vartheta = 60^\circ$ (Approach C). We choose $N = 502$. In each box plot, the central line denotes the median, the box spans the first and third quartiles, and the whiskers extend to the most extreme observations.

therefore consider a modified fourth-order ROD-type construction based on the same local polynomial approximation framework used for Approach C. However, the stencil associated with a ghost point does not contain any additional ghost points other than the ghost point for which the equation is constructed. Therefore, the resulting ghost equations are mutually decoupled, as in the original ROD method.

The results are reported in Table ???. The Coco–Russo method employs substantially larger stencils, both in terms of cardinality and maximum normalized diameter. The larger spatial extent increases the number of data dependencies and is therefore expected to increase communication costs in distributed-memory implementations, where compact stencils are generally preferable. An important advantage of the Coco–Russo construction is that the stencil coefficients can be computed efficiently by means of tensor-product one-dimensional interpolations. Furthermore, the relative positions of the ghost point and its associated boundary point provide sufficient geometric information to modify the stencil so that Requirements 1–3 are satisfied. However, the construction does not generally satisfy Requirement 4, since the resulting stencil may have a relatively large spatial extent.

The modified ROD method is similar in both construction and numerical behaviour to Approach A, since both belong to the explicit (decoupled)

class of ghost-point methods, whereas Approach C is implicit (coupled), as neighbouring ghost unknowns may enter the same boundary equation. Decoupled formulations allow the ghost equations to be treated independently, which is particularly attractive for parallel implementations, but may require interior points farther from the boundary, resulting in wider stencils and potentially larger communication costs. In contrast, Approach C introduces algebraic coupling among ghost unknowns but produces substantially more compact stencils without degrading accuracy. Moreover, the proposed local and global conditioning criteria lead, for the benchmarks considered, to a better-conditioned discrete system. The original ROD method typically employs larger stencils than the modified version considered here, since its reconstruction may involve additional interior points and more than one boundary point; this additional information is used to achieve the higher accuracy of the original sixth-order formulation.

Overall, the proposed Approach C achieves an accuracy comparable to that of the considered existing methods while producing more compact stencils and a better-conditioned linear system. Unlike the other constructions, Approach C systematically controls both the conditioning of the local reconstruction and the relative magnitude of the coefficients in the coupled ghost-point equations. The numerical results indicate that this conditioning control can be achieved without compromising accuracy, while substantially reducing the spatial extent of the stencils. These features make Approach C particularly attractive for large-scale computations and parallel implementations, especially in capturing boundary layer effects, as we will see in Section 4.4, Fig. 12.

Method	$\ e\ _\infty$	$ \overline{\mathcal{S}^k} $	$\delta_{\max}$	$\chi(A)$
Coco-Russo method [20]	5.54×10^{-8}	25	11.3	3.95×10^8
ROD method [23]	3.21×10^{-8}	15.82	7.13	4.12×10^8
Approach C	4.38×10^{-8}	15.71	6.11	3.15×10^8

Comparison of the proposed Approach C (S4.3 with $\vartheta = 60^\circ$) with representative high-order ghost-point methods from the literature: the fourth-order Coco-Russo method [20] and the fourth-order version of the finite-difference ROD method [23]. All methods are tested on the same benchmark problem (24) and the same Cartesian mesh ($N = 502$). We report the L^∞ error defined in (18), the average cardinality of the stencils adopted for the ghost points, the maximum stencil diameter $\delta_{\max} = \max_k \delta_k$ of (23), and the condition number of the global matrix A .

4.3. Poisson problem and complex domains

We address the ability of the numerical method to handle complex domains that do not fit with the Cartesian grids. In particular, we consider domains with different types of curvature (locally concave or convex), or presenting some cusps or corners. This leads to challenges in prescribing the correct boundary conditions at the discrete level, while preserving the optimal order. We assess the accuracy of the proposed method across different geometries (taken from [18]): the rotated leaf, the flower, and the hourglass. These four domains are shown in Fig. 9 and represented by a level set function, as described in Section 2.4.

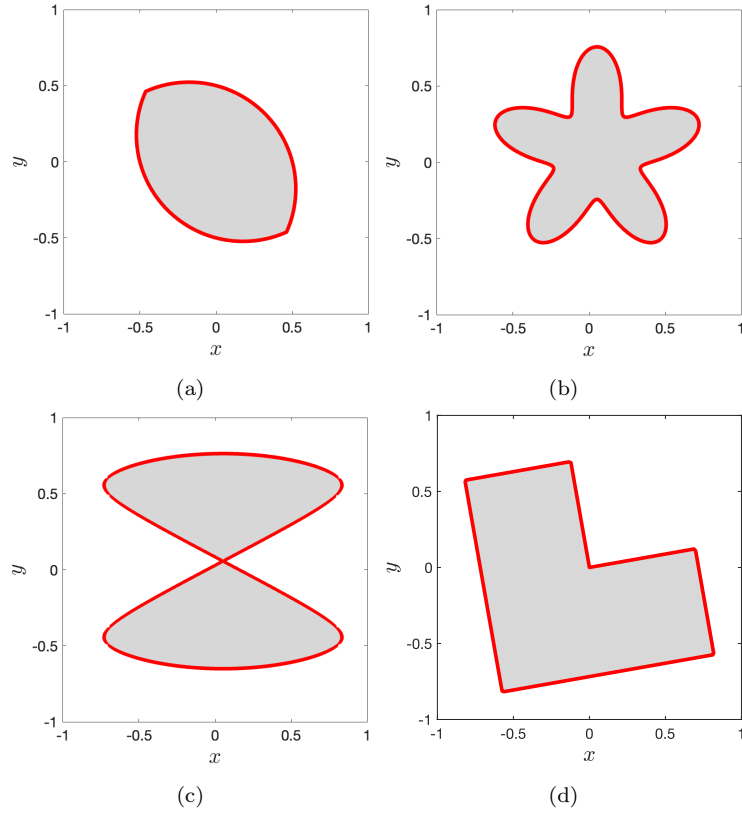


Figure 9: Complex geometries that do not fit with the Cartesian grid: the rotated leaf (a), the flower (b), the hourglass (c), and the rotated L -shaped domain (d).

- The rotated leaf. We use the following level-set function:

$$\begin{aligned}\tilde{x}_1 &= -0.25, \quad x_1 = \tilde{x}_1 \cos(\pi/4), \quad y_1 = \tilde{x}_1 \sin(\pi/4), \quad R_1 = \sqrt{(x - x_1)^2 + (y - y_1)^2} \\ \tilde{x}_2 &= 0.25, \quad x_2 = \tilde{x}_2 \cos(\pi/4), \quad y_2 = \tilde{x}_2 \sin(\pi/4), \quad R_2 = \sqrt{(x - x_2)^2 + (y - y_2)^2} \\ r_0 &= 0.7, \quad \varphi_1 = R_1 - r_0, \quad \varphi_2 = R_2 - r_0, \quad \varphi = \max\{\varphi_1, \varphi_2\}.\end{aligned}$$

- The flower-shaped domain. The level-set function is:

$$\begin{aligned}X &= x - 0.03\sqrt{3}, \quad Y = y - 0.04\sqrt{2}, \quad R = \sqrt{X^2 + Y^2} \\ \varphi &= R - 0.5 - \frac{Y^5 + 5X^4Y - 10X^2Y^3}{5R^5}.\end{aligned}$$

- The hourglass-shaped domain. It is a domain with a saddle point, whose level-set function is:

$$X = x - 0.03\sqrt{3}, \quad Y = y - 0.04\sqrt{2}, \quad \varphi = 256Y^4 - 16X^4 - 128Y^2 + 36X^2.$$

- The rotated L -shaped domain. It is a domain with a re-entrant corner, obtained by rotating an axis-aligned L -shaped geometry by an angle $\theta = \pi/18$ about the origin. To define its level-set representation, we introduce the rotated coordinates $\tilde{x} = x \cos \theta + y \sin \theta$, $\tilde{y} = -x \sin \theta + y \cos \theta$. In the $(\tilde{x}, \tilde{y})$ coordinate system, the outer square is represented by

$$\varphi_1(x, y) = \max \left\{ \tilde{x}^2 - \frac{1}{2}, \tilde{y}^2 - \frac{1}{2} \right\},$$

while the square removed from its upper-right quadrant is represented by

$$\varphi_2(x, y) = \max \left\{ \left(\tilde{x} - \frac{\sqrt{2}}{4} \right)^2 - \frac{1}{8}, \left(\tilde{y} - \frac{\sqrt{2}}{4} \right)^2 - \frac{1}{8} \right\}.$$

The level-set function defining the rotated L -shaped domain is then $\varphi = \max\{\varphi_1, -\varphi_2\}$. The domain has a re-entrant corner at the origin with opening angle $3\pi/2$.

Remark 8: For the rotated-leaf and L -shaped domains, the level-set function is constructed by combining two sub-level-set functions, say for simplicity as $\varphi(\mathbf{x}) = \max\{\varphi_1(\mathbf{x}), \varphi_2(\mathbf{x})\}$. Accordingly, the boundary can be regarded as the union of distinct boundary branches, each associated with the zero level set of one of the sub-level-set functions. Although the individual functions φ_1 and φ_2 are smooth in the regions of interest, their maximum is generally non-

smooth along the set where the active branch changes, i.e., where $\varphi_1 = \varphi_2$. Consequently, a high-order reconstruction of $\nabla\varphi$ may lose accuracy in the vicinity of such locations. This is particularly relevant near corners, since the gradient of the level-set function is used to determine the normal direction required for the computation of the closest boundary point and for the discretization of Neumann boundary conditions [20]. To avoid reconstructing the gradient across this non-smooth interface, for each ghost point $\mathbf{x}_k$ we select the sub-level-set function whose zero level is locally closest to $\mathbf{x}_k$. More precisely, we define

$$i_k = \arg \min_{i=1,2} |\varphi_i(\mathbf{x}_k)|$$

and use φ_{i_k} , rather than the composite level-set function φ , to reconstruct the gradient and determine the corresponding normal direction. This provides a smooth local level-set representation of the boundary branch associated with the ghost point. At a corner, the boundary normal and, depending on the position of the ghost point, the association with a boundary branch may not be unique. The above selection criterion removes this ambiguity at the discrete level by associating each ghost point with a single smooth boundary branch, from which the normal direction and the corresponding boundary point are determined.

We solve the Poisson equation with source term and mixed boundary conditions:

$$\begin{cases} -\Delta\phi(\mathbf{x}) = f & \text{for } \mathbf{x} \in \Omega \\ \phi(\mathbf{x}) = g_D & \text{for } \mathbf{x} \in \Gamma_D \\ \nabla\phi(\mathbf{x}) \cdot \mathbf{n}(\mathbf{x}) = g_N & \text{for } \mathbf{x} \in \Gamma_N \end{cases} \quad (24)$$

where $\Gamma_D = \{(x, y) \in \partial\Omega \text{ such that } x \geq 0\}$ and $\Gamma_N = \partial\Omega \setminus \Gamma_D$. We choose f , g_D and g_N in such a way that the analytical solution is $\phi^a(x, y) = \sin(2x)\sin(5y)$, namely $f = -\Delta\phi^a$, $g_D = \phi^a$ and $g_N = \nabla\phi^a \cdot \frac{\nabla\varphi}{|\nabla\varphi|}$.

We display in Fig. 10 the L^1 and L^∞ errors along with the convergence curves. The error decays monotonically and aligns closely with the optimal fourth-order reference slope, reaffirming that the method is robust across different geometries and under mixed boundary conditions.

4.4. Steady convection–diffusion problem with source term in an annulus

Once again, the domain Ω is the annulus defined in Section 4.2, and we seek ϕ solution of the linear convection–diffusion equation $-\kappa\Delta\phi + U \cdot \nabla\phi = g$, with $\kappa \neq 0$. We prescribe homogeneous Dirichlet boundary conditions to assess the ability of the scheme to catch boundary layers. To this end, the

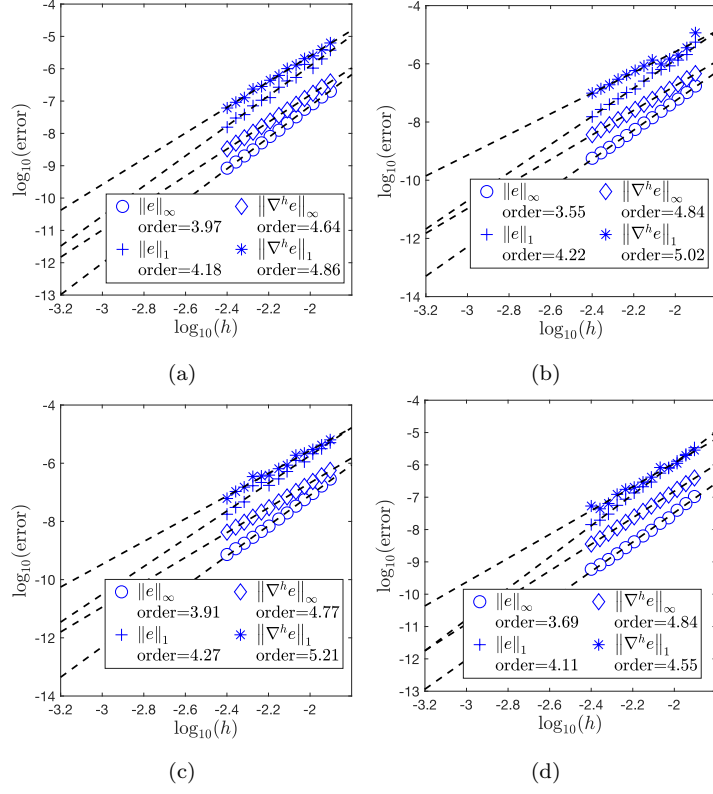


Figure 10: Numerical errors for the test of Section 4.3, using the domains of Fig. 9. Orders (solution: L^1 , L^∞ ; gradient: L^1 , L^∞). (a) 4.18, 3.97, 4.86, 4.64. (b) 4.22, 3.55, 5.02, 4.84. (c) 4.27, 3.91, 5.21, 4.77. (d) 4.11, 3.69, 4.55, 4.84.

velocity has the following expression:

$$U(x, y) = \frac{u_0}{r^2} \begin{pmatrix} x \\ y \end{pmatrix},$$

while $g(x, y) = g(r) = \frac{1}{r}$ depends only on r . Therefore, the solution ϕ is invariant by rotation and the equation reads in polar coordinates

$$-\frac{\kappa}{r} \frac{d}{dr} \left(r \frac{d\phi}{dr} \right) + \frac{u_0}{r} \frac{d\phi}{dr} = \frac{1}{r}.$$

Integration in r yields $-\kappa r \frac{d\phi}{dr} + u_0 \phi(r) = r + r_0$, with $r_0 \in \mathbb{R}$, and the homogeneous solution reads

$$\tilde{\phi}_0(r) = A\phi_0(r) = Ar^{u_0/\kappa}, \quad A \in \mathbb{R}.$$

On the other hand, we assume that the general solution can be written as $\phi(r) = A(r)\phi_0(r)$ and get

$$-\kappa r^{1+u_0/\kappa} \frac{dA}{dr} = r + r_0.$$

We have to consider several cases regarding the values of u_0 and κ .

Case 1: $u_0 \neq 0$ and $u_0 \neq \kappa$. Integrating once again gets the relation

$$A(r) = \frac{r^{1-u_0/\kappa}}{u_0 - \kappa} + r_0 \frac{r^{-u_0/\kappa}}{u_0} + C,$$

and the general solution is given by

$$\phi(r) = A(r)\phi_0(r) = \frac{r}{u_0 - \kappa} + \frac{r_0}{u_0} + Cr^{u_0/\kappa}, \quad (25)$$

where $C, r_0 \in \mathbb{R}$ will be defined to satisfy the boundary condition. Assuming $\phi(R_1) = \phi(R_2) = 0$, we obtain the real constant values

$$r_0 = \frac{u_0}{\kappa - u_0} \frac{R_1 R_2^{u_0/\kappa} - R_2 R_1^{u_0/\kappa}}{R_2^{u_0/\kappa} - R_1^{u_0/\kappa}}, \quad C = \frac{1}{\kappa - u_0} \frac{R_2 - R_1}{R_2^{u_0/\kappa} - R_1^{u_0/\kappa}}.$$

Case 2: $u_0 = 0$. For the pure diffusion problem, we get

$$\phi(r) = A(r)\phi_0(r) = A(r) = -\frac{1}{\kappa} (r + r_0 \log |r|) + C,$$

where r_0 and C are defined to satisfy the boundary condition. We assume that $\phi(R_1) = \phi(R_2) = 0$ and deduce that

$$r_0 = -\frac{R_2 - R_1}{\log(R_2) - \log(R_1)}, \quad C = -\frac{R_2 \log(R_1) - R_1 \log(R_2)}{\kappa(\log(R_2) - \log(R_1))}.$$

Case 3: $u_0 = \kappa \neq 0$. For this very specific case, we get an alternative solution that reads

$$A(r) = -\frac{1}{\kappa} \left(\log |r| - \frac{r_0}{r} \right) + C$$

and the general solution is then given by

$$\phi(r) = -\frac{1}{\kappa} (r \log |r| - r_0) + Cr$$

where r_0 and C will be defined to satisfy the boundary condition. From $\phi(R_1) = \phi(R_2) = 0$, we deduce that

$$r_0 = -\frac{R_1 R_2 (\log(R_2) - \log(R_1))}{R_2 - R_1}, \quad C = \frac{R_2 \log(R_2) - R_1 \log(R_1)}{\kappa(R_2 - R_1)}.$$

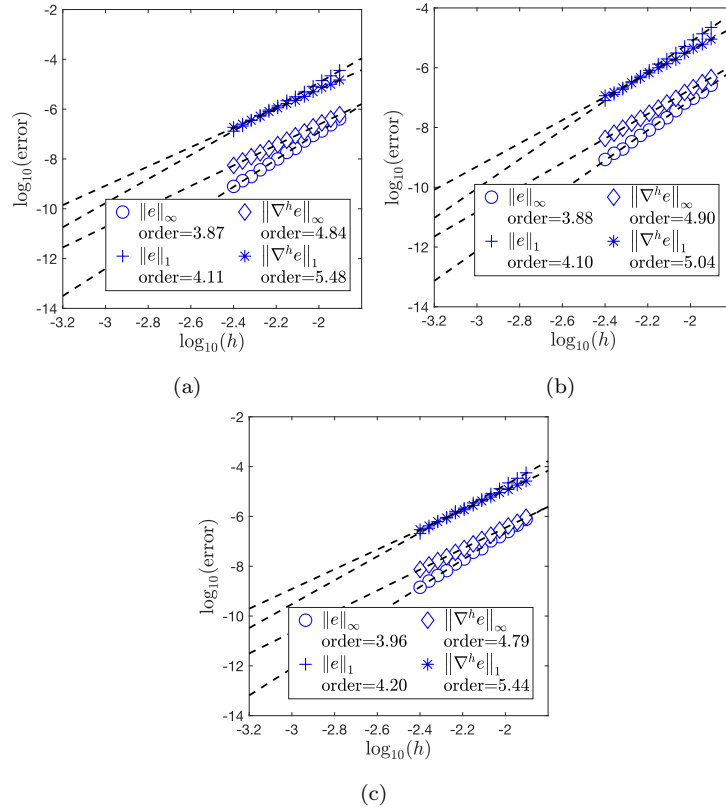


Figure 11: Numerical errors for the tests of Section 4.4: Case 1 (a), Case 2 (b), and Case 3 (c). Orders (solution: L^1 , L^∞ ; gradient: L^1 , L^∞). (a) 4.11, 3.87, 5.48, 4.84. (b) 4.10, 3.88, 5.04, 4.90. (c) 4.20, 3.96, 5.44, 4.79.

We plot in Fig. 11 the error convergence curves for the three cases. More specifically, the left panel corresponds to Case 1 with $\kappa = 2$ and $u_0 = 1$, the

middle panel displays Case 2 with $\kappa = 1$ and $u_0 = 0$, and the right panel represents Case 3 with $\kappa = u_0 = 1$. The major conclusion is that we recover the optimal fourth order method in all the cases with very straightforward convergence curves and low deviation regarding the slope. This point is not so easy to achieve in the literature since the stencil configuration change for each grid and usually brings some instability/non monotony decay of the order, similar to the one we have observed with S1 or S2 in L^∞ norm, for instance (Fig. 4).

Another issue concerns the Case 1, that exhibits pronounced boundary layer effects when $u_0 R_2 \gg \kappa \neq 0$ as shown in Fig. 12, right panel. These arise from the dominance of the convective term over diffusion, combined with fixed Dirichlet boundary conditions, as evidenced by the analytical solution (25). The length $\delta = \frac{\kappa}{u_0}$ represents the characteristic size of the boundary layer to compare with the characteristic length of the geometry R_2 , leading to the definition of the global dimensionless Péclet number $Pe = \frac{u_0 R_2}{\kappa}$. We perform two tests: benchmark 1 with $Pe = 8$ and $\delta = 1/10$, and benchmark 2 with $Pe = 20$ and $\delta = 1/25$. Since our objective is to investigate how the boundary conditions' discretization performs in the presence of such boundary layers, we introduce the dimensionless local (cell) Péclet number, defined as $Pe_{\text{loc}} = \frac{u_0 \Delta x}{\kappa}$, which depends on the grid size. Such a number describes the trade-off between boundary layer and grid size at the discrete level.

Taking $\Delta x = \frac{2}{N}$ with N ranging from 160 to 502, we have $Pe_{\text{loc}} \in [4.0 \times 10^{-2}, 1.6 \times 10^{-1}]$ for the first benchmark (Fig. 12, left panel), and $Pe_{\text{loc}} \in [1.0 \times 10^{-2}, 4.2 \times 10^{-1}]$ for the second benchmark (Fig. 12, middle panel), where we report the numerical errors and convergence orders. Once again, we observe the errors decrease monotonically and in accordance with the expected fourth-order convergence rate.

We point out that for problems with a higher cell Péclet number, namely in convection-dominated regimes, a stable and accurate discretization of the convective term would typically require an upwind scheme rather than central differences, unless a sufficiently fine grid is employed to satisfy the Péclet condition.

We point out that for problems with a higher cell Péclet number, namely in convection-dominated regimes, a stable and accurate discretization of the convective term would typically require an upwind or stabilized scheme rather than central differences, unless a sufficiently fine grid is employed to satisfy the Péclet condition. Since the present work focuses on the boundary discretization, the proposed ghost-point construction can be combined with

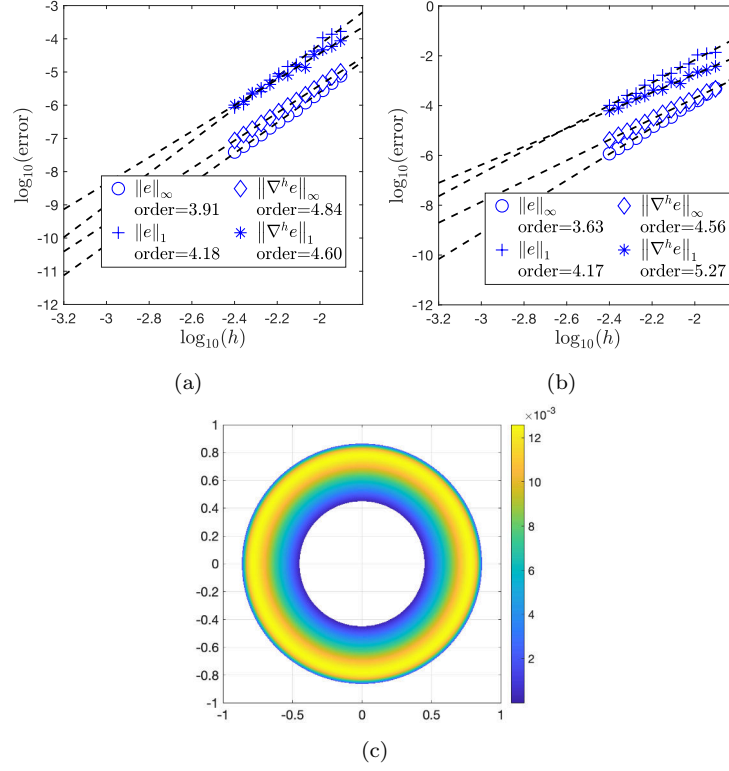


Figure 12: Numerical errors for the tests of Section 4.4: Benchmark 1: $\kappa = 1$ and $u_0 = 10$ (a). Benchmark 2: $\kappa = 1$ and $u_0 = 25$ (b). The numerical solution of the case $\kappa = 1$ and $u_0 = 25$ is shown in the last plot (c). Orders (solution: L^1 , L^∞ ; gradient: L^1 , L^∞). (a) 4.18, 3.91, 4.60, 4.84. (b) 4.17, 3.63, 5.27, 4.56.

different interior convection schemes depending on the flow regime.

Remark 9 (Cell Péclet number): Convection-dominated regimes discretization with higher cell Péclet numbers *i.e.* $Pe_h = |u_0|h/\kappa \geq 2$ are not unconditionally stable since we diagonal dominated property no longer holds. In general, stable and accurate discretization of the convective term would typically require an upwind or stabilized scheme rather than central differences, unless a sufficiently fine grid is employed to restore a cell Péclet number not exceeding 2. Of course, the compactness of the proposed boundary stencils does not remove such an issue, which originates from the interior discretization.

Nevertheless, the standard-centered discretization is still available even for large cell Péclet numbers when dealing with a smooth enough solution. In that case,

the parasitic modes are not activated, and the small dissipation manages to stabilize the solution. All the numerical simulations we have carried out do not present such non-physical oscillations, and we preserve the optimal convergence rate with compact stencils.

However, compact stencils are particularly advantageous in boundary-layer regions with sharp gradients, since the boundary reconstruction involves points closer to the boundary and therefore samples the solution over a smaller region where strong gradients may be present. Since the present work focuses on the boundary discretization, the proposed ghost-point construction can be combined with upwind or stabilized interior convection schemes when larger cell Péclet numbers are considered.

5. Conclusive Remarks

In this work, we introduced a novel boundary operator framework for ghost-point finite-difference methods on unfitted Cartesian grids, with the primary goal of achieving high-order accuracy while maintaining compact and well-conditioned stencils. The proposed approach reformulates the enforcement of boundary conditions through a local linear operator constructed via constrained least-squares reconstruction. This strategy provides significant flexibility in stencil design.

A key contribution of the paper lies in the identification of two criteria, which guide the construction of admissible and stable stencils: local and global conditioning. Building upon these criteria, we developed several strategies, finally leading to the cone-based adaptive procedure. This approach successfully balances the requirements of admissibility, conditioning, and compactness, overcoming limitations of traditional wide-stencil ghost-point methods.

The numerical experiments confirm several important findings. First, the proposed method achieves consistent fourth-order accuracy for both the solution and its gradient across a wide range of test cases, including mixed boundary conditions and complex geometries represented implicitly via level-set functions. Second, unlike many existing ghost-point techniques, the method exhibits smooth and monotonic convergence behaviour with respect to grid refinement, reducing the occurrence of error oscillations typically associated with irregular stencil configurations. Third, the compactness of the selected stencils makes the method particularly suitable for large-scale and parallel simulations, where communication overhead is a critical factor. Moreover, the method demonstrates robustness in challenging regimes, including convection-diffusion problems with boundary layers.

Even in such cases, the approach maintains stability and accuracy without requiring excessive stencil enlargement.

Despite these advantages, some limitations of the present approach should be mentioned. The numerical tests cover several smooth and moderately complex two-dimensional geometries, but extremely sharp cusps, corners, or very thin geometric features may still produce unfavourable local stencils. In such situations, the admissibility and conditioning criteria may require larger stencils or additional geometric treatment. Moreover, the least-squares reconstruction introduces a local preprocessing cost for each ghost point. This cost remains moderate because the stencils are compact and the local systems are small, and it can be amortized for fixed geometries; however, it may become more relevant for moving domains, where the boundary projection and stencil coefficients must be updated. Finally, for strongly convection-dominated regimes, centred discretizations of the convective term may need to be replaced or complemented by stabilized schemes. Future work will focus on three main directions: the extension to three-dimensional unfitted Cartesian grids, the treatment of time-dependent PDEs and moving domains, and the coupling of the proposed boundary operator with stabilized discretizations for convection-dominated flows. Additional work will also address adaptive stencil strategies for highly curved or nearly singular geometries.

As a natural extension of the present work, future research will investigate the application of the proposed approach to nonlinear problems, including the two-dimensional viscous Burgers equation.

Overall, the boundary operator formulation provides a flexible framework for high-order discretizations on unfitted domains, contributing to the design of efficient and robust numerical methods on complex geometries, particularly interesting in applications involving moving interfaces or multiscale features.

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Conflicts of Interest

The authors declare that there is no conflict of interest.

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Appendix A. The S4 algorithm.

In the algorithm below, we denote by $\chi(B^k)$ the local condition number $\chi(B^{k\dagger})$ introduced in Definition 2. If B^k does not satisfy the maximum-rank property, we set $\chi(B^k) = +\infty$. The quantity R_k is the global indicator defined in Definition 3. In all the numerical tests, we use $\Lambda_{\text{LOC}} = 10^6$, $\Lambda_{\text{GLO}} = 10$, and we allow at most three replacement steps in the S4.2 correction.

Algorithm 1 Construction of the S4.1, S4.2 and S4.3 isosceles triangular stencils (vertex angle ϑ)

```

1: Define the set of candidate grid points  $\mathbf{y}_j \in \mathcal{A}_h$  such that the angle between
    $\overline{\mathbf{x}_k \mathbf{y}_j}$  and  $\overline{\mathbf{x}_k \mathbf{p}_k}$  is  $\leq \vartheta/2$ , and order them from the closest to the ghost point:
    $\{\mathbf{y}_1, \mathbf{y}_2, \mathbf{y}_3, \dots\}$ .
2: Define  $q = N_o$  and the stencil  $\mathcal{S}^k = \{\mathbf{y}_1, \dots, \mathbf{y}_q\}$ .
3: while  $\chi(B^k) > \Lambda_{\text{LOC}}$  do
4:    $\mathcal{S}^k \leftarrow \mathcal{S}^k \cup \{\mathbf{y}_{q+1}\}$ .
5:    $q \leftarrow q + 1$ .
6: end while
7: Set  $\mathcal{S}_{\text{S4.1}}^k = \mathcal{S}^k$ .
8: Compute the coefficients  $a_\ell^k$  and the value  $R_k$ .
9: Set  $i = 0$ .
10: while  $R_k > \Lambda_{\text{GLO}}$  and  $i < 3$  do
11:   Identify the point  $\mathbf{x}_\ell \in \mathcal{S}^k \cap \mathcal{G}_h$ , with  $\mathbf{x}_\ell \neq \mathbf{x}_k$ , corresponding to the largest
   coefficient  $|a_\ell^k|$  entering the definition of  $R_k$ .
12:    $\mathcal{S}^k \leftarrow \mathcal{S}^k \setminus \{\mathbf{x}_\ell\}$ .
13:    $\mathcal{S}^k \leftarrow \mathcal{S}^k \cup \{\mathbf{y}_{q+1}\}$ .
14:    $q \leftarrow q + 1$ .
15:   while  $\kappa(B^k) > 10^6$  do
16:      $\mathcal{S}^k \leftarrow \mathcal{S}^k \cup \{\mathbf{y}_{q+1}\}$ .
17:      $q \leftarrow q + 1$ .
18:   end while
19:   Compute again the coefficients  $a_\ell^k$  and the value  $R_k$ .
20:    $i \leftarrow i + 1$ .
21: end while
22: Set  $\mathcal{S}_{\text{S4.2}}^k = \mathcal{S}^k$ .
23: if  $R_k > \Lambda_{\text{GLO}}$  then
24:   Modify the boundary point  $\mathbf{p}_k$  by projecting the ghost point horizontally
   or vertically onto the boundary, choosing the closer projection if both exist.
25:   Rebuild the ordered set of candidate grid points  $\{\mathbf{y}_1, \mathbf{y}_2, \mathbf{y}_3, \dots\}$  using the
   modified point  $\mathbf{p}_k$ .
26:   Define  $q = N_o$  and the stencil  $\mathcal{S}^k = \{\mathbf{y}_1, \dots, \mathbf{y}_q\}$ .
27:   while  $\kappa(B^k) > 10^6$  do
28:      $\mathcal{S}^k \leftarrow \mathcal{S}^k \cup \{\mathbf{y}_{q+1}\}$ .
29:      $q \leftarrow q + 1$ .
30:   end while
31:   Compute the coefficients  $a_\ell^k$  and the value  $R_k$ .
32:   Set  $i = 0$ .
33:   while  $R_k > \Lambda_{\text{GLO}}$  and  $i < 3$  do
34:     Identify the point  $\mathbf{x}_\ell \in \mathcal{S}^k \cap \mathcal{G}_h$ , with  $\mathbf{x}_\ell \neq \mathbf{x}_k$ , corresponding to the
     largest coefficient  $|a_\ell^k|$  entering the definition of  $R_k$ .
35:      $\mathcal{S}^k \leftarrow \mathcal{S}^k \setminus \{\mathbf{x}_\ell\}$ .
36:      $\mathcal{S}^k \leftarrow \mathcal{S}^k \cup \{\mathbf{y}_{q+1}\}$ .
37:      $q \leftarrow q + 1$ .
38:     while  $\kappa(B^k) > 10^6$  do
39:        $\mathcal{S}^k \leftarrow \mathcal{S}^k \cup \{\mathbf{y}_{q+1}\}$ .
40:        $q \leftarrow q + 1$ .
41:     end while
42:     Compute again the coefficients  $a_\ell^k$  and the value  $R_k$ .
43:      $i \leftarrow i + 1$ .
44:   end while
45: end if
46: Set  $\mathcal{S}_{\text{S4.3}}^k = \mathcal{S}^k$ .

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