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ABSTRACT

Higher-order tangent bundles possess a rich tower of fibrations, suggesting the existence of geometric structures compatible with their iterated bundle structure. In this paper, we introduce the notion of a connection tower on a higher-order tangent bundle and study the geometric structures induced by such towers. In particular, we show that connection towers determine natural multiconnections, adapted splittings of the tangent bundle, and canonical vector bundle structures on higher-order tangent bundles. We then construct a specific connection tower induced by the Levi-Civita connection of a Riemannian manifold. This construction extends the classical Dombrowski connection map on the tangent bundle and leads naturally to a family of higher-order Sasaki metrics. We study the associated lifts of vector fields and derive explicit Lie bracket formulas for these lifts, together with structural identities for the induced multiconnection. Finally, we determine the Levi-Civita connection of the higher-order Sasaki metrics and derive explicit geodesic equations on the second- and third-order tangent bundles. We also obtain characterization results relating geodesics of the higher-order Sasaki metrics to geodesics on the base manifold.

1. Introduction

Nonlinear connections on higher-order tangent bundles, as well as on fiber bundles in general, play a fundamental role in modern differential geometry. They provide a natural way to distinguish between vertical directions, corresponding to motions within the fibers, and horizontal directions, representing motion along the base manifold. This decomposition gives rise, in a purely geometric manner, to important notions such as covariant differentiation, curvature, and torsion. The relevance of these geometric structures extends well beyond pure mathematics, particularly when considered in the context of higher-order tangent bundles.

Higher-order tangent bundles are natural generalizations of the notion of the tangent bundle, where equivalence classes of higher-order contact curves replace the first-order ones. They serve as evolution spaces for higher-order differential equations and, therefore, provide a natural framework for their study. Thanks to the contributions of Morimoto [22], León and Rodrigues [12], Crampin et al. [10], Saunders [25], Andrés et al. [11], Miron [19], [20], Bucătaru [7], [8], Suri [26] and others, who have highlighted the importance of these fiber bundles in areas such as the calculus of variations, Lagrangian and Hamiltonian mechanics, classical field theory and Finsler geometry, these geometric structures have been attracting increasing attention in recent years.

In general, nonlinear connections do not arise from linear connections, as no linear structure is assumed on the fibers of a general fiber bundle. However, in the case of a Riemannian manifold, the Levi-Civita connection naturally induces a nonlinear connection in the tangent bundle. In fact, this can be done in a more general setting, when the fiber bundle is vectorial, or even principal, and endowed with a linear connection ([17],[29],[28]). What is particularly interesting in the case of a Riemannian manifold is that the induced structure can be described in terms of a connection map. Furthermore, the induced connection map, when combined with the Riemannian metric, defines a natural Riemannian metric on the tangent bundle, known as the Sasaki metric ([24],[15],[1]). This metric is characterized by the orthogonality of the horizontal and vertical distributions determined by the induced connection. Analogous constructions also appear in Finsler geometry ([3],[4]).

Connection maps in the tangent bundle were originally introduced by Dombrowski [13] and extended to vector bundles by Miron [21]. Later, Bucătaru characterized nonlinear connections in higher-order tangent bundles using connection maps and developed a geometric setting that associates nonlinear connections with higher-order differential

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equations([5],[6],[9]). The pionner work of Kosambi is also an important reference regarding the geometry of higher-order differential equations and induced connections on higher-order tangent bundles ([18]). Higher-order tangent bundles can naturally be viewed as forming a tower of fiber bundle submersions, due to the variety of fibrations they possess, with the projection onto the base manifold arising in the limit. This perspective suggests the possibility of adapting the notion of connection maps in order to define a tower of nonlinear connections. In this context, many questions may be raised. In the present paper, our purpose is to study a specific connection tower induced by the Levi–Civita connection and, subsequently, the resulting Riemannian structure in higher-order tangent bundles.

The main contributions of this paper are the following.

1. We introduce the notion of a connection tower on a higher-order tangent bundle and develop its basic geometric properties. In particular, we show that connection towers induce canonical multiconnection structures and vector bundle structures on higher-order tangent bundles.
2. We construct a specific connection tower induced by the Levi–Civita connection of a Riemannian manifold. This construction extends the classical Dombrowski connection map on the tangent bundle to the higher-order setting.
3. Using the induced multiconnection, we study horizontal and vertical lifts of vector fields and derive explicit formulas for the Lie brackets of these lifts.
4. We define a family of higher-order Sasaki metrics associated with the Levi–Civita-induced connection tower and determine the corresponding Levi–Civita connections.
5. We derive explicit geodesic equations on the second- and third-order tangent bundles and obtain characterization results relating higher-order Sasaki geodesics to geodesics on the base manifold.

The paper is organized as follows. In Section 2, we briefly recall basic definitions and notations of the theory of higher-order tangent bundles. We also provide a more detailed exposition of the theory of nonlinear connections and connection maps. Then we introduce the concept of nonlinear connection tower in terms of connection maps. The main results of this paper are stated and proved in Sections 3 and 4. In these two sections, the presentation is structured to emphasize the analogy with the results formulated in Subsections 2.1, 3.2 and 4.1. In Section 3, we construct a specific connection tower induced by the Levi–Civita connection. We then examine how vector fields can be lifted by this connection tower and describe the Lie bracket in terms of these lifts. In Section 4, we define a family of higher-order Sasaki metrics that recover the classical Sasaki metric in the base case. We also determine the Levi–Civita connection and derive the geodesic equations for the second- and third-order tangent bundles. Finally, we extend classical results concerning Sasaki geodesics to higher-order bundles. Additional background information, computations, and technical lemmas are placed in the appendices. Table 1 containing the notation frequently used in the paper can be found at the end of the manuscript.

2. The geometry of higher-order tangent bundles

This section contains the geometric background on higher-order tangent bundles and nonlinear connections used throughout the paper. We adopt the formalism of describing nonlinear connections by means of connection maps. We first recall the tangent-bundle model, in the form needed later for comparison with the higher-order constructions. We then review the basic geometry of $T^{(k)}M$, including its almost-tangent structure, vertical flag, connection maps, multiconnections, and adapted bases. Finally, we introduce connection towers, which are special connection maps compatible with the canonical projections of higher-order tangent bundles, and we record the lift conventions that will be used in the sequel.

Additional background and auxiliary details are collected in Appendix A. This includes the fuller tangent-bundle discussion, the adapted-basis inversion formulas, the proof that connection towers induce vector bundle structures on the higher-order tangent bundles, and the basic technical lemmas for lifted vector fields. For convenience, Table 1 at the end contains the notation commonly used throughout the article.

2.1. The tangent-bundle model

Let M be an n -dimensional smooth manifold and let $\tau : TM \rightarrow M$ denote the tangent-bundle projection. We write a vector $u \in T_p M$ as $u = (p, v)$ when it is useful to distinguish the base point and fiber component, and we denote by $(q^{(0)i}, q^{(1)i})$ the tangent-lifted local coordinates on TM induced by local coordinates (q^i) on M .

For $u \in TM$, the vertical space is

$$V_u = \ker(\tau_*|_u) \subset T_u TM,$$

and the vertical bundle is $V = \bigsqcup_{u \in TM} V_u$. If $u = (p, v)$ and $z \in T_p M$, the vertical lift of z at u is the tangent vector z^v at $t = 0$ of the curve $t \mapsto v + tz$. Thus the vertical lift at u gives a linear isomorphism $T_p M \cong V_u$. The vertical lift of a vector field $X \in \mathfrak{X}(M)$ is the vector field X^v on TM given by

$$X^v(u) = (X(p))^v.$$

In local coordinates, if $X = X^i \partial_i$, then

$$X^v = X^i \frac{\partial}{\partial q^{(1)i}}.$$

The canonical almost-tangent structure J on TM is the $(1, 1)$ -tensor field

$$J(X) = (\tau_* X)^v, \quad X \in T_u TM.$$

In local coordinates,

$$J = \frac{\partial}{\partial q^{(1)i}} \otimes dq^{(0)i},$$

so that $J^2 = 0$ and

$$V_u = \text{Im}(J|_u) = \ker(J|_u).$$

A nonlinear connection on TM is a vector subbundle $H \subset TTM$ complementary to the vertical bundle

$$TTM = H \oplus V. \quad (1)$$

Equivalently, a nonlinear connection can be defined via a *connection map*. That is, a vector bundle morphism $K : TTM \rightarrow TM$ with base function τ such that

$$K \circ J = \tau_*,$$

in which case the corresponding horizontal bundle is given by $H = \ker(K)$. Conversely, every nonlinear connection is realized as the kernel of a unique connection map. Locally,

$$K = \left(dq^{(1)i} + K_j^i dq^{(0)j} \right) \otimes \partial_i,$$

where K_j^i are the connection coefficients. The connection map induces a canonical identification

$$\Phi_u : T_u TM \longrightarrow T_p M \oplus T_p M, \quad \Phi_u(X) = (\tau_* X, K(X)),$$

for each $u \in T_p M$ and $X \in T_u TM$.

A tangent vector $z \in T_p M$ has a unique horizontal lift $z^h \in H_u$ at $u = (p, v) \in T_p M$, characterized by

$$\tau_*(z^h) = z, \quad K(z^h) = 0.$$

Vector fields are lifted similarly: if $X \in \mathfrak{X}(M)$, then $X^h(u) = (X(p))^h$ for $u \in T_p M$.

Remark 1. We note that the horizontal lift depends on the chosen fiber velocity $v \in T_p M$ such that $u = (p, v)$. A more geometrically invariant construction is therefore to define lifts (and connection maps) on the *pullback bundle* $\text{pr}_1 : \tau^* TM \rightarrow M$. However, we choose to proceed without making use of this formalism here, and often suppress the dependence on the fiber velocity when it is sufficiently clear from context.

If ∇ is a linear connection on M , then it induces a connection map on TM by the Dombrowski construction [13]. Namely, for $X \in T_u TM$, choose a family of curves $\gamma_s(t) = \gamma(s, t)$ on M adapted to X , and set

$$K(X) = \nabla_{\frac{\partial \gamma}{\partial t}} \frac{\partial \gamma}{\partial s} \Big|_{(s,t)=(0,0)}. \quad (2)$$

For torsion-free ∇ this agrees with the equivalent expression obtained by interchanging the two derivatives. In the present paper, the connection of interest is the Levi–Civita connection associated to a Riemannian metric g on M , which is torsion-free by definition. We note, however, that many of the constructions and results obtained in the paper can be studied for general linear connections as well, with additional torsion components appearing when commutation of covariant derivatives are applied.

2.2. Higher-order tangent bundles

We now recall the definitions and notation for tangent bundles of order k that will be used in the rest of the paper; see [12] for further details. Let M be a smooth manifold of dimension n . Two curves q and $\tilde{q}$ on M with a common initial point p are said to be equivalent up to order k at p if there is a chart (U, φ) around p such that

$$\frac{d^\alpha(\varphi \circ q)}{dt^\alpha}(0) = \frac{d^\alpha(\varphi \circ \tilde{q})}{dt^\alpha}(0), \quad \alpha = 1, \dots, k.$$

Denote by $j_0^k(q)$ the equivalence class of q , and define the *tangent space of order k at p* by

$$T_p^{(k)} M = \{j_0^k(q) \mid q(0) = p\}.$$

The *tangent bundle of order k* is then given by the disjoint union of all k^{th} -order tangent spaces

$$T^{(k)} M = \bigsqcup_{p \in M} T_p^{(k)} M,$$

together with the fiber bundle structure $\tau_k : T^{(k)} M \rightarrow M$ defined by $\tau_k(j_0^k(q)) = q(0)$. These fiber bundles were introduced by Ehresmann [14] within the broader setting of jet manifolds.

A chart (U, φ) on M , with local coordinates (q^i) , induces local coordinates $(q^{(0)i}, q^{(1)i}, \dots, q^{(k)i})$ on $\tau_k^{-1}(U)$ by

$$q^{(0)i} = q^i, \quad q^{(\alpha)i}(j_0^k(q)) := \frac{1}{\alpha!} \frac{d^\alpha q^i}{dt^\alpha}(0), \quad \alpha = 1, \dots, k.$$

Given a curve q on M , its k -jet is the curve $j^k q : I \rightarrow T^{(k)} M$ defined by

$$(j^k q)(t) = j_0^k(q_t), \quad q_t(s) = q(t + s).$$

In local coordinates,

$$(j^k q)(t) = \left(q^i(t), \frac{dq^i}{dt}(t), \frac{1}{2} \frac{d^2 q^i}{dt^2}(t), \dots, \frac{1}{k!} \frac{d^k q^i}{dt^k}(t) \right).$$

For $u \in T^{(k)} M$, every tangent vector $X \in T_u T^{(k)} M$ can be written locally as

$$X = X^{(0)i} \frac{\partial}{\partial q^{(0)i}} \Big|_u + \dots + X^{(k)i} \frac{\partial}{\partial q^{(k)i}} \Big|_u.$$

A family of curves $\gamma_s(t) = \gamma(s, t)$ on M is said to be k -adapted to X if the curve $s \mapsto (j^k \gamma_s)(0)$ on $T^{(k)} M$ is adapted to X . That is,

$$(j^k \gamma_s)(0) \Big|_{s=0} = u, \quad \frac{\partial}{\partial s} \Big|_{s=0} (j^k \gamma_s)(0) = X.$$

In local coordinates this gives

$$X^{(\alpha)i} = \frac{1}{\alpha!} \frac{\partial^{\alpha+1} \gamma^i}{\partial s \partial t^\alpha} \Big|_{(s,t)=(0,0)}, \quad \alpha = 0, \dots, k.$$

The almost-tangent structure on $T^{(k)} M$ is the $(1, 1)$ -tensor field given in coordinates by

$$J = \frac{\partial}{\partial q^{(1)i}} \otimes dq^{(0)i} + \dots + \frac{\partial}{\partial q^{(k)i}} \otimes dq^{(k-1)i}.$$

Thus

$$J(X) = X^{(0)i} \frac{\partial}{\partial q^{(1)i}} \Big|_u + \dots + X^{(k-1)i} \frac{\partial}{\partial q^{(k)i}} \Big|_u.$$

The endomorphism $J|_u : T_u T^{(k)} M \rightarrow T_u T^{(k)} M$ is nilpotent of index $k + 1$, and

$$\text{Im}(J^\alpha) = \ker(J^{k+1-\alpha}), \quad \alpha = 1, \dots, k.$$

We denote the *vertical space of order α* at $u \in T^{(k)}M$ by

$$V_\alpha(u) = \text{Im}((J|_u)^\alpha) = \text{span} \left\{ \frac{\partial}{\partial q^{(\alpha)i}} \Big|_u, \dots, \frac{\partial}{\partial q^{(k)i}} \Big|_u \right\}.$$

In particular, $V_1(u) = \ker(\tau_{k*}|_u)$ is the vertical subspace with respect to τ_k . It is easily seen that the vertical spaces satisfy the following chain of inclusions

$$0 \subset V_k(u) \subset V_{k-1}(u) \subset \dots \subset V_1(u).$$

For $0 \leq \alpha \leq k$, let

$$\tau_\alpha^{(k)} : T^{(k)}M \rightarrow T^{(\alpha)}M, \quad \tau_\alpha^{(k)}(j_0^k q) = j_0^\alpha q,$$

where $T^{(0)}M := M$ and $\tau_0^{(k)} = \tau_k$. Then the vertical bundle of the fibration

$$T^{(k)}M \xrightarrow{\tau_\alpha^{(k)}} T^{(\alpha)}M$$

is $V_{\alpha+1}$, for $\alpha = 0, \dots, k-1$, i.e.

$$V_{\alpha+1}(u) = \ker(\tau_{\alpha*}^{(k)}|_u).$$

This gives exact sequences

$$0 \longrightarrow V_{\alpha+1} \longrightarrow TT^{(k)}M \xrightarrow{\tau_{\alpha*}^{(k)}} TT^{(\alpha)}M \longrightarrow 0, \quad \alpha = 0, \dots, k-1.$$

2.3. Connection maps and multiconnections

A nonlinear connection on $T^{(k)}M$ is a vector subbundle $H \subset TT^{(k)}M$, again referred to as the horizontal bundle, which is complementary to the vertical bundle $V := V_1$

$$TT^{(k)}M = H \oplus V. \quad (3)$$

As before, nonlinear connections can be equivalently described by connection maps. A connection map on $T^{(k)}M$ is a τ_k -morphism

$$K = (K_1, \dots, K_k) : TT^{(k)}M \rightarrow (TM)_\oplus^k$$

such that

1. $K_\alpha : T_u T^{(k)}M \rightarrow T_{\tau_k(u)}M$ is linear for $\alpha = 1, \dots, k$;
2. $K_{\alpha+1} \circ J = K_\alpha$ for $\alpha = 1, \dots, k-1$;
3. $K_1 \circ J = \tau_{k*}$.

Here, the notation $(TM)_\oplus^k$ refers to the Whitney sum of k tangent bundles, i.e. $(TM)_\oplus^k = \bigoplus_{i=1}^k TM$. The corresponding horizontal bundle is then given by $H = \ker(K)$, and

$$TT^{(k)}M = \ker(K) \oplus \text{Im}(J). \quad (4)$$

In local coordinates a connection map is determined by kn^2 smooth functions $(K_\alpha)_j^i : T^{(k)}M \rightarrow \mathbb{R}$, called connection coefficients, through

$$K_\alpha = \left(dq^{(\alpha)i} + (K_1)_j^i dq^{(\alpha-1)j} + \dots + (K_\alpha)_j^i dq^{(0)j} \right) \otimes \partial_i, \quad \alpha = 1, \dots, k. \quad (5)$$

Equivalently,

$$K_\alpha \left(\frac{\partial}{\partial q^{(\alpha)i}} \Big|_u \right) = \partial_i \Big|_{\tau_k(u)}, \quad K_\alpha \left(\frac{\partial}{\partial q^{(\beta)i}} \Big|_u \right) = 0 \quad \text{for } \beta > \alpha.$$

Theorem 1 (Bucătaru, [5]). *A nonlinear connection on $T^{(k)}M$ uniquely determines a connection map of order k in M . Conversely, if*

$$K : TT^{(k)}M \rightarrow (TM)_{\oplus}^k$$

satisfies the three conditions above, then $\ker(K)$ is a nonlinear connection.

A connection map gives a canonical identification

$$\Phi : TT^{(k)}M \rightarrow (TM)_{\oplus}^{k+1}, \quad \Phi_u(X) = (\tau_{k*}X, K_1(X), \dots, K_k(X)).$$

For each $u \in T^{(k)}M$, the map Φ_u is a linear isomorphism. Under this isomorphism, horizontal vectors are identified with vectors of the form $(v, 0, \dots, 0)$, whereas vertical vectors with respect to τ_k are identified with vectors of the form $(0, v_1, \dots, v_k)$.

Given a nonlinear connection H , the associated multiconnection is the family of subspaces

$$H_0(u), \dots, H_{k-1}(u) \subset T_u T^{(k)}M$$

defined by

$$H_0(u) = H(u), \quad H_i(u) = J^i(H_0(u)), \quad i = 1, \dots, k-1.$$

Theorem 2 (Bucătaru, [9]). *Suppose that H is a nonlinear connection. Then, for all $\alpha = 1, \dots, k$,*

$$TT^{(k)}M = \bigoplus_{i=0}^{\alpha-1} H_i \oplus V_{\alpha}. \quad (6)$$

Taking $\alpha = k$ in Equation (6), every vector $X \in T_u T^{(k)}M$ can be uniquely decomposed as

$$X = h_0(X) + h_1(X) + \dots + h_k(X),$$

where h_{α} is the projection onto H_{α} for $\alpha = 0, \dots, k-1$, and $h_k = v_k$ is the projection onto V_k . If

$$X = X_0 + J(X_1) + \dots + J^{k-1}(X_{k-1}) + J^k(X_k), \quad X_i \in H_0,$$

then

$$\Phi_u(X) = (\tau_{k*}X_0, \tau_{k*}X_1, \dots, \tau_{k*}X_k).$$

For each $\alpha = 1, \dots, k$, define

$$N_{\alpha} = \bigoplus_{i=0}^{\alpha-1} H_i.$$

Then

$$TT^{(k)}M = N_{\alpha} \oplus V_{\alpha}, \quad (7)$$

where

$$N_{\alpha} = \ker(K_{\alpha}, \dots, K_k), \quad V_{\alpha} = \text{Im}(J^{\alpha}) = \ker(\tau_{k*}, K_1, \dots, K_{\alpha-1}).$$

A vector $X \in T_u T^{(k)}M$ is called α -vertical if $X \in V_{\alpha}$, and α -horizontal if $X \in N_{\alpha}$. We denote by n_{α} and v_{α} the projections onto N_{α} and V_{α} , respectively. These definitions are extended to vector fields and curves in the usual way.

2.4. Adapted bases

Define covectors $\delta q^{(\alpha)i}$ by

$$\delta q^{(\alpha)i} = dq^{(\alpha)i} + (K_1)_j^i dq^{(\alpha-1)j} + \dots + (K_{\alpha})_j^i dq^{(0)j}, \quad \alpha = 1, \dots, k,$$

and set $\delta q^{(0)i} = dq^{(0)i}$. We refer to the collection $\{\delta q^{(\alpha)i} \mid 0 \leq \alpha \leq k, 1 \leq i \leq n\}$ as the *adapted basis*. With respect to the adapted basis, the connection map and canonical projection take the simplified forms:

$$K_{\alpha} = \delta q^{(\alpha)i} \otimes \partial_i, \quad \tau_{k*} = \delta q^{(0)i} \otimes \partial_i.$$

Consider the local frame of vector fields $\left\{ \frac{\delta}{\delta q^{(\alpha)i}} \right\}$ dual to the adapted basis. This frame is adapted to the decomposition given in Equation (6) in the sense that, for each fixed α , the vector fields $\left\{ \frac{\delta}{\delta q^{(\alpha)i}} \right\}_{i=1}^n$ span H_α , while $\left\{ \frac{\partial}{\partial q^{(k)i}} \right\}_{i=1}^n$ span V_k . With respect to these bases, the projections similarly take the form

$$h_\alpha = \delta q^{(\alpha)i} \otimes \frac{\delta}{\delta q^{(\alpha)i}},$$

while the almost-tangent structure is given by

$$J = \frac{\delta}{\delta q^{(1)i}} \otimes \delta q^{(0)i} + \dots + \frac{\delta}{\delta q^{(k)i}} \otimes \delta q^{(k-1)i}.$$

The coordinate basis and adapted basis are related by dual coefficient functions $(C_\alpha)_j^i$. The details are recorded in Appendix A.1; the formulas most often used later are

$$dq^{(\alpha)i} = \delta q^{(\alpha)i} - (C_1)_j^i \delta q^{(\alpha-1)j} - \dots - (C_\alpha)_j^i \delta q^{(0)j}, \quad \alpha = 1, \dots, k, \quad (8)$$

and

$$\frac{\partial}{\partial q^{(\alpha)i}} = \frac{\delta}{\delta q^{(\alpha)i}} + (K_1)_i^j \frac{\delta}{\delta q^{(\alpha+1)j}} + \dots + (K_{k-\alpha})_i^j \frac{\delta}{\delta q^{(k)j}}. \quad (9)$$

2.5. Connection towers

Throughout the remainder of the article, we use the notation $\cdot^{(\alpha)}$ to signify that an object is being considered with respect to $T^{(\alpha)}M$, with the exception of the projection τ_k and the coefficients of certain objects in local coordinates for simplicity. In some places, the overset notation may be dropped when it is sufficiently clear from context which space an object pertains to.

The higher-order tangent bundle $T^{(k)}M$ carries a natural tower of fibrations

$$T^{(k)}M \xrightarrow{\tau_{k-1}^{(k)}} T^{(k-1)}M \xrightarrow{\tau_{k-2}^{(k-1)}} \dots \xrightarrow{\tau_1^{(2)}} TM \xrightarrow{\tau_0^{(1)}} M. \quad (10)$$

A standard nonlinear connection on $T^{(k)}M$ is adapted only to the projection onto the base manifold. However, the iterated bundle structure suggests the existence of compatible connection data simultaneously adapted to all intermediate fibrations. This leads to the following definition.

We say that a connection map $K = (K_1, \dots, K_k)$ on $T^{(k)}M$ is a *connection tower of order k* provided that, for each $\alpha = 1, \dots, k$, there exists a unique map

$$K_\alpha^{(\alpha)} : TT^{(\alpha)}M \rightarrow TM$$

such that

$$K_\alpha^{(k)} = K_\alpha^{(\alpha)} \circ \tau_{\alpha*}^{(k)}. \quad (11)$$

Given a connection tower K of order k , define

$$K^{(\alpha)} = (K_1^{(\alpha)}, \dots, K_\alpha^{(\alpha)}), \quad K_\mu^{(\alpha)} = K_\mu^{(\mu)} \circ \tau_{\mu*}^{(\alpha)}, \quad \mu = 1, \dots, \alpha.$$

Then K is a connection map on $T^{(\alpha)}M$ for each $\alpha = 1, \dots, k$. By construction, $K_\mu^{(k)}$ and $K_\mu^{(\alpha)}$ have the same connection coefficients pulled back through $\tau_\alpha^{(k)}$. In local coordinates, this compatibility means that

$$\frac{\partial (K_\alpha)_j^i}{\partial q^{(\mu)l}} = 0 \quad \text{whenever } \mu > \alpha.$$

Conversely, a connection map whose connection coefficients satisfy such a relation uniquely determines a connection tower.

The adapted bases on $T^{(k)}M$ and $T^{(\alpha)}M$ are naturally related through the projections. For $\mu \leq \alpha$,

$$\tau_{\alpha*}^{(k)} \left(\frac{\binom{(k)}{\delta}}{\delta q^{(\mu)i}} \right) = \frac{\binom{(\alpha)}{\delta}}{\delta q^{(\mu)i}},$$

whereas the adapted basis vectors with $\mu > \alpha$ are annihilated by $\tau_{\alpha*}^{(k)}$. Equivalently,

$$\tau_{\alpha*}^{(k)} = \frac{\binom{(\alpha)}{\delta}}{\delta q^{(0)i}} \otimes \binom{(k)}{\delta} q^{(0)i} + \dots + \frac{\binom{(\alpha)}{\delta}}{\delta q^{(\alpha)i}} \otimes \binom{(k)}{\delta} q^{(\alpha)i}.$$

A connection tower gives the commutative diagram:

$$\begin{array}{ccccccc} TT^{(k)}M & \xrightarrow{\tau_{k-1*}^{(k)}} & TT^{(k-1)}M & \xrightarrow{\tau_{k-2*}^{(k-1)}} & TT^{(k-2)}M & \xrightarrow{\tau_{k-3*}^{(k-2)}} & \dots \xrightarrow{\tau_{2*}^{(3)}} TT^{(2)}M & \xrightarrow{\tau_{1*}^{(2)}} & TTM \\ \downarrow \binom{(k)}{K} & & \downarrow \binom{(k-1)}{K} & & \downarrow \binom{(k-2)}{K} & & & \downarrow \binom{(2)}{K} & \downarrow \binom{(1)}{K} \\ (TM)_{\oplus}^k & \xrightarrow{\text{pr}_{k-1}^{(k)}} & (TM)_{\oplus}^{k-1} & \xrightarrow{\text{pr}_{k-2}^{(k-1)}} & (TM)_{\oplus}^{k-2} & \xrightarrow{\text{pr}_{k-3}^{(k-2)}} & \dots \xrightarrow{\text{pr}_2^{(3)}} & (TM)_{\oplus}^2 & \xrightarrow{\text{pr}_1^{(2)}} & TM \end{array}$$

where $\text{pr}_{\alpha-1}^{(\alpha)}$ denotes the projection of $(TM)_{\oplus}^{\alpha}$ onto $(TM)_{\oplus}^{\alpha-1}$ obtained by dropping the last component.

Applying $\tau_{\alpha*}^{(k)}$ to the multiconnection decomposition on $T^{(k)}M$ gives the corresponding decomposition on $T^{(\alpha)}M$. In particular, for all $\mu = 0, \dots, \alpha$,

$$\tau_{\alpha*}^{(k)}(H_{\mu}^{(k)}(u)) = H_{\mu}^{(\alpha)}(\tau_{\alpha}^{(k)}(u)), \quad \tau_{\alpha*}^{(k)}(V_{\alpha+1}^{(k)}(u)) = 0.$$

Thus a connection tower induces a tower of α -horizontal subbundles

$$N_k^{(k)} \xrightarrow{\tau_{k-1*}^{(k)}} N_{k-1}^{(k-1)} \xrightarrow{\tau_{k-2*}^{(k-1)}} \dots \xrightarrow{\tau_{1*}^{(2)}} N_1^{(1)},$$

where $N_{\alpha}^{(\alpha)} = \bigoplus_{i=0}^{\alpha-1} H_i^{(\alpha)}$. This justifies the terminology connection tower.

2.6. Vector bundle structures induced by connection towers

Connection towers also induce vector bundle structures on the higher-order tangent bundles themselves. For $\alpha \geq 1$,

we define the *Tulczyjew operator* $d_T^{(\alpha)} : T^{(\alpha)}M \longrightarrow TT^{(\alpha-1)}M$ by

$$d_T^{(\alpha)}(u) = \frac{d}{dt} \Big|_{t=0} (j^{\alpha-1} q)(t), \quad u = j_0^{\alpha}(q),$$

which is well-defined since $d_T^{(\alpha)}(u)$ depends only on the α -jet of q [27]. In local coordinates,

$$d_T^{(\alpha)} = q^{(1)i} \frac{\partial}{\partial q^{(0)i}} + 2q^{(2)i} \frac{\partial}{\partial q^{(1)i}} + \dots + \alpha q^{(\alpha)i} \frac{\partial}{\partial q^{(\alpha-1)i}}.$$

Theorem 3. Let $K^{(k)}$ be a connection tower of order k in M . For each $\alpha = 1, \dots, k$, let

$$\sigma : (TM)_{\oplus}^{\alpha} \rightarrow M$$

denote the natural projection, and define

$$F^{(\alpha)} : T^{(\alpha)}M \rightarrow (TM)_{\oplus}^{\alpha}$$

by

$$F(u) = \left(d_T^{(1)}(\tau_1^{(\alpha)}(u)), K_1(d_T^{(2)}(\tau_2^{(\alpha)}(u))), 2! K_2(d_T^{(3)}(\tau_3^{(\alpha)}(u))), \dots, (\alpha-1)! K_{\alpha-1}^{(\alpha)}(d_T(u)) \right). \quad (12)$$

Then F is a diffeomorphism for all $\alpha = 1, \dots, k$. Moreover,

$$\sigma \circ F = \tau_\alpha.$$

Consequently, the vector bundle structure of $(TM)_\oplus^\alpha$ over M pulls back through F to a vector bundle structure on $T^{(\alpha)}M$.

PROOF. Let

$$(X_0, \dots, X_{\alpha-1}) \in (TM)_\oplus^\alpha$$

be a point over $p \in M$. In local coordinates, the equation

$$F(u) = (X_0, \dots, X_{\alpha-1})$$

is triangular. Indeed, if $u = (q^{(0)}, \dots, q^{(\alpha)})$, then $X_0^i = q^{(1)i}$, and for $\mu = 1, \dots, \alpha-1$,

$$X_\mu^i = \mu! \left((\mu+1)q^{(\mu+1)i} + \mu q^{(\mu)j} (K_1)_j^i + \dots + q^{(1)j} (K_\mu)_j^i \right).$$

Here $(K_l)_j^i$, for $l = 1, \dots, \mu$, denotes the corresponding coefficient of the connection map on $T^{(\mu)}M$, pulled back to $T^{(\alpha)}M$ through $\tau_\mu^{(\alpha)}$. The first equations are

$$X_0^i = q^{(1)i},$$

$$X_1^i = 2q^{(2)i} + q^{(1)j} (K_1)_j^i,$$

$$X_2^i = 2! \left(3q^{(3)i} + 2q^{(2)j} (K_1)_j^i + q^{(1)j} (K_2)_j^i \right).$$

Since the coefficient of $q^{(\mu+1)i}$ in the μ th equation is $(\mu+1)!$, these equations can be solved recursively:

$$q^{(1)i} = X_0^i,$$

$$q^{(2)i} = \frac{1}{2} \left(X_1^i - (K_1)_j^i X_0^j \right),$$

and, in general,

$$q^{(\mu+1)i} = \frac{1}{(\mu+1)!} \left(X_\mu^i - H_{\mu+1}^i(X_0, \dots, X_{\mu-1}) \right)$$

for smooth functions $H_{\mu+1}^i$ determined by the connection coefficients and the previously solved coordinates. Hence the fiber coordinates $q^{(1)i}, \dots, q^{(\alpha)i}$ are uniquely and smoothly determined by $(X_0, \dots, X_{\alpha-1})$, while the base coordinates $q^{(0)i}$ are determined by the common base point p . Therefore F is a bijective local diffeomorphism, hence a diffeomorphism.

Finally, each component of $F(u)$ lies in $T_{\tau_\alpha(u)}M$, so $\sigma \circ F = \tau_\alpha$. The vector bundle structure of $(TM)_\oplus^\alpha$ over M therefore pulls back through F to a vector bundle structure on $T^{(\alpha)}M$. $\square$

2.7. Lifting vector fields on M to higher-order tangent bundles

Let $K^{(k)}$ be a connection tower of order k . By Theorem 2,

$$TT^{(k)}M = \bigoplus_{\alpha=0}^{k-1} H_{\alpha}^{(k)} \oplus V_k^{(k)}. \quad (13)$$

We use the convention

$$K_0^{(k)} := \tau_{k*}, \quad h_k^{(k)} := v_k^{(k)}.$$

Let $z \in T_p M$ and let $u \in T^{(k)}M$ with $\tau_k(u) = p$. For each $\alpha = 0, \dots, k$, there exists a unique vector

$$z^{h_{\alpha}^{(k)}} \in T_u T^{(k)}M$$

such that

$$K_{\alpha}^{(k)}(z^{h_{\alpha}^{(k)}}) = z, \quad K_{\beta}^{(k)}(z^{h_{\alpha}^{(k)}}) = 0 \quad \text{for all } \beta \neq \alpha.$$

For $\alpha = 0, \dots, k-1$, the vector $z^{h_{\alpha}^{(k)}}$ belongs to $H_{\alpha}^{(k)}$ and is called the $h_{\alpha}^{(k)}$ -lift of z to $T^{(k)}M$ at u . In particular, we see that $z^{h_0^{(k)}}$ is 1-horizontal while $z^{h_k^{(k)}}$ is k -vertical, so that we alternatively refer to the $h_0^{(k)}$ -lift as the 1-horizontal lift and the $h_k^{(k)}$ -lift as the k -vertical lift.

Similarly, for vector fields $X \in \mathfrak{X}(M)$, we define the $h_{\alpha}^{(k)}$ -lift of X as the vector field $X^{h_{\alpha}^{(k)}} \in \mathfrak{X}(T^{(k)}M)$ satisfying

$$K_{\alpha}^{(k)} \circ X^{h_{\alpha}^{(k)}} = X \circ \tau_k, \quad K_{\beta}^{(k)} \circ X^{h_{\alpha}^{(k)}} = 0 \quad \text{for } \beta \neq \alpha.$$

As before, we refer to the $h_k^{(k)}$ -lift of X as the k -vertical lift, and the $h_0^{(k)}$ lift as the 1-horizontal lift. The adapted basis is described by lifts of coordinate vector fields

$$\frac{\delta^{(k)}}{\delta q^{(\alpha)j}} = (\partial_j)^{h_{\alpha}^{(k)}}, \quad \alpha = 0, \dots, k.$$

Remark 2. Horizontal lifts for connections induced by a linear connection were previously studied by Bucataru ([6, Theorem 3.1]). The authors use the concept of *semisprays* to introduce the horizontal lifts, and then prove that this definition is independent of the choice of the semispray.

Lemma 4. Let $X, Y \in \mathfrak{X}(M)$ and $f \in C^{\infty}(M)$. Then

$$(X + Y)^{h_{\alpha}^{(k)}} = X^{h_{\alpha}^{(k)}} + Y^{h_{\alpha}^{(k)}}, \quad (fX)^{h_{\alpha}^{(k)}} = (f \circ \tau_k)X^{h_{\alpha}^{(k)}},$$

for $\alpha = 0, \dots, k$. If $X = X^j \partial_j$, then

$$X^{h_{\alpha}^{(k)}} = (X^j \circ \tau_k) \frac{\delta^{(k)}}{\delta q^{(\alpha)j}}, \quad \alpha = 0, \dots, k.$$

Moreover,

$$X^{h_0^{(k)}}(f \circ \tau_k) = (Xf) \circ \tau_k, \quad X^{h_{\alpha}^{(k)}}(f \circ \tau_k) = 0, \quad \alpha = 1, \dots, k.$$

PROOF. See Appendix A.2. □

Every vector field W on $T^{(k)}M$ admits the decomposition

$$W = (\tau_{k*} \circ W)^{(k)}_{h_0} + (K_1 \circ W)^{(k)}_{h_1} + \dots + (K_{k-1} \circ W)^{(k)}_{h_{k-1}} + (K_k \circ W)^{(k)}_{v_k}. \quad (14)$$

Similarly, if Γ is a curve on $T^{(k)}M$ and $q = \tau_k \circ \Gamma$, then

$$\dot{q} = \tau_{k*} \circ \dot{\Gamma}, \quad Y^{(\alpha)} = K_\alpha \circ \dot{\Gamma}, \quad \alpha = 1, \dots, k,$$

so that

$$\dot{\Gamma} = \dot{q}^{(k)}_{h_0} + (Y^{(1)})^{(k)}_{h_1} + \dots + (Y^{(k-1)})^{(k)}_{h_{k-1}} + (Y^{(k)})^{(k)}_{v_k}. \quad (15)$$

The curve Γ is α -horizontal if and only if

$$Y^{(\beta)} = 0, \quad \beta = \alpha, \dots, k,$$

and is α -vertical if and only if

$$Y^{(\beta)} = 0, \quad \beta = 1, \dots, \alpha - 1.$$

A large structural advantage to connection towers is that order- k computations can be reused in all higher orders, which is described by the following lemmas.

Lemma 5. *Let $X \in \mathfrak{X}(M)$. Then $X^{(k)}_{h_\alpha}$ is τ_{k-1} -related to $X^{(k-1)}_{h_\alpha}$:*

$$X^{(k-1)}_{h_\alpha} \circ \tau_{k-1}^{(k)} = \tau_{k-1*}^{(k)} \circ X^{(k)}_{h_\alpha}, \quad \alpha = 0, \dots, k-1. \quad (16)$$

PROOF. This follows immediately from the connection tower compatibility

$$K_\alpha \circ \tau_{k-1*}^{(k)} = K_\alpha, \quad \alpha = 0, \dots, k-1. \quad (17)$$

□

Moreover, for the Lie brackets of lifts, we have the following lemma.

Lemma 6. *For each $X, Y \in \mathfrak{X}(M)$,*

$$K_\mu \circ [X^{(k)}_{h_\alpha}, Y^{(k)}_{h_\beta}] = \left(K_\mu \circ [X^{(k-1)}_{h_\alpha}, Y^{(k-1)}_{h_\beta}] \right) \circ \tau_{k-1}^{(k)},$$

for $\alpha, \beta, \mu = 0, \dots, k-1$.

PROOF. By Lemma 5, the lifted vector fields $X^{(k)}_{h_\alpha}$ and $Y^{(k)}_{h_\beta}$ are τ_{k-1} -related to $X^{(k-1)}_{h_\alpha}$ and $Y^{(k-1)}_{h_\beta}$, respectively. Their brackets are therefore also τ_{k-1} -related. Applying the connection tower compatibility $K_\mu \circ \tau_{k-1*}^{(k)} = K_\mu$ gives the identity. □

In particular, the h_μ -components of the Lie brackets $[X^{(k)}_{h_\alpha}, Y^{(k)}_{h_\beta}]$ with $\mu < k$ are obtained from the corresponding brackets on $T^{(k-1)}M$, so that the genuinely new information in order k is the k -vertical components. Hence, if

$$\left[\frac{\delta^{(k)}}{\delta q^{(\alpha)i}}, \frac{\delta^{(k)}}{\delta q^{(\beta)l}} \right] = \sum_{j=1}^n \sum_{\mu=0}^k {}^k S_{(\alpha)i,(\beta)l}^{(\mu)j} \frac{\delta^{(k)}}{\delta q^{(\mu)j}},$$

then, for $\mu = 0, \dots, k-1$,

$${}^k S_{(\alpha)i,(\beta)l}^{(\mu)j} = {}^{k-1} S_{(\alpha)i,(\beta)l}^{(\mu)j} \circ \tau_{k-1}^{(k)},$$

and only the coefficients ${}^k S_{(\alpha)i,(\beta)l}^{(k)j}$ must be computed anew.

The Lie brackets of lifted vector fields will be studied in more detail in the next section.

3. Connection maps on $T^{(k)}M$ induced by the Riemannian metric

In this section, we introduce a particular connection tower on $T^{(k)}M$ induced by the Riemannian metric g on M . This construction extends the Dombrowski splitting of TTM to higher-order tangent bundles. In Section 4, we use this tower to orthogonalize the adapted splitting and thereby obtain a family of higher-order Sasaki metrics, recovering the classical Sasaki metric when $k = 1$.

The curvature identities used in this section are recorded in Appendix B.1. We keep the construction of the Levi-Civita-induced connection tower and the full second-order bracket verification in the main text, since these are the basic computations on which the later formulas rest. The longer third-order bracket computations and the general coordinate reduction for the k -vertical bracket components are collected in Appendix C.

3.1. Riemannian conventions

Consider a Riemannian manifold $(M, \langle \cdot, \cdot \rangle)$, equipped with its Levi-Civita connection ∇ . Thus ∇ is torsion-free and compatible with the metric:

1. $\nabla_X Y - \nabla_Y X = [X, Y]$,
2. $X\langle Y, Z \rangle = \langle \nabla_X Y, Z \rangle + \langle Y, \nabla_X Z \rangle$, for all $X, Y, Z \in \mathfrak{X}(M)$.

The Levi-Civita connection is uniquely defined by the *Koszul formula*

$$\begin{aligned} 2\langle \nabla_X Y, Z \rangle &= X\langle Y, Z \rangle + Y\langle X, Z \rangle - Z\langle X, Y \rangle \\ &\quad + \langle [X, Y], Z \rangle - \langle [X, Z], Y \rangle - \langle [Y, Z], X \rangle, \end{aligned} \quad (18)$$

for all $X, Y, Z \in \mathfrak{X}(T^{(k)}M)$.

We denote by ∇Y the linear map defined by $(\nabla Y)(X) = \nabla_X Y$, and we define the second covariant derivative by

$$\nabla_{XY}^2 = \nabla_X \nabla_Y - \nabla_{\nabla_X Y}.$$

The curvature tensor is defined by

$$R(X, Y)Z = \nabla_X \nabla_Y Z - \nabla_Y \nabla_X Z - \nabla_{[X, Y]} Z. \quad (19)$$

We also write $R(X, Y)$ for the endomorphism $Z \mapsto R(X, Y)Z$.

We use the standard curvature symmetries, the corresponding identities for ∇R and $\nabla^2 R$, and the basic commutation formula for covariant derivatives along a two-parameter family of curves. These identities are listed in Appendix B.1. Throughout the paper, Γ_{ij}^k denotes the Christoffel symbols of ∇ , so that $\nabla_{\partial_i} \partial_j = \Gamma_{ij}^k \partial_k$, and R_{ijk}^l denotes the components of the curvature tensor, defined by $R(\partial_i, \partial_j) \partial_k = R_{ijk}^l \partial_l$. The curvature tensor components of the Levi-Civita connection can be expressed in terms of the Christoffel symbols via

$$R_{jkl}^i = \partial_j \Gamma_{kl}^i - \partial_k \Gamma_{jl}^i + \Gamma_{kl}^m \Gamma_{jm}^i - \Gamma_{jl}^m \Gamma_{km}^i. \quad (20)$$

The components of the covariant derivative of the curvature tensor are denoted by $R_{i;jkl}^m$ and are given by

$$(\nabla_{\partial_i} R)(\partial_j, \partial_k) \partial_l = R_{i;jkl}^m \partial_m.$$

3.2. Connection map on TM induced by the Riemannian structure

Given a Riemannian metric g on M , there is a canonical choice for a connection map on TM . The Levi-Civita connection of (M, g) induces the connection map $K : TTM \rightarrow TM$ given by

$$K(X) = \nabla_{\frac{\partial \gamma}{\partial t}} \frac{\partial \gamma}{\partial s} \Big|_{(s,t)=(0,0)}, \quad (21)$$

where $\gamma_s : t \mapsto \gamma_s(t) = \gamma(s, t)$ is a family of curves on M adapted to $X \in TTM$. The connection map is well-defined, since $K(X)$ does not depend on the particular choice of γ . The connection coefficients of K are given by $K_j^i = q^{(1)l} \Gamma_{jl}^i$, and hence

$$K = \left(dq^{(1)i} + q^{(1)l} \Gamma_{jl}^i dq^{(0)j} \right) \otimes \partial_i.$$

The nonlinear connection $H = \ker(K)$ leads to the decomposition given in Equation (1). To standardize the notation with the general case on $T^{(k)}M$, we write $\overset{(1)}{H}_0 = H$ and $\overset{(1)}{V}_1 = V$. The horizontal and vertical lifts of a vector field X on M are denoted by $X^{\overset{(1)}{h}_0}$ and $X^{\overset{(1)}{v}_1}$, respectively.

Theorem 7 ([13]). *Let $X, Y \in \mathfrak{X}(M)$. Then*

$$[X^{\overset{(1)}{h}_0}, Y^{\overset{(1)}{v}_1}] = (\nabla_X Y)^{\overset{(1)}{v}_1}, \quad [X^{\overset{(1)}{v}_1}, Y^{\overset{(1)}{v}_1}] = 0, \quad [X^{\overset{(1)}{h}_0}, Y^{\overset{(1)}{h}_0}] = [X, Y]^{\overset{(1)}{h}_0} - (R(X, Y)u)^{\overset{(1)}{v}_1},$$

with $u \in TM$.

Remark 3. The last equation is written in abbreviated form. Precisely, if $u \in T_p M$, then

$$[X^{\overset{(1)}{h}_0}, Y^{\overset{(1)}{h}_0}]_u = [X, Y]_u^{\overset{(1)}{h}_0} - (R(X_p, Y_p)u)_u^{\overset{(1)}{v}_1}.$$

We use the same convention throughout the paper for tensorial expressions evaluated at the point $u \in T^{(k)}M$.

3.3. The Levi-Civita-induced connection tower

The goal of this subsection is to define a connection tower depending only on the Riemannian metric g , as happens with the connection map $\overset{(1)}{K}$ on TM . This tower should be related to the connection map on TM by the rule

$$\overset{(\alpha)}{K}_1 = \overset{(1)}{K} \circ \overset{(\alpha)}{\tau}_{1*}.$$

Equivalently, the connection maps $\overset{(\alpha)}{K} = (\overset{(\alpha)}{K}_1, \dots, \overset{(\alpha)}{K}_\alpha)$ on $T^{(\alpha)}M$ should satisfy

$$\overset{(\alpha)}{K}_\beta = \overset{(\alpha-1)}{K}_\beta \circ \overset{(\alpha)}{\tau}_{\alpha-1*}, \quad \beta = 1, \dots, \alpha - 1, \quad (22)$$

for $\alpha = 2, \dots, k$, starting with $\overset{(1)}{K}_1 = \overset{(1)}{K}$. Thus, it remains to specify the highest component $\overset{(\alpha)}{K}_\alpha$ at each order.

Let $u \in T^{(k)}M$ and $X \in T_u T^{(k)}M$. Choose a k -adapted family of curves $\gamma_s(t) = \gamma(s, t)$ on M , so that

$$(j^k \gamma_s)(0) \Big|_{s=0} = u, \quad \frac{\partial}{\partial s} \Big|_{s=0} (j^k \gamma_s)(0) = X.$$

For $\alpha = 1, \dots, k$, define

$$\overset{(k)}{K}_\alpha(X) = \frac{1}{\alpha!} \nabla^\alpha \frac{\partial \gamma}{\partial s} \Big|_{(s,t)=(0,0)}. \quad (23)$$

Here $\nabla^\alpha_{\frac{\partial \gamma}{\partial t}}$ denotes repeated covariant differentiation along the t -curves. The main result of this subsection is the following theorem.

Theorem 8. *The map $\overset{(k)}{K} = (\overset{(k)}{K}_1, \dots, \overset{(k)}{K}_k)$ defined by Equation (23) forms a connection map on $T^{(k)}M$. Moreover, the family of maps $\{\overset{(\alpha)}{K}\}_{\alpha=1}^k$ obtained by applying the same construction at each order is a connection tower.*

The proof proceeds in three steps. We first show that each component of $\overset{(k)}{K}$ is a well-defined linear map with the required normalization on vertical directions. The same computation also identifies the corresponding connection coefficients. We then record the resulting recursion among these coefficients, and finally use this recursion to prove that $\overset{(k)}{K}$ has the local form of a connection map and is compatible with the tower projections.

Lemma 9. *The map $\overset{(k)}{K} = (\overset{(k)}{K}_1, \dots, \overset{(k)}{K}_k)$ satisfies the following conditions:*

1. $K_\alpha^{(k)} : T_u T^{(k)} M \rightarrow T_{\tau_k(u)} M$ is linear for all $\alpha = 1, \dots, k$.
2. $K_\alpha^{(k)} \circ J^\alpha = \tau_{k*}$ for all $\alpha = 1, \dots, k$.

PROOF. We define the connection coefficients $(K_\alpha)_{(\mu)j}^i$ of K_α implicitly by $K_\alpha \left(\frac{\partial}{\partial q^{(\mu)i}} \right) = (K_\alpha)_{(\mu)j}^i \partial_j$ for all $\mu = 0, \dots, k$ and $i, j = 1, \dots, n$.

We prove by induction on α that

$$\nabla_{\frac{\partial \gamma}{\partial t}}^\alpha \frac{\partial \gamma}{\partial s} = \alpha! \sum_{\mu=0}^{\alpha} (L_\alpha)_{(\mu)j}^i \frac{1}{\mu!} \frac{\partial^{\mu+1} \gamma^j}{\partial s \partial t^\mu} \partial_i,$$

where, for $\alpha = 1, \dots, k$,

$$(L_\alpha)_{(\alpha)j}^i = \delta_j^i, \quad (L_1)_{(0)j}^i = \frac{\partial \gamma^i}{\partial t} \Gamma_{lj}^i.$$

For $\alpha = 1$, we have

$$\nabla_{\frac{\partial \gamma}{\partial t}} \frac{\partial \gamma}{\partial s} = \left(\frac{\partial^2 \gamma^i}{\partial s \partial t} + \frac{\partial \gamma^l}{\partial t} \frac{\partial \gamma^j}{\partial s} \Gamma_{lj}^i \right) \partial_i,$$

so the asserted form follows immediately.

Now suppose that the formula holds for some fixed $\alpha - 1 < k$, so that

$$\nabla_{\frac{\partial \gamma}{\partial t}}^{\alpha-1} \frac{\partial \gamma}{\partial s} = (\alpha - 1)! \sum_{\mu=0}^{\alpha-1} (L_{\alpha-1})_{(\mu)j}^i \frac{1}{\mu!} \frac{\partial^{\mu+1} \gamma^j}{\partial s \partial t^\mu} \partial_i,$$

with $(L_{\alpha-1})_{(\alpha-1)j}^i = \delta_j^i$. Applying one more covariant derivative in the t -direction gives

$$\begin{aligned} \nabla_{\frac{\partial \gamma}{\partial t}}^\alpha \frac{\partial \gamma}{\partial s} &= (\alpha - 1)! \sum_{\mu=0}^{\alpha-1} \frac{1}{\mu!} \left[\frac{\partial (L_{\alpha-1})_{(\mu)j}^i}{\partial t} \frac{\partial^{\mu+1} \gamma^j}{\partial s \partial t^\mu} + (L_{\alpha-1})_{(\mu)j}^i \frac{\partial^{\mu+2} \gamma^j}{\partial s \partial t^{\mu+1}} + (L_{\alpha-1})_{(\mu)j}^m \frac{\partial^{\mu+1} \gamma^j}{\partial s \partial t^\mu} \frac{\partial \gamma^l}{\partial t} \Gamma_{lm}^i \right] \partial_i \\ &= \frac{\partial^{\alpha+1} \gamma^i}{\partial s \partial t^\alpha} \partial_i + (\alpha - 1)! \sum_{\mu=0}^{\alpha-1} \frac{1}{\mu!} \left[\frac{\partial (L_{\alpha-1})_{(\mu)j}^i}{\partial t} + \mu (L_{\alpha-1})_{(\mu-1)j}^i + (L_{\alpha-1})_{(\mu)j}^m \frac{\partial \gamma^l}{\partial t} \Gamma_{lm}^i \right] \frac{\partial^{\mu+1} \gamma^j}{\partial s \partial t^\mu} \partial_i \\ &= \frac{\partial^{\alpha+1} \gamma^i}{\partial s \partial t^\alpha} \partial_i + (\alpha - 1)! \sum_{\mu=0}^{\alpha-1} \frac{1}{\mu!} \left[\frac{\partial (L_{\alpha-1})_{(\mu)j}^i}{\partial t} + (L_{\alpha-1})_{(\mu)j}^m (L_1)_{(0)m}^i + \mu (L_{\alpha-1})_{(\mu-1)j}^i \right] \frac{\partial^{\mu+1} \gamma^j}{\partial s \partial t^\mu} \partial_i. \end{aligned}$$

Comparison with the asserted form at order α gives the induction step.

Evaluating at $(s, t) = (0, 0)$, we obtain

$$K_\alpha^{(k)}(X) = \sum_{\mu=0}^{\alpha} (L_\alpha)_{(\mu)j}^i \Big|_{(s,t)=(0,0)} X^{(\mu)j} \partial_i \quad (24)$$

Moreover,

$$\begin{aligned} (L_\alpha)_{(\mu)j}^i \Big|_{(s,t)=(0,0)} &= \frac{1}{\alpha} \left[\sum_{v=0}^{\alpha-1} (v+1) q^{(v+1)m} \frac{\partial (L_{\alpha-1})_{(\mu)j}^i}{\partial q^{(v)m}} + \mu (L_{\alpha-1})_{(\mu-1)j}^i + (L_{\alpha-1})_{(\mu)j}^m (L_1)_{(0)m}^i \right] \\ &= \frac{1}{\alpha} \left[d_T (L_{\alpha-1})_{(\mu)j}^i + (L_{\alpha-1})_{(\mu)j}^m (L_1)_{(0)m}^i + \mu (L_{\alpha-1})_{(\mu-1)j}^i \right]. \end{aligned}$$

Thus the coefficients $(L_\alpha)_{(\mu)j}^i \Big|_{(s,t)=(0,0)}$ depend only on u , so that the maps

$$K_\alpha^{(k)} : T_u T^{(k)} M \rightarrow T_{\tau_k(u)} M$$

are well-defined and linear. In coordinates, the connection coefficients satisfy

$$(K_\alpha)_{(\mu)j}^i = (L_\alpha)_{(\mu)j}^i \Big|_{(s,t)=(0,0)} \quad \text{for } \mu \leq \alpha, \quad (K_\alpha)_{(\mu)j}^i = 0 \quad \text{for } \mu > \alpha$$

Since $(L_\alpha)_{(\alpha)j}^i = \delta_j^i$, it follows that $(K_\alpha)_{(\alpha)j}^i = \delta_j^i$. Therefore $\overset{(k)}{K}_\alpha \circ J^\alpha = \tau_{k*}$, proving the second condition. $\square$

The preceding computation also gives the following recursion satisfied by the connection coefficients.

Lemma 10. *The connection coefficients of $\overset{(k)}{K}$ satisfy*

$$(K_\alpha)_{(\mu)j}^i = \frac{1}{\alpha} \left[d_T(K_{\alpha-1})_{(\mu)j}^i + (K_{\alpha-1})_{(\mu)j}^m (K_1)_{(0)m}^i + \mu (K_{\alpha-1})_{(\mu-1)j}^i \right], \quad (25)$$

for $\alpha = 2, \dots, k$, $\mu = 0, \dots, \alpha - 1$, and $1 \leq i, j \leq n$, where

$$(K_{\alpha-1})_{(-1)j}^i := 0, \quad (K_1)_{(0)j}^i = q^{(1)l} \Gamma_{lj}^i.$$

PROOF. By Equation (24), the coefficients of $\overset{(k)}{K}_\alpha$ are obtained by evaluating the coefficients $(L_\alpha)_{(\mu)j}^i$ at $(s, t) = (0, 0)$. The final recursion derived in the proof of Lemma 9 therefore gives

$$(K_\alpha)_{(\mu)j}^i = \frac{1}{\alpha} \left[d_T(K_{\alpha-1})_{(\mu)j}^i + (K_{\alpha-1})_{(\mu)j}^m (K_1)_{(0)m}^i + \mu (K_{\alpha-1})_{(\mu-1)j}^i \right],$$

for $\alpha = 2, \dots, k$ and $\mu = 0, \dots, \alpha - 1$, with the convention $(K_{\alpha-1})_{(-1)j}^i = 0$. The expression for $(K_1)_{(0)j}^i$ follows from the base case $\alpha = 1$. $\square$

In particular, for $(K_\alpha)_j^i = (K_\alpha)_{(0)j}^i$, we have

$$(K_\alpha)_j^i = \frac{1}{\alpha} \left(d_T(K_{\alpha-1})_j^i + (K_{\alpha-1})_j^l (K_1)_l^i \right), \quad \alpha = 2, \dots, k. \quad (26)$$

This relation was introduced by Miron in [19, Theorem 9.1.1], where it was shown to describe a connection map for any smooth choice of $(K_1)_j^i$ satisfying the corresponding transformation laws under changes of coordinates.

We now complete the proof of Theorem 8.

PROOF (PROOF OF THEOREM 8). By Lemma 9, each

$$\overset{(k)}{K}_\alpha : T_u T^{(k)} M \rightarrow T_{\tau_k(u)} M$$

is well-defined and linear, and satisfies $\overset{(k)}{K}_\alpha \circ J^\alpha = \tau_{k*}$. It remains to prove that $\overset{(k)}{K} = (\overset{(k)}{K}_1, \dots, \overset{(k)}{K}_k)$ has the local form of a connection map and that the resulting family is compatible with the tower projections.

For $\alpha = 1, \dots, k$, let $P(\alpha)$ denote the statement

$$(K_\alpha)_{(\alpha-\beta)j}^i = (K_\beta)_{(0)j}^i, \quad \beta = 1, \dots, \alpha - 1. \quad (27)$$

We prove $P(\alpha)$ by induction on α . The case $\alpha = 1$ is vacuous. Suppose that $P(\alpha - 1)$ holds for some $1 < \alpha \leq k$.

First consider $\beta = 1$. By Equation (25),

$$\begin{aligned} (K_\alpha)_{(\alpha-1)j}^i &= \frac{1}{\alpha} \left[d_T(K_{\alpha-1})_{(\alpha-1)j}^i + (K_{\alpha-1})_{(\alpha-1)j}^m (K_1)_{(0)m}^i + (\alpha - 1)(K_{\alpha-1})_{(\alpha-2)j}^i \right] \\ &= \frac{1}{\alpha} \left[(K_1)_{(0)j}^i + (\alpha - 1)(K_1)_{(0)j}^i \right] = (K_1)_{(0)j}^i. \end{aligned}$$

Now let $\beta \geq 2$. The induction hypothesis gives

$$(K_{\alpha-1})_{(\alpha-\beta)j}^i = (K_{\beta-1})_{(0)j}^i, \quad \text{and} \quad (K_{\alpha-1})_{(\alpha-\beta-1)j}^i = (K_\beta)_{(0)j}^i.$$

when $\beta \leq \alpha - 2$, while the case $\beta = \alpha - 1$ is tautological for the latter. Therefore, again by Equation (25),

$$\begin{aligned} (K_\alpha)_{(\alpha-\beta)j}^i &= \frac{1}{\alpha} \left[d_T(K_{\beta-1})_{(0)j}^i + (K_{\beta-1})_{(0)j}^m (K_1)_{(0)m}^i + (\alpha - \beta)(K_\beta)_{(0)j}^i \right] \\ &= \frac{1}{\alpha} \left[\beta(K_\beta)_{(0)j}^i + (\alpha - \beta)(K_\beta)_{(0)j}^i \right] = (K_\beta)_{(0)j}^i, \end{aligned}$$

where the second equality uses Equation (26). This proves Equation (27).

Together with

$$(K_\alpha)_{(\alpha)j}^i = \delta_j^i, \quad (K_\alpha)_{(\mu)j}^i = 0 \quad \text{for } \mu > \alpha,$$

Equation (27) shows that

$$K_\alpha^{(k)} = \left(dq^{(\alpha)i} + (K_1)_j^i dq^{(\alpha-1)j} + \dots + (K_\alpha)_j^i dq^{(0)j} \right) \otimes \partial_i.$$

Hence $K = (K_1, \dots, K_k)$ is a connection map on $T^{(k)}M$.

Finally, we have already shown that $(K_\alpha)_j^i$ depends only on the coordinates $q^{(\mu)a}$ with $0 \leq \mu \leq \alpha$. Therefore $K_\alpha^{(k)}$ descends through $\tau_\alpha^{(k)}$ for each α , and the resulting family is compatible with the tower projections. Hence $\{K\}_{\alpha=1}^k$ is a connection tower. $\square$

The connection coefficients satisfy

$$\frac{\partial (K_\alpha)_i^j}{\partial q^{(\alpha-1)l}} = \Gamma_{li}^j, \quad \alpha = 1, \dots, k, \quad (28)$$

and, for $\alpha \geq 2$,

$$\frac{\partial (K_\alpha)_i^j}{\partial q^{(\alpha-1)l}} = \frac{1}{\alpha} q^{(1)a} dq^j \left(\nabla_{\partial_l} \nabla_{\partial_a} \partial_i + (\alpha - 1) \nabla_{\partial_a} \nabla_{\partial_l} \partial_i \right). \quad (29)$$

In general, the connection coefficients admit the intrinsic expression

$$(K_\alpha)_i^j = dq^j \left(K_\alpha^{(k)} \left(\frac{\partial}{\partial q^{(0)i}} \right) \right) = \frac{1}{\alpha!} dq^j \left(\nabla_{\frac{\partial}{\partial t}}^\alpha \partial_i \right) \Big|_{t=0}.$$

For the first values of α ,

$$\begin{aligned} (K_1)_i^j &= q^{(1)a} \Gamma_{ai}^j, \\ (K_2)_i^j &= q^{(2)a} dq^j (\nabla_{\partial_a} \partial_i) + \frac{1}{2} q^{(1)a} q^{(1)b} dq^j (\nabla_{\partial_a} \nabla_{\partial_b} \partial_i), \\ (K_3)_i^j &= q^{(3)a} dq^j (\nabla_{\partial_a} \partial_i) + q^{(2)a} q^{(1)b} dq^j (\nabla_{\partial_b} \nabla_{\partial_a} \partial_i) \\ &\quad + \frac{1}{6} q^{(1)a} q^{(1)b} q^{(1)c} dq^j (\nabla_{\partial_c} \nabla_{\partial_a} \nabla_{\partial_b} \partial_i) + \frac{1}{3} q^{(2)a} q^{(1)b} R_{abi}^j. \end{aligned}$$

We shall also use the diffeomorphism $F^{(k)}$ from Equation (12) associated with the present connection tower. Let $F_\beta^{(k)} : T^{(k)}M \rightarrow TM$ be its components functions, i.e.

$$F(u)^{(k)} = (F_0(u)^{(k)}, F_1(u)^{(k)}, \dots, F_{k-1}(u)^{(k)}).$$

Then, if $u = j_0^k(q)$,

$$F_\alpha(u)^{(k)} = \nabla_{\frac{dq}{dt}}^\alpha \frac{dq}{dt} \Big|_{t=0}, \quad \alpha = 0, \dots, k-1.$$

If F_α^i are the coordinate functions of $F_\alpha^{(k)}$, then

$$F_\alpha^i = d_T^{(k)} F_{\alpha-1}^i + F_{\alpha-1}^l q^{(1)j} \Gamma_{lj}^i, \quad F_0^i = q^{(1)i}. \quad (30)$$

Moreover,

$$\frac{\partial F_\alpha^i}{\partial q^{(\alpha)j}} = (\alpha+1)! q^{(1)l} \Gamma_{lj}^i, \quad \frac{\partial F_\alpha^i}{\partial q^{(\alpha+1)j}} = (\alpha+1)! \delta_j^i, \quad \frac{\partial F_\alpha^i}{\partial q^{(\mu)j}} = 0 \quad (31)$$

for $\alpha+2 \leq \mu \leq k$, and consequently

$$\frac{\delta F_\alpha^i}{\delta q^{(\alpha)j}} = 0. \quad (32)$$

We denote $F_\alpha^{(k)}(u)$ by $u^{(\alpha+1)}$. Thus, through $F^{(k)}$, we identify a point $u \in T^{(k)}M$ with

$$(u^{(1)}, u^{(2)}, \dots, u^{(k)}) \in (TM)_\oplus^k.$$

Remark 4. The above construction is not the only formal way one might try to extend the first-order Dombrowski connection map, since there is no a priori preference for the order of covariant differentiation in the components of the connection maps. For instance, a natural alternative candidate is

$$\tilde{K}_\alpha^{(k)}(X) = \frac{1}{\alpha!} \nabla_{\frac{\partial \gamma}{\partial s}} \nabla^{\alpha-1} \frac{\partial \gamma}{\partial t} \Big|_{(s,t)=(0,0)},$$

which agrees with the usual Dombrowski connection map in order one. However, we claim that already for $k=3$, this candidate fails to produce a valid connection map without additional curvature correction terms, due to the failure of the compatibility condition with the canonical almost-tangent structure J . Thus, among the simple iterated-covariant-derivative constructions without correction terms, the order of differentiation used in this manuscript is the natural one. It is in this sense that the resulting Levi-Civita-induced connection tower can be regarded as canonical. We leave a detailed analysis of alternative constructions to future work.

Remark 5. The coefficients F_α^i for this specific Levi-Civita-induced connection tower have previously been introduced by Kosambi [18, 2.1], and later by Miron and Atanasiu ([21]). At second order, the connection map studied by Anastasiei and Bucătaru [2] provides an instance of the connection towers introduced here.

3.4. Lie brackets of lifted vector fields

We now calculate Lie brackets of $h_\alpha^{(k)}$ -lifts of vector fields on M for the Levi-Civita-induced connection tower.

The k -vertical components of the brackets between $h_0^{(k)}$ -lifts are the most involved; the full computation for $k=2$ is given below, while the longer $k=3$ computation is collected in Appendix C.2.

We begin with the brackets whose second factor is the k -vertical lift of a vector field on M .

Lemma 11. *Let $X, Y \in \mathfrak{X}(M)$. Then*

$$[X^{(k)h_0}, Y^{(k)v_k}] = (\nabla_X Y)^{(k)v_k}, \quad [X^{(k)h_\alpha}, Y^{(k)v_k}] = 0, \quad \forall \alpha = 1, 2, \dots, k-1, \quad [X^{(k)v_k}, Y^{(k)v_k}] = 0.$$

PROOF. From Equation (9), the adapted basis vectors $\delta/\delta q^{(\alpha)i}$ are independent of $q^{(k)l}$ for $\alpha = 1, \dots, k-1$. Hence

$$\left[\frac{\delta}{\delta q^{(\alpha)i}}, \frac{\partial}{\partial q^{(k)l}} \right] = 0, \quad \alpha = 1, \dots, k-1.$$

It follows from Lemma 4 that $[X^{(k)h_\alpha}, Y^{(k)v_k}] = 0$ for $\alpha = 1, \dots, k-1$. The identity $[X^{(k)v_k}, Y^{(k)v_k}] = 0$ is immediate. Finally,

$$\begin{aligned} \left[\frac{\delta}{\delta q^{(0)i}}, \frac{\partial}{\partial q^{(k)l}} \right] &= \frac{\partial (K_k)_i^j}{\partial q^{(k)l}} \frac{\partial}{\partial q^{(k)j}} \\ &= \Gamma_{li}^j \frac{\partial}{\partial q^{(k)j}} = \left(\nabla_{\partial_i} \partial_l \right)^{(k)v_k}. \end{aligned}$$

Therefore, by Lemma A.3,

$$[X^{(k)h_0}, Y^{(k)v_k}] = \left(X^i \partial_i Y^j + \Gamma_{il}^j X^i Y^l \right) \frac{\partial}{\partial q^{(k)j}} = (\nabla_X Y)^{(k)v_k}.$$

□

For $k = 2$, the complete bracket table is as follows.

Theorem 12. *Let $X, Y \in \mathfrak{X}(M)$. Then*

$$\begin{aligned} [X^{(2)h_0}, Y^{(2)v_2}] &= (\nabla_X Y)^{(2)v_2}, \quad [X^{(2)h_1}, Y^{(2)v_2}] = [X^{(2)v_2}, Y^{(2)v_2}] = 0 \\ [X^{(2)h_0}, Y^{(2)h_1}] &= (\nabla_X Y)^{(2)h_1} + \frac{1}{2} (R(u^{(1)}, X)Y - R(X, Y)u^{(1)})^{(2)v_2}, \quad [X^{(2)h_1}, Y^{(2)h_1}] = 0, \\ [X^{(2)h_0}, Y^{(2)h_0}] &= [X, Y]^{(2)h_0} - (R(X, Y)u^{(1)})^{(2)h_1} - \left(\frac{1}{2} R(X, Y)u^{(2)} + (\nabla_{u^{(1)}} R)(X, Y)u^{(1)} \right)^{(2)v_2}, \end{aligned}$$

where $u^{(1)} = F_0^{(2)}(u)$, $u^{(2)} = F_1^{(2)}(u)$, $u \in T^{(2)}M$.

From Lemma 6, many of the Lie brackets are inherited directly from the first-order case studied by Dombrowski (Theorem 7). The remaining brackets will be handled now in a series of technical lemmas.

Lemma 13. *Let $X, Y \in \mathfrak{X}(M)$. Then $K_2^{(2)}([X^{(2)h_1}, Y^{(2)h_1}]_u) = 0$, for $u \in T^{(2)}M$.*

PROOF. Using Equation (9) and Lemma A.3, we get

$$\begin{aligned} \left[\frac{\delta}{\delta q^{(1)i}}, \frac{\delta}{\delta q^{(1)l}} \right] &= \left[\frac{\partial}{\partial q^{(1)i}} - (K_1)_i^j \frac{\delta}{\delta q^{(2)j}}, \frac{\delta}{\delta q^{(1)l}} \right] \\ &= \left[\frac{\partial}{\partial q^{(1)i}}, \frac{\delta}{\delta q^{(1)l}} \right] + \frac{\delta(K_1)_i^j}{\delta q^{(1)l}} \frac{\delta}{\delta q^{(2)j}} \\ &= \left[\frac{\partial}{\partial q^{(1)i}}, \frac{\partial}{\partial q^{(1)l}} - (K_1)_l^j \frac{\delta}{\delta q^{(2)j}} \right] + \frac{\delta(K_1)_i^j}{\delta q^{(1)l}} \frac{\delta}{\delta q^{(2)j}} \\ &= -(K_1)_l^j \left[\frac{\partial}{\partial q^{(1)i}}, \frac{\delta}{\delta q^{(2)j}} \right] + \left(\frac{\partial(K_1)_i^j}{\partial q^{(1)l}} - \frac{\partial(K_1)_l^j}{\partial q^{(1)i}} \right) \frac{\delta}{\delta q^{(2)j}} \\ &= \left(\frac{\partial(K_1)_i^j}{\partial q^{(1)l}} - \frac{\partial(K_1)_l^j}{\partial q^{(1)i}} \right) \frac{\delta}{\delta q^{(2)j}}. \end{aligned}$$

Thus, since ∇ is torsion-free, $K_2^{(2)} \left[\frac{\delta}{\delta q^{(1)i}}, \frac{\delta}{\delta q^{(1)l}} \right] = 0$. Taking into account Lemma A.3, the result follows. □

Lemma 14. *Let $X, Y \in \mathfrak{X}(M)$. Then $K_2^{(2)}([X^{(2)h_0}, Y^{(2)h_1}]_u) = \frac{1}{2} (R(u^{(1)}, X_p)Y_p - R(X_p, Y_p)u^{(1)})$, where $u^{(1)} = F_0^{(2)}(u)$, $u \in T_p^{(2)}M$, $p \in M$.*

PROOF. Using again Equation (9) and Lemma A.3, we get

$$\begin{aligned} \left[\frac{\delta}{\delta q^{(0)i}}, \frac{\delta}{\delta q^{(1)l}} \right] &= \left[\frac{\partial}{\partial q^{(0)i}} - (K_1)_i^j \frac{\delta}{\delta q^{(1)j}} - (K_2)_i^j \frac{\delta}{\delta q^{(2)j}}, \frac{\delta}{\delta q^{(1)l}} \right] \\ &= \left[\frac{\partial}{\partial q^{(0)i}}, \frac{\delta}{\delta q^{(1)l}} \right] + \frac{\delta(K_1)_i^j}{\delta q^{(1)l}} \frac{\delta}{\delta q^{(1)j}} + \frac{\delta(K_2)_i^j}{\delta q^{(1)l}} \frac{\delta}{\delta q^{(2)j}} - (K_1)_i^j \left[\frac{\delta}{\delta q^{(1)j}}, \frac{\delta}{\delta q^{(1)l}} \right] \end{aligned}$$

$$= \left[\frac{\partial}{\partial q^{(0)i}}, \frac{\delta}{\delta q^{(1)l}} \right] + \frac{\delta(K_1)_i^j}{\delta q^{(1)l}} \frac{\delta}{\delta q^{(1)j}} + \frac{\delta(K_2)_i^j}{\delta q^{(1)l}} \frac{\delta}{\delta q^{(2)j}}.$$

Moreover,

$$\begin{aligned} \left[\frac{\partial}{\partial q^{(0)i}}, \frac{\delta}{\delta q^{(1)l}} \right] &= \left[\frac{\partial}{\partial q^{(0)i}}, \frac{\partial}{\partial q^{(1)l}} - (K_1)_l^j \frac{\delta}{\delta q^{(2)j}} \right] \\ &= -\frac{\partial(K_1)_l^j}{\partial q^{(0)i}} \frac{\delta}{\delta q^{(2)j}} - (K_1)_l^j \left[\frac{\partial}{\partial q^{(0)i}}, \frac{\delta}{\delta q^{(2)j}} \right] \\ &= -\frac{\partial(K_1)_l^j}{\partial q^{(0)i}} \frac{\delta}{\delta q^{(2)j}}. \end{aligned}$$

Hence, we obtain

$$\left[\frac{\delta}{\delta q^{(0)i}}, \frac{\delta}{\delta q^{(1)l}} \right] = \frac{\partial(K_1)_i^j}{\partial q^{(1)l}} \frac{\delta}{\delta q^{(1)j}} + \left(\frac{\delta(K_2)_i^j}{\delta q^{(1)l}} - \frac{\partial(K_1)_l^j}{\partial q^{(0)i}} \right) \frac{\delta}{\delta q^{(2)j}},$$

from which it follows that

$$\begin{aligned} {}^{(2)}K_2 \circ \left[\frac{\delta}{\delta q^{(0)i}}, \frac{\delta}{\delta q^{(1)l}} \right] &= \left(\frac{\delta(K_2)_i^j}{\delta q^{(1)l}} - \frac{\partial(K_1)_l^j}{\partial q^{(0)i}} \right) \partial_j \\ &= \left(\frac{\partial(K_2)_i^j}{\partial q^{(1)l}} - (K_1)_l^m \frac{\partial(K_1)_i^j}{\partial q^{(1)m}} - \frac{\partial(K_1)_l^j}{\partial q^{(0)i}} \right) \partial_j. \end{aligned}$$

By direct calculation,

$$\begin{aligned} \frac{\partial(K_2)_i^j}{\partial q^{(1)l}} &= \frac{1}{2} q^{(1)a} d q^j (\nabla_{\partial_a} \nabla_{\partial_l} \partial_i) + \frac{1}{2} q^{(1)a} d q^j (\nabla_{\partial_l} \nabla_{\partial_a} \partial_i) \\ &= q^{(1)a} d q^j (\nabla_{\partial_a} \nabla_{\partial_l} \partial_i) + \frac{1}{2} q^{(1)a} R_{lai}^j, \end{aligned}$$

and

$$(K_1)_l^m \frac{\partial(K_1)_i^j}{\partial q^{(1)m}} + \frac{\partial(K_1)_l^j}{\partial q^{(0)i}} = q^{(1)a} d q^j (\nabla_{\partial_i} \nabla_{\partial_a} \partial_l).$$

Therefore,

$$\begin{aligned} {}^{(2)}K_2 \left(\left[\frac{\delta}{\delta q^{(0)i}}, \frac{\delta}{\delta q^{(1)l}} \right]_u \right) &= q^{(1)a} \left(d q^j (\nabla_{\partial_a} \nabla_{\partial_l} \partial_i - \nabla_{\partial_l} \nabla_{\partial_a} \partial_i) + \frac{1}{2} R_{lai}^j \right) \partial_j \Big|_p \\ &= q^{(1)a} (R_{ail}^j + \frac{1}{2} R_{lai}^j) \partial_j \Big|_p \\ &= \frac{1}{2} q^{(1)a} (R_{ail}^j - R_{ila}^j) \partial_j \Big|_p. \end{aligned}$$

Finally, taking into account Lemma A.3, the result follows. $\square$

Lemma 15. Let $X, Y \in \mathfrak{X}(M)$. Then ${}^{(2)}K_2([X^{(2)h_0}, Y^{(2)h_0}]_u) = -\frac{1}{2}R(X_p, Y_p)u^{(2)} - (\nabla_{u^{(1)}}R)(X_p, Y_p)u^{(1)}$, where $u^{(1)} = {}^{(2)}F_0(u)$, $u^{(2)} = {}^{(2)}F_1(u)$, $u \in T_p^{(2)}M$, $p \in M$.

PROOF. It suffices to calculate $\overset{(2)}{v}_2 \circ \left[\frac{\delta}{\delta q^{(0)i}}, \frac{\delta}{\delta q^{(0)j}} \right]$. Due to the complexity of the calculations involving the derivatives in order to $q^{(0)i}$, we use another method to determine it. Since $\overset{(2)}{v}_2 \circ \left[\frac{\delta}{\delta q^{(0)i}}, \frac{\delta}{\delta q^{(0)j}} \right]$ is the last component of $\left[\frac{\delta}{\delta q^{(0)i}}, \frac{\delta}{\delta q^{(0)j}} \right]$ to determine, we will use the Jacobi identity and combine it with all the other components.

We first observe that

$$\left[\frac{\delta}{\delta q^{(0)i}}, \frac{\delta}{\delta q^{(0)j}} \right]_u = R_{ija}^l(p) q^{(1)a} \frac{\delta}{\delta q^{(1)l}} \Big|_u + S_{(0)i(0)j}^{(2)l}(u) \frac{\delta}{\delta q^{(2)l}} \Big|_u, \quad u \in T_p^{(2)} M, p \in M, \quad (33)$$

where $S_{(0)i(0)j}^{(2)l} \in C^\infty(T^{(2)} M)$. Hence, it is enough to obtain the smooth functions $S_{(0)i(0)j}^{(2)l}$.

From the Jacobi identity, we get

$$\begin{aligned} \left[\frac{\delta}{\delta q^{(2)m}}, \left[\frac{\delta}{\delta q^{(0)i}}, \frac{\delta}{\delta q^{(0)j}} \right] \right] &= \left[\frac{\delta}{\delta q^{(0)j}}, \left[\frac{\delta}{\delta q^{(0)i}}, \frac{\delta}{\delta q^{(2)m}} \right] \right] - \left[\frac{\delta}{\delta q^{(0)i}}, \left[\frac{\delta}{\delta q^{(0)j}}, \frac{\delta}{\delta q^{(2)m}} \right] \right] \\ &= \left[\frac{\delta}{\delta q^{(0)j}}, \left(\nabla_{\partial_i} \partial_m \right)^{\overset{(2)}{v}_2} \right] - \left[\frac{\delta}{\delta q^{(0)i}}, \left(\nabla_{\partial_j} \partial_m \right)^{\overset{(2)}{v}_2} \right] \\ &= (R(\partial_j, \partial_i) \partial_m)^{\overset{(2)}{v}_2} \end{aligned}$$

On the other hand, using Equation (33), we obtain

$$\begin{aligned} \left[\frac{\delta}{\delta q^{(2)m}}, \left[\frac{\delta}{\delta q^{(0)i}}, \frac{\delta}{\delta q^{(0)j}} \right] \right] &= R_{ija}^l(p) q^{(1)a} \left[\frac{\delta}{\delta q^{(2)m}}, \frac{\delta}{\delta q^{(1)l}} \right] + \frac{\delta S_{(0)i(0)j}^{(2)l}}{\delta q^{(2)m}} \frac{\delta}{\delta q^{(2)l}} + S_{(0)i(0)j}^{(2)l} \left[\frac{\delta}{\delta q^{(2)m}}, \frac{\delta}{\delta q^{(2)l}} \right] \\ &= \frac{\partial S_{(0)i(0)j}^{(2)l}}{\partial q^{(2)m}} \frac{\partial}{\partial q^{(2)l}}. \end{aligned}$$

Combining the two last equations, it follows that

$$S_{(0)i(0)j}^{(2)l} = -\frac{1}{2} R_{ijm}^l F_1^m + \Phi_{ij}^l,$$

where the functions $\Phi_{ij}^l \in C^\infty(T^{(2)} M)$ satisfy $\frac{\partial \Phi_{ij}^l}{\partial q^{(2)\lambda}} = 0$, for all $\lambda = 1, \dots, n$.

From the Jacobi identity, we also have

$$\left[\frac{\delta}{\delta q^{(1)m}}, \left[\frac{\delta}{\delta q^{(0)i}}, \frac{\delta}{\delta q^{(0)j}} \right] \right] = \left[\frac{\delta}{\delta q^{(0)j}}, \left[\frac{\delta}{\delta q^{(0)i}}, \frac{\delta}{\delta q^{(1)m}} \right] \right] - \left[\frac{\delta}{\delta q^{(0)i}}, \left[\frac{\delta}{\delta q^{(0)j}}, \frac{\delta}{\delta q^{(1)m}} \right] \right].$$

Let $u \in T_p^{(2)} M$, with $p \in M$. Then, from Lemma 14,

$$\left[\frac{\delta}{\delta q^{(0)j}}, \left[\frac{\delta}{\delta q^{(0)i}}, \frac{\delta}{\delta q^{(1)m}} \right] \right]_u = \left[\frac{\delta}{\delta q^{(0)j}}, \left(\nabla_{\partial_i} \partial_m \right)^{\overset{(2)}{h}_1} \right]_u + \frac{1}{2} \left[\frac{\delta}{\delta q^{(0)j}}, (R(u^{(1)}, \partial_i) \partial_m - R(\partial_i, \partial_m) u^{(1)})^{\overset{(2)}{v}_2} \right]_u.$$

We again use Lemma 14 to simplify the first term.

$$\left[\frac{\delta}{\delta q^{(0)j}}, \left(\nabla_{\partial_i} \partial_m \right)^{\overset{(2)}{h}_1} \right]_u = (\nabla_{\partial_j} \nabla_{\partial_i} \partial_m)_u^{\overset{(2)}{h}_1} + \frac{1}{2} \left(R(u^{(1)}, \partial_j) \nabla_{\partial_i} \partial_m - R(\partial_j, \nabla_{\partial_i} \partial_m) u^{(1)} \right)_u^{\overset{(2)}{v}_2}$$

On the other hand, to calculate the Lie bracket $\left[\frac{\delta}{\delta q^{(0)j}}, \left(R(u^{(1)}, \partial_i) \partial_m - R(\partial_i, \partial_m) u^{(1)} \right)^{(2)} \right]_u$, we first need to rewrite it as $\left[\frac{\delta}{\delta q^{(0)j}}, F_0^a \left(R(\partial_a, \partial_i) \partial_m - R(\partial_i, \partial_m) \partial_a \right)^{(2)} \right]_u$. Then

$$\begin{aligned} \left[\frac{\delta}{\delta q^{(0)j}}, F_0^a \left(R(\partial_a, \partial_i) \partial_m - R(\partial_i, \partial_m) \partial_a \right)^{(2)} \right] &= \frac{\delta F_0^a}{\delta q^{(0)j}} \left(R(\partial_a, \partial_i) \partial_m - R(\partial_i, \partial_m) \partial_a \right)^{(2)} \\ &\quad + F_0^a \left[\frac{\delta}{\delta q^{(0)j}}, \left(R(\partial_a, \partial_i) \partial_m - R(\partial_i, \partial_m) \partial_a \right)^{(2)} \right]. \end{aligned}$$

Now, we simplify the first term using $\frac{\delta F_0^a}{\delta q^{(0)j}}(u) = -dq^j(\nabla_{u^{(1)}} \partial_a)$ and apply Lemma 11 to the second term.

$$\begin{aligned} \left[\frac{\delta}{\delta q^{(0)j}}, \left(R(u^{(1)}, \partial_i) \partial_m - R(\partial_i, \partial_m) u^{(1)} \right)^{(2)} \right]_u &= \left((\nabla_{\partial_j} R)(u^{(1)}, \partial_i) \partial_m + R(u^{(1)}, \nabla_{\partial_j} \partial_i) \partial_m \right. \\ &\quad + R(u^{(1)}, \partial_i) \nabla_{\partial_j} \partial_m - (\nabla_{\partial_j} R)(\partial_i, \partial_m) u^{(1)} \\ &\quad \left. - R(\nabla_{\partial_j} \partial_i, \partial_m) u^{(1)} - R(\partial_i, \nabla_{\partial_j} \partial_m) u^{(1)} \right)_u^{(2)}. \end{aligned}$$

Combining the two Lie brackets we obtain

$$\begin{aligned} {}^{(2)}_{v_2} \left(\left[\frac{\delta}{\delta q^{(0)j}}, \left[\frac{\delta}{\delta q^{(0)i}}, \frac{\delta}{\delta q^{(1)m}} \right] \right]_u \right) &= \frac{1}{2} \left(R(u^{(1)}, \partial_j) \nabla_{\partial_i} \partial_m - R(\partial_j, \nabla_{\partial_i} \partial_m) u^{(1)} + (\nabla_{\partial_j} R)(u^{(1)}, \partial_i) \partial_m + R(u^{(1)}, \nabla_{\partial_j} \partial_i) \partial_m \right. \\ &\quad \left. + R(u^{(1)}, \partial_i) \nabla_{\partial_j} \partial_m - (\nabla_{\partial_j} R)(\partial_i, \partial_m) u^{(1)} - R(\nabla_{\partial_j} \partial_i, \partial_m) u^{(1)} - R(\partial_i, \nabla_{\partial_j} \partial_m) u^{(1)} \right)_u^{(2)}. \end{aligned}$$

Applying this result twice, we finally conclude that

$$\begin{aligned} &{}^{(2)}_{K_2} \left(\left[\frac{\delta}{\delta q^{(1)m}}, \left[\frac{\delta}{\delta q^{(0)i}}, \frac{\delta}{\delta q^{(0)j}} \right] \right]_u \right) \\ &= {}^{(2)}_{K_2} \left(\left[\frac{\delta}{\delta q^{(0)j}}, \left[\frac{\delta}{\delta q^{(0)i}}, \frac{\delta}{\delta q^{(1)m}} \right] \right]_u - \left[\frac{\delta}{\delta q^{(0)i}}, \left[\frac{\delta}{\delta q^{(0)j}}, \frac{\delta}{\delta q^{(1)m}} \right] \right]_u \right) \\ &= \frac{1}{2} \left[(\nabla_{\partial_j} R)(u^{(1)}, \partial_i) \partial_m - (\nabla_{\partial_i} R)(u^{(1)}, \partial_j) \partial_m - (\nabla_{\partial_j} R)(\partial_i, \partial_m) u^{(1)} + (\nabla_{\partial_i} R)(\partial_j, \partial_m) u^{(1)} \right]_p \\ &= -\frac{1}{2} \left[(\nabla_{u^{(1)}} R)(\partial_i, \partial_j) \partial_m + (\nabla_{\partial_m} R)(\partial_i, \partial_j) u^{(1)} \right]_p. \end{aligned}$$

On the other hand, using again Equation (33), we have

$$\begin{aligned} {}^{(2)}_{v_2} \circ \left[\frac{\delta}{\delta q^{(1)m}}, \left[\frac{\delta}{\delta q^{(0)i}}, \frac{\delta}{\delta q^{(0)j}} \right] \right] &= \frac{\delta S_{(0)i(0)j}^{(2)l}}{\delta q^{(1)m}} \frac{\delta}{\delta q^{(2)l}} \\ &= \frac{\partial \Phi_{ij}^l}{\partial q^{(1)m}} \frac{\delta}{\delta q^{(2)l}}. \end{aligned}$$

Hence, combining the two last equations, it follows that

$$\begin{aligned} \Phi_{ij}^l(u) &= -\frac{1}{2} q^{(1)m} dq^l \left[(\nabla_{u^{(1)}} R)(\partial_i, \partial_j) \partial_m + (\nabla_{\partial_m} R)(\partial_i, \partial_j) u^{(1)} \right]_p + h_{ij}^l(u) \\ &= -dq^l \left[(\nabla_{u^{(1)}} R)(\partial_i, \partial_j) u^{(1)} \right]_p + h_{ij}^l(u). \end{aligned}$$

where the functions $h_{ij}^l \in C^\infty(T^{(2)}M)$ satisfy $\frac{\partial h_{ij}^l}{\partial q^{(2)\lambda}} = \frac{\partial h_{ij}^l}{\partial q^{(1)\lambda}} = 0$, for all $\lambda = 1, \dots, n$.
Therefore,

$$K_2^{(2)} \left(\left[\frac{\delta}{\delta q^{(0)i}}, \frac{\delta}{\delta q^{(0)j}} \right]_u \right) = \left[-\frac{1}{2} R(\partial_i, \partial_j) u^{(2)} - (\nabla_{u^{(1)}} R)(\partial_i, \partial_j) u^{(1)} \right]_p + h_{ij}^l(u).$$

Finally, using Lemma C.1, we conclude that

$$K_2^{(2)} \left(\left[\frac{\delta}{\delta q^{(0)i}}, \frac{\delta}{\delta q^{(0)j}} \right]_u \right) = \left[-\frac{1}{2} R(\partial_i, \partial_j) u^{(2)} - (\nabla_{u^{(1)}} R)(\partial_i, \partial_j) u^{(1)} \right]_p,$$

which, taking into account Lemma A.3, leads to the result. $\square$

PROOF (PROOF OF THEOREM 12). The components below order 2 are inherited from the first-order bracket computations given in Theorem 7 together with Lemma 6, whereas the identities involving $v_2^{(2)}$ follow from Lemma 11. The remaining components are given by Lemmas 13, 14, and 15, together with Lemma A.3 from Appendix A.2. $\square$

Remark 6. The second-order bracket formulas involving the horizontal distribution were previously obtained by Anastasiei and Bucătaru [2, Prop. 3.1-3.3], in terms of their decomposition $T \text{Osc}^2 M = N_0 \oplus N_1 \oplus V_2$. Under the identification of N_0, N_1, V_2 with the distributions determined by our h_0 -, h_1 -, and v_2 -lifts, these give the corresponding cases of Theorem 12. We include the complete bracket table and its proof here because it provides the basis for the higher-order calculations.

For $k = 3$, one obtains the following higher-order analogue.

Theorem 16. *Let $X, Y \in \mathfrak{X}(M)$. Then*

$$\begin{aligned} [X^{(3)h_0}, Y^{(3)v_3}] &= (\nabla_X Y)^{(3)v_3}, \quad [X^{(3)h_1}, Y^{(3)v_3}] = [X^{(3)h_2}, Y^{(3)v_3}] = [X^{(3)h_3}, Y^{(3)v_3}] = 0, \\ [X^{(3)h_0}, Y^{(3)h_2}] &= (\nabla_X Y)^{(3)h_2} - \left(R(X, u^{(1)})Y + \frac{1}{3} R(u^{(1)}, Y)X \right)^{(3)v_3}, \quad [X^{(3)h_1}, Y^{(3)h_2}] = [X^{(3)h_2}, Y^{(3)h_2}] = 0, \\ [X^{(3)h_0}, Y^{(3)h_1}] &= (\nabla_X Y)^{(3)h_1} + \frac{1}{2} \left(R(u^{(1)}, X)Y - R(X, Y)u^{(1)} \right)^{(3)h_2} \\ &\quad - \frac{1}{2} \left(R(X, Y)u^{(2)} + \frac{1}{3} R(Y, u^{(2)})X + (\nabla_{u^{(1)}} R)(X, Y)u^{(1)} + \frac{1}{3} (\nabla_{u^{(1)}} R)(Y, u^{(1)})X \right)^{(3)v_3}, \\ [X^{(3)h_1}, Y^{(3)h_1}] &= -\frac{1}{2} \left(R(X, Y)u^{(1)} \right)^{(3)v_3}, \\ [X^{(3)h_0}, Y^{(3)h_0}] &= [X, Y]^{(3)h_0} - \left(R(X, Y)u^{(1)} \right)^{(3)h_1} - \left(\frac{1}{2} R(X, Y)u^{(2)} + (\nabla_{u^{(1)}} R)(X, Y)u^{(1)} \right)^{(3)h_2} \\ &\quad - \left(\frac{1}{3!} R(X, Y)u^{(3)} + \frac{1}{3} (\nabla_{u^{(1)}} R)(X, Y)u^{(2)} - \frac{1}{3} (\nabla_{u^{(2)}} R)(X, Y)u^{(1)} + \frac{1}{2} (\nabla_{u^{(1)}}^2 R)(X, Y)u^{(1)} \right)^{(3)v_3}, \end{aligned}$$

where $u^{(1)} = F_0^{(3)}(u)$, $u^{(2)} = F_1^{(3)}(u)$, $u^{(3)} = F_2^{(3)}(u)$, $u \in T^{(3)}M$.

PROOF (PROOF SKETCH). The components below order 3 are inherited from the second-order bracket formulas given in Theorem 12 together with Lemma 6. Thus only the 3-vertical components have to be computed. These are obtained by applying the adapted-basis bracket formula to the pairs $(h_0^{(3)}, h_2^{(3)})$, $(h_1^{(3)}, h_1^{(3)})$, $(h_0^{(3)}, h_1^{(3)})$, and $(h_0^{(3)}, h_0^{(3)})$. The first three computations use only the coefficient identities given in Equations (28)–(29) and the first Bianchi identity. The last computation, namely the 3-vertical component of $[X^{(3)h_0}, Y^{(3)h_0}]$, requires repeated use of the Jacobi identity and introduces the terms involving ∇R and $\nabla^2 R$. These longer reductions are given in Appendix C.2. Combining them with Lemma A.3 gives the displayed formulas. $\square$

Notice that, already in order 3, covariant derivatives of curvature and second covariant derivatives of curvature appear. This is the main source of the combinatorial growth in the general bracket formulas. We conclude with a general reduction method for the k -vertical components of Lie brackets of h_α -lifts on $T^{(k)}M$. The method uses auxiliary vector fields in $T^{(k)}M$ that decompose the h_α -lifts into lifts from $T^{(k-1)}M$ and k -vertical vector fields. The coordinate formula is recorded in Appendix C.3. We will need the following two consequences later.

Proposition 17. *Let $X, Y \in \mathfrak{X}(M)$. Then*

$$[X^{(k)h_{k-\alpha}}, Y^{(k)h_{k-1}}] = 0, \alpha = 1, 2, \dots, k-1.$$

The proof is given in Appendix C.3.

Proposition 18. *Let $X, Y \in \mathfrak{X}(M)$. Then*

$$[X^{(k)h_0}, Y^{(k)h_{k-1}}] = (\nabla_X Y)^{(k)h_{k-1}} - \frac{1}{k} (R(X, Y)u^{(1)} + (k-1)R(X, u^{(1)})Y)^{(k)v_k}, \quad (34)$$

where $u^{(1)} = F_0(u)$, $u \in T^{(k)}M$.

The proof is given in Appendix C.3.

4. Higher-order Sasaki metrics

In this section, we use the Levi–Civita-induced connection tower constructed in Subsection 3.3 to define a higher-order Sasaki metric on each higher-order tangent bundle. We first recall the classical Sasaki metric on TM , since it provides the model for the higher-order construction. We then define the k -Sasaki metric on $T^{(k)}M$, record the basic metric properties needed later, and compute the corresponding Levi–Civita connection in the cases $k = 2$ and $k = 3$. Finally, we derive the geodesic equations on $(T^{(2)}M, g^{(2)})$ and $(T^{(3)}M, g^{(3)})$, and prove the general jet lift characterization for arbitrary order.

Throughout this section, $K = (K_1, \dots, K_k)$ denotes the Levi–Civita-induced connection tower on $T^{(k)}M$.

4.1. The Sasaki metric on TM

There is also a canonical choice for the Riemannian metric on TM , called the *Sasaki metric*, with respect to which the splitting Equation (1) is an orthogonal decomposition. The Sasaki metric $g^{(1)}$ is defined by

$$g^{(1)}(X, Y) = g(\tau_{1*}X, \tau_{1*}Y) + g(K(X), K(Y)),$$

where $X, Y \in T_u TM$, $u \in TM$.

The following standard formulas will serve as the first-order model for the higher-order computations.

Theorem 19 (Dombrowski, [13]). *Let $X, Y \in \mathfrak{X}(M)$. Then*

$$\begin{aligned} \nabla_{X^{(1)v_1}}^{(1)} Y^{(1)v_1} &= 0, \\ \nabla_{X^{(1)h_0}}^{(1)} Y^{(1)v_1} &= \frac{1}{2} (R(u, Y)X)^{(1)h_0} + (\nabla_X Y)^{(1)v_1}, \\ \nabla_{X^{(1)h_0}}^{(1)} Y^{(1)h_0} &= (\nabla_X Y)^{(1)h_0} - \frac{1}{2} (R(X, Y)u)^{(1)v_1}, \end{aligned}$$

where $u \in TM$.

PROOF. Applying Koszul's formula (18) to triples of horizontal and vertical lifts, and using the bracket identities in Theorem 7, gives the displayed components after pairing with arbitrary horizontal and vertical lifts. The metric nondegeneracy of the horizontal–vertical splitting then determines the covariant derivatives uniquely. $\square$

Let Γ be a curve on TM . Denote by q the curve obtained by projecting Γ onto M , $q := \tau_1 \circ \Gamma$, and by V the vector field along q defined by the curve Γ . The velocity vector field $\dot{\Gamma}$ gives rise to the vector fields along q

$$\dot{q} = \tau_{1*} \circ \dot{\Gamma}, \quad Y = \overset{(1)}{K} \circ \dot{\Gamma},$$

with respect to which

$$\dot{\Gamma} = \dot{q}^{\overset{(1)}{h}_0} + Y^{\overset{(1)}{v}_1}.$$

Theorem 20. *A curve Γ on TM is a Riemannian geodesic of the Sasaki metric if and only if*

$$0 = \nabla_{\dot{q}} \dot{q} + R(V, Y)\dot{q},$$

$$0 = \nabla_{\dot{q}} Y.$$

PROOF. Using the decomposition $\dot{\Gamma} = \dot{q}^{\overset{(1)}{h}_0} + Y^{\overset{(1)}{v}_1}$ and the formulas of Theorem 19, together with torsion-freeness to compute $\overset{(1)}{\nabla}_{Y^{\overset{(1)}{v}_1}} \dot{q}^{\overset{(1)}{h}_0}$, one obtains

$$\overset{(1)}{\nabla}_{\dot{\Gamma}} \dot{\Gamma} = (\nabla_{\dot{q}} \dot{q} + R(V, Y)\dot{q})^{\overset{(1)}{h}_0} + (\nabla_{\dot{q}} Y)^{\overset{(1)}{v}_1}.$$

Since the horizontal and vertical distributions are orthogonal and complementary, the covariant acceleration vanishes if and only if both displayed components vanish. $\square$

Remark 7. If Γ is a horizontal geodesic on TM , then $\tau_1 \circ \Gamma$ is a geodesic on M .

Given a curve q on M , the curve $j^1 q = (q, \dot{q})$ is called the *tangent lift* of q (or *first-jet lift*). If $\Gamma = j^1 q$, then

$$\dot{\Gamma} = \dot{q}^{\overset{(1)}{h}_0} + (\nabla_{\dot{q}} \dot{q})^{\overset{(1)}{v}_1}.$$

Corollary 21. *Let q be a curve on M . The first-jet lift $\Gamma = j^1 q$ is a geodesic on TM if and only if q is a geodesic on M .*

4.2. The k -Sasaki metric

For each $u \in T^{(k)}M$, we consider the inner product $\langle\langle \cdot, \cdot \rangle\rangle_u$ on $T_u T^{(k)}M$ defined by

$$\langle\langle X, Y \rangle\rangle_u = \langle \tau_{k*} X, \tau_{k*} Y \rangle + \langle K_1(X), K_1(Y) \rangle + \cdots + \langle K_k(X), K_k(Y) \rangle, \quad (35)$$

where $X, Y \in T_u T^{(k)}M$. These inner products define a Riemannian metric on $T^{(k)}M$ [19, Theorem 9.1.3]. We call this metric the *k -Sasaki metric* associated with the connection tower $\overset{(k)}{K}$, and denote it by $\overset{(k)}{g}$.

The k -Sasaki metric makes the multiconnection decomposition given in Equation (6) orthogonal. Indeed, in terms of the adapted covector basis,

$$\overset{(k)}{g} = \sum_{\alpha=0}^k \sum_{i,j=1}^{\dim M} g_{ij} \delta q^{(\alpha)i} \otimes \delta q^{(\alpha)j},$$

where g_{ij} are the coordinates of the Riemannian metric on M .

Remark 8. The maps $\overset{(k)}{\tau}_\alpha$ are Riemannian submersions from $(T^{(k)}M, \overset{(k)}{g})$ to $(T^{(\alpha)}M, \overset{(\alpha)}{g})$, for $\alpha = 0, 1, \dots, k-1$; see [23]. This will be relevant to the study of geodesics on these jet bundles.

Remark 9. If the base manifold is flat, then all curvature terms in the formulas below vanish. In this case the k -Sasaki metric is locally the product metric induced by the connection coordinates, and the geodesic equations reduce to

$$\nabla_{\dot{q}} \dot{q} = 0, \quad \nabla_{\dot{q}} Y^{(\alpha)} = 0, \quad \alpha = 1, \dots, k.$$

This provides a useful check on the curvature terms in the second- and third-order formulas.

We now study Riemannian geodesics on $(T^{(k)}M, g^{(k)})$, especially when the curves are k -horizontal or k -jets of curves on M . If q is a curve on M and $\Gamma = j^k q$, then the velocity vector field of Γ is

$$\dot{\Gamma} = \dot{q}^{(k)} + (\nabla_{\dot{q}} \dot{q})^{(k)} + \frac{1}{2} (\nabla_{\dot{q}}^2 \dot{q})^{(k)} + \dots + \frac{1}{k!} (\nabla_{\dot{q}}^k \dot{q})^{(k)}. \quad (36)$$

Let $\nabla^{(k)}$ denote the Levi-Civita connection of $(T^{(k)}M, g^{(k)})$. Our aim is to now establish results relating the Levi-Civita connections on tangent bundles of different orders, as we previously did for Lie brackets in Lemma 6. The main computational tool here is the Koszul formula 18, together with the Lie bracket identities we established in the previous section.

Due to the direct sum decomposition given in Equation (6) and the $C^\infty(T^{(k)}M)$ -bilinearity and non-degeneracy of the k -Sasaki metric, it suffices to apply the Koszul formula when X, Y, Z are lifts of vector fields on M . The following three results isolate the general computations that are used repeatedly. Their proofs are given in Appendix D.1.

Lemma 22. *Let $X, Y, Z \in \mathfrak{X}(M)$. Then*

$$\left(X^{(k)} \langle Y^{(k)}, Z^{(k)} \rangle \right) \circ \tau_k = \delta_{\alpha 0} \delta_{\beta \mu} (\langle \nabla_X Y, Z \rangle + \langle Y, \nabla_X Z \rangle).$$

Lemma 23. *Let $X, Y \in \mathfrak{X}(M)$. Then*

$$K_\mu^{(k)} \circ \left(\nabla_{X^{(k)}} Y^{(k)} \right) = K_\mu^{(k-1)} \circ \left(\nabla_{X^{(k-1)}} Y^{(k-1)} \right) \circ \tau_{k-1}^{(k)},$$

for $\mu = 0, 1, \dots, k-1$, and

$$K_k^{(k)} \circ \left(\nabla_{X^{(k)}} Y^{(k)} \right) = \frac{1}{2} K_k^{(k)} \circ \left([X^{(k)}, Y^{(k)}] \right),$$

for all $\alpha, \beta = 0, 1, \dots, k-1$.

Proposition 24. *Let $\nabla^{(k)}$ be the Levi-Civita connection of $(T^{(k)}M, g^{(k)})$. Then*

$$\begin{aligned} K_k^{(k)} \circ \left(\nabla_{X^{(k)}} Y^{(k)} \right) &= K_k^{(k)} \circ \left(\nabla_{X^{(k)}} Y^{(k)} \right) = \dots = K_k^{(k)} \circ \left(\nabla_{X^{(k)}} Y^{(k)} \right) = 0, \\ K_k^{(k)} \circ \left(\nabla_{X^{(k)}} Y^{(k)} \right) &= (\nabla_X Y) \circ \tau_k, \\ K_\mu^{(k)} \circ \left(\nabla_{X^{(k)}} Y^{(k)} \right) &= 0, \quad \mu = 0, 1, \dots, k-1, \end{aligned}$$

for all $X, Y \in \mathfrak{X}(M)$.

4.3. Geodesics on $(T^{(2)}M, g^{(2)})$

We first calculate the covariant derivatives of $\nabla^{(2)}$ and then obtain the geodesic equations on $(T^{(2)}M, g^{(2)})$. The proof of the following theorem is included below, since it gives the model for the higher-order Koszul computations.

Theorem 25. *Let $\nabla^{(2)}$ be the Levi-Civita connection of $(T^{(2)}M, g^{(2)})$. Then*

$$\begin{aligned}
 \nabla_{X^{(2)}v_2}^{(2)} Y^{(2)} &= 0, \\
 \nabla_{X^{(2)h_1}}^{(2)} Y^{(2)} &= \frac{1}{4} (R(X, Y)u^{(1)} + R(u^{(1)}, Y)X)^{(2)}_{h_0}, \\
 \nabla_{X^{(2)h_0}}^{(2)} Y^{(2)} &= \left(\frac{1}{4} R(u^{(2)}, Y)X + \frac{1}{2} (\nabla_{u^{(1)}} R)(u^{(1)}, Y)X \right)^{(2)}_{h_0} \\
 &\quad + \left(\frac{1}{4} R(u^{(1)}, X)Y + \frac{1}{4} R(u^{(1)}, Y)X \right)^{(2)}_{h_1} + (\nabla_X Y)^{(2)}_{v_2}, \\
 \nabla_{X^{(2)h_1}}^{(2)} Y^{(2)} &= 0, \\
 \nabla_{X^{(2)h_0}}^{(2)} Y^{(2)} &= \frac{1}{2} (R(u^{(1)}, Y)X)^{(2)}_{h_0} + (\nabla_X Y)^{(2)}_{h_1} - \left(\frac{1}{4} R(X, Y)u^{(1)} + \frac{1}{4} R(X, u^{(1)})Y \right)^{(2)}_{v_2}, \\
 \nabla_{X^{(2)h_0}}^{(2)} Y^{(2)} &= (\nabla_X Y)^{(2)}_{h_0} - \frac{1}{2} (R(X, Y)u^{(1)})^{(2)}_{h_1} - \left(\frac{1}{4} R(X, Y)u^{(2)} + \frac{1}{2} (\nabla_{u^{(1)}} R)(X, Y)u^{(1)} \right)^{(2)}_{v_2},
 \end{aligned}$$

for all $X, Y \in \mathfrak{X}(M)$, where $u^{(1)} = F_0(u)$, $u^{(2)} = F_1(u)$, and $u \in T^{(2)}M$.

PROOF. The components involving only h_0 - and h_1 -lifts follow from Lemma 23 and Theorem 12. The 2-vertical components involving one v_2 -lift are given by Proposition 24. It remains to obtain the horizontal components of the derivatives involving one 2-vertical lift.

For example, for $Z \in \mathfrak{X}(M)$,

$$\begin{aligned}
 \langle \nabla_{X^{(2)h_0}}^{(2)} Y^{(2)}, Z^{(2)h_1} \rangle &= -\langle Y^{(2)v_2}, \nabla_{X^{(2)h_0}}^{(2)} Z^{(2)h_1} \rangle \\
 &= \left\langle Y, \frac{1}{4} R(X, Z)u^{(1)} + \frac{1}{4} R(X, u^{(1)})Z \right\rangle \\
 &= \left\langle \frac{1}{4} R(u^{(1)}, X)Y + \frac{1}{4} R(u^{(1)}, Y)X, Z \right\rangle.
 \end{aligned}$$

Thus

$$K_1^{(2)} \left(\nabla_{X^{(2)h_0}}^{(2)} Y^{(2)} \right) = \frac{1}{4} R(u^{(1)}, X)Y + \frac{1}{4} R(u^{(1)}, Y)X.$$

The remaining components are obtained analogously by applying the Koszul formula against h_0 -, h_1 -, and v_2 -lifts and using the second-order bracket formulas from Theorem 12. Combining these components gives the displayed formulas. $\square$

Let Γ be a curve on $T^{(2)}M$. Let $q = \tau_2 \circ \Gamma$, and let

$$V^{(1)} = F_0 \circ \Gamma, \quad V^{(2)} = F_1 \circ \Gamma.$$

We also write

$$Y^{(\alpha)} = K_\alpha^{(2)} \circ \dot{\Gamma}, \quad \alpha = 1, 2.$$

Then

$$\dot{\Gamma} = \dot{q}^{(2)h_0} + (Y^{(1)})^{(2)h_1} + (Y^{(2)})^{(2)v_2}. \quad (37)$$

Theorem 26. *A curve Γ is a Riemannian geodesic on $(T^{(2)}M, g^{(2)})$ if and only if*

$$\begin{aligned} 0 &= \nabla_{\dot{q}} \dot{q} + R(V^{(1)}, Y^{(1)})\dot{q} + \frac{1}{2}R(V^{(2)}, Y^{(2)})\dot{q} + (\nabla_{V^{(1)}}R)(V^{(1)}, Y^{(2)})\dot{q} \\ &\quad + \frac{1}{2}R(Y^{(1)}, Y^{(2)})V^{(1)} + \frac{1}{2}R(V^{(1)}, Y^{(2)})Y^{(1)}, \\ 0 &= \nabla_{\dot{q}} Y^{(1)} + \frac{1}{2}R(V^{(1)}, \dot{q})Y^{(2)} + \frac{1}{2}R(V^{(1)}, Y^{(2)})\dot{q}, \\ 0 &= \nabla_{\dot{q}} Y^{(2)}. \end{aligned}$$

PROOF. Using the decomposition given in Equation (37) and the torsion-free property of $\nabla^{(2)}$, the covariant acceleration of Γ can be written as

$$\begin{aligned} \nabla_{\dot{\Gamma}}^{(2)} \dot{\Gamma} &= \nabla_{\dot{q}^{(2)h_0}}^{(2)} \dot{q}^{(2)h_0} + 2\nabla_{\dot{q}^{(2)h_0}}^{(2)} (Y^{(1)})^{(2)h_1} - [\dot{q}^{(2)h_0}, (Y^{(1)})^{(2)h_1}] \\ &\quad + 2\nabla_{\dot{q}^{(2)h_0}}^{(2)} (Y^{(2)})^{(2)v_2} - [\dot{q}^{(2)h_0}, (Y^{(2)})^{(2)v_2}] + 2\nabla_{(Y^{(1)})^{(2)h_1}}^{(2)} (Y^{(2)})^{(2)v_2}. \end{aligned}$$

Applying Theorem 25 and the corresponding bracket formulas gives

$$\begin{aligned} \tau_{2*} \left(\nabla_{\dot{\Gamma}}^{(2)} \dot{\Gamma} \right) &= \nabla_{\dot{q}} \dot{q} + R(V^{(1)}, Y^{(1)})\dot{q} + \frac{1}{2}R(V^{(2)}, Y^{(2)})\dot{q} + (\nabla_{V^{(1)}}R)(V^{(1)}, Y^{(2)})\dot{q} \\ &\quad + \frac{1}{2}R(Y^{(1)}, Y^{(2)})V^{(1)} + \frac{1}{2}R(V^{(1)}, Y^{(2)})Y^{(1)}, \\ K_1 \left(\nabla_{\dot{\Gamma}}^{(2)} \dot{\Gamma} \right) &= \nabla_{\dot{q}} Y^{(1)} + \frac{1}{2}R(V^{(1)}, \dot{q})Y^{(2)} + \frac{1}{2}R(V^{(1)}, Y^{(2)})\dot{q}, \\ K_2 \left(\nabla_{\dot{\Gamma}}^{(2)} \dot{\Gamma} \right) &= \nabla_{\dot{q}} Y^{(2)}. \end{aligned}$$

Since the maps τ_{2*}, K_1, K_2 determine the decomposition of $TT^{(2)}M$, the vanishing of the covariant acceleration is equivalent to the three displayed equations in Theorem 26. $\square$

Remark 10. If Γ is a 2-horizontal geodesic on $T^{(2)}M$, then $\tau_1^{(2)} \circ \Gamma$ is a geodesic on TM .

Now let $\Gamma = j^2 q$ be the second-jet lift of a curve q on M . Then

$$V^{(1)} = \dot{q}, \quad V^{(2)} = \nabla_{\dot{q}} \dot{q},$$

and

$$\dot{\Gamma} = \dot{q}^{(2)h_0} + (\nabla_{\dot{q}} \dot{q})^{(2)h_1} + \frac{1}{2} \left(\nabla_{\dot{q}}^2 \dot{q} \right)^{(2)v_2}. \quad (38)$$

Corollary 27. *Let q be a curve on M . The second-jet lift $\Gamma = j^2 q$ is a geodesic on $T^{(2)}M$ if and only if q is a geodesic on M .*

PROOF. If q is a geodesic on M , then Equation (38) gives $\dot{\Gamma} = \dot{q}^{(2)h_0}$ and Theorem 25 gives $\nabla_{\Gamma}^{(2)}\dot{\Gamma} = 0$. Conversely, suppose that $\Gamma = j^2q$ is a geodesic. Then Theorem 26 gives, after substituting

$$V^{(1)} = \dot{q}, \quad V^{(2)} = \nabla_{\dot{q}}\dot{q}, \quad Y^{(1)} = \nabla_{\dot{q}}\dot{q}, \quad Y^{(2)} = \frac{1}{2}\nabla_{\dot{q}}^2\dot{q},$$

a system whose final two equations imply

$$\nabla_{\dot{q}}^2\dot{q} + \frac{1}{4}R(\dot{q}, \nabla_{\dot{q}}^2\dot{q})\dot{q} = 0, \quad \nabla_{\dot{q}}^3\dot{q} = 0.$$

Let $A = \nabla_{\dot{q}}^2\dot{q}$. Taking the inner product of the first equation with $\dot{q}$ gives $\langle A, \dot{q} \rangle = 0$, and the second equation gives $\nabla_{\dot{q}}A = 0$. Differentiating $\langle A, \dot{q} \rangle = 0$ once gives $\langle A, \nabla_{\dot{q}}\dot{q} \rangle = 0$. The first component equation, paired with $\dot{q}$, then gives $\langle \nabla_{\dot{q}}\dot{q}, \dot{q} \rangle = 0$. Differentiating this last identity and using $\langle A, \dot{q} \rangle = 0$ yields $\|\nabla_{\dot{q}}\dot{q}\|^2 = 0$. Hence q is a geodesic. $\square$

4.4. Geodesics on $(T^{(3)}M, g^{(3)})$

We now calculate the covariant derivatives of $\nabla^{(3)}$ and obtain the geodesic equations on $(T^{(3)}M, g^{(3)})$. The proof of the next theorem is given in Appendix D.2.

Theorem 28. Let $\nabla^{(3)}$ be the Levi-Civita connection of $(T^{(3)}M, g^{(3)})$. Then

$$\begin{aligned} \nabla_{X^{v_3}}^{(3)} Y^{v_3} &= 0, \\ \nabla_{X^{h_2}}^{(3)} Y^{v_3} &= \left(\frac{1}{2}R(X, Y)u^{(1)} + \frac{1}{6}R(u^{(1)}, X)Y \right)^{h_0}, \\ \nabla_{X^{h_1}}^{(3)} Y^{v_3} &= \left(\frac{1}{4}R(u^{(2)}, Y)X + \frac{1}{4}(\nabla_{u^{(1)}}R)(u^{(1)}, Y)X + \frac{1}{12}R(X, u^{(2)})Y \right. \\ &\quad \left. + \frac{1}{12}(\nabla_{u^{(1)}}R)(X, u^{(1)})Y \right)^{h_0} + \frac{1}{4}(R(u^{(1)}, Y)X)^{h_1}, \\ \nabla_{X^{h_0}}^{(3)} Y^{v_3} &= \left(\frac{1}{12}R(u^{(3)}, Y)X + \frac{1}{6}(\nabla_{u^{(1)}}R)(u^{(2)}, Y)X - \frac{1}{6}(\nabla_{u^{(2)}}R)(u^{(1)}, Y)X \right. \\ &\quad \left. + \frac{1}{4}(\nabla_{u^{(1)}}^2R)(u^{(1)}, Y)X \right)^{h_0} \\ &\quad + \left(\frac{1}{4}R(u^{(2)}, Y)X + \frac{1}{4}(\nabla_{u^{(1)}}R)(u^{(1)}, Y)X - \frac{1}{12}R(X, Y)u^{(2)} \right. \\ &\quad \left. - \frac{1}{12}(\nabla_{u^{(1)}}R)(X, Y)u^{(1)} \right)^{h_1} \\ &\quad + \left(\frac{1}{6}R(X, Y)u^{(1)} + \frac{1}{2}R(u^{(1)}, X)Y \right)^{h_2} + (\nabla_X Y)^{v_3}, \\ \nabla_{X^{h_2}}^{(3)} Y^{h_2} &= 0, \\ \nabla_{X^{h_1}}^{(3)} Y^{h_2} &= \frac{1}{4}(R(X, Y)u^{(1)} + R(u^{(1)}, Y)X)^{h_0}, \\ \nabla_{X^{h_0}}^{(3)} Y^{h_2} &= \left(\frac{1}{4}R(u^{(2)}, Y)X + \frac{1}{2}(\nabla_{u^{(1)}}R)(u^{(1)}, Y)X \right)^{h_0} \end{aligned}$$

$$\begin{aligned}
 & + \left(\frac{1}{4} R(u^{(1)}, X)Y + \frac{1}{4} R(u^{(1)}, Y)X \right)^{(3)}_{h_1} + (\nabla_X Y)^{(3)}_{h_2} \\
 & + \left(\frac{1}{2} R(u^{(1)}, X)Y - \frac{1}{6} R(u^{(1)}, Y)X \right)^{(3)}_{v_3}, \\
 \nabla^{(3)}_{X^{(3)}_{h_1}} Y^{(3)}_{h_1} & = -\frac{1}{4} (R(X, Y)u^{(1)})^{(3)}_{v_3}, \\
 \nabla^{(3)}_{X^{(3)}_{h_0}} Y^{(3)}_{h_1} & = \frac{1}{2} (R(u^{(1)}, Y)X)^{(3)}_{h_0} + (\nabla_X Y)^{(3)}_{h_1} - \left(\frac{1}{4} R(X, Y)u^{(1)} + \frac{1}{4} R(X, u^{(1)})Y \right)^{(3)}_{h_2} \\
 & - \left(\frac{1}{4} R(X, Y)u^{(2)} + \frac{1}{4} (\nabla_{u^{(1)}} R)(X, Y)u^{(1)} + \frac{1}{12} R(Y, u^{(2)})X \right. \\
 & \quad \left. + \frac{1}{12} (\nabla_{u^{(1)}} R)(Y, u^{(1)})X \right)^{(3)}_{v_3}, \\
 \nabla^{(3)}_{X^{(3)}_{h_0}} Y^{(3)}_{h_0} & = (\nabla_X Y)^{(3)}_{h_0} - \frac{1}{2} (R(X, Y)u^{(1)})^{(3)}_{h_1} - \left(\frac{1}{4} R(X, Y)u^{(2)} + \frac{1}{2} (\nabla_{u^{(1)}} R)(X, Y)u^{(1)} \right)^{(3)}_{h_2} \\
 & - \left(\frac{1}{12} R(X, Y)u^{(3)} + \frac{1}{6} (\nabla_{u^{(1)}} R)(X, Y)u^{(2)} - \frac{1}{6} (\nabla_{u^{(2)}} R)(X, Y)u^{(1)} \right. \\
 & \quad \left. + \frac{1}{4} (\nabla_{u^{(1)}}^2 R)(X, Y)u^{(1)} \right)^{(3)}_{v_3},
 \end{aligned}$$

for all $X, Y \in \mathfrak{X}(M)$, where $u^{(1)} = F_0(u)$, $u^{(2)} = F_1(u)$, $u^{(3)} = F_2(u)$, and $u \in T^{(3)}M$.

Let Γ be a curve on $T^{(3)}M$. Let $q = \tau_3 \circ \Gamma$, and let

$$V^{(1)} = F_0 \circ \Gamma, \quad V^{(2)} = F_1 \circ \Gamma, \quad V^{(3)} = F_2 \circ \Gamma.$$

We also write

$$Y^{(\alpha)} = K_\alpha \circ \dot{\Gamma}, \quad \alpha = 1, 2, 3.$$

Then

$$\dot{\Gamma} = \dot{q}^{(3)}_{h_0} + (Y^{(1)})^{(3)}_{h_1} + (Y^{(2)})^{(3)}_{h_2} + (Y^{(3)})^{(3)}_{v_3}. \quad (39)$$

Theorem 29. A curve Γ on $T^{(3)}M$ is a Riemannian geodesic of the 3-Sasaki metric if and only if

$$\begin{aligned}
 0 & = \nabla_{\dot{q}} \dot{q} + R(V^{(1)}, Y^{(1)})\dot{q} + \frac{1}{2} R(V^{(2)}, Y^{(2)})\dot{q} + (\nabla_{V^{(1)}} R)(V^{(1)}, Y^{(2)})\dot{q} \\
 & + \frac{1}{6} R(V^{(3)}, Y^{(3)})\dot{q} + \frac{1}{3} (\nabla_{V^{(1)}} R)(V^{(2)}, Y^{(3)})\dot{q} - \frac{1}{3} (\nabla_{V^{(2)}} R)(V^{(1)}, Y^{(3)})\dot{q} \\
 & + \frac{1}{2} (\nabla_{V^{(1)}}^2 R)(V^{(1)}, Y^{(3)})\dot{q} + \frac{1}{2} R(Y^{(1)}, Y^{(2)})V^{(1)} + \frac{1}{2} R(V^{(1)}, Y^{(2)})Y^{(1)} \\
 & + \frac{1}{2} R(V^{(2)}, Y^{(3)})Y^{(1)} + \frac{1}{2} (\nabla_{V^{(1)}} R)(V^{(1)}, Y^{(3)})Y^{(1)} + \frac{1}{6} R(Y^{(1)}, V^{(2)})Y^{(3)} \\
 & + \frac{1}{6} (\nabla_{V^{(1)}} R)(Y^{(1)}, V^{(1)})Y^{(3)} + R(Y^{(2)}, Y^{(3)})V^{(1)} + \frac{1}{3} R(V^{(1)}, Y^{(2)})Y^{(3)}, \\
 0 & = \nabla_{\dot{q}} Y^{(1)} + \frac{1}{2} R(V^{(1)}, \dot{q})Y^{(2)} + \frac{1}{2} R(V^{(1)}, Y^{(2)})\dot{q} + \frac{1}{2} R(V^{(2)}, Y^{(3)})\dot{q} \\
 & + \frac{1}{2} (\nabla_{V^{(1)}} R)(V^{(1)}, Y^{(3)})\dot{q} - \frac{1}{6} (\nabla_{V^{(1)}} R)(\dot{q}, Y^{(3)})V^{(1)} - \frac{1}{6} R(\dot{q}, Y^{(3)})V^{(2)} \\
 & + \frac{1}{2} R(V^{(1)}, Y^{(3)})Y^{(1)},
 \end{aligned}$$

$$\begin{aligned} 0 &= \nabla_{\dot{q}} Y^{(2)} + R(V^{(1)}, \dot{q})Y^{(3)} + \frac{1}{3}R(\dot{q}, Y^{(3)})V^{(1)}, \\ 0 &= \nabla_{\dot{q}} Y^{(3)}. \end{aligned}$$

The proof is given in Appendix E.1.

Remark 11. If Γ is a 3-horizontal geodesic on $T^{(3)}M$, then $\tau_2 \circ \Gamma$ is a geodesic on $T^{(2)}M$.

If $\Gamma = j^3 q$ is the third-jet lift of a curve q on M , then

$$V^{(1)} = \dot{q}, \quad V^{(2)} = \nabla_{\dot{q}} \dot{q}, \quad V^{(3)} = \nabla_{\dot{q}}^2 \dot{q},$$

and

$$\begin{aligned} {}^{(3)}K_1(\dot{\Gamma}) &= \nabla_{\dot{q}} \dot{q}, & {}^{(3)}K_2(\dot{\Gamma}) &= \frac{1}{2!} \nabla_{\dot{q}}^2 \dot{q}, & {}^{(3)}K_3(\dot{\Gamma}) &= \frac{1}{3!} \nabla_{\dot{q}}^3 \dot{q}. \end{aligned}$$

The following corollary to Theorem 29 can be proven by similar arguments to those used in the proof of Corollary 27.

Corollary 30. Let q be a curve on M . The third-jet lift $\Gamma = j^3 q$ is a geodesic on $T^{(3)}M$ if and only if q is a geodesic on M .

4.5. Geodesics on $(T^{(k)}M, g^{(k)})$

The full geodesic system on $T^{(k)}M$ becomes combinatorially large. However, the final two components admit a uniform closed form, which is sufficient for the jet lift characterization below.

Lemma 31. Let $\nabla^{(k)}$ be the Levi-Civita connection of $(T^{(k)}M, g^{(k)})$. Then, for all $W \in \mathfrak{X}(T^{(k)}M)$,

$${}^{(k)}K_k \circ \left(\nabla_W^{(k)} W \right) = \nabla_{\tau_{k*} \circ W} ({}^{(k)}K_k \circ W), \quad (40)$$

$$\begin{aligned} {}^{(k)}K_{k-1} \circ \left(\nabla_W^{(k)} W \right) &= \nabla_{\tau_{k*} \circ W} ({}^{(k)}K_{k-1} \circ W) + \frac{1}{k} R(u^{(1)}, {}^{(k)}K_{k-1} \circ W) (\tau_{k*} \circ W) \\ &\quad + \frac{k-1}{k} R(u^{(1)}, \tau_{k*} \circ W) ({}^{(k)}K_{k-1} \circ W), \end{aligned} \quad (41)$$

where $u^{(1)} = F_0^{(k)}(u)$, $u \in T^{(k)}M$.

PROOF. Let $W \in \mathfrak{X}(T^{(k)}M)$. Using Equation (59), the bracket formulas involving $v_k^{(k)}$ -lifts, and Proposition 24, we obtain

$$\begin{aligned} {}^{(k)}K_k \circ \left(\nabla_W^{(k)} W \right) &= 2 {}^{(k)}K_k \circ \left(\nabla_{h_0 \circ W}^{(k)} (v_k \circ W) \right) + {}^{(k)}K_k \circ \left([v_k \circ W, h_0 \circ W] \right) \\ &= \nabla_{\tau_{k*} \circ W} ({}^{(k)}K_k \circ W). \end{aligned}$$

This proves Equation (40).

For the next component, let $Z \in \mathfrak{X}(M)$. Applying the Koszul formula gives

$$\begin{aligned} \langle \nabla_W^{(k)} W, Z^{(k)} \rangle &= (h_0 \circ W) \langle W, Z^{(k)} \rangle - \langle h_{k-1} \circ W, (\nabla_{\tau_{k*} \circ W} Z)^{h_{k-1}} \rangle \\ &\quad + \frac{1}{k} \langle v_k \circ W, (R({}^{(k)}K_{k-1} \circ W, Z)u^{(1)} + (k-1)R({}^{(k)}K_{k-1} \circ W, u^{(1)})Z)^{v_k} \rangle. \end{aligned}$$

Using the curvature symmetries, this becomes

$$\begin{aligned} \langle\langle \nabla_W^{(k)} W, Z^{(k)} \rangle\rangle &= \left\langle \nabla_{\tau_{k*} \circ W}^{(k)} (K_{k-1} \circ W), Z \right\rangle \\ &+ \frac{1}{k} \left\langle R(u^{(1)}, \tau_{k*} \circ W) (K_{k-1} \circ W), Z \right\rangle \\ &+ \frac{k-1}{k} \left\langle R(u^{(1)}, K_{k-1} \circ W) (\tau_{k*} \circ W), Z \right\rangle. \end{aligned}$$

Since this holds for all Z , Equation (41) follows. $\square$

Let Γ be a curve on $T^{(k)}M$ and let $q = \tau_k \circ \Gamma$. We denote by

$$V^{(\alpha)} = F_{\alpha-1}^{(k)} \circ \Gamma, \quad \alpha = 1, \dots, k,$$

the vector fields along q determined by the connection tower vector bundle structure, and by

$$Y^{(\alpha)} = K_{\alpha}^{(k)} \circ \dot{\Gamma}, \quad \alpha = 1, \dots, k,$$

the components of the velocity vector field. Then

$$\dot{\Gamma} = \dot{q}^{(k)} + (Y^{(1)})^{(k)} + \dots + (Y^{(k)})^{(k)}. \quad (42)$$

Lemma 32. *Let Γ be a curve on $T^{(k)}M$. If Γ is a geodesic on $T^{(k)}M$, then*

$$\nabla_{\dot{q}} Y^{(k-1)} + \frac{1}{k} R(V^{(1)}, Y^{(k-1)}) \dot{q} + \frac{k-1}{k} R(V^{(1)}, \dot{q}) Y^{(k-1)} = 0, \quad (43)$$

$$\nabla_{\dot{q}} Y^{(k)} = 0. \quad (44)$$

PROOF. This follows immediately by applying Lemma 31 to $W = \dot{\Gamma}$ and using $\nabla_{\dot{\Gamma}} \dot{\Gamma} = 0$. $\square$

Corollary 33. *Let q be a curve on M . If the k -jet lift $\Gamma = j^k q$ is a geodesic on $T^{(k)}M$, then Γ is k -horizontal.*

PROOF. Suppose that $\Gamma = j^k q$ is a geodesic. Then, by Lemma 32 and Equation (36), the vector field

$$A := \nabla_{\dot{q}}^k \dot{q}$$

satisfies

$$A + \frac{1}{k} R(\dot{q}, A) \dot{q} = 0, \quad \nabla_{\dot{q}} A = 0.$$

Taking the inner product of the first equation with $\dot{q}$ gives $\langle A, \dot{q} \rangle = 0$, since $R(\dot{q}, A)$ is skew-adjoint. Since A is parallel along q , repeated differentiation yields

$$0 = \frac{d^r}{dt^r} \langle A, \dot{q} \rangle = \left\langle A, \nabla_{\dot{q}}^r \dot{q} \right\rangle, \quad r = 0, 1, \dots, k.$$

Taking $r = k$ gives $\|A\|^2 = 0$. Hence $A = 0$, which is precisely the condition that Γ is k -horizontal. $\square$

Proposition 34. *If Γ is a k -horizontal geodesic on $T^{(k)}M$, then $\tau_{k-1}^{(k)} \circ \Gamma$ is a geodesic on $T^{(k-1)}M$.*

PROOF. Suppose that Γ is a k -horizontal geodesic on $T^{(k)}M$, that is, $\nabla_{\dot{\Gamma}} \dot{\Gamma} = 0$ and $K_k^{(k)} \circ \dot{\Gamma} = 0$. Then $\dot{\Gamma} = n_k^{(k)} \circ \dot{\Gamma}$. Let $\tilde{\Gamma} = \tau_{k-1}^{(k)} \circ \Gamma$. Its velocity vector field is

$$\dot{\tilde{\Gamma}} = \tau_{k-1*}^{(k)} \circ n_k^{(k)} \circ \dot{\Gamma}.$$

By Lemma 23,

$$0 = K_\mu \circ \left(\nabla_{\tau_{k-1} \circ \Gamma}^{(k)} (n_k \circ \Gamma) \right) = K_\mu \circ \left(\nabla_{\tilde{\Gamma}}^{(k-1)} \right),$$

for $\mu = 0, 1, \dots, k-1$. Hence $\nabla_{\tilde{\Gamma}}^{(k-1)} \Gamma = 0$. $\square$

Remark 12. It follows that an α -horizontal geodesic on $T^{(k)}M$ projects onto a geodesic on $T^{(\alpha-1)}M$, for each $\alpha = 1, \dots, k$.

Remark 13. The results in Proposition 34 and Remark 12 also follow from the fact that the projections $\tau_\alpha^{(k)}$ are Riemannian submersions; see [16, Proposition 3.1]. The proof above records the corresponding connection map argument.

Proposition 35. Let q be a curve on M . The k -jet $\Gamma = j^k q$ is a geodesic on $T^{(k)}M$ if and only if q is a geodesic on M .

PROOF. Suppose that $\Gamma = j^k q$ is a geodesic on $T^{(k)}M$. By Corollary 33, Γ is k -horizontal. Therefore, by Proposition 34, the curve $\tau_{k-1}^{(k)} \circ \Gamma$ is a geodesic on $T^{(k-1)}M$. Since this projection is the $(k-1)$ -jet of q , the argument can be repeated. Inductively, all lower jet lifts are geodesics, and in particular q is a geodesic on M .

Conversely, if q is a geodesic on M , then all higher covariant derivatives $\nabla_q^r \dot{q}$ vanish for $r \geq 1$. Hence the velocity field of $j^k q$ is 1-horizontal, and Lemma 23 gives $\nabla_{\tilde{\Gamma}}^{(k)} \Gamma = 0$. $\square$

A. Background material for connection towers

A.1. Adapted-basis details

This appendix records the coordinate relations between the coordinate bases and the adapted bases associated with a connection map.

The adapted covectors are

$$\delta q^{(\alpha)i} = dq^{(\alpha)i} + (K_1)_j^i dq^{(\alpha-1)j} + \dots + (K_\alpha)_j^i dq^{(0)j}, \quad \alpha = 1, \dots, k,$$

and $\delta q^{(0)i} = dq^{(0)i}$. Let $\delta/\delta q^{(\alpha)i}$ denote the dual basis. The dual coefficients $(C_\alpha)_j^i$ are defined by

$$\frac{\delta}{\delta q^{(\alpha)i}} = \frac{\partial}{\partial q^{(\alpha)i}} - (C_1)_i^j \frac{\partial}{\partial q^{(\alpha+1)j}} - \dots - (C_{k-\alpha})_i^j \frac{\partial}{\partial q^{(k)j}}.$$

Proposition A.1 (Miron [19]). The dual coefficients satisfy

$$dq^{(\alpha)i} \left(\frac{\delta}{\delta q^{(\mu)j}} \right) = -(C_{\alpha-\mu})_j^i$$

for all $\alpha = 1, \dots, k$, $\mu = 0, \dots, \alpha-1$, and $i, j = 1, \dots, n$. In particular,

$$dq^{(\alpha)i} = \delta q^{(\alpha)i} - (C_1)_j^i \delta q^{(\alpha-1)j} - \dots - (C_\alpha)_j^i \delta q^{(0)j},$$

for all $\alpha = 1, \dots, k$. Similarly,

$$\frac{\partial}{\partial q^{(\alpha)i}} = \frac{\delta}{\delta q^{(\alpha)i}} + (K_1)_i^j \frac{\delta}{\delta q^{(\alpha+1)j}} + \dots + (K_{k-\alpha})_i^j \frac{\delta}{\delta q^{(k)j}}.$$

PROOF. The result follows by inverting the triangular change-of-basis matrix relating the coordinate covectors $dq^{(\alpha)i}$ to the adapted covectors $\delta q^{(\alpha)i}$. The vector-field identity is the dual triangular relation. $\square$

Proposition A.2 (Miron [19]). The dual coefficients are constructed recursively in terms of the connection coefficients by

$$(C_\alpha)_j^i = (K_\alpha)_j^i - \sum_{\mu=1}^{\alpha-1} (K_\mu)_l^i (C_{\alpha-\mu})_j^l, \quad (C_1)_j^i = (K_1)_j^i. \quad (45)$$

PROOF. From the definition of the adapted covector basis,

$$\delta q^{(\alpha)i} = dq^{(\alpha)i} + (K_1)_l^i dq^{(\alpha-1)l} + \dots + (K_\alpha)_l^i dq^{(0)l}.$$

Evaluating this on $\delta/\delta q^{(0)j}$ gives

$$0 = dq^{(\alpha)i} \left(\frac{\delta}{\delta q^{(0)j}} \right) + (K_1)_l^i dq^{(\alpha-1)l} \left(\frac{\delta}{\delta q^{(0)j}} \right) + \dots + (K_\alpha)_j^i.$$

Using Proposition A.1 yields

$$-(C_\alpha)_j^i - (K_1)_l^i (C_{\alpha-1})_j^l - \dots - (K_{\alpha-1})_l^i (C_1)_j^l + (K_\alpha)_j^i = 0,$$

which is precisely Equation (45). $\square$

A.2. Technical lemmas for lifted vector fields

This appendix contains the technical lift identities used in the computation of Lie brackets.

PROOF (PROOF OF LEMMA 4). The linearity identities follow directly from the defining characterization of the lift.

Indeed, the maps $\tau_{k*}, K_1, \dots, K_k$ are fiberwise linear on each tangent space $T_u T^{(k)} M$. Hence the unique vector whose α -component is prescribed and whose remaining components are zero depends linearly on the prescribed vector in $T_{\tau_k(u)} M$. This gives

$$(X + Y)^{(k)h_\alpha} = X^{(k)h_\alpha} + Y^{(k)h_\alpha}, \quad (cX)^{(k)h_\alpha} = cX^{(k)h_\alpha}.$$

Similarly, multiplying the prescribed vector by $f(\tau_k(u))$ gives

$$(fX)^{(k)h_\alpha} = (f \circ \tau_k)X^{(k)h_\alpha}.$$

Since

$$\frac{\delta}{\delta q^{(\alpha)j}} = (\partial_j)^{(k)h_\alpha}, \quad \alpha = 0, \dots, k,$$

the coordinate formula follows by applying the preceding identity to $X = X^j \partial_j$:

$$X^{(k)h_\alpha} = (X^j \circ \tau_k)(\partial_j)^{(k)h_\alpha} = (X^j \circ \tau_k) \frac{\delta}{\delta q^{(\alpha)j}}.$$

The final formulas follow immediately from the fact that $Y(f \circ \tau_k) = df(\tau_{k*} Y)$ for all $Y \in \mathfrak{X}(T^{(k)} M)$. $\square$

Lemma A.3. *Let $X, Y \in \mathfrak{X}(M)$ and $f, g \in C^\infty(M)$. Then*

$$[(fX)^{(k)h_\alpha}, (gY)^{(k)h_\beta}] = \delta_{0\alpha}(f \circ \tau_k)(Xg)Y^{(k)h_\beta} - \delta_{0\beta}(g \circ \tau_k)(Yf)X^{(k)h_\alpha} + (f \circ \tau_k)(g \circ \tau_k)[X^{(k)h_\alpha}, Y^{(k)h_\beta}],$$

for $\alpha, \beta = 0, \dots, k$.

PROOF. Using Lemma 4,

$$[(fX)^{(k)h_\alpha}, (gY)^{(k)h_\beta}] = (f \circ \tau_k)X^{(k)h_\alpha}(g \circ \tau_k)Y^{(k)h_\beta} - (g \circ \tau_k)Y^{(k)h_\beta}(f \circ \tau_k)X^{(k)h_\alpha} + (f \circ \tau_k)(g \circ \tau_k)[X^{(k)h_\alpha}, Y^{(k)h_\beta}].$$

The result follows from Lemma 4. $\square$

The $h_\alpha^{(k)}$ -lifts of vector fields on M to $T^{(k)} M$ are compatible with the projections of the connection tower.

B. Riemannian identities and the Levi–Civita-induced tower

B.1. Riemannian identities used in Section 3

This appendix records the curvature identities used in the computations of Section 3. We use the convention

$$R(X, Y)Z = \nabla_X \nabla_Y Z - \nabla_Y \nabla_X Z - \nabla_{[X, Y]} Z.$$

Lemma B.1. *The following identities hold for all $X, Y, Z, W \in \mathfrak{X}(M)$:*

1. (First skew-symmetry identity) $R(X, Y) + R(Y, X) = 0$;
2. (Second skew-symmetry identity) $\langle R(X, Y)Z, W \rangle + \langle R(X, Y)W, Z \rangle = 0$;
3. (Symmetry by pairs identity) $\langle R(X, Y)Z, W \rangle = \langle R(W, Z)Y, X \rangle$;
4. (First Bianchi identity) $R(X, Y)Z + R(Y, Z)X + R(Z, X)Y = 0$.

The covariant derivative of R in the direction of X is defined by

$$\begin{aligned} (\nabla_X R)(Y, Z)W &= \nabla_X(R(Y, Z)W) - R(\nabla_X Y, Z)W \\ &\quad - R(Y, \nabla_X Z)W - R(Y, Z)\nabla_X W. \end{aligned} \quad (46)$$

We regard ∇R as the $(4, 1)$ -tensor field determined by $(\nabla R)(X; \cdot) = \nabla_X R$. Similarly, $\nabla^2 R$ denotes the $(5, 1)$ -tensor field

$$(\nabla^2 R)(X, Y; \cdot) = \nabla_{XY}^2 R.$$

The curvature endomorphism also acts on tensor fields; in particular,

$$\begin{aligned} (R(X, Y)R)(A, B)C &= R(X, Y)R(A, B)C - R(R(X, Y)A, B)C \\ &\quad - R(A, R(X, Y)B)C - R(A, B)R(X, Y)C. \end{aligned}$$

Lemma B.2. *The following identities hold for all $X, Y, V, Z, W \in \mathfrak{X}(M)$:*

1. (First skew-symmetry identity) $(\nabla_X R)(Y, Z)W + (\nabla_X R)(Z, Y)W = 0$;
2. (Second skew-symmetry identity) $\langle (\nabla_X R)(Y, Z)W, V \rangle + \langle (\nabla_X R)(Y, Z)V, W \rangle = 0$;
3. (Symmetry by pairs identity) $\langle (\nabla_X R)(Y, Z)W, V \rangle = \langle (\nabla_X R)(V, W)Z, Y \rangle$;
4. (First Bianchi identity) $(\nabla_X R)(Y, Z)W + (\nabla_X R)(Z, W)Y + (\nabla_X R)(W, Y)Z = 0$;
5. (Second Bianchi identity) $(\nabla_X R)(Y, Z)W + (\nabla_Y R)(Z, X)W + (\nabla_Z R)(X, Y)W = 0$.

Lemma B.3. *The following identities hold for all $X, Y, A, B, C, D \in \mathfrak{X}(M)$:*

1. (First skew-symmetry identity) $(\nabla^2 R)(X, Y; A, B) + (\nabla^2 R)(X, Y; B, A) = 0$;
2. (Second skew-symmetry identity) $\langle (\nabla^2 R)(X, Y; A, B)C, D \rangle + \langle (\nabla^2 R)(X, Y; A, B)D, C \rangle = 0$;
3. (Symmetry by pairs identity) $\langle (\nabla^2 R)(X, Y; A, B)C, D \rangle + \langle (\nabla^2 R)(X, Y; D, C)B, A \rangle = 0$;
4. (First Bianchi identity) $(\nabla^2 R)(X, Y; A, B)C + (\nabla^2 R)(X, Y; B, C)A + (\nabla^2 R)(X, Y; C, A)B = 0$;
5. (Second Bianchi identity) $(\nabla^2 R)(X, Y; A, B)C + (\nabla^2 R)(X, A; B, Y)C + (\nabla^2 R)(X, B; Y, A)C = 0$;
6. (Nested curvature identity) $(\nabla^2 R)(X, Y; A, B)C - (\nabla^2 R)(Y, X; A, B)C = (R(X, Y)R)(A, B)C$.

Lemma B.4. *Let $\gamma_s : t \mapsto \gamma_s(t) = \gamma(s, t)$ be a smooth family of curves on M , and let Z be any smooth family of vector fields along γ . Then*

$$\nabla_{\frac{\partial \gamma}{\partial s}} \nabla_{\frac{\partial \gamma}{\partial t}} Z = \nabla_{\frac{\partial \gamma}{\partial t}} \nabla_{\frac{\partial \gamma}{\partial s}} Z + R\left(\frac{\partial \gamma}{\partial s}, \frac{\partial \gamma}{\partial t}\right) Z.$$

PROOF (PROOF OF LEMMAS B.1–B.4). The identities in Lemma B.1 are the standard curvature symmetries of the Levi–Civita connection. Lemma B.2 follows by covariantly differentiating these identities and using $\nabla g = 0$, together with the second Bianchi identity. Lemma B.3 follows by applying a second covariant derivative and commuting the two covariant derivatives; the commutator is precisely the curvature action on the tensor R . Finally, Lemma B.4 is the defining commutation identity for covariant derivatives along a two-parameter family, obtained from the definition of the curvature tensor and the torsion-freeness of the Levi–Civita connection. $\square$

C. Lie bracket computations

C.1. Weak fiberwise-linearity

Let 0_p be the null element in the fiber $\tau_k^{-1}(p)$ for all $p \in M$, that is, $0_p = j_0^k(q)$, where q is the constant curve through p . We say that a function $f \in C^\infty(T^{(k)}M)$ is *weakly fiberwise linear* if $f(0_p) = 0$ for all $p \in M$. Similarly, a vector field $X \in \Gamma(TT^{(k)}M)$ is weakly fiberwise linear if $X_{0_p} = 0$ for all $p \in M$.

It follows from the definition of the Levi-Civita-induced connection coefficients that $(K_\alpha)_j^i$ and $\partial(K_\alpha)_j^i/\partial q^{(0)l}$ are weakly fiberwise linear for all $\alpha = 1, \dots, k$ and all i, j, l .

Lemma C.1. *The vector field*

$$\left[\frac{\delta}{\delta q^{(0)i}}, \frac{\delta}{\delta q^{(0)j}} \right]$$

is weakly fiberwise linear.

PROOF. Let $f \in C^\infty(T^{(k)}M)$. At 0_p , all connection coefficients vanish, and therefore

$$\frac{\delta f}{\delta q^{(0)i}}(0_p) = \frac{\partial f}{\partial q^{(0)i}}(0_p).$$

Thus $\delta f/\delta q^{(0)i}$ is weakly fiberwise linear if and only if $\partial f/\partial q^{(0)i}$ is weakly fiberwise linear. Using also the weak fiberwise linearity of $\partial(K_\alpha)_j^i/\partial q^{(0)l}$, we obtain

$$\frac{\delta}{\delta q^{(0)i}} \left(\frac{\delta f}{\delta q^{(0)j}} \right) (0_p) = \frac{\partial^2 f}{\partial q^{(0)i} \partial q^{(0)j}} (0_p).$$

Interchanging i and j and subtracting gives

$$\left[\frac{\delta}{\delta q^{(0)i}}, \frac{\delta}{\delta q^{(0)j}} \right]_{0_p} (f) = 0.$$

Since this holds for every f , the bracket vanishes at 0_p . □

C.2. Third-order bracket computations

We now record the computations that lead to Theorem 16.

Lemma C.2. *Let $X, Y \in \mathfrak{X}(M)$. Then*

$$K_3^{(3)}([X^{(3)h_0}, Y^{(3)h_2}]_u) = -R(X_p, u^{(1)})Y_p - \frac{1}{3}R(u^{(1)}, Y_p)X_p, \quad K_3^{(3)}([X^{(3)h_1}, Y^{(3)h_2}]_u) = K_3^{(3)}([X^{(3)h_2}, Y^{(3)h_2}]_u) = 0,$$

where $u^{(1)} = F_0^{(3)}(u)$, $u \in T_p^{(3)}M$, $p \in M$.

PROOF. Using Equation (9) and Lemma 11, it is easily verified that $[X^{(3)h_2}, Y^{(3)h_2}] = 0$ for all $X, Y \in \mathfrak{X}(M)$. Moreover,

$$\begin{aligned} \left[\frac{\delta}{\delta q^{(1)i}}, \frac{\delta}{\delta q^{(2)l}} \right] &= \left[\frac{\partial}{\partial q^{(1)i}} - (K_1)_i^j \frac{\delta}{\delta q^{(2)j}} - (K_2)_i^j \frac{\partial}{\partial q^{(3)j}}, \frac{\delta}{\delta q^{(2)l}} \right] \\ &= \left[\frac{\partial}{\partial q^{(1)i}}, \frac{\delta}{\delta q^{(2)l}} \right] + \frac{\delta(K_2)_i^j}{\delta q^{(2)l}} \frac{\partial}{\partial q^{(3)j}} \\ &= \left[\frac{\partial}{\partial q^{(1)i}}, \frac{\partial}{\partial q^{(2)l}} - (K_1)_l^j \frac{\partial}{\partial q^{(3)j}} \right] + \frac{\partial(K_1)_i^j}{\partial q^{(1)l}} \frac{\partial}{\partial q^{(3)j}} \\ &= \left(\frac{\partial(K_1)_i^j}{\partial q^{(1)l}} - \frac{\partial(K_1)_l^j}{\partial q^{(1)i}} \right) \frac{\partial}{\partial q^{(3)j}} \end{aligned}$$

$$= 0.$$

Finally,

$$\begin{aligned} \left[\frac{\delta}{\delta q^{(0)i}}, \frac{\delta}{\delta q^{(2)l}} \right] &= \left[\frac{\partial}{\partial q^{(0)i}} - (K_1)_i^j \frac{\delta}{\delta q^{(1)j}} - (K_2)_i^j \frac{\delta}{\delta q^{(2)j}} - (K_3)_i^j \frac{\partial}{\partial q^{(3)j}}, \frac{\delta}{\delta q^{(2)l}} \right] \\ &= \left[\frac{\partial}{\partial q^{(0)i}}, \frac{\delta}{\delta q^{(2)l}} \right] + \frac{\partial(K_2)_i^j}{\partial q^{(2)l}} \frac{\delta}{\delta q^{(2)j}} + \frac{\delta(K_3)_i^j}{\delta q^{(2)l}} \frac{\partial}{\partial q^{(3)j}} \\ &= \left[\frac{\partial}{\partial q^{(0)i}}, \frac{\partial}{\partial q^{(2)l}} - (K_1)_l^j \frac{\partial}{\partial q^{(3)j}} \right] + \frac{\partial(K_2)_i^j}{\partial q^{(2)l}} \frac{\delta}{\delta q^{(2)j}} + \left(\frac{\partial(K_3)_i^j}{\partial q^{(2)l}} - (K_1)_l^m \frac{\partial(K_3)_i^j}{\partial q^{(3)m}} \right) \frac{\partial}{\partial q^{(3)j}} \\ &= \frac{\partial(K_1)_i^j}{\partial q^{(1)l}} \frac{\delta}{\delta q^{(2)j}} + \left(\frac{\partial(K_3)_i^j}{\partial q^{(2)l}} - (K_1)_l^m \frac{\partial(K_1)_i^j}{\partial q^{(1)m}} - \frac{\partial(K_1)_l^j}{\partial q^{(0)i}} \right) \frac{\partial}{\partial q^{(3)j}}. \end{aligned}$$

Hence,

$$K_3^{(3)} \circ \left[\frac{\delta}{\delta q^{(0)i}}, \frac{\delta}{\delta q^{(2)l}} \right] = \left(\frac{\partial(K_3)_i^j}{\partial q^{(2)l}} - (K_1)_l^m \frac{\partial(K_1)_i^j}{\partial q^{(1)m}} - \frac{\partial(K_1)_l^j}{\partial q^{(0)i}} \right) \partial_j.$$

By direct calculation,

$$\frac{\partial(K_3)_i^j}{\partial q^{(2)l}} = q^{(1)a} d q^j \left(\nabla_{\partial_a} \nabla_{\partial_l} \partial_i \right) + q^{(1)a} \frac{1}{3} R_{lai}^j,$$

$$(K_1)_l^m \frac{\partial(K_1)_i^j}{\partial q^{(1)m}} + \frac{\partial(K_1)_l^j}{\partial q^{(0)i}} = d q^j (\nabla_{\partial_i} \nabla_{\partial_a} \partial_l)$$

Therefore,

$$\begin{aligned} K_3^{(3)} \left(\left[\frac{\delta}{\delta q^{(0)i}}, \frac{\delta}{\delta q^{(2)l}} \right]_u \right) &= \left[\frac{1}{3} q^{(1)a} R_{lai}^j + q^{(1)a} d q^j (\nabla_{\partial_a} \nabla_{\partial_l} \partial_i - \nabla_{\partial_i} \nabla_{\partial_a} \partial_l) \right] \partial_j \Big|_p \\ &= q^{(1)a} \left[R_{ail}^j + \frac{1}{3} R_{lai}^j \right] \partial_j \Big|_p, \end{aligned}$$

from which the result follows upon an application of the first Bianchi identity. $\square$

Lemma C.3. *Let $X, Y \in \mathfrak{X}(M)$. Then*

$$\begin{aligned} K_3^{(3)}([X^{h_0}, Y^{h_1}]_u) &= -\frac{1}{2} \left(R(X_p, Y_p) u^{(2)} + \frac{1}{3} R(Y_p, u^{(2)}) X_p + (\nabla_{u^{(1)}} R)(X_p, Y_p) u^{(1)} + \frac{1}{3} (\nabla_{u^{(1)}} R)(Y_p, u^{(1)}) X_p \right), \\ K_3^{(3)}([X^{h_1}, Y^{h_1}]_u) &= -\frac{1}{2} R(X_p, Y_p) u^{(1)}, \end{aligned}$$

where $u^{(1)} = F_0^{(3)}(u)$, $u^{(2)} = F_1^{(3)}(u)$, $u \in T_p^{(3)} M$, $p \in M$.

PROOF. Applying the same method as in the previous lemma, we get

$$\begin{aligned} \left[\frac{\delta}{\delta q^{(1)i}}, \frac{\delta}{\delta q^{(1)l}} \right] &= \left[\frac{\partial}{\partial q^{(1)i}} - (K_1)_i^j \frac{\delta}{\delta q^{(2)j}} - (K_2)_i^j \frac{\partial}{\partial q^{(3)j}}, \frac{\delta}{\delta q^{(1)l}} \right] \\ &= \left[\frac{\partial}{\partial q^{(1)i}}, \frac{\delta}{\delta q^{(1)l}} \right] + \frac{\delta(K_1)_i^j}{\delta q^{(1)l}} \frac{\delta}{\delta q^{(2)j}} + \frac{\delta(K_2)_i^j}{\delta q^{(1)l}} \frac{\partial}{\partial q^{(3)j}} \end{aligned}$$

$$\begin{aligned}
 &= \left[\frac{\partial}{\partial q^{(1)i}}, \frac{\partial}{\partial q^{(1)l}} - (K_1)_l^j \frac{\delta}{\delta q^{(2)j}} - (K_2)_l^j \frac{\partial}{\partial q^{(3)j}} \right] + \frac{\delta(K_1)_i^j}{\delta q^{(1)l}} \frac{\delta}{\delta q^{(2)j}} + \frac{\delta(K_2)_i^j}{\delta q^{(1)l}} \frac{\partial}{\partial q^{(3)j}} \\
 &= -(K_1)_l^j \left[\frac{\partial}{\partial q^{(1)i}}, \frac{\delta}{\delta q^{(2)j}} \right] + \left(\frac{\partial(K_1)_i^j}{\partial q^{(1)l}} - \frac{\partial(K_1)_l^j}{\partial q^{(1)i}} \right) \frac{\delta}{\delta q^{(2)j}} + \left(\frac{\delta(K_2)_i^j}{\delta q^{(1)l}} - \frac{\partial(K_2)_l^j}{\partial q^{(1)i}} \right) \frac{\partial}{\partial q^{(3)j}} \\
 &= \left(\frac{\partial(K_1)_i^j}{\partial q^{(1)l}} - \frac{\partial(K_1)_l^j}{\partial q^{(1)i}} \right) \frac{\delta}{\delta q^{(2)j}} + \left(\frac{\delta(K_2)_i^j}{\delta q^{(1)l}} + (K_1)_l^m \frac{\partial(K_1)_m^j}{\partial q^{(1)i}} - \frac{\partial(K_2)_l^j}{\partial q^{(1)i}} \right) \frac{\partial}{\partial q^{(3)j}}.
 \end{aligned}$$

Hence,

$${}^{(3)}K_3 \circ \left[\frac{\delta}{\delta q^{(1)i}}, \frac{\delta}{\delta q^{(1)l}} \right] = \left(\frac{\partial(K_2)_i^j}{\partial q^{(1)l}} - \frac{\partial(K_2)_l^j}{\partial q^{(1)i}} \right) \partial_j.$$

Thus, we obtain

$$\begin{aligned}
 {}^{(3)}K_3 \left(\left[\frac{\delta}{\delta q^{(1)i}}, \frac{\delta}{\delta q^{(1)l}} \right]_u \right) &= \frac{1}{2} q^{(1)a} (R_{lai}^j + R_{ail}^j) \partial_j \Big|_p \\
 &= \frac{1}{2} q^{(1)a} R_{lia}^j \partial_j \Big|_p,
 \end{aligned}$$

after applying the first Bianchi identity.

Next,

$$\begin{aligned}
 \left[\frac{\delta}{\delta q^{(0)i}}, \frac{\delta}{\delta q^{(1)l}} \right] &= \left[\frac{\partial}{\partial q^{(0)i}} - (K_1)_i^j \frac{\delta}{\delta q^{(1)j}} - (K_2)_i^j \frac{\delta}{\delta q^{(2)j}} - (K_3)_i^j \frac{\partial}{\partial q^{(3)j}}, \frac{\delta}{\delta q^{(1)l}} \right] \\
 &= \left[\frac{\partial}{\partial q^{(0)i}}, \frac{\delta}{\delta q^{(1)l}} \right] + \frac{\delta(K_1)_i^j}{\delta q^{(1)l}} \frac{\delta}{\delta q^{(1)j}} + \frac{\delta(K_2)_i^j}{\delta q^{(1)l}} \frac{\delta}{\delta q^{(2)j}} + \frac{\delta(K_3)_i^j}{\delta q^{(1)l}} \frac{\partial}{\partial q^{(3)j}} - (K_1)_i^j \left[\frac{\delta}{\delta q^{(1)j}}, \frac{\delta}{\delta q^{(1)l}} \right] \\
 &= \left[\frac{\partial}{\partial q^{(0)i}}, \frac{\delta}{\delta q^{(1)l}} \right] + \frac{\delta(K_1)_i^j}{\delta q^{(1)l}} \frac{\delta}{\delta q^{(1)j}} + \frac{\delta(K_2)_i^j}{\delta q^{(1)l}} \frac{\delta}{\delta q^{(2)j}} + \left(\frac{\delta(K_3)_i^j}{\delta q^{(1)l}} + \frac{1}{2} (K_1)_i^m q^{(1)a} R_{mla}^j \right) \frac{\partial}{\partial q^{(3)j}}.
 \end{aligned}$$

Moreover,

$$\begin{aligned}
 \left[\frac{\partial}{\partial q^{(0)i}}, \frac{\delta}{\delta q^{(1)l}} \right] &= \left[\frac{\partial}{\partial q^{(0)i}}, \frac{\partial}{\partial q^{(1)l}} - (K_1)_l^j \frac{\delta}{\delta q^{(2)j}} - (K_2)_l^j \frac{\partial}{\partial q^{(3)j}} \right] \\
 &= -\frac{\partial(K_1)_l^j}{\partial q^{(0)i}} \frac{\delta}{\delta q^{(2)j}} - \frac{\partial(K_2)_l^j}{\partial q^{(0)i}} \frac{\partial}{\partial q^{(3)j}} - (K_1)_l^j \left[\frac{\partial}{\partial q^{(0)i}}, \frac{\delta}{\delta q^{(2)j}} \right] \\
 &= -\frac{\partial(K_1)_l^j}{\partial q^{(0)i}} \frac{\delta}{\delta q^{(2)j}} + \left((K_1)_l^m \frac{\partial(K_1)_m^j}{\partial q^{(0)i}} - \frac{\partial(K_2)_l^j}{\partial q^{(0)i}} \right) \frac{\partial}{\partial q^{(3)j}}.
 \end{aligned}$$

Hence, we obtain

$$\begin{aligned}
 \left[\frac{\delta}{\delta q^{(0)i}}, \frac{\delta}{\delta q^{(1)l}} \right] &= \frac{\partial(K_1)_i^j}{\partial q^{(1)l}} \frac{\delta}{\delta q^{(1)j}} + \left(\frac{\delta(K_2)_i^j}{\delta q^{(1)l}} - \frac{\partial(K_1)_l^j}{\partial q^{(0)i}} \right) \frac{\delta}{\delta q^{(2)j}} \\
 &\quad + \left((K_1)_l^m \frac{\partial(K_1)_m^j}{\partial q^{(0)i}} - \frac{\partial(K_2)_l^j}{\partial q^{(0)i}} + \frac{\delta(K_3)_i^j}{\delta q^{(1)l}} + \frac{1}{2} (K_1)_i^m q^{(1)a} R_{mla}^j \right) \frac{\partial}{\partial q^{(3)j}},
 \end{aligned}$$

from which it follows that

$${}^{(3)}K_3 \circ \left[\frac{\delta}{\delta q^{(0)i}}, \frac{\delta}{\delta q^{(1)l}} \right] = \left((K_1)_l^m \frac{\partial(K_1)_m^j}{\partial q^{(0)i}} - \frac{\partial(K_2)_l^j}{\partial q^{(0)i}} + \frac{\delta(K_3)_i^j}{\delta q^{(1)l}} + \frac{1}{2} (K_1)_i^m q^{(1)a} R_{mla}^j \right) \partial_j$$

$$\begin{aligned}
 &= \left((K_1)_l^m \frac{\partial(K_1)_m^j}{\partial q^{(0)i}} - \frac{\partial(K_2)_l^j}{\partial q^{(0)i}} + \frac{1}{2} (K_1)_i^m q^{(1)a} R_{mli}^j + \frac{\partial(K_3)_i^j}{\partial q^{(1)l}} \right. \\
 &\quad \left. - (K_1)_l^m \frac{\partial(K_3)_i^j}{\partial q^{(2)m}} + ((K_1)_r^m (K_1)_l^r - (K_2)_l^m) \frac{\partial(K_1)_i^j}{\partial q^{(1)m}} \right) \partial_j.
 \end{aligned}$$

By direct calculation, we get

$$\begin{aligned}
 (K_1)_l^m \frac{\partial(K_1)_m^j}{\partial q^{(0)i}} &= q^{(1)a} q^{(1)b} \left(dq^m (\nabla_{\partial_a} \partial_l) dq^j (\nabla_{\partial_i} \nabla_{\partial_b} \partial_m) - dq^m (\nabla_{\partial_a} \partial_l) dq^r (\nabla_{\partial_b} \partial_m) dq^j (\nabla_{\partial_i} \partial_r) \right), \\
 \frac{\partial(K_2)_l^j}{\partial q^{(0)i}} &= q^{(2)a} \left[dq^j (\nabla_{\partial_i} \nabla_{\partial_a} \partial_l) - dq^m (\nabla_{\partial_a} \partial_l) dq^j (\nabla_{\partial_i} \partial_m) \right] \\
 &\quad + \frac{1}{2} q^{(1)a} q^{(1)b} \left[dq^j (\nabla_{\partial_i} \nabla_{\partial_a} \nabla_{\partial_b} \partial_l) - dq^m (\nabla_{\partial_a} \nabla_{\partial_b} \partial_l) dq^j (\nabla_{\partial_i} \partial_m) \right], \\
 \frac{\partial(K_3)_i^j}{\partial q^{(1)l}} &= q^{(2)a} \left[dq^j (\nabla_{\partial_l} \nabla_{\partial_a} \partial_i) + \frac{1}{3} R_{ali}^j \right] + \frac{1}{6} q^{(1)a} q^{(1)b} dq^j (\nabla_{\partial_a} \nabla_{\partial_l} \nabla_{\partial_b} \partial_i + \nabla_{\partial_b} \nabla_{\partial_a} \nabla_{\partial_l} \partial_i + \nabla_{\partial_l} \nabla_{\partial_a} \nabla_{\partial_b} \partial_i) \\
 &= q^{(2)a} \left[dq^j (\nabla_{\partial_l} \nabla_{\partial_a} \partial_i) + \frac{1}{3} R_{ali}^j \right] + \frac{1}{2} q^{(1)a} q^{(1)b} dq^j (\nabla_{\partial_a} \nabla_{\partial_l} \nabla_{\partial_b} \partial_i) \\
 &\quad + \frac{1}{6} q^{(1)a} q^{(1)b} dq^j (\nabla_{\partial_a} (R(\partial_b, \partial_l) \partial_i) + R(\partial_l, \partial_a) \nabla_{\partial_b} \partial_i), \\
 (K_1)_l^m \frac{\partial(K_3)_i^j}{\partial q^{(2)m}} &= q^{(1)a} q^{(1)b} \left(dq^m (\nabla_{\partial_a} \partial_l) dq^j (\nabla_{\partial_b} \nabla_{\partial_m} \partial_i) + \frac{1}{3} dq^m (\nabla_{\partial_a} \partial_l) R_{mbi}^j \right), \\
 (K_1)_r^m (K_1)_l^r - (K_2)_l^m &= -q^{(2)a} dq^m (\nabla_{\partial_a} \partial_l) + q^{(1)a} q^{(1)b} \left(dq^m (\nabla_{\partial_a} \partial_r) dq^r (\nabla_{\partial_b} \partial_l) - \frac{1}{2} dq^m (\nabla_{\partial_a} \nabla_{\partial_b} \partial_l) \right).
 \end{aligned}$$

Collecting all terms with leading coefficients $q^{(2)a}$, we obtain

$$q^{(2)a} \left[dq^j (\nabla_{\partial_l} \nabla_{\partial_a} \partial_i) - dq^j (\nabla_{\partial_i} \nabla_{\partial_a} \partial_l) + \frac{1}{3} R_{ali}^j \right] \partial_j = q^{(2)a} \left[R_{lia}^j + \frac{1}{3} R_{ali}^j \right] \partial_j.$$

Moreover, we note that $q^{(2)m} = \frac{1}{2} \left(F_1^m - q^{(1)a} q^{(1)b} dq^m (\nabla_{\partial_a} \partial_b) \right)$, so that

$$q^{(2)a} \left[R_{lia}^j + \frac{1}{3} R_{ali}^j \right] \partial_j = \frac{1}{2} F_1^m \left[R_{lim}^j + \frac{1}{3} R_{mli}^j \right] \partial_j - \frac{1}{2} q^{(1)a} q^{(1)b} dq^m (\nabla_{\partial_a} \partial_b) \left[R_{lim}^j + \frac{1}{3} R_{mli}^j \right] \partial_j.$$

Collecting all the terms with leading coefficients $q^{(1)a} q^{(1)b}$ now, we obtain

$$\begin{aligned}
 &q^{(1)a} q^{(1)b} \left[dq^m (\nabla_{\partial_a} \partial_l) dq^j (\nabla_{\partial_i} \nabla_{\partial_b} \partial_m - \nabla_{\partial_b} \nabla_{\partial_m} \partial_i) - \frac{1}{2} dq^j (\nabla_{\partial_i} \nabla_{\partial_a} \nabla_{\partial_b} \partial_l - \nabla_{\partial_a} \nabla_{\partial_l} \nabla_{\partial_b} \partial_i) \right. \\
 &\quad + \frac{1}{2} dq^m (\nabla_{\partial_b} \partial_l) R_{mla}^j + \frac{1}{6} dq^m (\nabla_{\partial_b} \partial_l) R_{lam}^j - \frac{1}{3} dq^m (\nabla_{\partial_b} \partial_l) R_{mai}^j + \frac{1}{6} dq^j (\nabla_{\partial_a} (R(\partial_b, \partial_l) \partial_i)) \\
 &\quad \left. - \frac{1}{2} dq^m (\nabla_{\partial_a} \partial_b) R_{lim}^j - \frac{1}{6} dq^m (\nabla_{\partial_a} \partial_b) R_{mli}^j \right] \partial_j \\
 &= q^{(1)a} q^{(1)b} \left[dq^m (\nabla_{\partial_b} \partial_l) R_{iam}^j + \frac{1}{2} R_{a;lib}^j + \frac{1}{2} dq^m (\nabla_{\partial_b} \partial_l) R_{mia}^j + \frac{1}{2} dq^m (\nabla_{\partial_b} \partial_l) R_{lma}^j + \frac{1}{2} dq^m (\nabla_{\partial_a} \partial_b) R_{lim}^j \right. \\
 &\quad - dq^m (\nabla_{\partial_b} \partial_l) R_{iam}^j + \frac{1}{2} dq^m (\nabla_{\partial_b} \partial_l) R_{mla}^j + \frac{1}{6} dq^m (\nabla_{\partial_b} \partial_l) R_{lam}^j - \frac{1}{3} dq^m (\nabla_{\partial_b} \partial_l) R_{mai}^j \\
 &\quad - \frac{1}{2} dq^m (\nabla_{\partial_a} \partial_b) R_{lim}^j - \frac{1}{6} dq^m (\nabla_{\partial_a} \partial_b) R_{mli}^j + \frac{1}{6} R_{a;bli} + \frac{1}{6} dq^m (\nabla_{\partial_a} \partial_b) R_{mli}^j \\
 &\quad \left. + \frac{1}{6} dq^m (\nabla_{\partial_b} \partial_l) R_{ami}^j + \frac{1}{6} dq^m (\nabla_{\partial_b} \partial_l) R_{alm}^j \right] \partial_j.
 \end{aligned}$$

Grouping terms with common leading coefficients, we see that all terms with leading coefficients $dq^m (\nabla_{\partial_b} \partial_l)$, $dq^m (\nabla_{\partial_a} \partial_b)$, and $dq^m (\nabla_{\partial_b} \partial_l)$ cancel due to the first Bianchi identity. Hence,

$$K_3^{(3)} \circ \left[\frac{\delta}{\delta q^{(0)i}}, \frac{\delta}{\delta q^{(1)l}} \right] = \left(\frac{1}{2} F_1^m \left[R_{lim}^j + \frac{1}{3} R_{mli}^j \right] + \frac{1}{2} F_0^a F_0^b \left[R_{a;lib}^j + \frac{1}{3} R_{a;bli}^j \right] \right) \partial_j,$$

from which the result follows. $\square$

Lemma C.4. *Let $X, Y \in \mathfrak{X}(M)$. Then*

$$K_3([X^{(3)h_0}, Y^{(3)h_0}]_u) = -\frac{1}{3!}R(X_p, Y_p)u^{(3)} - \frac{1}{3}(\nabla_{u^{(1)}}R)(X_p, Y_p)u^{(2)} + \frac{1}{3}(\nabla_{u^{(2)}}R)(X_p, Y_p)u^{(1)} - \frac{1}{2}(\nabla_{u^{(1)}}^2R)(X_p, Y_p)u^{(1)},$$

where $u^{(1)} = F_0^{(3)}(u)$, $u^{(2)} = F_1^{(3)}(u)$, $u^{(3)} = F_2^{(3)}(u)$, $u \in T_p^{(3)}M$, $p \in M$.

PROOF. First, observe that

$$\left[\frac{\delta}{\delta q^{(0)i}}, \frac{\delta}{\delta q^{(0)j}} \right] = F_0^m R_{ijm}^l \frac{\delta}{\delta q^{(1)l}} - \left(\frac{1}{2} F_1^m R_{ijm}^l + F_0^a F_0^b R_{a;ijb}^l \right) \frac{\delta}{\delta q^{(2)l}} + S_{(0)i(0)j}^{(3)l} \frac{\partial}{\partial q^{(3)l}}, \quad (47)$$

where $S_{(0)i(0)j}^{(3)l} \in C^\infty(T^{(3)}M)$. Then

$$\left[\frac{\partial}{\partial q^{(3)m}}, \left[\frac{\delta}{\delta q^{(0)i}}, \frac{\delta}{\delta q^{(0)j}} \right] \right] = \frac{\partial S_{(0)i(0)j}^{(3)l}}{\partial q^{(3)m}} \frac{\partial}{\partial q^{(3)l}},$$

On the other hand, from the Jacobi identity,

$$\begin{aligned} \left[\frac{\partial}{\partial q^{(3)m}}, \left[\frac{\delta}{\delta q^{(0)i}}, \frac{\delta}{\delta q^{(0)j}} \right] \right] &= \left[\frac{\delta}{\delta q^{(0)j}}, \left[\frac{\delta}{\delta q^{(0)i}}, \frac{\partial}{\partial q^{(3)m}} \right] \right] - \left[\frac{\delta}{\delta q^{(0)i}}, \left[\frac{\delta}{\delta q^{(0)j}}, \frac{\partial}{\partial q^{(3)m}} \right] \right] \\ &= \left[\frac{\delta}{\delta q^{(0)j}}, (\nabla_{\partial_i} \partial_m)^{(3)} \right] - \left[\frac{\delta}{\delta q^{(0)i}}, (\nabla_{\partial_j} \partial_m)^{(3)} \right] \\ &= (\nabla_{\partial_j} \nabla_{\partial_i} \partial_m - \nabla_{\partial_i} \nabla_{\partial_j} \partial_m)^{(3)} \\ &= -R_{ijm}^l \frac{\partial}{\partial q^{(3)l}}, \end{aligned}$$

so that $\frac{\partial S_{(0)i(0)j}^{(3)l}}{\partial q^{(3)m}} = -R_{ijm}^l$. It follows that

$$S_{(0)i(0)j}^{(3)l} = -\frac{1}{3!}R_{ijm}^l F_2^m + \Phi_{ij}^l, \quad (48)$$

for some functions $\Phi_{ij}^l \in C^\infty(T^{(3)}M)$ satisfying $\frac{\partial \Phi_{ij}^l}{\partial q^{(3)\lambda}} = 0$, for all $\lambda = 1, \dots, n$.

Similarly,

$$\begin{aligned} \left[\frac{\delta}{\delta q^{(2)m}}, \left[\frac{\delta}{\delta q^{(0)i}}, \frac{\delta}{\delta q^{(0)j}} \right] \right]_u &= \left[\frac{\delta}{\delta q^{(0)j}}, \left[\frac{\delta}{\delta q^{(0)i}}, \frac{\delta}{\delta q^{(2)m}} \right] \right]_u - \left[\frac{\delta}{\delta q^{(0)i}}, \left[\frac{\delta}{\delta q^{(0)j}}, \frac{\delta}{\delta q^{(2)m}} \right] \right]_u \\ &= \left[\frac{\delta}{\delta q^{(0)j}}, (\nabla_{\partial_i} \partial_m)^{(3)} \right]_u - \left[\frac{\delta}{\delta q^{(0)i}}, (\nabla_{\partial_j} \partial_m)^{(3)} \right]_u \\ &\quad - \left[\frac{\delta}{\delta q^{(0)i}}, (\nabla_{\partial_j} \partial_m)^{(3)} \right]_u - \left[\frac{\delta}{\delta q^{(0)j}}, (\nabla_{\partial_i} \partial_m)^{(3)} \right]_u \\ &= (R(\partial_j, \partial_i) \partial_m)^{(3)}_u - (R(\partial_j, u^{(1)}) \nabla_{\partial_i} \partial_m \\ &\quad + \frac{1}{3} R(u^{(1)}, \nabla_{\partial_i} \partial_m) \partial_j - R(\partial_i, u^{(1)}) \nabla_{\partial_j} \partial_m - \frac{1}{3} R(u^{(1)}, \nabla_{\partial_j} \partial_m) \partial_i)^{(3)}_u \\ &= \left[\frac{\delta}{\delta q^{(0)j}}, (\nabla_{\partial_i} \partial_m)^{(3)} \right]_u - \left[\frac{\delta}{\delta q^{(0)i}}, (\nabla_{\partial_j} \partial_m)^{(3)} \right]_u \end{aligned}$$

$$- \left[\frac{\delta}{\delta q^{(0)j}}, (R(\partial_i, u^{(1)})\partial_m + \frac{1}{3}R(u^{(1)}, \partial_m)\partial_i)^{(3)} \right]_u + \left[\frac{\delta}{\delta q^{(0)i}}, (R(\partial_j, u^{(1)})\partial_m + \frac{1}{3}R(u^{(1)}, \partial_m)\partial_j)^{(3)} \right]_u$$

Then, we calculate the first two Lie brackets.

$$\begin{aligned} \left[\frac{\delta}{\delta q^{(0)j}}, (\nabla_{\partial_i} \partial_m)^{(3)} \right]_u - \left[\frac{\delta}{\delta q^{(0)i}}, (\nabla_{\partial_j} \partial_m)^{(3)} \right]_u &= (R(\partial_j, \partial_i)\partial_m)^{(3)}_u - \left(R(\partial_j, u^{(1)})\nabla_{\partial_i} \partial_m + \frac{1}{3}R(u^{(1)}, \nabla_{\partial_i} \partial_m)\partial_j \right. \\ &\quad \left. - R(\partial_i, u^{(1)})\nabla_{\partial_j} \partial_m - \frac{1}{3}R(u^{(1)}, \nabla_{\partial_j} \partial_m)\partial_i \right)^{(3)}_u \end{aligned}$$

Note that we cannot directly use Lemma 11 to calculate the two final Lie brackets, as also happened in the proof of Lemma 15. Rather, we must use the following similar identity.

$$\left[X^{(3)}_{h_0}, (R(Y, u^{(1)})Z)^{(3)}_{v_3} \right]_u = ((\nabla_X R)(Y, u^{(1)})Z + R(\nabla_X Y, u^{(1)})Z + R(Y, u^{(1)})\nabla_X Z)^{(3)}_{v_3}, X, Y, Z \in \mathfrak{X}(M) \quad (49)$$

By making use of the symmetries of the curvature tensor, we obtain similar results when $u^{(1)}$ is in any of the other arguments of R . Then, we apply these results to calculate the two final Lie brackets. Combining the four Lie brackets, we obtain

$$^{(3)}_{v_3} \left(\left[\frac{\delta}{\delta q^{(2)m}}, \left[\frac{\delta}{\delta q^{(0)i}}, \frac{\delta}{\delta q^{(0)j}} \right] \right]_u \right) = \left((\nabla_{\partial_i} R)(\partial_j, u^{(1)})\partial_m - (\nabla_{\partial_j} R)(\partial_i, u^{(1)})\partial_m + \frac{1}{3}(\nabla_{\partial_i} R)(u^{(1)}, \partial_m)\partial_j - \frac{1}{3}(\nabla_{\partial_j} R)(u^{(1)}, \partial_m)\partial_i \right)^{(3)}_{v_3}.$$

On the other hand, from Equation (47), we also have

$$\begin{aligned} K_3^{(3)} \circ \left[\frac{\delta}{\delta q^{(2)m}}, \left[\frac{\delta}{\delta q^{(0)i}}, \frac{\delta}{\delta q^{(0)j}} \right] \right] &= \frac{\delta S_{ij}^{(3)l}}{\delta q^{(2)m}} \frac{\partial}{\partial q^{(3)l}} \\ &= \left(-\frac{1}{3!} R^l_{ija} \frac{\delta F_2^a}{\delta q^{(2)m}} + \frac{\partial \Phi_{ij}^{(3)l}}{\partial q^{(2)m}} \right) \frac{\partial}{\partial q^{(3)l}} \\ &= \frac{\partial \Phi_{ij}^{(3)l}}{\partial q^{(2)m}} \frac{\partial}{\partial q^{(3)l}}. \end{aligned}$$

Then, it follows that

$$\Phi_{ij}^l = \frac{1}{2} F_0^a F_1^m \left(R^l_{i;jam} - R^l_{j;iam} + \frac{1}{3} R^l_{i;amj} - \frac{1}{3} R^l_{j;ami} \right) + \Psi_{ij}^l, \quad (50)$$

for some functions $\Psi_{ij}^l \in C^\infty(T^{(3)}M)$ satisfying $\frac{\partial \Psi_{ij}^l}{\partial q^{(2)\lambda}} = \frac{\partial \Psi_{ij}^l}{\partial q^{(3)\lambda}} = 0$, for all $\lambda = 1, \dots, n$. For the sake of simplicity, in what follows, we will use the notation $Q^l_{ijam} = R^l_{i;jam} - R^l_{j;iam} + \frac{1}{3} R^l_{i;amj} - \frac{1}{3} R^l_{j;ami}$. Hence,

$$S_{(0)i(0)j}^{(3)l} = -\frac{1}{3!} R^l_{ijm} F_2^m + \frac{1}{2} F_0^a F_1^m Q^l_{ijam} + \Psi_{ij}^l. \quad (51)$$

Finally,

$$\begin{aligned} ^{(3)}_{v_3} \circ \left[\frac{\delta}{\delta q^{(1)m}}, \left[\frac{\delta}{\delta q^{(0)i}}, \frac{\delta}{\delta q^{(0)j}} \right] \right] &= \frac{\delta S_{ij}^{(3)l}}{\delta q^{(1)m}} \frac{\partial}{\partial q^{(3)l}} - F_0^b R^k_{ijb} \left[\frac{\delta}{\delta q^{(1)m}}, \frac{\delta}{\delta q^{(1)k}} \right] \\ &= \left[R^l_{ijk} \left((K_2)_m^k - \frac{1}{3!} \frac{\partial F_2^k}{\partial q^{(1)m}} \right) + \frac{1}{2} F_1^k Q^l_{ijmk} + \frac{\partial \Psi_{ij}^l}{\partial q^{(1)m}} + \frac{1}{2} F_0^a F_0^b R^l_{mka} R^k_{ijb} \right] \frac{\partial}{\partial q^{(3)l}}. \end{aligned}$$

Moreover, since

$$(K_2)_m^k - \frac{1}{3!} \frac{\partial F_2^k}{\partial q^{(1)m}} = \frac{1}{3!} F_0^a F_0^b R^k_{amb},$$

we conclude that

$$\stackrel{(3)}{\nu_3} \circ \left[\frac{\delta}{\delta q^{(1)m}}, \left[\frac{\delta}{\delta q^{(0)i}}, \frac{\delta}{\delta q^{(0)j}} \right] \right] = \left(\frac{1}{2} F_1^k Q_{ijmk}^l + \frac{1}{6} F_0^a F_0^b R_{amb}^k + \frac{1}{2} F_0^a F_0^b R_{mka}^l R_{ijb}^k + \frac{\partial \Psi_{ij}^l}{\partial q^{(1)m}} \right) \frac{\partial}{\partial q^{(3)l}}.$$

Hence, using the notation

$$C = \stackrel{(3)}{K_3} \circ \left[\frac{\delta}{\delta q^{(1)m}}, \left[\frac{\delta}{\delta q^{(0)i}}, \frac{\delta}{\delta q^{(0)j}} \right] \right]$$

with $C = C_{mij}^l \partial_l$, we have

$$\frac{\partial \Psi_{ij}^l}{\partial q^{(1)m}} = -\frac{1}{2} F_1^k Q_{ijmk}^l - \frac{1}{6} F_0^a F_0^b R_{amb}^k - \frac{1}{2} F_0^a F_0^b R_{mka}^l R_{ijb}^k + C_{mij}^l,$$

from which it follows that

$$\Psi_{ij}^l = -\frac{1}{2} F_0^m F_1^k Q_{ijmk}^l - \frac{1}{6} F_0^M F_0^a F_0^b R_{amb}^k - \frac{1}{2} F_0^M F_0^a F_0^b R_{mka}^l R_{ijb}^k + F_0^m C_{mij}^l + h_{ij}^l, \quad (52)$$

for some functions $h_{ij}^l \in C^\infty(T^{(3)}M)$ satisfying $\frac{\partial h_{ij}^l}{\partial q^{(1)m}} = \frac{\partial h_{ij}^l}{\partial q^{(2)m}} = \frac{\partial h_{ij}^l}{\partial q^{(3)m}} = 0$, for all $m = 1, \dots, n$.

In summary, from Equations (51)-(52), we have

$$S_{(0)i(0)j}^{(3)l} = -\frac{1}{3!} R_{ijm}^l F_2^m + \frac{1}{2} F_0^a F_1^m Q_{ijam}^l - \frac{1}{2} F_0^m F_1^k Q_{ijmk}^l - \frac{1}{6} F_0^M F_0^a F_0^b R_{amb}^k - \frac{1}{2} F_0^M F_0^a F_0^b R_{mka}^l R_{ijb}^k + F_0^m C_{mij}^l + h_{ij}^l,$$

where C_{mij}^l are the coefficient functions of C .

Finally, through Lemma C.1, we know that $S_{(0)i(0)j}^{(3)l}$ is weakly fiberwise linear, which implies that

$$S_{(0)i(0)j}^{(3)l} = -\frac{1}{3!} R_{ijm}^l F_2^m + \frac{1}{2} F_0^a F_1^m Q_{ijam}^l - \frac{1}{2} F_0^m F_1^k Q_{ijmk}^l - \frac{1}{6} F_0^M F_0^a F_0^b R_{amb}^k - \frac{1}{2} F_0^M F_0^a F_0^b R_{mka}^l R_{ijb}^k + F_0^m C_{mij}^l. \quad (53)$$

On the other hand, to obtain the functions C_{mij}^l , we apply once again the Jacobi identity. Thus, we get

$$\left[\frac{\delta}{\delta q^{(1)m}}, \left[\frac{\delta}{\delta q^{(0)i}}, \frac{\delta}{\delta q^{(0)j}} \right] \right] = \left[\frac{\delta}{\delta q^{(0)j}}, \left[\frac{\delta}{\delta q^{(0)i}}, \frac{\delta}{\delta q^{(1)m}} \right] \right] - \left[\frac{\delta}{\delta q^{(0)i}}, \left[\frac{\delta}{\delta q^{(0)j}}, \frac{\delta}{\delta q^{(1)m}} \right] \right]. \quad (54)$$

Now, we calculate

$$\begin{aligned} \left[\frac{\delta}{\delta q^{(0)j}}, \left[\frac{\delta}{\delta q^{(0)i}}, \frac{\delta}{\delta q^{(1)m}} \right] \right] &= \left[\frac{\delta}{\delta q^{(0)j}}, (\nabla_{\partial_i} \partial_m)^{(3)} \right]^{h_1} + \frac{1}{2} \left[\frac{\delta}{\delta q^{(0)j}}, (R(u^{(1)}, \partial_i) \partial_m - R(\partial_i, \partial_m) u^{(1)})^{(3)} \right]^{h_2} \\ &\quad - \frac{1}{2} \left[\frac{\delta}{\delta q^{(0)j}}, \left(R(\partial_i, \partial_m)^{(2)} + \frac{1}{3} R(\partial_m, u^{(2)}) \partial_i + (\nabla_{u^{(1)}} R)(\partial_i, \partial_m) u^{(1)} + \frac{1}{3} (\nabla_{u^{(1)}} R)(\partial_m, u^{(1)}) \partial_i \right)^{(3)} \right]^{v_3}. \end{aligned}$$

The first Lie bracket simplifies as follows.

$$\begin{aligned} \left[\frac{\delta}{\delta q^{(0)j}}, (\nabla_{\partial_i} \partial_m)^{(3)} \right]^{h_1} &= (\nabla_{\partial_j} \nabla_{\partial_i} \partial_m)^{(3)} + \frac{1}{2} (R(u^{(1)}, \partial_j) \nabla_{\partial_i} \partial_m - R(\partial_j, \nabla_{\partial_i} \partial_m) u^{(1)})^{(3)} \\ &\quad - \frac{1}{2} \left(R(\partial_j, \nabla_{\partial_i} \partial_m) u^{(2)} + \frac{1}{3} R(\nabla_{\partial_i} \partial_m, u^{(2)}) \partial_j + (\nabla_{u^{(1)}} R)(\partial_j, \nabla_{\partial_i} \partial_m) u^{(1)} + \frac{1}{3} (\nabla_{u^{(1)}} R)(\nabla_{\partial_i} \partial_m, u^{(1)}) \partial_j \right)^{(3)}_{v_3} \end{aligned}$$

With regard to the second Lie bracket, we have

$$\left[\frac{\delta}{\delta q^{(0)j}}, (R(u^{(1)}, \partial_i) \partial_m - R(\partial_i, \partial_m) u^{(1)})^{(3)} \right]^{h_2} =$$

$$\begin{aligned} & \left[(\nabla_{\partial_j} R)(u^{(1)}, \partial_i) \partial_m + R(u^{(1)}, \nabla_{\partial_j} \partial_i) \partial_m + R(u^{(1)}, \partial_i) \nabla_{\partial_j} \partial_m - (\nabla_{\partial_j} R)(\partial_i, \partial_m) u^{(1)} - R(\nabla_{\partial_j} \partial_i, \partial_m) u^{(1)} - R(\partial_i, \nabla_{\partial_j} \partial_m) u^{(1)} \right]^{(3)}_{h_2} \\ & - \left[R(\partial_j, u^{(1)}) R(u^{(1)}, \partial_i) \partial_m - R(\partial_j, u^{(1)}) R(\partial_i, \partial_m) u^{(1)} + \frac{1}{3} R(u^{(1)}, R(u^{(1)}, \partial_i) \partial_m) \partial_j - \frac{1}{3} R(u^{(1)}, R(\partial_i, \partial_m) u^{(1)}) \partial_j \right]^{(3)}_{v_3}. \end{aligned}$$

Regarding the third Lie bracket, we first need to see how terms involving $u^{(3)}$ transform within the Lie bracket. For all $X, Y, Z \in \mathfrak{X}(M)$, we have

$$\begin{aligned} \left[X^{(3)}_{h_0}, (R(Y, Z)u^{(2)})^{(3)}_{v_3} \right] &= \left[X^{(3)}_{h_0}, F_1^\lambda (R(Y, Z) \partial_\lambda)^{(3)}_{v_3} \right] \\ &= X^{(3)}_{h_0} (F_1^\lambda) (R(Y, Z) \partial_\lambda)^{(3)}_{v_3} + F_1^\lambda \left[X^{(3)}_{h_0}, (R(Y, Z) \partial_\lambda)^{(3)}_{v_3} \right] \\ &= X^i \frac{\delta F_1^\lambda}{\delta q^{(0)i}} (R(Y, Z) \partial_\lambda)^{(3)}_{v_3} + F_1^\lambda \left[X^{(3)}_{h_0}, (R(Y, Z) \partial_\lambda)^{(3)}_{v_3} \right]. \end{aligned}$$

Then, we use Equations (31)-(32) to obtain

$$\frac{\delta F_1^\lambda}{\delta q^{(0)i}} = \frac{\partial F_1^\lambda}{\partial q^{(0)i}} - 2(K_2)_i^\lambda,$$

which we simplify due to

$$\frac{\partial F_1^\lambda}{\partial q^{(0)i}} = 2(K_2)_i^\lambda - F_1^\lambda \Gamma_{ai}^\lambda + q^{(1)a} q^{(1)b} R_{iab}^\lambda.$$

Thus, we apply the result to the first term and use Lemma 11 to calculate the second term in order to have

$$\left[X^{(3)}_{h_0}, (R(Y, Z)u^{(2)})^{(3)}_{v_3} \right] = (-F_1^\lambda \Gamma_{ai}^\lambda + q^{(1)a} q^{(1)b} R_{iab}^\lambda) X^i (R(Y, Z) \partial_\lambda)^{(3)}_{v_3} + F_1^\lambda (\nabla_X (R(Y, Z) \partial_\lambda))^{(3)}_{v_3}.$$

Hence,

$$\left[X^{(3)}_{h_0}, (R(Y, Z)u^{(2)})^{(3)}_{v_3} \right] = (R(Y, Z)R(X, u^{(1)})u^{(1)} + (\nabla_X R)(Y, Z)u^{(2)} + R(\nabla_X Y, Z)u^{(2)} + R(Y, \nabla_X Z)u^{(2)})^{(3)}_{v_3} \quad (55)$$

Similarly,

$$\left[X^{(3)}_{h_0}, (R(Y, u^{(2)})Z)^{(3)}_{v_3} \right] = (R(Y, R(X, u^{(1)})u^{(1)})Z + (\nabla_X R)(Y, u^{(2)})Z + R(\nabla_X Y, u^{(2)})Z + R(Y, u^{(2)})\nabla_X Z)^{(3)}_{v_3} \quad (56)$$

Next,

$$\begin{aligned} [X^{(3)}_{h_0}, (\nabla_{u^{(1)}} R)(Y, Z)u^{(1)})^{(3)}_{v_3}] &= [X^{(3)}_{h_0}, F_0^a F_0^b ((\nabla_{\partial_a} R)(Y, Z) \partial_b)^{(3)}_{v_3}] \\ &= X^{(3)}_{h_0} (F_0^a F_0^b) ((\nabla_{\partial_a} R)(Y, Z) \partial_b)^{(3)}_{v_3} + q^{(1)a} q^{(1)b} [X^{(3)}_{h_0}, ((\nabla_{\partial_a} R)(Y, Z) \partial_b)^{(3)}_{v_3}] \\ &= [-q^{(1)a} (\nabla_{\nabla_X \partial_a} R)(Y, Z) - q^{(1)b} (\nabla_{\nabla_X} R)(Y, Z) \nabla_X \partial_b + q^{(1)a} q^{(1)b} \nabla_X ((\nabla_{\partial_a} R)(Y, Z) \partial_b)]^{(3)}_{v_3} \end{aligned}$$

and we obtain

$$[X^{(3)}_{h_0}, (\nabla_{u^{(1)}} R)(Y, Z)u^{(1)})^{(3)}_{v_3}] = [(\nabla^2 R)(X, u^{(1)}; Y, Z)u^{(1)} + (\nabla_{u^{(1)}} R)(\nabla_X Y, Z)u^{(1)} + (\nabla_{u^{(1)}} R)(Y, \nabla_X Z)u^{(1)}]^{(3)}_{v_3}. \quad (57)$$

Now, we apply the identities (55)-(56)-(57) to simplify the third Lie bracket and use the identities given in Lemma B.2.

$$\left[\frac{\delta}{\delta q^{(0)j}}, \left(R(\partial_i, \partial_m) u^{(2)} + \frac{1}{3} R(\partial_m, u^{(2)}) \partial_i + (\nabla_{u^{(1)}} R)(\partial_i, \partial_m) u^{(1)} + \frac{1}{3} (\nabla_{u^{(1)}} R)(\partial_m, u^{(1)}) \partial_i \right)^{(3)}_{v_3} \right] =$$

$$\begin{aligned}
 & \left[R(\partial_i, \partial_m) R(\partial_j, u^{(1)}) u^{(1)} + (\nabla_{\partial_j} R)(\partial_i, \partial_m) u^{(2)} + R(\nabla_{\partial_j} \partial_i, \partial_m) u^{(2)} + R(\partial_i, \nabla_{\partial_j} \partial_m) u^{(2)} \right. \\
 & + \frac{1}{3} R(\partial_m, R(\partial_j, u^{(1)}) u^{(1)}) \partial_i + \frac{1}{3} (\nabla_{\partial_j} R)(\partial_m, u^{(2)}) \partial_i + \frac{1}{3} R(\nabla_{\partial_j} \partial_m, u^{(2)}) \partial_i + \frac{1}{3} R(\partial_i, u^{(2)}) \nabla_{\partial_j} \partial_i \\
 & + (\nabla_{u^{(1)}} R)(\partial_i, \nabla_{\partial_j} \partial_m) u^{(1)} + (\nabla_{u^{(1)}} R)(\nabla_{\partial_j} \partial_i, \partial_m) u^{(1)} + \frac{1}{3} (\nabla_{u^{(1)}} R)(\nabla_{\partial_j} \partial_m, u^{(1)}) \partial_i \\
 & \left. + \frac{1}{3} (\nabla_{u^{(1)}} R)(\partial_m, u^{(1)}) \nabla_{\partial_j} \partial_i + (\nabla^2 R)(\partial_j, u^{(1)}; \partial_i, \partial_m) u^{(1)} + \frac{1}{3} (\nabla^2 R)(\partial_j, u^{(1)}; \partial_m, u^{(1)}) \partial_i \right] \Bigg|_{\mathcal{V}_3}^{(3)}
 \end{aligned}$$

Combining the three Lie brackets to simplify the two Lie brackets in Equation (54), we obtain

$$\begin{aligned}
 C(u) = \frac{1}{2} & \left[-R(\partial_j, u^{(1)}) R(u^{(1)}, \partial_i) \partial_m + R(\partial_j, V) R(\partial_i, \partial_m) u^{(1)} - \frac{1}{3} R(u^{(1)}, R(u^{(1)}, \partial_i) \partial_m) \partial_j + \frac{1}{3} R(u^{(1)}, R(\partial_i, \partial_m) u^{(1)}) \partial_j \right. \\
 & - R(\partial_i, \partial_m) R(\partial_j, u^{(1)}) u^{(1)} - (\nabla_{\partial_j} R)(\partial_i, \partial_m) u^{(2)} - \frac{1}{3} R(\partial_m, R(\partial_j, u^{(1)}) u^{(1)}) \partial_i - \frac{1}{3} (\nabla_{\partial_j} R)(\partial_m, u^{(2)}) \partial_i \\
 & + R(\partial_i, u^{(1)}) R(u^{(1)}, \partial_j) \partial_m - R(\partial_i, u^{(1)}) R(\partial_j, \partial_m) u^{(1)} + \frac{1}{3} R(u^{(1)}, R(u^{(1)}, \partial_j) \partial_m) \partial_i - \frac{1}{3} R(u^{(1)}, R(\partial_j, \partial_m) u^{(1)}) \partial_i \\
 & + R(\partial_j, \partial_m) R(\partial_i, u^{(1)}) u^{(1)} + (\nabla_{\partial_j} R)(\partial_i, \partial_m) u^{(2)} + \frac{1}{3} R(\partial_m, R(\partial_i, u^{(1)}) u^{(1)}) \partial_j + \frac{1}{3} (\nabla_{\partial_i} R)(\partial_m, u^{(2)}) \partial_j \\
 & \left. - (\nabla^2 R)(\partial_j; u^{(1)}, \partial_i, \partial_m) u^{(1)} - \frac{1}{3} (\nabla^2 R)(\partial_j, u^{(1)}; \partial_m, u^{(1)}) \partial_i + (\nabla^2 R)(\partial_i, u^{(1)}; \partial_j, \partial_m) u^{(1)} + \frac{1}{3} (\nabla^2 R)(\partial_i, u^{(1)}; \partial_m, u^{(1)}) \partial_j \right]_p
 \end{aligned}$$

Now, if we conjugate Equation (53) with the previous equation and take into account Lemma 4, we conclude that

$$\begin{aligned}
 K_3 \circ [X^{\binom{3}{h_0}}, Y^{\binom{3}{h_0}}] &= \frac{1}{2} \left[-\frac{1}{3} R(X, Y) u^{(3)} + 3R(Y, u^{(1)}) R(X, u^{(1)}) u^{(1)} - 3R(X, u^{(1)}) R(Y, u^{(1)}) u^{(1)} + R(u^{(1)}, R(X, u^{(1)}) u^{(1)}) Y \right. \\
 & - R(u^{(1)}, R(Y, u^{(1)}) u^{(1)}) X - R(u^{(1)}, R(X, Y) u^{(1)}) u^{(1)} + (\nabla_X R)(Y, u^{(1)}) u^{(2)} - (\nabla_Y R)(X, u^{(1)}) u^{(2)} \\
 & \left. + \frac{1}{3} (\nabla_X R)(u^{(1)}, u^{(2)}) Y - \frac{1}{3} (\nabla_Y R)(u^{(1)}, u^{(2)}) X + (\nabla^2 R)(X, u^{(1)}; Y, u^{(1)}) u^{(1)} - (\nabla^2 R)(Y, u^{(1)}; X, u^{(1)}) u^{(1)} \right].
 \end{aligned}$$

We will now make use of the identities given in Lemma B.1 to group as many terms together as possible. Hence,

$$\begin{aligned}
 & 3R(Y, u^{(1)}) R(X, u^{(1)}) u^{(1)} - 3R(X, u^{(1)}) R(Y, u^{(1)}) u^{(1)} + R(u^{(1)}, R(X, u^{(1)}) u^{(1)}) Y - R(u^{(1)}, R(Y, u^{(1)}) u^{(1)}) X \\
 & - R(u^{(1)}, R(X, Y) u^{(1)}) u^{(1)} \\
 & = (R(Y, u^{(1)}) R)(X, u^{(1)}) u^{(1)} - (R(X, u^{(1)}) R)(Y, u^{(1)}) u^{(1)}.
 \end{aligned}$$

Then, using the identities given in Lemma B.3, we obtain

$$\begin{aligned}
 & (R(Y, u^{(1)}) R)(X, u^{(1)}) u^{(1)} - (R(X, u^{(1)}) R)(Y, u^{(1)}) u^{(1)} \\
 & = (\nabla^2 R)(Y, u^{(1)}; X, u^{(1)}) u^{(1)} - (\nabla^2 R)(X, u^{(1)}; Y, u^{(1)}) u^{(1)} - (\nabla^2 R)(u^{(1)}, u^{(1)}; X, Y) u^{(1)}.
 \end{aligned}$$

Therefore,

$$\begin{aligned}
 K_3 \circ [X^{\binom{3}{h_0}}, Y^{\binom{3}{h_0}}] &= -\frac{1}{3!} R(X, Y) u^{(3)} - \frac{1}{2} (\nabla^2 R)(u^{(1)}, u^{(1)}; X, Y) u^{(1)} \\
 & + \frac{1}{2} \left[(\nabla_X R)(Y, u^{(1)}) u^{(2)} - (\nabla_Y R)(X, u^{(1)}) u^{(2)} + \frac{1}{3} (\nabla_X R)(u^{(1)}, u^{(2)}) Y - \frac{1}{3} (\nabla_Y R)(u^{(1)}, u^{(2)}) X \right]
 \end{aligned}$$

Next, from the first and second Bianchi identities given in Lemma B.2, we get

$$\begin{aligned}
 & (\nabla_X R)(Y, u^{(1)}) u^{(2)} - (\nabla_Y R)(X, u^{(1)}) u^{(2)} + \frac{1}{3} (\nabla_X R)(u^{(1)}, u^{(2)}) Y - \frac{1}{3} (\nabla_Y R)(u^{(1)}, u^{(2)}) X \\
 & = -\frac{2}{3} (\nabla_{u^{(1)}} R)(X, Y) u^{(2)} - \frac{1}{3} (\nabla_{u^{(2)}} R)(X, Y) u^{(1)}
 \end{aligned}$$

Finally, using the definition of the second covariant derivative of R , we conclude that

$$K_3 \circ [X^{\binom{3}{h_0}}, Y^{\binom{3}{h_0}}] = -\frac{1}{3!} R(X, Y) u^{(3)} - \frac{1}{3} (\nabla_{u^{(1)}} R)(X, Y) u^{(2)} + \frac{1}{3} (\nabla_{u^{(2)}} R)(X, Y) u^{(1)} - \frac{1}{2} (\nabla^2_{u^{(1)}} R)(X, Y) u^{(1)}$$

□

PROOF (PROOF OF THEOREM 16). The identities involving $\overset{(3)}{v}_3$ follow from Lemma 11. The brackets involving $\overset{(3)}{h}_2$ are given by Lemma C.2; the brackets involving $\overset{(3)}{h}_1$ are given by Lemma C.3; and the $\overset{(3)}{h}_0$ - $\overset{(3)}{h}_0$ bracket is given by Lemma C.4. The lower components are inherited from the second-order bracket formulas through Lemma 6. $\square$

C.3. General reduction of k -vertical bracket components

This appendix gives the coordinate reduction used to obtain the general partial bracket formulas in Propositions 17 and 18.

Lemma C.5. *Consider the vector fields defined by*

$$\frac{\overset{(k-1)}{\delta}_{k-1}}{\delta q^{(\alpha)i}} = \frac{\overset{(k)}{\delta}}{\delta q^{(\alpha)i}} + (C_{k-\alpha})_i^j \frac{\overset{(k)}{\delta}}{\delta q^{(k)j}}, \quad \alpha = 0, 1, \dots, k-1,$$

where $(C_\alpha)_i^l$ are the dual coefficients given by (45). Then

$$\begin{aligned} K_k \circ \left[\frac{\overset{(k)}{\delta}}{\delta q^{(\alpha)i}}, \frac{\overset{(k)}{\delta}}{\delta q^{(\beta)l}} \right] &= \left(\frac{\overset{(k-1)}{\delta} (C_{k-\alpha})_i^j}{\delta q^{(\beta)l}} - \frac{\overset{(k-1)}{\delta} (C_{k-\beta})_l^j}{\delta q^{(\alpha)i}} + (C_{k-\alpha})_i^a \frac{\overset{(k)}{\delta} (C_{k-\beta})_l^j}{\delta q^{(k)a}} - (C_{k-\beta})_l^a \frac{\overset{(k)}{\delta} (C_{k-\alpha})_i^j}{\delta q^{(k)a}} \right) \partial_j \\ &+ K_k \circ \left[\frac{\overset{(k-1)}{\delta}_{k-1}}{\delta q^{(\alpha)i}}, \frac{\overset{(k-1)}{\delta}_{k-1}}{\delta q^{(\beta)l}} \right], \end{aligned}$$

for all $\alpha, \beta = 0, 1, \dots, k-1$.

PROOF. First of all, we observe that each vector field $\frac{\overset{(k)}{\delta}}{\delta q^{(\alpha)i}}$ decomposes into the auxiliary vector field $\frac{\overset{(k-1)}{\delta}_{k-1}}{\delta q^{(\alpha)i}}$ and the k -vertical vector fields $\frac{\overset{(k)}{\delta}}{\delta q^{(k)j}}$, $j = 1, \dots, n$, as follows.

$$\frac{\overset{(k)}{\delta}}{\delta q^{(\alpha)i}} = \frac{\overset{(k-1)}{\delta}_{k-1}}{\delta q^{(\alpha)i}} - (C_{k-\alpha})_i^j \frac{\overset{(k)}{\delta}}{\delta q^{(k)j}}$$

Hence, we have

$$\begin{aligned} \left[\frac{\overset{(k)}{\delta}}{\delta q^{(\alpha)i}}, \frac{\overset{(k)}{\delta}}{\delta q^{(\beta)l}} \right] &= \left[\frac{\overset{(k)}{\delta}}{\delta q^{(\alpha)i}}, \frac{\overset{(k-1)}{\delta}_{k-1}}{\delta q^{(\beta)l}} - (C_{k-\beta})_l^j \frac{\overset{(k)}{\delta}}{\delta q^{(k)j}} \right] \\ &= \left[\frac{\overset{(k)}{\delta}}{\delta q^{(\alpha)i}}, \frac{\overset{(k-1)}{\delta}_{k-1}}{\delta q^{(\beta)l}} \right] - \frac{\overset{(k)}{\delta} (C_{k-\beta})_l^j}{\delta q^{(\alpha)i}} \frac{\delta}{\delta q^{(k)j}} - (C_{k-\beta})_l^j \left[\frac{\overset{(k)}{\delta}}{\delta q^{(\alpha)i}}, \frac{\overset{(k)}{\delta}}{\delta q^{(k)j}} \right]. \end{aligned}$$

Moreover,

$$\begin{aligned} \left[\frac{\overset{(k)}{\delta}}{\delta q^{(\alpha)i}}, \frac{\overset{(k-1)}{\delta}_{k-1}}{\delta q^{(\beta)l}} \right] &= \left[\frac{\overset{(k-1)}{\delta}_{k-1}}{\delta q^{(\alpha)i}} - (C_{k-\alpha})_i^j \frac{\overset{(k)}{\delta}}{\delta q^{(k)j}}, \frac{\overset{(k-1)}{\delta}_{k-1}}{\delta q^{(\beta)l}} \right] \\ &= \left[\frac{\overset{(k-1)}{\delta}_{k-1}}{\delta q^{(\alpha)i}}, \frac{\overset{(k-1)}{\delta}_{k-1}}{\delta q^{(\beta)l}} \right] + \frac{\overset{(k-1)}{\delta}_{k-1} (C_{k-\alpha})_i^j}{\delta q^{(\beta)l}} \frac{\delta}{\delta q^{(k)j}} - (C_{k-\alpha})_i^j \left[\frac{\overset{(k)}{\delta}}{\delta q^{(k)i}}, \frac{\overset{(k-1)}{\delta}_{k-1}}{\delta q^{(\beta)l}} \right] \end{aligned}$$

and, since

$$\frac{\delta_{k-1}^{(k-1)} (C_{k-\alpha})_i^j}{\delta q^{(\beta)l}} = \frac{\delta^{(k)} (C_{k-\alpha})_i^j}{\delta q^{(\beta)i}} + (C_{k-\beta})_i^j \frac{\delta^{(k)} (C_{k-\alpha})_i^j}{\delta q^{(k)j}}$$

and

$$\left[\frac{\delta^{(k)}}{\delta q^{(k)i}}, \frac{\delta_{k-1}^{(k-1)}}{\delta q^{(\beta)i}} \right] = 0,$$

we conclude that

$$\begin{aligned} \left[\frac{\delta^{(k)}}{\delta q^{(\alpha)i}}, \frac{\delta_{k-1}^{(k-1)}}{\delta q^{(\beta)i}} \right] &= \left[\frac{\delta_{k-1}^{(k-1)}}{\delta q^{(\alpha)i}} - (C_{k-\alpha})_i^j \frac{\delta^{(k)}}{\delta q^{(k)j}}, \frac{\delta_{k-1}^{(k-1)}}{\delta q^{(\beta)i}} \right] \\ &= \left[\frac{\delta_{k-1}^{(k-1)}}{\delta q^{(\alpha)i}}, \frac{\delta_{k-1}^{(k-1)}}{\delta q^{(\beta)i}} \right] + \frac{\delta_{k-1}^{(k-1)} (C_{k-\alpha})_i^j}{\delta q^{(\beta)l}} \frac{\delta}{\delta q^{(k)j}} - (C_{k-\alpha})_i^j \left[\frac{\delta^{(k)}}{\delta q^{(k)i}}, \frac{\delta_{k-1}^{(k-1)}}{\delta q^{(\beta)i}} \right] \\ &= \left(\frac{\delta^{(k)} (C_{k-\alpha})_i^j}{\delta q^{(\beta)i}} + (C_{k-\beta})_i^j \frac{\delta^{(k)} (C_{k-\alpha})_i^j}{\delta q^{(k)j}} + \frac{\delta_{k-1}^{(k-1)} (C_{k-\alpha})_i^j}{\delta q^{(\beta)l}} \right) \frac{\delta}{\delta q^{(k)j}}. \end{aligned}$$

On the other hand, we have

$$\begin{aligned} \left[\frac{\delta^{(k)}}{\delta q^{(\alpha)i}}, \frac{\delta^{(k)}}{\delta q^{(k)j}} \right] &= \left[\frac{\delta_{k-1}^{(k-1)}}{\delta q^{(\alpha)i}} - (C_{k-\alpha})_i^j \frac{\delta^{(k)}}{\delta q^{(k)j}}, \frac{\delta^{(k)}}{\delta q^{(k)j}} \right] \\ &= \left[\frac{\delta_{k-1}^{(k-1)}}{\delta q^{(\alpha)i}}, \frac{\delta^{(k)}}{\delta q^{(k)j}} \right] + \frac{\delta^{(k)} (C_{k-\alpha})_i^j}{\delta q^{(k)j}} \frac{\delta^{(k)}}{\delta q^{(k)j}} \\ &= \frac{\delta^{(k)} (C_{k-\alpha})_i^j}{\delta q^{(k)j}} \frac{\delta^{(k)}}{\delta q^{(k)j}}. \end{aligned}$$

Hence,

$$\begin{aligned} \left[\frac{\delta^{(k)}}{\delta q^{(\alpha)i}}, \frac{\delta^{(k)}}{\delta q^{(\beta)l}} \right] &= \left(\frac{\delta_{k-1}^{(k-1)} (C_{k-\alpha})_i^j}{\delta q^{(\beta)l}} - \frac{\delta_{k-1}^{(k-1)} (C_{k-\beta})_l^j}{\delta q^{(\alpha)i}} + (C_{k-\alpha})_i^a \frac{\delta^{(k)} (C_{k-\beta})_l^j}{\delta q^{(k)a}} - (C_{k-\beta})_l^a \frac{\delta^{(k)} (C_{k-\alpha})_i^j}{\delta q^{(k)a}} \right) \frac{\delta^{(k)}}{\delta q^{(k)j}} \\ &\quad + \left[\frac{\delta_{k-1}^{(k-1)}}{\delta q^{(\alpha)i}}, \frac{\delta_{k-1}^{(k-1)}}{\delta q^{(\beta)l}} \right], \end{aligned}$$

and the result follows immediately. $\square$

Proposition C.6. Let $S_{(\alpha)i,(\beta)l}^{(\mu)j}$ be the j -th-coordinate function of $K_{\mu} \circ \left[\frac{\delta}{\delta q^{(\alpha)i}}, \frac{\delta}{\delta q^{(\beta)l}} \right]$, for each $\alpha, \beta, \mu = 0, 1, \dots, k$. Then

$$S_{(\alpha)i,(\beta)l}^{(k)j} = \frac{\delta^{(k-1)} (C_{k-\alpha})_i^j}{\delta q^{(\beta)l}} - \frac{\delta^{(k-1)} (C_{k-\beta})_l^j}{\delta q^{(\alpha)i}} + (C_{k-\alpha})_i^a \frac{\delta^{(k)} (C_{k-\beta})_l^j}{\delta q^{(k)a}} - (C_{k-\beta})_l^a \frac{\delta^{(k)} (C_{k-\alpha})_i^j}{\delta q^{(k)a}} + \sum_{a=1}^n \sum_{\mu=0}^{k-1} S_{(\alpha)i,(\beta)l}^{(\mu)a} (C_{k-\mu})_a^j,$$

for all $\alpha, \beta = 0, 1, \dots, k-1$.

PROOF. Note that $\tau_{k-1}^{(k)} \left(\frac{\delta^{(k-1)}}{\delta q^{(\alpha)i}} \right) = \frac{\delta^{(k-1)}}{\delta q^{(\alpha)i}}$. Then

$$K_k \circ \left[\frac{\delta^{(k-1)}}{\delta q^{(\alpha)i}}, \frac{\delta^{(k-1)}}{\delta q^{(\beta)l}} \right] = \sum_{\mu=0}^{k-1} S_{(\alpha)i,(\beta)l}^{(\mu)j} K_k \left(\frac{\delta^{(k-1)}}{\delta q^{(\mu)j}} \right) = \sum_{a=1}^n \sum_{\mu=0}^{k-1} S_{(\alpha)i,(\beta)l}^{(\mu)a} (C_{k-\mu})_a^j \partial_j.$$

Now, if we combine this equation with Lemma C.5, the result follows. $\square$

PROOF (PROOF OF PROPOSITION 17). Using Proposition C.6, we get

$$K_k \circ \left[\frac{\delta}{\delta q^{(k-1)i}}, \frac{\delta}{\delta q^{(k-1)l}} \right] = \left(\frac{\partial (C_1)_i^j}{\partial q^{(k-1)l}} - \frac{\partial (C_1)_l^j}{\partial q^{(k-1)i}} + (C_1)_i^a \frac{\partial (C_1)_l^j}{\partial q^{(k)a}} - (C_1)_l^a \frac{\partial (C_1)_i^j}{\partial q^{(k)a}} + \sum_{a=1}^n \sum_{\mu=0}^{k-1} S_{(k-1)i,(k-1)l}^{(\mu)a} (C_{k-\mu})_a^j \right) \partial_j$$

and, since $S_{(k-1)i,(k-1)l}^{(\mu)a} = 0$, for $\mu = 0, 1, \dots, k-1$, and $(C_1)_i^j$ do not depend on $q^{(\lambda)a}$, for $\lambda > 1$, the result is valid for $\alpha = 1$ (for all $k \in \mathbb{N}$). Now, we suppose that the result is valid for $\alpha - 1$ (for all $k \in \mathbb{N}$) and $\alpha < k - 1$. Then, ${}^k S_{(k-\alpha)i,(k-1)l}^{(\mu)a} = {}^{k-1} S_{(k-(\alpha-1))i,(k-1)l}^{(\mu)a} = 0$ and

$$K_k \circ \left[\frac{\delta}{\delta q^{(k-\alpha)i}}, \frac{\delta}{\delta q^{(k-1)l}} \right] = \left(\frac{\partial (C_{\alpha})_i^j}{\partial q^{(k-1)l}} - \frac{\delta^{(k-1)} (C_1)_l^j}{\delta q^{(k-\alpha)i}} + (C_{\alpha})_i^a \frac{\partial (C_1)_l^j}{\partial q^{(k)a}} - (C_1)_l^a \frac{\partial (C_{\alpha})_i^j}{\partial q^{(k)a}} + \sum_{a=1}^n \sum_{\mu=0}^{k-1} S_{(k-\alpha)i,(k-1)l}^{(\mu)a} (C_{k-\mu})_a^j \right) \partial_j$$

is still zero, so the result is valid for α . This means that the identity is hereditary in α (up to $\alpha = k - 2$). Therefore, we conclude the result is valid for $\alpha < k - 1$.

Finally, for $\alpha = k - 1$, and because ${}^k S_{(1)i,(k-1)l}^{(\mu)a} = {}^{k-1} S_{(1)i,(k-1)l}^{(\mu)a} = 0$, for $\mu = 0, 1, \dots, k - 1$, we have

$$\begin{aligned} K_k \circ \left[\frac{\delta}{\delta q^{(1)i}}, \frac{\delta}{\delta q^{(k-1)l}} \right] &= \left(\frac{\partial (C_{k-1})_i^j}{\partial q^{(k-1)l}} - \frac{\delta^{(k-1)} (C_1)_l^j}{\delta q^{(1)i}} + (C_{k-1})_i^a \frac{\partial (C_1)_l^j}{\partial q^{(k)a}} - (C_1)_l^a \frac{\partial (C_{k-1})_i^j}{\partial q^{(k)a}} + \sum_{a=1}^n \sum_{\mu=0}^{k-1} S_{(1)i,(k-1)l}^{(\mu)a} (C_{k-\mu})_a^j \right) \partial_j \\ &= \left(\frac{\partial (K_{k-1})_i^j}{\partial q^{(k-1)l}} - \frac{\delta^{(k-1)} (K_1)_l^j}{\delta q^{(1)i}} \right) \partial_j \\ &= \left(\frac{\partial (K_{k-1})_i^j}{\partial q^{(k-1)l}} - \frac{\partial (K_1)_l^j}{\partial q^{(1)i}} \right) \partial_j, \end{aligned}$$

which is zero due to Equation 28. $\square$

PROOF (PROOF OF PROPOSITION 18). Due to Lemma 11, we have that ${}^k S_{(0)i,(k-1)l}^{(k-1)j} = {}^{k-1} S_{(0)i,(k-1)l}^{(k-1)j} = \Gamma_{il}^j$ and ${}^k S_{(0)i,(k-1)l}^{(\mu)j} = {}^{k-1} S_{(0)i,(k-1)l}^{(\mu)j} = 0$, for $\mu = 0, 1, \dots, k - 2$.

On the other hand, using Proposition C.6 and Equations (28)–(29), we get

$${}^k S_{(0)i,(k-1)l}^{(k-1)j} = \frac{\partial (C_k)_i^j}{\partial q^{(k-1)l}} - \frac{\partial (C_1)_l^j}{\partial q^{(0)i}} + (K_1)_i^a \frac{\partial (C_1)_l^j}{\partial q^{(1)a}} - (C_1)_l^a \frac{\partial (C_k)_i^j}{\partial q^{(k)a}} + S_{(0)i,(k-1)l}^{(k-1)a} (C_1)_a^j$$

$$\begin{aligned}
 &= -q^{(1)a} dq^j (\nabla_{\partial_i} \nabla_{\partial_a} \partial_l) + \frac{1}{k} q^{(1)a} dq^j (\nabla_{\partial_l} \nabla_{\partial_a} \partial_i + (k-1) \nabla_{\partial_a} \nabla_{\partial_l} \partial_i) \\
 &= \frac{1}{k} q^{(1)a} dq^j (\nabla_{\partial_l} \nabla_{\partial_i} \partial_a - \nabla_{\partial_i} \nabla_{\partial_l} \partial_a) + \frac{k-1}{k} q^{(1)a} dq^j (\nabla_{\partial_a} \nabla_{\partial_l} \partial_i - \nabla_{\partial_l} \nabla_{\partial_a} \partial_i) \\
 &= -\frac{1}{k} q^{(1)a} (R_{ila}^j + (k-1) R_{ial}^j).
 \end{aligned}$$

□

D. Koszul computations for the k -Sasaki metric

D.1. Metric reductions for the k -Sasaki metric

This appendix contains the general Koszul-formula reductions used in Section 4.

PROOF (PROOF OF LEMMA 22). The result follows from the metric compatibility of the Levi-Civita connection on M , together with Lemma 4 and the identity

$$\langle\langle Y^{(k)h_\beta}, Z^{(k)h_\mu} \rangle\rangle = \delta_{\beta\mu} \langle Y, Z \rangle \circ \tau_k.$$

Indeed, only the $h_0^{(k)}$ -lift differentiates functions pulled back from M non-trivially, and so

$$X^{(k)h_\alpha} (\langle\langle Y^{(k)h_\beta}, Z^{(k)h_\mu} \rangle\rangle) = \delta_{\alpha 0} \delta_{\beta\mu} X \langle Y, Z \rangle \circ \tau_k.$$

Metric compatibility on M gives the stated formula. □

PROOF (PROOF OF LEMMA 23). To prove the first $k-1$ identities, it is enough to show that

$$\langle\langle \nabla_{X^{(k)h_\alpha}} Y^{(k)h_\beta}, Z^{(k)h_\mu} \rangle\rangle_k = \langle\langle \nabla_{X^{(k-1)h_\alpha}} Y^{(k-1)h_\beta}, Z^{(k-1)h_\mu} \rangle\rangle_{k-1}, \quad (58)$$

for all $X, Y, Z \in \mathfrak{X}(M)$ and $\alpha, \beta, \mu = 0, 1, \dots, k-1$. This follows by applying the Koszul formula (18), Lemma 22, and the compatibility of the lift brackets under $\tau_{k-1}^{(k)}$.

For the last identity, let W be k -horizontal. By the Koszul formula, Lemma 22, and the bracket formulas involving $U_k^{(k)}$ -lifts,

$$\langle\langle \nabla_W W, Z^{(k)U_k} \rangle\rangle_k = 0, \quad Z \in \mathfrak{X}(M).$$

Thus

$$K_k^{(k)} \circ \left(\nabla_W W \right) = 0. \quad (59)$$

Polarizing this identity with $W = X^{(k)h_\alpha} + Y^{(k)h_\beta}$ gives

$$2K_k^{(k)} \circ \left(\nabla_{X^{(k)h_\alpha}} Y^{(k)h_\beta} \right) = K_k^{(k)} \circ \left([X^{(k)h_\alpha}, Y^{(k)h_\beta}] \right),$$

which proves the claim. □

PROOF (PROOF OF PROPOSITION 24). Let $X, Y, Z \in \mathfrak{X}(M)$. Using the Koszul formula and the bracket identities involving $U_k^{(k)}$ -lifts, one obtains

$$\langle\langle \nabla_{X^{(k)h_\alpha}} Y^{(k)U_k}, Z^{(k)U_k} \rangle\rangle = 0, \quad \alpha = 1, \dots, k-1,$$

and

$$\begin{aligned} 2\langle\langle \nabla_{X^{h_0}}^{(k)} Y^{v_k}, Z^{v_k} \rangle\rangle &= X^{h_0} \langle\langle Y^{v_k}, Z^{v_k} \rangle\rangle + \langle\langle [X^{h_0}, Y^{v_k}], Z^{v_k} \rangle\rangle - \langle\langle [X^{h_0}, Z^{v_k}], Y^{v_k} \rangle\rangle \\ &= 2\langle \nabla_X Y, Z \rangle. \end{aligned}$$

This gives the k -vertical components. Finally, applying the same formula with Z^{h_μ} , $\mu = 0, \dots, k-1$, gives

$$\langle\langle \nabla_{X^{v_k}}^{(k)} Y^{v_k}, Z^{h_\mu} \rangle\rangle = 0,$$

which yields the remaining components. □

D.2. Third-order Levi-Civita computations

PROOF (PROOF OF THEOREM 28). The proof is obtained by the same method as Theorem 25. Lemma 23 gives the components involving only h_α -lifts from the third-order bracket formulas in Theorem 16. Proposition 24 gives the v_3 -components of the derivatives involving one 3-vertical lift. It remains only to compute the lower components of these latter derivatives.

For example, the h_0 -component of $\nabla_{X^{h_0}}^{(3)} Y^{v_3}$ is obtained by pairing against Z^{h_0} :

$$\begin{aligned} \langle\langle \nabla_{X^{h_0}}^{(3)} Y^{v_3}, Z^{h_0} \rangle\rangle &= -\langle\langle Y^{v_3}, \nabla_{X^{h_0}}^{(3)} Z^{h_0} \rangle\rangle \\ &= \left\langle Y, \frac{1}{12} R(X, Z)u^{(3)} + \frac{1}{6} (\nabla_{u^{(1)}} R)(X, Z)u^{(2)} - \frac{1}{6} (\nabla_{u^{(2)}} R)(X, Z)u^{(1)} \right. \\ &\quad \left. + \frac{1}{4} (\nabla_{u^{(1)}}^2 R)(X, Z)u^{(1)} \right\rangle \\ &= \left\langle \frac{1}{12} R(u^{(3)}, Y)X + \frac{1}{6} (\nabla_{u^{(1)}} R)(u^{(2)}, Y)X - \frac{1}{6} (\nabla_{u^{(2)}} R)(u^{(1)}, Y)X \right. \\ &\quad \left. + \frac{1}{4} (\nabla_{u^{(1)}}^2 R)(u^{(1)}, Y)X, Z \right\rangle. \end{aligned}$$

Thus

$$\tau_{3*} \left(\nabla_{X^{h_0}}^{(3)} Y^{v_3} \right) = \frac{1}{12} R(u^{(3)}, Y)X + \frac{1}{6} (\nabla_{u^{(1)}} R)(u^{(2)}, Y)X - \frac{1}{6} (\nabla_{u^{(2)}} R)(u^{(1)}, Y)X + \frac{1}{4} (\nabla_{u^{(1)}}^2 R)(u^{(1)}, Y)X.$$

The remaining components are obtained analogously by applying the Koszul formula against h_0 -, h_1 -, h_2 -, and v_3 -lifts and using the bracket formulas of Section 3. Combining all components gives the theorem. □

E. Geodesic computations

E.1. Third-order geodesic computations

PROOF (PROOF OF THEOREM 29). Using Equation (39) and the torsion-free property of ∇ , the covariant acceleration is

$$\begin{aligned} \nabla_{\dot{\Gamma}} \dot{\Gamma} &= \nabla_{\dot{q}^{h_0}}^{(3)} \dot{q}^{h_0} + 2 \nabla_{\dot{q}^{h_0}}^{(3)} (Y^{(1)})^{h_1} - [\dot{q}^{h_0}, (Y^{(1)})^{h_1}] \\ &\quad + 2 \nabla_{\dot{q}^{h_0}}^{(3)} (Y^{(2)})^{h_2} - [\dot{q}^{h_0}, (Y^{(2)})^{h_2}] + 2 \nabla_{\dot{q}^{h_0}}^{(3)} (Y^{(3)})^{v_3} - [\dot{q}^{h_0}, (Y^{(3)})^{v_3}] \end{aligned}$$

$$\begin{aligned}
 & + \nabla_{(Y^{(1)})^{h_1}}^{(3)} (Y^{(1)})^{h_1} + 2 \nabla_{(Y^{(1)})^{h_1}}^{(3)} (Y^{(2)})^{h_2} - [(Y^{(1)})^{h_1}, (Y^{(2)})^{h_2}] \\
 & + 2 \nabla_{(Y^{(1)})^{h_1}}^{(3)} (Y^{(3)})^{v_3} - [(Y^{(1)})^{h_1}, (Y^{(3)})^{v_3}] + 2 \nabla_{(Y^{(2)})^{h_2}}^{(3)} (Y^{(3)})^{v_3} - [(Y^{(2)})^{h_2}, (Y^{(3)})^{v_3}].
 \end{aligned}$$

The terms involving $\nabla_{(Y^{(2)})^{h_2}}^{(3)} (Y^{(2)})^{h_2}$ and $\nabla_{(Y^{(3)})^{v_3}}^{(3)} (Y^{(3)})^{v_3}$ vanish by Theorem 28. Applying Theorem 28 and the bracket formulas from Section 3 to the remaining terms gives the four component equations displayed in Theorem 29. $\square$

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Table 1

Notation used throughout the paper.

Notation	Meaning
$T^{(k)}M$	The k -th order tangent bundle of M , whose elements are k -jets $j_0^k(q)$ of curves q at 0.
$\tau_k : T^{(k)}M \rightarrow M$	The canonical projection onto the base point, given by $\tau_k(j_0^k(q)) = q(0)$.
$\tau_\alpha^{(k)} : T^{(k)}M \rightarrow T^{(\alpha)}M$	The canonical truncation map, given by $\tau_\alpha^{(k)}(j_0^k(q)) = j_0^\alpha(q)$, for $0 \leq \alpha \leq k$.
$(q^{(0)i}, q^{(1)i}, \dots, q^{(k)i})$	Induced coordinates on $T^{(k)}M$, with $q^{(\alpha)i} = \frac{1}{\alpha!} \frac{d^\alpha q^i}{dt^\alpha}(0)$.
$j^k q$	The k -jet of a curve q , given by $t \mapsto j_0^k(q_t)$, where $q_t(s) = q(t+s)$.
J	The canonical almost-tangent structure on $T^{(k)}M$. In coordinates,
$J = \frac{\partial}{\partial q^{(1)i}} \otimes dq^{(0)i} + \dots + \frac{\partial}{\partial q^{(k)i}} \otimes dq^{(k-1)i}.$	
$TM_\oplus^k$	The Whitney sum of k tangent bundles, i.e. $TM_\oplus^k := \bigoplus_{i=1}^k TM$.
V_α	The α -vertical bundle, defined as $V_\alpha = \text{Im}(J^\alpha) = \ker(\tau_{k*}^{\alpha-1})$, for $\alpha = 1, \dots, k$.
$K = (K_1, \dots, K_k)$	A connection map on $T^{(k)}M$. When compatible with the tower of projections, it is called a connection tower.
$H_0^{(k)}$	The horizontal bundle $\ker(K) \subset TT^{(k)}M$.
$H_\alpha^{(k)}$	The α -th component of the associated multiconnection, defined by $H_\alpha^{(k)} = J^\alpha(H_0^{(k)})$, for $\alpha = 0, \dots, k-1$.
$N_\alpha^{(k)}$	The α -horizontal bundle $N_\alpha^{(k)} = H_0^{(k)} \oplus \dots \oplus H_{\alpha-1}^{(k)}$.
$h_\alpha^{(k)}, v_\alpha^{(k)}, n_\alpha^{(k)}$	Projection operators onto $H_\alpha^{(k)}$, $V_\alpha^{(k)}$, and $N_\alpha^{(k)}$, respectively.
$\delta q^{(\alpha)i}$	The adapted coframe determined by a connection map, defined such that
$K_\alpha^{(k)} = \delta q^{(\alpha)i} \otimes \partial_i.$	
$\frac{\delta}{\delta q^{(\alpha)i}}$	The adapted frame dual to $\delta q^{(\alpha)i}$.
$X^{(k)}_{h_\alpha}$	The $h_\alpha^{(k)}$ -lift of a vector field $X \in \mathfrak{X}(M)$ to $T^{(k)}M$, characterized by $K_\alpha^{(k)}(X^{h_\alpha}) = X \circ \tau_k$ and $K_\beta^{(k)}(X^{h_\alpha}) = 0$ for $\beta \neq \alpha$.
$X^{(k)}_{v_k}$	The k -vertical lift of X to $T^{(k)}M$. This is also denoted by X^{h_k} when treating all lift components uniformly.
$d_T^{(k)}$	Tulczyjew's operator $d_T^{(k)} : T^{(k)}M \rightarrow TT^{(k-1)}M$ defined by $d_T^{(k)}(j_0^k q) = \frac{d}{dt} \Big _{t=0} (j^{k-1} q)(t)$.
$F^{(k)} : T^{(k)}M \rightarrow (TM)_\oplus^k$	The diffeomorphism induced by a connection tower which pulls back the vector bundle structure of $(TM)_\oplus^k$ to $T^{(k)}M$.
∇	The Levi-Civita connection of the Riemannian metric g on M .
R	The curvature tensor of ∇ , with the convention: $R(X, Y)Z = \nabla_X \nabla_Y Z - \nabla_Y \nabla_X Z - \nabla_{[X, Y]} Z$.
$g^{(k)}$	The k -Sasaki metric on $T^{(k)}M$ induced by the adapted splitting and the base metric g .
$\bar{\nabla}^{(k)}$	The Levi-Civita connection of the k -Sasaki metric $g^{(k)}$.