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DIFFERENTIAL VARIETIES OF UPPER TRIANGULAR MATRICES

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Abstract. Let L be a Lie algebra acting by derivations on an associative algebra A over a field F of characteristic zero. The polynomial identities satisfied by A with respect to this action are called differential identities, or L-identities. In this paper, we study the differential identities of the algebra UT_k of $k \times k$ upper triangular matrices and take a first step toward the classification of minimal L-varieties of differential exponent 3.

We first prove that, whenever UT_k generates a minimal variety of algebras with derivations, the L-action can be replaced by its semisimple part. More precisely, it is enough to consider inner derivations induced by diagonal elements. We then apply this reduction to UT_3 and explicitly determine the T_L-ideal of differential identities and the corresponding differential codimension sequence for every such action on UT_3. Finally, we show that every L-variety generated by UT_k, with $k \geq 3$, contains UT_3 endowed with one of these L-actions.

1. Introduction

The theory of polynomial identities studies associative algebras through the identical relations they satisfy. Among the numerical invariants introduced to measure such relations quantitatively, the codimension sequence $c_n(A)$ of an algebra A, due to Regev [17], plays a central role. When A satisfies a nontrivial polynomial identity, $c_n(A)$ is exponentially bounded, and Giambruno and Zaicev proved that the limit $\exp(A) := \lim_{n\to\infty} \sqrt[n]{c_n(A)}$ always exists and is a nonnegative integer, the *PI-exponent* of A (see [9, Chapter 6]), settling a conjecture of Amitsur. This integer provides a scale on which to compare varieties of algebras, and a natural problem is to classify, among the varieties of a given exponent, the *minimal* ones. Giambruno and Zaicev showed that, for $d \geq 2$, minimal varieties of exponent d are generated by upper block-triangular matrix algebras [8]; in particular, among these, the algebra UT_k of $k \times k$ upper triangular matrices, with trivial (one-dimensional) diagonal blocks, realizes a minimal variety of exponent k. The case $k = 2$ is especially distinguished: by a classical theorem of Kemer [14], UT_2, together with the infinite dimensional Grassmann algebra G, generates the only variety of *almost polynomial growth*, i.e., of exponential growth all of whose proper subvarieties have polynomial growth.

Differential identities generalize ordinary polynomial identities to the setting of algebras equipped with an action of a Lie algebra by derivations. Given a Lie algebra L acting on A by derivations, one obtains, via the universal enveloping algebra $U(L)$, the notion of *L-identity* (or *differential identity*) of A, first introduced by Kharchenko in [13]. As in the ordinary case, the differential codimension sequence $c_n^L(A)$ of a finite dimensional L-algebra A grows exponentially, and Gordienko in [10] proved that $\exp^L(A) := \lim_{n\to\infty} \sqrt[n]{c_n^L(A)}$ exists and is a nonnegative integer, the *L-exponent* of A. Since every ordinary polynomial identity of A is, in particular, an L-identity, we have $c_n(A) \leq c_n^L(A)$ for all n, and hence one always has $\exp(A) \leq \exp^L(A)$. Gordienko and Kochetov conjectured in [11] that equality always holds and proved this conjecture when L is a finite-dimensional semisimple Lie algebra. More recently, Rizzo in [19] established the equality for an arbitrary Lie algebra L, namely,

$$\exp^L(A) = \exp(A).$$

Thus, although the presence of a Lie algebra action may substantially enlarge the space of polynomial identities, it does not affect the exponential rate of growth of the corresponding codimension sequence.

This equality suggests that the classical theory of minimal varieties, and of varieties of almost polynomial growth, should admit a differential counterpart closely related to the ordinary one, generated this time by algebras with derivations. A first piece of evidence in this direction comes from the case

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$k = 2$: Giambruno and Rizzo [7] showed that UT_2, endowed with the inner derivation induced by e_{22}, still generates an L-variety of almost polynomial growth, and Rizzo [18] showed that, up to T_L-equivalence, UT_2 with these two actions (the trivial action and the action induced by this inner derivation) are the only two families of 2×2 upper triangular matrix L-algebras that generate varieties of algebras with derivations with this property. Moreover, in [19], it was shown that, for solvable L, UT_2 with either one of these two L-actions by derivations are the only two families of finite-dimensional algebras generating L-varieties with almost polynomial growth. See also [15] for a complete classification of the L-subvarieties generated by UT_2 with these two actions.

Together, these results provide a complete differential analogue of Kemer's theorem for exponent 2 in the solvable case. The picture changes markedly when L is semisimple. In [3], Brox and Rizzo showed that, for every $k \geq 2$, the algebra $M_k(F)$ of $k \times k$ matrices, endowed with the natural action of its Lie algebra of derivations $\mathrm{Der}(M_k(F))$, which is a simple Lie algebra, generates an L-variety of almost polynomial growth. Moreover, Argenti [2] completely classified the L-varieties of almost polynomial growth when L is the special linear Lie algebra of order 2. These results highlight the significant influence of the structural properties of the acting Lie algebra on the growth of the varieties it determines.

It is then natural to ask what happens one step further, namely, for algebras of exponent 3: since UT_3 realizes a minimal variety of exponent 3 in the ordinary setting, and since $\exp^L(UT_3^\theta) = \exp(UT_3) = 3$ for *every* L-action θ by Rizzo's theorem, one expects UT_3, suitably endowed with a nontrivial derivation action, to still generate minimal L-varieties of exponent 3. We refer the reader to [5,16] for partial results concerning the differential identities of UT_k for $k \geq 3$.

This paper takes a first step toward the classification of minimal L-varieties. Our key structural result places a strong restriction, *a priori*, on the actions that can give rise to a minimal L-variety. Namely, we prove that, for every minimal L-variety of exponent k generated by UT_k, the L-action on UT_k may be assumed, without loss of generality, to be semisimple; equivalently, it may be generated by derivations induced by diagonal elements of UT_k. In the case $k = 3$, this reduction allows us to restrict our analysis to semisimple L-actions on UT_3. Thus, these actions are not merely more tractable from a computational point of view: up to replacing the action by its semisimple part, they are precisely the actions that can generate a minimal L-variety of exponent 3.

We determine explicitly, for every one-dimensional Lie subalgebra and for the unique two-dimensional Lie subalgebra of diagonal derivations of UT_3, the T_L-ideal of differential identities and the corresponding differential codimension sequence. Moreover, we prove that, for arbitrary k and every L-action, the L-variety generated by UT_k contains the L-algebra UT_3 endowed with one of these actions. Thus, our computations identify the building blocks that occur in L-varieties generated by upper triangular matrices. Combined with the structural reduction above, these results provide the candidates for minimal L-varieties of exponent 3.

2. Differential identities and L-algebras

Throughout this paper, F will denote a field of characteristic zero, A an associative F-algebra, and $(L, [\cdot\,,\cdot]_L)$ a Lie algebra over F that we will denote simply as L when confusion does not arise.

Recall that A^- denotes the Lie algebra obtained from A by endowing its underlying vector space with the *commutator bracket* $[a, b] := ab - ba$ for all $a, b \in A$.

Let $\mathrm{End}_F(A)$ denote the associative algebra of F-linear endomorphisms of A acting on the right of A. Accordingly, composition is taken from left to right, and we adopt exponential notation for their action.

We say that an F-linear endomorphism $\delta : A \to A$ is a *derivation* on A if it satisfies the *Leibniz rule*: for all $a, b \in A$,

$$(ab)^\delta = a^\delta b + ab^\delta.$$

If $a \in A$, then the F-linear map $\mathrm{ad}_a : A \to A$ defined by $b^{\mathrm{ad}_a} = [b, a] = ba - ab$ for all $b \in A$ is a derivation on A called *inner derivation* induced by the element a.

The vector space $\mathrm{Der}(A)$ of all derivations of A, equipped with the commutator bracket, forms a Lie subalgebra of $\mathrm{End}_F(A)^-$. Moreover, the subspace $\mathrm{ad}(A)$ of inner derivations of A is a Lie ideal of $\mathrm{Der}(A)$.

Definition 2.1. We say that A is an *L-algebra* (or an *algebra with derivations*), or equivalently that L *acts on A by derivations*, if there exists a Lie algebra homomorphism

$$\theta\colon L \to \mathrm{Der}(A).$$

We write $\bar{L} = \theta(L) \subseteq \mathrm{Der}(A)$ for the image of L under θ.

An algebra A may admit several L-algebra structures corresponding to distinct Lie algebra homomorphisms $\theta\colon L \to \mathrm{Der}(A)$. When it is necessary to specify the homomorphism defining the action, we write A^θ for the algebra A endowed with the L-action determined by θ.

Since $\bar{L} = \theta(L)$ is a Lie subalgebra of $\mathrm{Der}(A)$, the inclusion $\bar{L} \hookrightarrow \mathrm{Der}(A)$ endows A with the structure of a $\bar{L}$-algebra. Thus, the action of L on A factors through $\bar{L}$. Moreover, distinct homomorphisms $\theta\colon L \to \mathrm{Der}(A)$ with the same image $\bar{L}$ induce the same $\bar{L}$-algebra structure on A. For this reason, throughout the paper we describe an L-action on A by means of its image $\bar{L}$, and say that L acts on A *via* $\bar{L}$.

Whenever it is necessary to specify the image Lie algebra of derivations, we denote by $A^{\bar{L}}$ the algebra A endowed with an L-action whose image is $\bar{L}$. Remark that this notation records only the Lie subalgebra $\bar{L} \subseteq \mathrm{Der}(A)$ and suppresses the homomorphism $\theta\colon L \to \mathrm{Der}(A)$ defining the action. Consequently, different homomorphisms with the same image give rise to the same notation $A^{\bar{L}}$, although they need not define the same L-algebra structure. A homomorphism θ is therefore always implicitly part of the data, even when the notation $A^{\bar{L}}$ does not display it; we will write θ explicitly only when needed to avoid ambiguity.

Following [1, Definition 4.1], we make the following definition.

Definition 2.2. We say that L-algebra structures A^θ and $A^{\theta'}$ are *equivalent* if $\theta(L) = \theta'(L)$ as subalgebras of $\mathrm{Der}(A)$.

This defines an equivalence relation on the set of L-algebra structures on A, with one class for each Lie subalgebra of $\mathrm{Der}(A)$ arising as an image. In particular, $A^{\bar{L}}$ always denotes a fixed representative of a single equivalence class.

Since associative algebras are often easier to study than Lie algebras, it is convenient to translate Lie algebra action into an associative setting via the universal enveloping algebra $U(L)$ of L. Recall that $U(L)$ is an unital associative algebra into which L embeds naturally as a Lie subalgebra of $U(L)^-$. Moreover, $U(L)$ is infinite dimensional whenever $L \neq 0$, even if L itself is finite-dimensional. For further details on the structure of $U(L)$ see [12, Section 17].

By the universal property of $U(L)$, an L-action on A can be uniquely extended to a right $U(L)$-action (which by abuse of notation we also call an L-action), by extending θ to the homomorphism of unital associative algebras $\Theta\colon U(L) \to \mathrm{End}_F(A)$ with $\Theta(1_{U(L)}) = \mathrm{id}_A$. In this way, A becomes a right $U(L)$-module algebra, where the action is written in exponential notation.

Note that when $\theta(L) = 0$, then $\Theta(U(L)) \cong F$, and the $U(L)$-action is just the linear action of F.

For fixed L, the class of L-algebras is a nontrivial variety as it contains $U(L)$. Ideals of L-algebras (*L-ideals*) are understood to be invariant under the $U(L)$-action, and homomorphisms $\psi : A \to B$ between L-algebras A, B (*L-homomorphisms*) must satisfy $\psi(a^u) = \psi(a)^u$ for all $a \in A$ and $u \in U(L)$.

The variety of L-algebras contains the *free (unital associative) L-algebra* $F\langle X|L\rangle$, freely generated by a countably infinite set of variables $X := \{x_1, x_2, \dots\}$, i.e., $F\langle X|L\rangle$ is uniquely determined up to an isomorphism by the following universal property: given a L-algebra A, any map $\varphi : X \to A$ can be uniquely extended to a homomorphism of L-algebras $\bar{\varphi} : F\langle X|L\rangle \to A$, which we call the *evaluation* of $F\langle X|L\rangle$ at elements $\varphi(x_1), \varphi(x_2), \dots$ from A. Notice that the evaluation $\bar{\varphi}$ of $F\langle X|L\rangle$ in A depends not only on the map φ but also on the right action of L on A. Hence, $\bar{\varphi}$ depends on the Lie algebra homomorphism $\theta : L \to \mathrm{Der}(A)$ that defines the L-action on A.

Since A can be a L-algebra with respect to different L-actions, we will denote the evaluation $\bar{\varphi}$ by $\bar{\varphi}_\theta$, or simply φ_θ, when necessary to clarify which action we are considering.

We can construct $F\langle X|L\rangle$ as follows: given the free vector space V_X spanned by X, we consider the tensor product $V_X \otimes U(L)$, then the free L-algebra is the tensor algebra generated by $V_X \otimes U(L)$, i.e,

$$F\langle X|L\rangle = \bigoplus_{n=0}^{\infty} \big(V_X \otimes U(L)\big)^{\otimes n},$$

where L acts on $V_X \otimes U(L)$ by right multiplication on the right component $U(L)$ and it is extended to the whole tensor algebra according to the Leibniz rule.

From now on, we write $x^u := x \otimes u$ and $x := x \otimes 1_{U(L)}$ for $x \in X$ and $u, 1_{U(L)} \in U(L)$. Then, given a basis $\mathcal{B}_{U(L)} := \{u_i \mid i \geq 1\}$ of $U(L)$, the set

$$\mathcal{B}_{F\langle X|L\rangle} := \big\{1, x_{i_1}^{u_{j_1}} \cdots x_{i_n}^{u_{j_n}} \mid n \geq 1,\ i_1, \dots, i_n \geq 1,\ u_{j_1}, \dots, u_{j_n} \in \mathcal{B}_{U(L)}\big\}$$

is a basis of $F\langle X|L\rangle$ and the L-action is determined by

$$(x_{i_1}^{u_{j_1}} x_{i_2}^{u_{j_2}} \dots x_{i_n}^{u_{j_n}})^d = x_{i_1}^{u_{j_1} d} x_{i_2}^{u_{j_2}} \dots x_{i_n}^{u_{j_n}} + x_{i_1}^{u_{j_1}} x_{i_2}^{u_{j_2} d} \dots x_{i_n}^{u_{j_n}} + \dots + x_{i_1}^{u_{j_1}} x_{i_2}^{u_{j_2}} \dots x_{i_n}^{u_{j_n} d}$$

for $u_{j_1}, \dots, u_{j_n} \in \mathcal{B}_{U(L)}$ and $d \in L$. The elements of the free L-algebra are called *differential polynomials* or *L-polynomials*.

A *T_L-ideal* of the free L-algebra is an L-ideal which in addition is invariant under all L-endomorphisms of $F\langle X|L\rangle$ or *substitutions*, which send variables of $x_i \in X$ to elements of $F\langle X|L\rangle$. Given a set $S \subseteq F\langle X|L\rangle$, by $\langle S\rangle_{T_L}$ we denote the smallest T_L-ideal containing S; also, we say that an L-polynomial $f \in F\langle X|L\rangle$ is a *consequence* of the L-polynomials in S, or *follows* from the L-polynomials in S, if $f \in \langle S\rangle_{T_L}$.

Given an L-algebra A^θ, a differential polynomial $f(x_1, \dots, x_n) \in F\langle X|L\rangle$ is an *L-identity* of A^θ, or a *differential identity*, when the role of L is clear, and we write $f \equiv 0$, if $f(a_1, \dots, a_n) = 0$ for any $a_1, \dots, a_n \in A$, where the evaluation replaces each exponent $u \in U(L)$ occurring in f by $\Theta(u) \in \mathrm{End}_F(A)$. We denote by $\mathrm{Id}^L(A)$ the set of L-identities of A, which is a T_L-ideal of the free L-algebra $F\langle X|L\rangle$. Note that $\mathrm{Id}^L(A)$ is the intersection of all kernels of evaluations of $F\langle X|L\rangle$ in A.

Remark that when $\theta(L) = 0$, then $\Theta(U(L)) \cong F$, and we are dealing with the ordinary polynomial identities.

For $n \geq 1$, we denote by P_n^L the vector space of *multilinear L-polynomials* of degree n in the variables $x_1, \dots, x_n$, that is

$$P_n^L := \mathrm{span}_F\{x_{\sigma(1)}^{u_{i_1}} \cdots x_{\sigma(n)}^{u_{i_n}} \mid \sigma \in S_n,\ u_{i_1}, \dots, u_{i_n} \in \mathcal{B}_{U(L)}\},$$

where S_n denotes the symmetric group acting on $\{1, \dots, n\}$. Since F has characteristic zero, a standard argument shows that the T_L-ideal $\mathrm{Id}^L(A)$ is completely determined by its multilinear L-polynomials. Thus it is reasonable to consider the quotient space

$$P_n^L(A) := \frac{P_n^L}{P_n^L \cap \mathrm{Id}^L(A)},$$

and so one can define the nth *L-codimension*, or *differential codimension*, of A as

$$c_n^L(A) := \dim_F P_n^L(A), \quad n \geq 1.$$

Note that $c_n^L(A)$ is not necessarily finite. However, if the action of $U(L)$ is finite dimensional, i.e., if $\Theta(U(L))$ is a finite-dimensional algebra, then $c_n^L(A)$ is finite for all $n \geq 1$.

In [10] Gordienko captured the exponential growth of the differential codimension sequence of a finite dimensional algebra A. More precisely, he proved that if A is a finite dimensional L-algebra, then the limit

$$\exp^L(A) := \lim_{n\to\infty} \sqrt[n]{c_n^L(A)}$$

exists and is a nonnegative integer called the *L-exponent of A*. In case $L = 0$, we are dealing with ordinary polynomial identities, and the L-exponent of A is the (ordinary) exponent of A, denoted by $\exp(A)$ (see [9, Chapter 6]). In [19] Rizzo proved that

$$\exp^L(A) = \exp(A) \tag{2.1}$$

in case A is a finite dimensional L-algebra.

A variety of L-algebras generated by an L-algebra A, denoted by $\mathrm{var}^L(A)$, is called *L-variety*, or *differential variety* and $\mathrm{Id}^L(\mathcal{V}) := \mathrm{Id}^L(A)$. The *growth* of $\mathcal{V} = \mathrm{var}^L(A)$ is the growth of the sequence $c_n^L(\mathcal{V}) := c_n^L(A)$, $n \geq 1$, and the L-exponent of $\mathcal{V}$ is $\exp^L(\mathcal{V}) := \exp^L(A)$. A L-variety $\mathcal{V}$ is said *minimal* of exponent $d \geq 2$ if $\exp^L(\mathcal{V}) = d$ and $\exp^L(\mathcal{U}) < d$ for all proper L-subvariety $\mathcal{U}$ of $\mathcal{V}$.

3. L-varieties generated by upper triangular matrices

For $k \geq 2$, let UT_k be the algebra of $k \times k$ upper triangular matrices over F, and e_{ij} be the (i, j)-unit matrix whose only nonzero entry is 1_F in position (i, j). In [4], the authors studied the derivations of UT_k and proved the following.

Theorem 3.1. *Any derivation of UT_k is inner.*

As a direct consequence of the above theorem we have the following.

Corollary 3.2. $\dim_F \mathrm{Der}(UT_k) = \frac{k^2+k-2}{2}$.

For $a \in UT_k$, write $a = a^{(s)} + a^{(n)}$, where $a^{(s)}$ denotes the diagonal part of a and $a^{(n)} := a - a^{(s)}$ is strictly upper triangular. For any $\delta \in \mathrm{Der}(UT_k)$, by Theorem 3.1, there exists $a \in UT_k$, such that $\delta = \mathrm{ad}_a$. We set

$$\delta = \delta^{(s)} + \delta^{(n)}, \tag{3.1}$$

where $\delta^{(s)} = \mathrm{ad}_{a^{(s)}}$ and $\delta^{(n)} = \mathrm{ad}_{a^{(n)}}$.

Notice that, for $1 \leq i \leq j \leq k$,

$$e_{ij}^{\delta^{(s)}} = \begin{cases} \gamma_{i,j} e_{ij} & \text{if } i \neq j \\ 0 & \text{if } i = j \end{cases} \qquad \text{and} \qquad e_{ij}^{\delta^{(n)}} \in J^{j-i+1} \tag{3.2}$$

where $\gamma_{i,j} \in F$.

Lemma 3.3. *Let $k \geq 2$. For $a = a^{(s)} + a^{(n)} \in UT_k$, define*

$$s : \mathrm{Der}(UT_k) \longrightarrow \mathrm{Der}(UT_k), \qquad s(\mathrm{ad}_a) = \mathrm{ad}_{a^{(s)}}.$$

Then s is a homomorphism of Lie algebras.

Proof. By Theorem 3.1 every derivation of UT_k is inner, so s is defined on all of $\mathrm{Der}(UT_k)$. We first check that s is well defined. If $\mathrm{ad}_a = \mathrm{ad}_b$ for some $a, b \in UT_k$, then $a - b$ lies in the center of UT_k, which consists of scalar matrices; in particular $a - b$ is diagonal, so $a^{(s)} - b^{(s)} = a - b$ and $\mathrm{ad}_{a^{(s)}} = \mathrm{ad}_{b^{(s)}}$. Linearity of s follows immediately from the linearity of the map $a \mapsto a^{(s)}$. It remains to check that s preserves the Lie bracket. Notice that for any a, b the commutator $[a, b] = ab - ba$ is strictly upper triangular, i.e., $[a, b]^{(s)} = 0$. Hence:

$$s([\mathrm{ad}_a, \mathrm{ad}_b]) = s(\mathrm{ad}_{[a,b]}) = \mathrm{ad}_{[a,b]^{(s)}} = \mathrm{ad}_0 = 0.$$

On the other hand, since $a^{(s)}$ and $b^{(s)}$ are diagonal matrices, they commute, so

$$[s(\mathrm{ad}_a), s(\mathrm{ad}_b)] = [\mathrm{ad}_{a^{(s)}}, \mathrm{ad}_{b^{(s)}}] = \mathrm{ad}_{[a^{(s)}, b^{(s)}]} = \mathrm{ad}_0 = 0.$$

Therefore $s([\mathrm{ad}_a, ad_b]) = [s(\mathrm{ad}_a), s(\mathrm{ad}_b)]$ for all $a, b \in UT_k$, proving that s is a Lie algebra homomorphism. □

Let us denote by $UT_k^{\delta_1,\ldots,\delta_r}$ the algebra UT_k in which L acts by derivations through the Lie subalgebra of $\mathrm{Der}(UT_k)$ spanned by $\{\delta_1, \ldots, \delta_r\}$, for $1 \leq r \leq \frac{k^2+k-2}{2}$.

We stress that $UT_k^{\delta_1,\ldots,\delta_r}$ always denotes a fixed representative of a single equivalence class in the sense of Definition (2.2). More precisely, there exists a Lie algebra homomorphism $\theta : L \to \mathrm{Der}(UT_k)$ with image $\mathrm{span}\{\delta_1, \ldots, \delta_r\}$, such that the L-algebra $UT_k^{\delta_1,\ldots,\delta_r}$ is precisely UT_k^{θ}. Thus, although the homomorphism θ is suppressed in the notation, it is always understood to be part of the underlying data.

Theorem 3.4. *Consider the L-algebra $UT_k^{\delta_1,\ldots,\delta_r}$ for some $1 \leq r \leq \frac{k^2+k-2}{2}$. Then there exists a Lie algebra homomorphism $\theta^{(s)} : L \to \mathrm{Der}(UT_k)$ such that $\theta^{(s)}(L)$ is spanned by $\{\delta_1^{(s)}, \ldots, \delta_r^{(s)}\}$ and*

$$UT_k^{\delta_1^{(s)},\ldots,\delta_r^{(s)}} \in \mathrm{var}^L\left(UT_k^{\delta_1,\ldots,\delta_r}\right),$$

where $UT_k^{\delta_1^{(s)},\ldots,\delta_r^{(s)}} = UT_k^{\theta^{(s)}}$.

Proof. Since L acts on $UT_k^{\delta_1,\ldots,\delta_r}$ by derivation, there exists a Lie algebra homomorphism $\theta : L \to \mathrm{Der}(UT_k)$ such that $\theta(L)$ is the Lie subalgebra of $\mathrm{Der}(UT_k)$ spanned by $\{\delta_1, \ldots, \delta_r\}$. Let $s : \mathrm{Der}(UT_k) \to \mathrm{Der}(UT_k)$ be the Lie algebra homomorphism of Lemma 3.3, and set $\theta^{(s)} := s \circ \theta$. Since s and θ are both Lie algebra homomorphisms, so is $\theta^{(s)}$, and $\theta^{(s)}(L) = s(\theta(L))$ is the Lie subalgebra of $\mathrm{Der}(UT_k)$ spanned by $\{\delta_1^{(s)}, \ldots, \delta_r^{(s)}\}$, since $\delta_i^{(s)} = s(\delta_i)$ for $i = 1, \ldots, r$.

In particular, if $d_1, \ldots, d_r$ are elements of L with $\theta(d_i) = \delta_i$ for $1 \leq i \leq r$, then $\theta^{(s)}(d_i) = s(\theta(d_i)) = \delta_i^{(s)}$ for $1 \leq i \leq r$, and $\theta^{(s)}(d) = 0$ whenever $\theta(d) = 0$.

Now, let $f \in \mathrm{Id}^L\left(UT_k^{\delta_1,\ldots,\delta_r}\right)$ with $\deg f = n$ and assume, as we may, that f is multilinear. Decompose f as $f = f^{(d)} + f^{(1)}$, where $f^{(d)} \in P_n^L$ is a differential polynomial in which in each monomial appears at least a variable with exponent different from $1_{U(L)}$, and $f^{(1)} \in P_n$ is a multilinear ordinary polynomial, i.e., in each monomial all the variables appear with exponent equal to $1_{U(L)}$.

Since f is multilinear, it is enough to restrict to evaluations on matrix units. Moreover, $f^{(1)}$ is a multilinear ordinary polynomial, and hence its evaluation in UT_k does not depend on the Lie algebra action on UT_k.

Since f is a differential identity of $UT_k^{\delta_1,\dots,\delta_r}$, for every evaluation $\varphi : X \longrightarrow UT_k$ we have

$$\varphi_\theta(f^{(d)}) = -\varphi(f^{(1)}). \tag{3.3}$$

Now fix an evaluation $\varphi : X \longrightarrow UT_k$ and let

$$\varphi_{\theta^{(s)}}(f^{(d)}) = \sum_{1\le i<j\le k} \alpha_{ij} e_{ij}$$

and

$$\varphi_\theta(f^{(d)}) = \sum_{1\le i<j\le k} \beta_{ij} e_{ij},$$

where $\alpha_{ij}, \beta_{ij} \in F$. We aim to show that $\alpha_{ij} = \beta_{ij}$ for all $1 \le i < j \le k$. It will then follow that $\varphi_{\theta^{(s)}}(f^{(d)}) = \varphi_\theta(f^{(d)})$, and, using (3.3), we conclude that $\varphi_{\theta^{(s)}}(f) = 0$. As this holds for any evaluation φ, we deduce that $f \in \mathrm{Id}^L\left(UT_k^{\delta_1^{(s)},\dots,\delta_r^{(s)}}\right)$, as required.

For distinct indices $p_1, \dots, p_l \in \{1, \dots, n\}$, let $M_{p_1,\dots,p_l}$ denote the set of all monomials m of $f^{(d)}$ with nonzero coefficient such that only the variables $x_{p_1}, \dots, x_{p_l}$, may appear with exponent different from $1_{U(L)}$.

Let $1 \le i < j \le k$ and suppose that $\beta_{ij} \neq 0$. Then, by (3.3), the coefficient of e_{ij} in $\varphi(f^{(1)})$ is nonzero. Since $f^{(1)}$ is an ordinary polynomial,there exists a monomial of $f^{(1)}$ whose evaluation is a nonzero multiple of e_{ij}. Then by the multiplication table of UT_k there exist distinct indices $p_1, \dots, p_l$ such that the corresponding variables are evaluated on off-diagonal matrix units whose ordered product is e_{ij}, while all the remaining variables are evaluated on diagonal matrix units.

By (3.1) and (3.2) if $m \in M_{p_1,\dots,p_l}$, then

$$\varphi_\theta(m) = \gamma_m^{(s)} e_{ij} + j_m$$

where $\gamma_m^{(s)} \in F$ depends on $\delta_1^{(s)}, \dots, \delta_r^{(s)}$ and $j_m \in J^{j-i+1}$ depends on $\delta_1^{(n)}, \dots, \delta_r^{(n)}$; if $m \notin M_{p_1,\dots,p_l}$, then $\varphi_\theta(m) \in J^{j-i+1}$. Since $e_{ij} \in J^{j-i}$ but $e_{ij} \notin J^{j-i+1}$, we obtain

$$\beta_{ij} = \sum_{m\in M_{p_1,\dots,p_l}} \tau_m \gamma_m^{(s)},$$

where $\tau_m \in F$ is the coefficient of m in $f^{(d)}$ for any $m \in M_{p_1,\dots,p_l}$.

On the other hand, since by (3.2) $\varphi_{\theta^{(s)}}(m) = \gamma_m^{(s)} e_{ij}$ for all $m \in M_{p_1,\dots,p_l}$, and $\varphi_{\theta^{(s)}}(m) = 0$ for all $m \notin M_{p_1,\dots,p_l}$, we get

$$\alpha_{ij} = \sum_{m\in M_{p_1,\dots,p_l}} \tau_m \gamma_m^{(s)}.$$

Therefore, $\alpha_{ij} = \beta_{ij}$, as claimed.

Now suppose that $\beta_{ij} = 0$ for some $1 \le i < j \le k$. We prove that $\alpha_{ij} = 0$. Assume, by contradiction, that $\alpha_{ij} \neq 0$. Then there exists at least a monomial m in $f^{(d)}$, with nonzero coefficient, such that

$$\varphi_{\theta^{(s)}}(m) = \gamma_m^{(s)} e_{ij} \neq 0,$$

where $\gamma_m^{(s)} \in F$ depends on $\delta_1^{(s)}, \dots, \delta_r^{(s)}$. Since by (3.2) each $\delta_l^{(s)}$ sends any e_{ab} with $a \neq b$ to a scalar multiple of itself, the multiplication table of UT_k implies that there exist distinct indices $p_1, \dots, p_l$ such that the corresponding variables are evaluated on off-diagonal matrix units whose ordered product is e_{ij}, while all the remaining variables are evaluated on diagonal matrix units.

By (3.2) and the multiplication table of UT_k we get that: if $m' \in M_{p_1,\dots,p_l}$, then $\varphi_{\theta^{(s)}}(m') = \gamma_{m'}^{(s)} e_{ij}$ where $\gamma_{m'}^{(s)} \in F$ depends on $\delta_1^{(s)}, \dots, \delta_r^{(s)}$; otherwise, if $m' \notin M_{p_1,\dots,p_l}$, $\varphi_{\theta^{(s)}}(m') = 0$. Consequently, the coefficient of e_{ij} in $\varphi_{\theta^{(s)}}(f^{(d)})$ is

$$\alpha_{ij} = \sum_{m'\in M_{p_1,\dots,p_l}} \tau_{m'} \gamma_{m'}^{(s)},$$

where $\tau_{m'}$ is the coefficient of m' in $f^{(d)}$ for any $m' \in M_{p_1,\dots,p_l}$.

On the other side, by (3.1) and (3.2) we have that $\varphi_\theta(m') = \gamma_{m'}^{(s)} e_{ij} + j_{m'}$ with $j_{m'} \in J^{j-i+1}$ for all $m' \in M_{p_1,\dots,p_l}$ and $\varphi_\theta(m') \in J^{j-i+1}$ for all $m' \notin M_{p_1,\dots,p_l}$. Thus, since $e_{ij} \in J^{j-i}$ but $e_{ij} \notin J^{j-i+1}$, the coefficient of e_{ij} in $\varphi_\theta(f^{(d)})$ is

$$\beta_{ij} = \sum_{m'\in M_{p_1,\dots,p_l}} \tau_{m'} \gamma_{m'}^{(s)}.$$

Therefore, $\alpha_{ij} = \beta_{ij}$, contradicting the assumption that $\beta_{ij} = 0$ and $\alpha_{ij} \neq 0$. Hence $\alpha_{ij} = 0$.

□

Observe that in case $k = 2$ the above result was proved in [19, Corollary 4.5] by using the explicit description of the T_L-ideals of differential identities of UT_2 with all the possible L-actions by derivations determined in [7, 18].

Remark 3.5. For any minimal L-variety of exponent k generated by UT_k, we may assume, without loss of generality, that the L-action on UT_k is semisimple, that is, generated by derivations induced by diagonal elements of UT_k. Indeed, given an L-action θ on UT_k, let $\theta^{(s)} := s \circ \theta$ denote its semisimple part, where s is the Lie algebra homomorphism defined in Lemma 3.3. Then, by (2.1), $\exp^L(UT_k^{\theta^{(s)}}) = \exp^L(UT_k^{\theta}) = \exp(UT_k) = k$. On the other end, the above theorem implies that $\mathrm{var}^L(UT_k^{\theta^{(s)}}) \subseteq \mathrm{var}^L(UT_k^{\theta})$. Therefore, if $\mathrm{var}^L(UT_k^{\theta})$ is minimal, the above inclusion cannot be proper; otherwise, $\mathrm{var}^L(UT_k^{\theta})$ would contain a proper L-subvariety with the same exponent, contradicting the minimality of $\mathrm{var}^L(UT_k^{\theta})$. Hence, $\mathrm{var}^L(UT_k^{\theta}) = \mathrm{var}^L(UT_k^{\theta^{(s)}})$.

4. Differential identities of UT_3

In this section, we determine the T_L-ideals of the algebra UT_3 under distinct L-actions by derivations. To this end, let

$$\varepsilon_1 := \mathrm{ad}_{(-e_{11})} \quad \text{and} \quad \varepsilon_2 := \mathrm{ad}_{e_{33}},$$

and consider the L-algebras:

(1) UT_3, where L acts trivially;
(2) $UT_3^{\varepsilon_1}$, where L acts via the one-dimensional Lie algebra generated by ε_1;
(3) $UT_3^{\varepsilon_2}$, where L acts via the one-dimensional Lie algebra generated by ε_2;
(4) $UT_3^{\varepsilon_1+\beta\varepsilon_2}$, where $\beta \in F \setminus \{0\}$, and L acts via the one-dimensional Lie algebra generated by $\varepsilon_1 + \beta\varepsilon_2$;
(5) $UT_3^{\varepsilon_1,\varepsilon_2}$, where L acts via the two-dimensional abelian Lie algebra with basis $\{\varepsilon_1, \varepsilon_2\}$.

Remark 4.1. Recall (Definition 2.2) that UT_3^{θ} and $UT_3^{\theta'}$ are equivalent if $\theta(L) = \theta'(L)$. Equivalent structures need not have the same T_L-ideal of identities, but they always have the same L-codimension sequence, as follows.

Let $\bar{L} = \mathrm{span}\{\delta_1, \ldots, \delta_r\}$ be the common image of θ, θ', and choose adapted bases $\{d_i\}$, $\{d_i'\}$ of L with $\theta(d_i) = \theta'(d_i') = \delta_i$ for $i \leq r$ and $d_j, d_j' \in \ker\theta, \ker\theta'$ for $j > r$. For an L-polynomial g whose exponents are written in terms of $d_1, \ldots, d_r$ only, let g' be obtained by relabelling $d_i \mapsto d_i'$. Since $\theta(d_i) = \theta'(d_i')$, evaluating g via θ and g' via θ' at the same $\varphi : X \to UT_3$ performs the same operations, so $\varphi_\theta(g) = \varphi_{\theta'}(g')$ for every φ; hence $g \equiv 0$ on UT_3^{θ} if and only if $g' \equiv 0$ on $UT_3^{\theta'}$.

Consequently, if $\mathrm{Id}^L(UT_3^{\theta})$ is generated by finitely many L-polynomials $g_1, \ldots, g_m$ of this form (exponents in $d_1, \ldots, d_r$ only) together with $x^{d_j} \equiv 0$ for $j > r$, then $\mathrm{Id}^L(UT_3^{\theta'})$ is generated by $g_1', \ldots, g_m'$ together with the same vanishing conditions $x^{d_j'} \equiv 0$, and $c_n^L(UT_3^{\theta}) = c_n^L(UT_3^{\theta'})$ for all n.

For this reason, we fix one representative θ for each of the five images spanned by ε_1 and ε_2, and write *the* L-algebra $UT_3^{\varepsilon_1}$ (etc.) for this fixed representative: as verified in each theorem below, its generators are of the form just described, so the resulting T_L-ideal (up to relabelling) and L-codimension sequence do not depend on the choice of representative.

Then we have the following result, which underlines the importance of determining the T_L-ideals of the L-algebras listed above.

Proposition 4.2. *Let L be a Lie algebra and $\theta : L \to \mathrm{Der}(UT_k)$ a Lie algebra homomorphism. Then, for every $k \geq 3$, the L-variety $\mathrm{var}^L(UT_k^{\theta})$ contains at least one of the following algebras:*

$$UT_3, \quad UT_3^{\varepsilon_1}, \quad UT_3^{\varepsilon_2}, \quad UT_3^{\varepsilon_1+\beta\varepsilon_2}, \quad UT_3^{\varepsilon_1,\varepsilon_2},$$

where $\beta \in F \setminus \{0\}$, for a suitable Lie algebra homomorphism $\overline{\theta} : L \to \mathrm{Der}(UT_3)$.

Proof. We first consider the case $k = 3$, so $\theta : L \to \mathrm{Der}(UT_3)$. If $\theta(L) = 0$, then $UT_3^{\theta} = UT_3$ is already in the list. So assume $\theta(L) \neq 0$ and let $r = \dim\theta(L) \geq 1$; fix a basis $\{\delta_1, \ldots, \delta_r\}$ of $\theta(L)$, so that $UT_3^{\theta} = UT_3^{\delta_1,\ldots,\delta_r}$. By Theorem 3.4, setting $\theta^{(s)} := s \circ \theta$, we have $\theta^{(s)}(L) = \mathrm{span}\{\delta_1^{(s)}, \ldots, \delta_r^{(s)}\}$ and $UT_3^{\delta_1^{(s)},\ldots,\delta_r^{(s)}} \in \mathrm{var}^L(UT_3^{\theta})$, where $UT_3^{\delta_1^{(s)},\ldots,\delta_r^{(s)}} = UT_3^{\theta^{(s)}}$.

By Lemma 3.3, $s(\mathrm{Der}(UT_3)) = \{\mathrm{ad}_b \mid b \text{ diagonal}\}$, and so $\{\varepsilon_1, \varepsilon_2\}$ is a basis of $s(\mathrm{Der}(UT_3))$, which therefore has dimension 2. Hence $\theta^{(s)}(L) \subseteq s(\mathrm{Der}(UT_3)) = \mathrm{span}\{\varepsilon_1, \varepsilon_2\}$, and $\dim \theta^{(s)}(L) \in \{0, 1, 2\}$. Clearly if $\dim \theta^{(s)}(L) = 0$, then $UT_3^{\delta_1^{(s)}, \ldots, \delta_r^{(s)}} = UT_3$.

If $\dim \theta^{(s)}(L) = 1$, then $\theta^{(s)}(L)$ is generated by $\alpha\varepsilon_1 + \beta\varepsilon_2$ for some $(\alpha, \beta) \neq (0, 0)$; up to rescaling, this is one of the algebras $UT_3^{\varepsilon_1}$, $UT_3^{\varepsilon_2}$, or $UT_3^{\varepsilon_1 + \beta'\varepsilon_2}$ for some $\beta' \neq 0$.

If $\dim \theta^{(s)}(L) = 2$, then $\theta^{(s)}(L) = \mathrm{span}\{\varepsilon_1, \varepsilon_2\}$, so $UT_3^{\delta_1^{(s)}, \ldots, \delta_r^{(s)}} = UT_3^{\varepsilon_1, \varepsilon_2}$.

Now suppose $k > 3$ and consider $I = \mathrm{span}\{e_{ij} \mid j - i > 0,\ j > 3\}$. Clearly I is an L-ideal of UT_k^θ, and in particular I is invariant under $\theta(L)$. Hence the quotient $A = UT_k/I$ has a natural structure of L-algebra given by the Lie homomorphism $\theta_A : L \to \mathrm{Der}(A)$ defined by

$$(a + I)^{\theta_A(d)} := a^{\theta(d)} + I,$$

for all $d \in L$ and $a + I \in A$.

Let $B = \mathrm{span}\{e_{ij} + I \mid i \leq j,\ j = 1, 2, 3\}$. Then B is an L-subalgebra of A^{θ_A} (i.e., invariant under $\theta_A(L)$), and B is isomorphic to UT_3 as an ordinary algebra via some isomorphism $\chi : B \to UT_3$, transporting this action along χ, i.e. setting $\tilde{\theta}(d) := \chi \circ \theta_A(d)|_B \circ \chi^{-1}$ for $d \in L$, yields a Lie algebra homomorphism $\tilde{\theta} : L \to \mathrm{Der}(UT_3)$ such that $\chi(a^{\theta_A(d)}) = \chi(a)^{\tilde{\theta}(d)}$ for all $a \in B,\ d \in L$; that is, χ is an L-isomorphism between B (with this action) and $UT_3^{\tilde{\theta}}$. Consequently, $\mathrm{Id}^L(B) = \mathrm{Id}^L(UT_3^{\tilde{\theta}})$, and hence $\mathrm{var}^L(B) = \mathrm{var}^L(UT_3^{\tilde{\theta}})$.

Applying the first part of the proof to $\tilde{\theta}$, we conclude that $\mathrm{var}^L(UT_3^{\tilde{\theta}})$ contains at least one of the algebras

$$UT_3, \quad UT_3^{\varepsilon_1}, \quad UT_3^{\varepsilon_2}, \quad UT_3^{\varepsilon_1 + \beta\varepsilon_2}, \quad UT_3^{\varepsilon_1, \varepsilon_2},$$

where $\beta \in F \setminus \{0\}$. Since $\mathrm{var}^L(UT_3^{\tilde{\theta}}) = \mathrm{var}^L(B) \subseteq \mathrm{var}^L(A) \subseteq \mathrm{var}^L(UT_k^\theta)$, it follows that $\mathrm{var}^L(UT_k^\theta)$ contains at least one of the algebras listed above. □

Next we determine the T_L-ideals of the L-algebras UT_3, $UT_3^{\varepsilon_1}$, $UT_3^{\varepsilon_2}$, $UT_3^{\varepsilon_1 + \beta\varepsilon_2}$, with $\beta \neq 0$, and $UT_3^{\varepsilon_1, \varepsilon_2}$.

For $k \geq 2$, we denote by $[x_1, x_2, \ldots, x_k]$ the left-normed commutator of length k in $x_1, x_2, \ldots, x_k$. Moreover, we call any variable with an exponent different from $1_{U(L)}$ a commutator of length 1. We begin with the following general technical results concerning left-normed commutators.

Lemma 4.3. *For any $n \geq 4$ and $3 \leq a \leq n - 1$, we have*

$$[x_1, \ldots, x_a, x_{a+1}, \ldots, x_n] = [x_1, \ldots, x_{a-1}, x_{a+1}, x_a, x_{a+2}, \ldots, x_n] + \Gamma,$$

where Γ is a linear combination of polynomials of the form

$$[x_1, x_2, x_{i_1}, \ldots, x_{i_k}][x_a, x_{a+1}, x_{j_1}, \ldots, x_{j_{n-k-4}}],$$
$$[x_a, x_{a+1}, x_{i_1}, \ldots, x_{i_k}][x_1, x_2, x_{j_1}, \ldots, x_{j_{n-k-4}}],$$

where $0 \leq k \leq n - 4$.

Proof. By the Jacobi identity, it follows that

$$[[x_1, x_2], [x_3, x_4], x_5] = [[x_1, x_2, x_5], [x_3, x_4]] + [[x_1, x_2], [x_3, x_4, x_5]].$$

As a consequence, for $n \geq 4$ and $3 \leq a \leq n - 1$, we have that $[[x_1, \ldots, x_{a-1}], [x_a, x_{a+1}], x_{a+2}, \ldots, x_n]$ is a linear combination of products of the form

$$[x_1, x_2, x_{i_1}, \ldots, x_{i_k}][x_a, x_{a+1}, x_{j_1}, \ldots, x_{j_{n-k-4}}],$$
$$[x_a, x_{a+1}, x_{i_1}, \ldots, x_{i_k}][x_1, x_2, x_{j_1}, \ldots, x_{j_{n-k-4}}],$$

where $0 \leq k \leq n - 4$. Hence, since

$$\begin{aligned}[x_1, \ldots, x_a, x_{a+1}, \ldots, x_n] =& [[x_1, \ldots, x_{a-1}], [x_a, x_{a+1}], x_{a+2}, \ldots, x_n] \\ &+ [x_1, \ldots, x_{a-1}, x_{a+1}, x_a, x_{a+2}, \ldots, x_n],\end{aligned}$$

for $n \geq 4$ and $3 \leq a \leq n - 1$, we obtain the desired conclusion. □

Lemma 4.4. *Let $u \in U(L)$. For any $n \geq 3$, the commutator $[x_1^u[x_2, x_3], x_4, \ldots, x_n]$ is a linear combination of L-polynomial of the form*

$$[x_1^u, x_{i_1}, \ldots, x_{i_k}][x_2, x_3, x_{j_1}, \ldots, x_{j_{n-k-3}}],$$

where $0 \leq k \leq n - 3$.

Proof. If $n = 3$, there is nothing to prove. So, assume $n = 4$. Then

$$\begin{aligned}[x_1^u[x_2, x_3], x_4] &= x_1^u[x_2, x_3]x_4 - x_4x_1^u[x_2, x_3] = x_1^u[x_2, x_3, x_4] + x_1^u x_4[x_2, x_3] - x_4x_1^u[x_2, x_3] \\ &= x_1^u[x_2, x_3, x_4] + [x_1^u, x_4][x_2, x_3],\end{aligned}$$

which establishes the claim for $n = 4$. Iterating this procedure for general n we obtain the desired conclusion. □

Similarly, we can prove the following.

Lemma 4.5. *Let $u \in U(L)$. For any $n \geq 3$, the commutator $[[x_1, x_2]x_3^u, x_4, \ldots, x_n]$ is a linear combination of L-polynomial of the form*

$$[x_1, x_2, x_{i_1}, \ldots, x_{i_k}][x_3^u, x_{j_1}, \ldots, x_{j_{n-k-3}}],$$

where $0 \leq k \leq n - 3$.

Lemma 4.6. *Let $u_1, u_2 \in U(L)$. For any $n \geq 2$, the commutator $[x_1^{u_1}x_2^{u_2}, x_3, \ldots, x_n]$ can be written as a linear combination of L-polynomial of the form*

$$[x_1^{u_1}, x_{i_1}, \ldots, x_{i_k}][x_2^{u_2}, x_{j_1}, \ldots, x_{j_{n-k-2}}],$$

where $0 \leq k \leq n - 2$.

Proof. If $n = 2$, there is nothing to prove. So, assume $n = 3$. Then

$$[x_1^{u_1}x_2^{u_2}, x_3] = x_1^{u_1}x_2^{u_2}x_3 - x_3x_1^{u_1}x_2^{u_2} = x_1^{u_1}[x_2^{u_2}, x_3] + x_1^{u_1}x_3x_2^{u_2} - x_3x_1^{u_1}x_2^{u_2} = x_1^{u_1}[x_2^{u_2}, x_3] + [x_1^{u_1}, x_3]x_2^{u_2},$$

which establishes the claim for $n = 3$. Iterating this procedure for general n we obtain the desired conclusion. □

Lemma 4.7. *Let $u \in U(L)$. For any $n \geq 3$ and $2 \leq a \leq n - 1$, we have*

$$[x_1^u, x_2, \ldots, x_a, x_{a+1}, \ldots, x_n] = [x_1^u, x_2, \ldots, x_{a-1}, x_{a+1}, x_a, x_{a+2}, \ldots, x_n] + \Gamma,$$

where Γ is a linear combination of L-polynomials of the form

$$\begin{aligned}&[x_1^u, x_{i_1}, \ldots, x_{i_k}][x_a, x_{a+1}, x_{j_1}, \ldots, x_{j_{n-k-3}}], \\ &[x_a, x_{a+1}, x_{i_1}, \ldots, x_{i_k}][x_1^u, x_{j_1}, \ldots, x_{j_{n-k-3}}],\end{aligned}$$

where $0 \leq k \leq n - 3$.

Proof. Let $n \geq 3$. If $3 \leq a \leq n - 1$, then the desired conclusion follows directly from Lemma 4.3.

It remains to consider the case $a = 2$. By the Jacobi identity, we obtain

$$[x_1^u, x_2, x_3, \ldots, x_n] = [x_1^u[x_2, x_3], x_4, \ldots, x_n] - [[x_2, x_3]x_1^u, x_4, \ldots, x_n] + [x_1^u, x_3, x_2, x_4, \ldots, x_n].$$

Applying Lemmas 4.4 and 4.5 to the first two terms yields the desired conclusion. □

Now, we determine the T_L-ideal of the differential identities of the L-algebra UT_3.

Theorem 4.8. *Let L be a Lie algebra and consider the algebra UT_3 where L acts trivially (i.e., $\theta = 0$). Then there exists a basis $\mathcal{B}_L = \{d_i \mid i \geq 1\}$ of L over F such that the T_L-ideal of L-identities of UT_3 is generated by the following L-polynomials*

$$x^{d_i}, \quad [x_1, x_2][x_3, x_4][x_5, x_6],$$

for all $i \geq 1$. Moreover, $c_n^L(UT_3) = (n^2 - 7n + 9)3^{n-2} + (3n - 6)2^{n-1} + 3$.

Proof. Since $x^d \equiv 0$ for all $d \in L$ is a differential identity of UT_3, then we are dealing with the ordinary polynomial identities, and by [6, Theorem 5.2.1] we get $\mathrm{Id}^L(UT_3) = \langle x^{d_i},\ [x_1, x_2][x_3, x_4][x_5, x_6] \mid i \geq 1\rangle_{T_L}$, and the following L-polynomials form a basis of P_n^L modulo $P_n^L \cap \mathrm{Id}^L(UT_3)$

$$x_{i_1} \ldots x_{i_r}[x_{j_1}, x_{j_2}, \ldots, x_{j_s}][x_{k_1}, x_{k_2}, \ldots, x_{k_t}],$$

where $r,s,t\geq 0$, $s,t\neq 1$, $i_1<\cdots<i_r$, $j_1>j_2<\cdots<j_s$, $k_1>k_2<\cdots<k_t$. Now, by counting these polynomials, we obtain

$$\begin{aligned}c_n^L(UT_3) =&1+\sum_{s=2}^{n}\binom{n}{s}(s-1)+\sum_{r=0}^{n-4}\binom{n}{r}\left(\sum_{s=2}^{n-r-2}\binom{n-r}{s}(s-1)(n-r-s-1)\right)\\ =&1+\sum_{s=2}^{n}\binom{n}{s}(s-1)+\sum_{m=4}^{n}\binom{n}{m}\left(\sum_{s=2}^{m-2}\binom{m}{s}\big(ms-s^2-(m-1)\big)\right)\\ =&1+\sum_{s=2}^{n}\binom{n}{s}(s-1)+\sum_{m=4}^{n}\binom{n}{m}\Big(m^2 2^{m-2}-5m2^{m-2}+2^m+2m-2\Big)\\ =&(n^2-7n+9)3^{n-2}+(3n-6)2^{n-1}+3,\end{aligned}$$

as claimed. □

Now, we determine the T_L-ideal of the differential identities of the L-algebra $UT_3^{\varepsilon_1}$. To this end, note that ε_1 acts on the matrix units in the following way:

$$(e_{11})^{\varepsilon_1}=(e_{22})^{\varepsilon_1}=(e_{33})^{\varepsilon_1}=(e_{23})^{\varepsilon_1}=0,\quad (e_{12})^{\varepsilon_1}=e_{12},\quad (e_{13})^{\varepsilon_1}=e_{13},\tag{4.1}$$

Now, we establish the following further useful lemmas.

Lemma 4.9. *Let $d\in L$. Then $x_1^d x_2^d,\ x_1^d x_2 x_3^d\in\langle x^{d^2}-x^d\rangle_{T_L}$.*

Proof. Using the Leibniz rule, we have that

$$(x_1x_2)^{d^2}-(x_1x_2)^d=x_1^{d^2}x_2+2x_1^dx_2^d+x_1x_2^{d^2}-x_1^dx_2-x_1x_2^d.$$

Since the characteristic of F is different from 2, it follows that $x_1^dx_2^d\in\langle x^{d^2}-x^d\rangle_{T_L}$. Moreover, since $(x_1x_2)^dx_3^d=x_1^dx_2x_3^d+x_1x_2^dx_3^d$, it follows that $x_1^dx_2x_3^d$ is a consequence of $x_1^dx_2^d$, and we are done. □

As a consequence of the above lemma, we obtain the following.

Lemma 4.10. *Let $d\in L$. Then*

$$x_1^d[x_2,x_3][x_4,x_5],\ x_1^dx_2[x_3,x_4][x_5,x_6]\in\langle x^{d^2}-x^d,\ \big([x_1,x_2]^d-[x_1,x_2]\big)[x_3,x_4]\rangle_{T_L}.$$

By straightforward computation, we obtain the following result.

Lemma 4.11. *Let $u\in U(L)$. Then $[x_1,x_2][x_3,x_4][x_5,x_6]\in\langle[x_1,x_2]x_3^u,\ \big([x_1,x_2]^u-[x_1,x_2]\big)[x_3,x_4]\rangle_{T_L}$.*

Theorem 4.12. *Let L be a Lie algebra and let $UT_3^{\varepsilon_1}$ be the L-algebra UT_3 where L acts via the one-dimensional Lie subalgebra of $\mathrm{Der}(UT_3)$ spanned by the inner derivation $\varepsilon_1=\mathrm{ad}_{(-e_{11})}$. Then there exists a basis $\mathcal{B}_L=\{d_i\mid i\geq 1\}$ of L over F such that the T_L-ideal of differential identities of $UT_3^{\varepsilon_1}$ is generated by the following L-polynomials*

$$x^{d_i},\quad x^{d_1^2}-x^{d_1},\quad [x_1,x_2]x_3^{d_1},\quad \big([x_1,x_2]^{d_1}-[x_1,x_2]\big)[x_3,x_4],$$

for all $i\geq 2$. Moreover, $c_n^L(UT_3^{\varepsilon_1})=(n^2-4n)3^{n-2}+(3n-2)2^{n-1}+2$.

Proof. Since L acts on $UT_3^{\varepsilon_1}$ by derivation, there exists a Lie algebra homomorphism $\theta\colon L\to\mathrm{Der}(UT_3)$ with image $\theta(L)=F\,\varepsilon_1$. Therefore, we may choose a basis $\mathcal{B}_L=\{d_i\mid i\geq 1\}$ of L such that

$$\theta(d_1)=\varepsilon_1\qquad\text{and}\qquad d_i\in\ker\theta\quad\text{for all } i\geq 2.\tag{4.2}$$

Let $I=\langle x^{d_i},\ x^{d_1^2}-x^{d_1},\ [x_1,x_2]x_3^{d_1},\ \big([x_1,x_2]^{d_1}-[x_1,x_2]\big)[x_3,x_4]\mid i\geq 2\rangle_{T_L}$. By (4.1), (4.2) and straightforward calculations, we have that $I\subseteq\mathrm{Id}^L(UT_3^{\varepsilon_1})$.

To prove the opposite inclusion, let f be a differential polynomial of P_n^L. By the Poincaré–Birkhoff–Witt (PBW) Theorem, f can be written as a linear combination of polynomials of the type

$$x_{i_1}^{u_{a_1}}\dots x_{i_r}^{u_{a_r}}c_1\dots c_b,\tag{4.3}$$

where $r,b\geq 0$, $i_1<\cdots<i_r$, $u_{a_1},\dots,u_{a_r}\in\mathcal{B}_{U(L)}$ and each $c_1,\dots,c_b$ is a left normed commutator in the variables $x_j^{u_e}$ with $u_e\in\mathcal{B}_{U(L)}$. Now, since $x^{d_i},\ x^{d_1^2}-x^{d_1}\in I$, $i\geq 2$, it follows that, modulo I, the exponents satisfy $u_{a_1},\dots,u_{a_r},u_e\in\{1_{U(L)},d_1\}$. Moreover, by Lemma 4.9, we have $x_1^{d_1}x_2^{d_1},\ x_1^{d_1}x_2x_3^{d_1}\in I$, and hence, modulo I, at most one variable in (4.3) may occur with exponent d_1. In addition, Lemma 4.11 yields $[x_1,x_2][x_3,x_4][x_5,x_6]\in I$, so that in (4.3) at most two left-normed commutators may appear modulo I; that is, $0\leq b\leq 2$. Furthermore, since $[x_1,x_2]x_3^{d_1}\in I$ and by the Leibniz rule also $[x_1,x_2]x_3x_4^{d_1}\in I$,

whenever a variable with exponent d_1 appears in (4.3), it must occur either in the tail $x_{i_1}^{u_{a_1}} \dots x_{i_r}^{u_{a_r}}$, i.e., $u_{a_l} = d_1$ for some $1 \leq l \leq r$, or inside the first commutator c_1. If d_1 appears as an exponent of a variable in c_1, then by Lemma 4.3 and by the Jacobi identity, we may assume that it occurs in the first entry of c_1.

Now, observe that for any $0 \leq k \leq r$,

$$x_{i_1} \cdots x_{i_k} x_{i_{k+1}}^{d_1} x_{i_{k+2}} \cdots x_{i_r} = x_{i_1} \cdots x_{i_k} x_{i_{k+2}} x_{i_{k+1}}^{d_1} x_{i_{k+3}} \cdots x_{i_r} + x_{i_1} \cdots x_{i_k} [x_{i_{k+1}}^{d_1}, x_{i_{k+2}}] x_{i_{k+3}} \cdots x_{i_r}.$$

Consequently, any monomial of the form $x_{i_1} \cdots x_{i_k} x_{i_{k+1}}^{d_1} x_{i_{k+2}} \cdots x_{i_r}$ can be expressed as a linear combination of polynomials of the type $x_{i_1} \cdots x_{i_k} [x_{j_1}^{d_1}, x_{j_2}, \dots, x_{j_{r-k}}]$. Therefore, since Lemma 4.10 gives $x_1^d [x_2, x_3][x_4, x_5]$, $x_1^d x_2 [x_3, x_4][x_5, x_6] \in I$, it follows that, modulo I, the polynomial f can be written as a linear combination of polynomials of the forms

$$x_{i_1} \dots x_{i_r} [x_{j_1}, x_{j_2}, \dots, x_{j_s}][x_{k_1}, x_{k_2}, \dots, x_{k_t}], \tag{4.4}$$

$$x_{i_1} \dots x_{i_r} [x_{h_1}^{d_1}, x_{h_2}, \dots, x_{h_q}][x_{m_1}, x_{m_2}, \dots, x_{m_w}], \tag{4.5}$$

where $r, s, t, w \geq 0$, $s, t, w \neq 1$, $q \geq 1$ such that $i_1 < \cdots < i_r$. By Lemma 4.3, we may assume, modulo I, that $j_3 < \cdots < j_s$, $k_3 < \cdots < k_t$ and $m_3 < \cdots < m_w$ in (4.4) and (4.5). Additionally, by applying the Jacobi identity, and, if necessary, Lemma 4.3 once more, we may further assume, modulo I, that $j_1 > j_2 < j_3$, $k_1 > k_2 < k_3$ and $m_1 > m_2 < m_3$ in (4.4) and (4.5). Moreover, by Lemma 4.7, we may suppose, modulo I, that $h_2 < \cdots < h_q$ in (4.5). Furthermore, since

$$[x_1^{d_1}, x_2][x_3, x_4] \equiv [x_2^{d_1}, x_1][x_3, x_4] + [x_1, x_2][x_3, x_4] \pmod{I},$$

if $w \geq 2$ in (4.5), we may also assume, modulo I, that $h_1 < h_2$. This implies that the following L-polynomials generate P_n^L modulo $P_n^L \cap I$:

$$\begin{aligned} &x_{i_1} \dots x_{i_r} [x_{j_1}, x_{j_2}, \dots, x_{j_s}][x_{k_1}, x_{k_2}, \dots, x_{k_t}], \\ &x_{i_1} \dots x_{i_r} [x_{h_1}^{d_1}, x_{h_2}, \dots, x_{h_q}][x_{m_1}, x_{m_2}, \dots, x_{m_w}], \end{aligned} \tag{4.6}$$

where $r, s, t, w \geq 0$, with $s, t, w \neq 1$, and $q \geq 1$ such that $i_1 < \cdots < i_r$, $j_1 > j_2 < \cdots < j_s$, $k_1 > k_2 < \cdots < k_t$, $m_1 > m_2 < \cdots < m_w$, $h_2 < \cdots < h_q$, and if $w \geq 2$, $h_1 < h_2$.

Next, we prove that these polynomials are also linearly independent modulo $\mathrm{Id}^L(UT_3^{\varepsilon_1})$. To this end, let $f \in \mathrm{Id}^L(UT_3^{\varepsilon_1})$ be a linear combination of the differential polynomials (4.6). By taking into account (4.1), we shall make suitable evaluations to prove that all the scalars appearing in f are zero, and this will complete the proof.

Since UT_3 has a unit element, we may, without loss of generality, disregard the tail $x_{i_1} \dots x_{i_r}$ ($i_1 < \cdots < i_r$), where all the variables have exponent $1_{U(L)}$; indeed, let $r_{\max}$ be the maximal tail length occurring in f, and fix a term with tail $x_{i_1} \dots x_{i_{r_{\max}}}$. For any evaluation φ with $\varphi(x_{i_1}) = \cdots = \varphi(x_{i_{r_{\max}}}) = 1_{UT_3}$: a term with this exact tail evaluates to the value of its commutator(s) alone, since 1_{UT_3} is a multiplicative identity; a term whose tail is different from $x_{i_1} \dots x_{i_{r_{\max}}}$ vanishes, since at least a variable in $\{x_{i_1}, \dots, x_{i_{r_{\max}}}\}$ occurs inside a commutator and evaluates to 1_{UT_3} there. As $f \equiv 0$, this yields $\varphi(f) = 0$ for every such φ, an equation involving only the coefficients of the terms with tail exactly $x_{i_1} \dots x_{i_{r_{\max}}}$; determining the coefficients as below shows they vanish. Repeating for every maximal tail, and then by decreasing induction on the tail length, shows all coefficients vanish, so we may assume that every variable $x_1, \dots, x_n$ that appears in f with exponent $1_{U(L)}$ occurs exclusively within a commutator, i.e., we may suppose that

$$\begin{aligned} f = &\sum_{\mathcal{J}_{j_1}, \mathcal{K}_{k_1}} \alpha_{\mathcal{J}_{j_1}, \mathcal{K}_{k_1}} [x_{j_1}, x_{j_2}, \dots, x_{j_s}][x_{k_1}, x_{k_2}, \dots, x_{k_t}] \\ &+ \sum_{\mathcal{H}_{h_1}, \mathcal{M}_{m_1}} \beta_{\mathcal{H}_{h_1}, \mathcal{M}_{m_1}} [x_{h_1}^{d_1}, x_{h_2}, \dots, x_{h_q}][x_{m_1}, x_{m_2}, \dots, x_{m_w}], \end{aligned}$$

where $\mathcal{J}_{j_1} = \{j_1, \dots, j_s\}$, $\mathcal{K}_{k_1} = \{k_1, \dots, k_t\}$, $\mathcal{H}_{h_1} = \{h_1, \dots, h_q\}$, $\mathcal{M}_{m_1} = \{m_1, \dots, m_w\}$, are subsets of indices in $\{1, \dots, n\}$ such that $\mathcal{J}_{j_1} \cap \mathcal{K}_{k_1} = \mathcal{H}_{h_1} \cap \mathcal{M}_{m_1} = \emptyset$, with $s \geq 2$, $t, w \geq 0$, $t, w \neq 1$, $q \geq 1$, $s + t = q + w = n$, and subjected to the conditions $j_1 > j_2 < \cdots < j_s$, $k_1 > k_2 < \cdots < k_t$, $m_1 > m_2 < \cdots < m_w$, $h_2 < \cdots < h_q$, and if $w \geq 2$, then $h_1 < h_2$.

For a fixed set $\mathcal{J}_{j_1} = \{j_1, \dots, j_n\}$ such that $j_1 > j_2 < \cdots < j_n$, if we evaluate $x_{j_1} = e_{23}$ and $x_{j_2} = \cdots = x_{j_n} = e_{33}$, then we get $\alpha_{\mathcal{J}_{j_1}, \emptyset} e_{23} = 0$ thus $\alpha_{\mathcal{J}_{j_1}, \emptyset} = 0$.

Now fixed the set $\mathcal{H}_{h_1} = \{h_1, \dots, h_n\}$ such that $h_2 < \cdots < h_n$, we make the evaluation $x_{h_1} = e_{13}$ and $x_{h_2} = \cdots = x_{h_n} = e_{33}$. In this case we get $\beta_{\mathcal{H}_{h_1}, \emptyset} e_{13} = 0$, so $\beta_{\mathcal{H}_{h_1}, \emptyset} = 0$.

Next, let $\mathcal{J}_{j_1} = \{j_1, \dots, j_s\}$ and $\mathcal{K}_{k_1} = \{k_1, \dots, k_t\}$ be two fixed disjoint subsets of $\{1, \dots, n\}$ with $s, t \geq 2$, $s + t = n$, and subject to the conditions $j_1 > j_2 < \cdots < j_s$, $k_1 > k_2 < \cdots < k_t$. From the evaluation $x_{j_1} = e_{12}$, $x_{j_2} = \cdots = x_{j_n} = e_{11}$, $x_{k_1} = e_{23}$ and $x_{k_2} = \cdots = x_{k_n} = e_{33}$, we get $\alpha_{\mathcal{J}_{j_1}, \mathcal{K}_{k_1}} e_{13} = 0$ thus $\alpha_{\mathcal{J}_{j_1}, \mathcal{K}_{k_1}} = 0$. Here remark that the polynomial $[x_{h_1}^{d_1}, x_{h_2}, \dots, x_{h_s}][x_{k_1}, x_{k_2}, \dots, x_{k_t}]$ evaluates to zero since $j_1 > j_2 < \cdots < j_s$ whereas $h_1 < h_2 < \cdots < h_s$ and $e_{11}^{\varepsilon_1} = 0$.

Finally, for any fixed $\mathcal{H}_{h_1} = \{h_1, \dots, h_q\}$ and $\mathcal{M}_{m_1} = \{m_1, \dots, m_w\}$ disjoint subset of $\{1, \dots, n\}$ with $q \geq 1$, $w \geq 2$, $q + w = n$, and subject to the conditions $h_1 < \cdots < h_q$, $m_1 > m_2 < \cdots < m_w$, we make the substitution $x_{h_1} = e_{12}$, $x_{h_2} = \cdots = x_{h_q} = e_{11}$, $x_{m_1} = e_{23}$ and $x_{m_2} = \cdots = x_{m_w} = e_{33}$. Then, we get $\beta_{\mathcal{H}_{h_1}, \mathcal{M}_{m_1}} e_{13} = 0$, that is $\beta_{\mathcal{H}_{h_1}, \mathcal{M}_{m_1}} = 0$.

Therefore, all the scalars appearing in f are zero, and as a consequence, the polynomials (4.6) are linearly independent modulo $\mathrm{Id}^L(UT_3^{\varepsilon_1})$. Since $P_n^L \cap I \subseteq P_n^L \cap \mathrm{Id}^L(UT_3^{\varepsilon_1})$, this proves that $\mathrm{Id}^L(UT_3^{\varepsilon_1}) = I$ and the L-polynomials (4.6) are a basis of P_n^L modulo $P_n^L \cap \mathrm{Id}^L(UT_3^{\varepsilon_1})$.

Finally, to determine the L-codimensions of $UT_3^{\varepsilon_1}$, we count the L-polynomials in (4.6). By Theorem 4.8, we have

$$\begin{aligned} c_n^L(UT_3^{\varepsilon_1}) =& c_n^L(UT_3) + \sum_{m=1}^{n} \binom{n}{m} m + \sum_{r=0}^{n-3} \binom{n}{r} \left(\sum_{w=2}^{n-r-1} \binom{n-r}{w} (w-1) \right) \\ =& (n^2 - 7n + 9)3^{n-2} + (3n-6)2^{n-1} + 3 + n2^{n-1} + (n-3)3^{n-1} + (4-n)2^{n-1} - 1 \\ =& (n^2 - 4n)3^{n-2} + (3n-2)2^{n-1} + 2, \end{aligned}$$

as claimed. □

With similar techniques, we can also compute the T_L-ideal of differential identities of $UT_3^{\varepsilon_2}$. Recall that ε_2 acts on the matrix units as follows:

$$(e_{11})^{\varepsilon_2} = (e_{22})^{\varepsilon_2} = (e_{33})^{\varepsilon_2} = (e_{12})^{\varepsilon_2} = 0, \quad (e_{23})^{\varepsilon_2} = e_{23}, \quad (e_{13})^{\varepsilon_2} = e_{13}. \tag{4.7}$$

In analogy with Lemmas 4.10 and 4.11, we establish the following results.

Lemma 4.13. *Let $d \in L$. Then*

$$[x_1, x_2][x_3, x_4]x_5^d,\ [x_1, x_2][x_3, x_4]x_5x_6^d \in \langle x^{d^2} - x^d,\ [x_1, x_2]\big([x_3, x_4]^d - [x_3, x_4]\big)\rangle_{T_L}.$$

Lemma 4.14. *Let $u \in U(L)$. Then $[x_1, x_2][x_3, x_4][x_5, x_6] \in \langle x_1^u[x_2, x_3],\ [x_1, x_2]\big([x_3, x_4]^u - [x_3, x_4]\big)\rangle_{T_L}$.*

Now, by using Lemmas 4.3, 4.7, 4.9, 4.13 and 4.14, and following step-by-step the lines of Theorem 4.12 with the necessary changes, we can also compute the T_L-ideal of differential identities of $UT_3^{\varepsilon_2}$.

Theorem 4.15. *Let L be a Lie algebra and $UT_3^{\varepsilon_2}$ be the L-algebra UT_3 where L acts via the one-dimensional Lie subalgebra of $\mathrm{Der}(UT_3)$ spanned by the inner derivation $\varepsilon_2 = \mathrm{ad}_{e_{33}}$. Then there exists a basis $\mathcal{B}_L = \{d_i \mid i \geq 1\}$ of L over F such that the T_L-ideal of L-identities of $UT_3^{\varepsilon_2}$ is generated by the following L-polynomials*

$$x^{d_i}, \quad x^{d_1^2} - x^{d_1}, \quad x_1^{d_1}[x_2, x_3], \quad [x_1, x_2]\big([x_3, x_4]^{d_1} - [x_3, x_4]\big),$$

for all $i \geq 2$. Moreover, $c_n^L(UT_3^{\varepsilon_2}) = (n^2 - 4n)3^{n-2} + (3n-2)2^{n-1} + 2$.

Remark 4.16 (General strategy)**.** We record here, once and for all, the five-step scheme used throughout this section to compute the T_L-ideal of identities and the L-codimension sequence of UT_3 endowed with an L-action; each theorem below only specifies the case-dependent data.

Step 1 (adapted basis). Fix $\theta : L \to \mathrm{Der}(UT_3)$ with the prescribed image, and a basis $\mathcal{B}_L = \{d_i \mid i \geq 1\}$ of L adapted to θ (i.e., $\theta(d_1) = \delta_1$, or $\theta(d_1) = \delta_1, \theta(d_2) = \delta_2$ in the two-dimensional case, and $d_i \in \ker\theta$ for the remaining indices).

Step 2 (inclusion $I \subseteq \mathrm{Id}^L(UT_3^{\theta})$). Let I be the T_L-ideal generated by the polynomials in the statement. Using the explicit action of $\theta(d_1)$ (and $\theta(d_2)$) on the matrix units, one checks directly that $I \subseteq \mathrm{Id}^L(UT_3^{\theta})$.

Step 3 (PBW reduction). By the PBW Theorem, every $f \in P_n^L$ is a linear combination of monomials $x_{i_1}^{u_{a_1}} \cdots x_{i_r}^{u_{a_r}} c_1 \cdots c_b$. Case-specific technical lemmas bound the exponents occurring modulo I, the number of commutators ($b \leq 2$), and force each non-trivial exponent into a specific position (tail or a prescribed entry of a commutator). In particular, for any $u \in \mathcal{B}_{U(L)}$ and $1 \leq k \leq r$, a variable with exponent u occurring in the tail can be moved into the first entry of a commutator via the identity

$$x_{i_1} \cdots x_{i_{k-1}} x_{i_k}^u x_{i_{k+1}} \cdots x_{i_r} = x_{i_1} \cdots x_{i_{k-1}} x_{i_{k+1}} x_{i_k}^u x_{i_{k+2}} \cdots x_{i_r} + x_{i_1} \cdots x_{i_{k-1}} [x_{i_k}^u, x_{i_{k+1}}] x_{i_{k+2}} \cdots x_{i_r}, \tag{4.8}$$

applied repeatedly (with k decreasing) to obtain a linear combination of polynomials with the variable of exponent u in the first entry of a commutator.

Step 4 (tail elimination). Since UT_3 has a unit, evaluating the variables of a maximal-length tail at 1_{UT_3} shows, by decreasing induction on the tail length, that every variable with exponent $1_{U(L)}$ may be assumed to occur inside a commutator.

Step 5 (linear independence via evaluations). For each family of indices indexing a candidate basis element, a suitable evaluation on matrix units, determined by the action of $\theta(d_1)$ (and $\theta(d_2)$), isolates its coefficient and shows it must vanish, proving linear independence; counting the resulting basis gives the L-codimension.

Next, we describe the T_L-ideal of differential identities of $UT_3^{\varepsilon_1+\beta\varepsilon_2}$ with $\beta \in F \setminus \{0\}$. To this end, notice that from (4.1) and (4.7), it follows that

$$(e_{11})^{\varepsilon_1+\beta\,\varepsilon_2} = (e_{22})^{\varepsilon_1+\beta\,\varepsilon_2} = (e_{33})^{\varepsilon_1+\beta\,\varepsilon_2} = 0,$$

$$(e_{12})^{\varepsilon_1+\beta\,\varepsilon_2} = e_{12}, \quad (e_{23})^{\varepsilon_1+\beta\,\varepsilon_2} = \beta e_{23}, \quad (e_{13})^{\varepsilon_1+\beta\,\varepsilon_2} = (1+\beta)e_{13}.$$

We begin with the case $\beta = -1$. In this situation, L acts on UT_3 as the one-dimensional Lie algebra generated by $\delta := \varepsilon_1 - \varepsilon_2 = \mathrm{ad}_{e_{22}}$. Moreover, if we set $\nu_{12} := 2^{-1}(\delta^2+\delta)$ and $\nu_{23} := 2^{-1}(\delta^2-\delta)$, we get

$$(e_{12})^{\nu_{12}} = e_{12}, \quad (e_{11})^{\nu_{12}} = (e_{22})^{\nu_{12}} = (e_{33})^{\nu_{12}} = (e_{23})^{\nu_{12}} = (e_{13})^{\nu_{12}} = 0, \tag{4.9}$$

and

$$(e_{23})^{\nu_{23}} = e_{23}, \quad (e_{11})^{\nu_{23}} = (e_{22})^{\nu_{23}} = (e_{33})^{\nu_{23}} = (e_{12})^{\nu_{23}} = (e_{13})^{\nu_{23}} = 0. \tag{4.10}$$

Now, we determine the T_L-ideal of the differential identities of the L-algebra $UT_3^{\varepsilon_1-\varepsilon_2}$. To this end, we first establish the following technical lemmas.

Lemma 4.17. *Let $d \in L$. If we denote $u_{12} := 2^{-1}(d^2+d)$ and $u_{23} := 2^{-1}(d^2-d)$, then*

$$x_1^{u_{12}}x_2^{u_{12}}, \quad x_1^{u_{23}}x_2^{u_{23}} \in \langle x^{d^3} - x^d \rangle_{T_L}.$$

Proof. By the Leibniz Rule, we have

$$(x_1x_2)^{d^3} - (x_1x_2)^d = x_1^{d^3}x_2 + x_1x_2^{d^3} + 3x_1^{d^2}x_2^d + 3x_1^dx_2^{d^2} - x_1^dx_2 - x_1x_2^d$$

and

$$(x_1x_2)^{d^4} - (x_1x_2)^{d^2} = x_1^{d^4}x_2 + x_1x_2^{d^4} + 4x_1^{d^3}x_2^d + 4x_1^dx_2^{d^3} + 6x_1^{d^2}x_2^{d^2} - x_1^{d^2}x_2 - x_1x_2^{d^2} - 2x_1^dx_2^d.$$

As a consequence, it follows that

$$x_1^{d^2}x_2^d + x_1^dx_2^{d^2},\ x_1^{d^2}x_2^{d^2} + x_1^dx_2^d \in \langle x^{d^3} - x^d \rangle_{T_L}.$$

Now, since $u_{12} = 2^{-1}(d^2+d)$ and $u_{23} := 2^{-1}(d^2-d)$, we have

$$x_1^{u_{12}}x_2^{u_{12}} = 4^{-1}(x_1^{d^2}x_2^{d^2} + x_1^{d^2}x_2^d + x_1^dx_2^{d^2} + x_1^dx_2^d) \in \langle x^{d^3} - x^d \rangle_{T_L}$$

and

$$x_1^{u_{23}}x_2^{u_{23}} = 4^{-1}(x_1^{d^2}x_2^{d^2} - x_1^{d^2}x_2^d - x_1^dx_2^{d^2} + x_1^dx_2^d) \in \langle x^{d^3} - x^d \rangle_{T_L},$$

as desired. □

As a consequence of the above lemma and by straightforward calculations, we obtain the following result.

Lemma 4.18. *Let $d \in L$. If we denote $u_{12} := 2^{-1}(d^2+d)$ and $u_{23} := 2^{-1}(d^2-d)$, then*

$$x_1^{u_{12}}[x_2,x_3]x_4^{u_{23}} \in \langle x^{d^3} - x^d,\ x_1^{u_{12}}\big([x_2,x_3]^{u_{23}} - [x_2,x_3]\big)\rangle_{T_L}.$$

Lemma 4.19. *Let $d \in L$. If we denote $u_{12} := 2^{-1}(d^2+d)$ and $u_{23} := 2^{-1}(d^2-d)$, then*

$$(x_1x_2)^{u_{12}} - x_1^{u_{12}}x_2 - x_1x_2^{u_{12}} + x_1^{u_{12}}x_2^{u_{23}} \quad \textit{and} \quad (x_1x_2)^{u_{23}} - x_1^{u_{23}}x_2 - x_1x_2^{u_{23}} + x_1^{u_{12}}x_2^{u_{23}}$$

are consequences of the L-polynomials in $\{x^{d^3} - x^d,\ x_1^{u_{23}}x_2^{u_{12}}\}$.

Proof. Since $d = u_{12} - u_{23}$, it follows that $x_1^d x_2^d = x_1^{u_{12}} x_2^{u_{12}} - x_1^{u_{12}} x_2^{u_{23}} - x_1^{u_{23}} x_2^{u_{12}} + x_1^{u_{23}} x_2^{u_{23}}$. Then, by Lemma 4.17, we get

$$x_1^d x_2^d \equiv -x_1^{u_{12}} x_2^{u_{23}} \pmod{\langle x^{d^3} - x^d,\ x_1^{u_{23}} x_2^{u_{12}} \rangle_{T_L}}.$$

As a consequence, since $u_{12} = 2^{-1}(d^2 + d)$ and $u_{23} = 2^{-1}(d^2 - d)$ and by the Leibniz rule, we obtain

$$(x_1 x_2)^{u_{12}} \equiv x_1^{u_{12}} x_2 + x_1 x_2^{u_{12}} - x_1^{u_{12}} x_2^{u_{23}} \pmod{\langle x^{d^3} - x^d,\ x_1^{u_{23}} x_2^{u_{12}} \rangle_{T_L}},$$

and

$$(x_1 x_2)^{u_{23}} \equiv x_1^{u_{23}} x_2 + x_1 x_2^{u_{23}} - x_1^{u_{12}} x_2^{u_{23}} \pmod{\langle x^{d^3} - x^d,\ x_1^{u_{23}} x_2^{u_{12}} \rangle_{T_L}},$$

as desired. □

Lemma 4.20. *Let $d \in L$. If we denote $u_{12} := 2^{-1}(d^2 + d)$ and $u_{23} := 2^{-1}(d^2 - d)$, then*

$$x_1^{u_{12}} x_2 x_3^{u_{12}},\ x_1^{u_{23}} x_2 x_3^{u_{23}},\ x_1^{u_{23}} x_2 x_3^{u_{12}} \in \langle x^{d^3} - x^d,\ x_1^{u_{23}} x_2^{u_{12}} \rangle_{T_L}.$$

Proof. From Lemma 4.19 it follows that

$$(x_1 x_2)^{u_{12}} x_3^{u_{12}} \equiv x_1^{u_{12}} x_2 x_3^{u_{12}} + x_1 x_2^{u_{12}} x_3^{u_{12}} - x_1^{u_{12}} x_2^{u_{23}} x_3^{u_{12}} \pmod{\langle x^{d^3} - x^d,\ x_1^{u_{23}} x_2^{u_{12}} \rangle_{T_L}}.$$

Consequently, by Lemma 4.17 we obtain that $x_1^{u_{12}} x_2 x_3^{u_{12}} \in \langle x^{d^3} - x^d,\ x_1^{u_{23}} x_2^{u_{12}} \rangle_{T_L}$.

Analogous arguments show that $x_1^{u_{23}} x_2 x_3^{u_{23}},\ x_1^{u_{23}} x_2 x_3^{u_{12}} \in \langle x^{d^3} - x^d,\ x_1^{u_{23}} x_2^{u_{12}} \rangle_{T_L}$. □

Lemma 4.21. *Let $d \in L$. If we denote $u_{12} := 2^{-1}(d^2 + d)$ and $u_{23} := 2^{-1}(d^2 - d)$, then*

$$[x_1^{u_{12}}, x_2][x_3, x_4] - [x_2^{u_{12}}, x_1][x_3, x_4] - [x_1, x_2][x_3, x_4]$$

is a consequence of the L-polynomials in

$$\big\{x^{d^3} - x^d,\ x_1^{u_{23}} x_2^{u_{12}},\ x_1^{u_{23}} [x_2, x_3],\ \big([x_1, x_2]^{u_{12}} - [x_1, x_2]\big)[x_3, x_4]\big\}.$$

Proof. From Lemma 4.19 we obtain

$$[x_1, x_2]^{u_{12}} \equiv [x_1^{u_{12}}, x_2] - [x_2^{u_{12}}, x_1] - x_1^{u_{12}} x_2^{u_{23}} + x_2^{u_{12}} x_1^{u_{23}} \pmod{\langle x^{d^3} - x^d,\ x_1^{u_{23}} x_2^{u_{12}} \rangle_{T_L}}.$$

Consequently,

$$\begin{aligned}\big([x_1, x_2]^{u_{12}} - [x_1, x_2]\big)[x_3, x_4] \equiv & [x_1^{u_{12}}, x_2][x_3, x_4] - [x_2^{u_{12}}, x_1][x_3, x_4] \\ & - [x_1, x_2][x_3, x_4] \pmod{\langle x^{d^3} - x^d,\ x_1^{u_{23}} x_2^{u_{12}},\ x_1^{u_{23}} [x_2, x_3] \rangle_{T_L}}.\end{aligned}$$

This proves the desired conclusion. □

Similarly, we can prove the following result.

Lemma 4.22. *Let $d \in L$. If we denote $u_{12} := 2^{-1}(d^2 + d)$ and $u_{23} := 2^{-1}(d^2 - d)$, then:*

(1) $[x_1^{u_{12}}, x_2] x_3^{u_{23}} - [x_2^{u_{12}}, x_1] x_3^{u_{23}} - [x_1, x_2] x_3^{u_{23}} \in \langle x^{d^3} - x^d,\ x_1^{u_{23}} x_2^{u_{12}},\ \big([x_1, x_2]^{u_{12}} - [x_1, x_2]\big) x_3^{u_{23}} \rangle_{T_L}$;

(2) $x_1^{u_{12}} [x_2^{u_{23}}, x_3] - x_1^{u_{12}} [x_3^{u_{23}}, x_2] - x_1^{u_{12}} [x_2, x_3] \in \langle x^{d^3} - x^d,\ x_1^{u_{23}} x_2^{u_{12}},\ x_1^{u_{12}} \big([x_2, x_3]^{u_{23}} - [x_2, x_3]\big) \rangle_{T_L}$.

Lemma 4.23. *Let $d \in L$. If we denote $u_{12} := 2^{-1}(d^2 + d)$ and $u_{23} := 2^{-1}(d^2 - d)$, then*

$$[x_1, x_2][x_3^{u_{23}}, x_4] - [x_1, x_2][x_4^{u_{23}}, x_3] - [x_1, x_2][x_3, x_4]$$

is a consequence of the L-polynomials in

$$\Big\{x^{d^3} - x^d,\ x_1^{u_{23}} x_2^{u_{12}},\ [x_1, x_2] x_3^{u_{12}},\ x_1^{u_{12}} \big([x_2, x_3]^{u_{23}} - [x_2, x_3]\big),\ \big([x_1, x_2]^{u_{12}} - [x_1, x_2]\big)[x_3, x_4]\Big\}.$$

Proof. Since

$$[x_1, x_2]\big([x_3, x_4]^{u_{23}} - [x_3, x_4]\big) \in \langle x_1^{u_{12}} \big([x_2, x_3]^{u_{23}} - [x_2, x_3]\big),\ \big([x_1, x_2]^{u_{12}} - [x_1, x_2]\big)[x_3, x_4] \rangle_{T_L},$$

the result follows by arguments analogous to those of Lemma 4.21. □

Remark 4.24. Since the commutator satisfies the Leibniz rule, for any $u_1, u_2 \in U(L)$ we have:

(1) $[x_1, x_2] x_3 x_4^{u_1} \in \langle [x_1, x_2] x_3^{u_1} \rangle_{T_L}$;

(2) $x_1^{u_2} x_2 [x_3, x_4] \in \langle x_1^{u_2} [x_2, x_3] \rangle_{T_L}$;

(3) $x_1^{u_1} x_2 [x_3, x_4] x_5^{u_2}, x_1^{u_1} [x_2, x_3] x_4 x_5^{u_2} \in \langle x_1^{u_1} [x_2, x_3] x_4^{u_2} \rangle_{T_L}$.

Now, by using the above results and following the scheme of Remark 4.16, we can compute the T_L-ideal of differential identities of $UT_3^{\varepsilon_1 - \varepsilon_2}$.

Theorem 4.25. *Let L be a Lie algebra and $UT_3^{\varepsilon_1-\varepsilon_2}$ be the L-algebra UT_3 where L acts via the one-dimensional Lie subalgebra of* $\mathrm{Der}(UT_3)$ *spanned by the inner derivation $\varepsilon_1-\varepsilon_2=\mathrm{ad}_{e_{22}}$. Then there exists a basis $\mathcal{B}_L=\{d_i \mid i\geq 1\}$ of L over F such that the T_L-ideal of L-identities of $UT_3^{\varepsilon_1-\varepsilon_2}$ is generated by the following L-polynomials*

$$x^{d_i},\quad x^{d_1^3}-x^{d_1},\quad x_1^{u_{23}}x_2^{u_{12}},\quad [x_1,x_2]x_3^{u_{12}},\quad x_1^{u_{23}}[x_2,x_3],\quad x_1^{u_{12}}\big([x_2,x_3]^{u_{23}}-[x_2,x_3]\big),$$
$$\big([x_1,x_2]^{u_{12}}-[x_1,x_2]\big)x_3^{u_{23}},\quad \big([x_1,x_2]^{u_{12}}-[x_1,x_2]\big)[x_3,x_4],$$

for all $i\geq 2$, where $u_{12}:=2^{-1}(d_1^2+d_1)$ and $u_{23}:=2^{-1}(d_1^2-d_1)$.

Moreover, $c_n^L(UT_3^{\varepsilon_1-\varepsilon_2})=(n^2-n)3^{n-2}+(3n-2)2^{n-1}+2$.

Proof. We follow the scheme of Remark 4.16, with $\theta(d_1)=\varepsilon_1-\varepsilon_2$; recall that $\varepsilon_1-\varepsilon_2$ acts on the matrix units as in (4.9)–(4.10), via $u_{12}=2^{-1}(d_1^2+d_1)$ and $u_{23}=2^{-1}(d_1^2-d_1)$, whose action annihilates every matrix unit except, respectively, e_{12} and e_{23}.

Step 2 is immediate from (4.9)–(4.10).

Step 3. Let $f\in P_n^L$. As in the proof of Theorem 4.12, the exponents occurring modulo I reduce to $\{1_{U(L)},d_1,d_1^2\}$ and as a consequence to $\{1_{U(L)},u_{12},u_{23}\}$, at most two left-normed commutators occur (Lemma 4.11), and, by Lemmas 4.17 and 4.20, at most one variable may carry exponent u_{12} and at most one may carry exponent u_{23}; so a monomial contains 0, 1, or 2 variables with exponent different from $1_{U(L)}$.

If none occurs, the argument is exactly as in Theorem 4.8 (the trivial action case), giving the ordinary family

$$x_{i_1}\cdots x_{i_r}[x_{j_1},\dots,x_{j_s}][x_{k_1},\dots,x_{k_t}],\qquad j_1>j_2<\cdots<j_s,\quad k_1>k_2<\cdots<k_t.$$

If exactly one occurs, with exponent u_{12} (resp. u_{23}), following the argument as in the proof of Theorem 4.12 for the single exponent d_1, using (4.8) (applied to u_{12}, resp. u_{23}) in place of the analogous identity there, and Lemma 4.21 (resp. Lemma 4.23) in place of the Leibniz rule argument fixing the order of the commutator in which appears u_{12} (resp. u_{23}); this gives the families

$$x_{i_1}\dots x_{i_r}[x_{h_1}^{u_{12}},x_{h_2},\dots,x_{h_{q_1}}][x_{m_1},x_{m_2},\dots,x_{m_{w_1}}],$$
$$x_{i_1}\dots x_{i_r}[x_{m_1'},x_{m_2'},\dots,x_{m_{w_2}'}][x_{h_1'}^{u_{23}},x_{h_2'},\dots,x_{h_{q_2}'}],$$

where $r,w_1,w_2\geq 0$, $w_1,w_2\neq 1$, $q_1,q_2\geq 1$, $i_1<\cdots<i_r$, $m_1>m_2<\cdots<m_{w_1}$, $m_1'>m_2'<\cdots<m_{w_2}'$, $h_2<\cdots<h_{q_1}$ $h_2'<\cdots<h_{q_2}'$. Additionally, if $w_1\geq 2$ then $h_1<h_2$ and if $w_2\geq 2$ then $h_1'<h_2'$.

The genuinely new case is when both u_{12} and u_{23} occur together in one monomial. Since $x_1^{u_{23}}x_2^{u_{12}}\in I$ and, by Lemma 4.20, also $x_1^{u_{23}}x_2x_3^{u_{12}}\in I$, the two corresponding variables must appear, modulo I, in the fixed order u_{12} before u_{23}. We examine the cases where the number of commutators b is equal to $2,1,0$ in turn.

Case $b=2$. Since $x_1^{u_{23}}[x_2,x_3]\in I$ and its Leibniz consequence (Remark 4.24), u_{23} must lie in the second commutator c_2; since

$$x_1^{u_{12}}[x_2,x_3]x_4^{u_{23}}\in I\qquad\text{(Lemma 4.18)}$$

and its Leibniz consequences (Remark 4.24), u_{12} must lie in the first commutator c_1; both are placed in the first entry of their commutator.

Case $b=1$. As above, u_{23} must lie in the single commutator, while u_{12} may initially lie in the tail or in the commutator. In the former case, applying (4.8) to u_{12} rewrites the polynomial as a linear combination of terms with two separate commutators, one containing u_{12} and the other containing u_{23}. In the latter case, Lemma 4.18 ensures that u_{12} and u_{23} occupy the first two positions of the commutator. Since $x_1^{u_{23}}x_2^{u_{12}}\in I$, we have

$$[x_1^{u_{12}},x_2^{u_{23}},x_3]\equiv[x_1^{u_{12}}x_2^{u_{23}},x_3]\pmod I,$$

and Lemma 4.6 splits this into two separate commutators, one for each exponent.

Case $b=0$. Applying (4.8) first moves the variable with exponent u_{23} into a commutator, giving a linear combination of terms

$$x_{i_1}\cdots x_{i_{a-1}}x_{i_a}^{u_{12}}x_{i_{a+1}}\cdots x_{i_l}[x_{j_1}^{u_{23}},x_{j_2},\dots,x_{j_{n-l}}];$$

applying again (4.8) to the remaining tail variable with exponent u_{12} then yields a linear combination of terms

$$x_{i_1}\cdots x_{i_r}[x_{k_1}^{u_{12}},x_{k_2},\dots,x_{k_m}][x_{j_1}^{u_{23}},x_{j_2},\dots,x_{j_{n-r-m}}].$$

In each of these three sub-cases, modulo I, both variables with exponent different from $1_{U(L)}$ lie in the first entries of two separate commutators, with u_{12} leading. Ordering the remaining entries via

Lemma 4.7, and fixing $l_1 < l_2$, $l'_1 < l'_2$ via Lemmas 4.21 and 4.23, we obtain, modulo I, the family $x_{i_1}\cdots x_{i_r}[x_{l_1}^{u_{12}},\dots,x_{l_{v_1}}][x_{l'_1}^{u_{23}},\dots,x_{l'_{v_2}}]$.

Combining the four families above, we conclude that the following L-polynomials generate P_n^L modulo $P_n^L \cap I$:

$$\begin{aligned}
&x_{i_1}\dots x_{i_r}[x_{j_1},x_{j_2},\dots,x_{j_s}][x_{k_1},x_{k_2},\dots,x_{k_t}],\\
&x_{i_1}\dots x_{i_r}[x_{h_1}^{u_{12}},x_{h_2},\dots,x_{h_{q_1}}][x_{m_1},x_{m_2},\dots,x_{m_{w_1}}],\\
&x_{i_1}\dots x_{i_r}[x_{m'_1},x_{m'_2},\dots,x_{m'_{w_2}}][x_{h'_1}^{u_{23}},x_{h'_2},\dots,x_{h'_{q_2}}],\\
&x_{i_1}\dots x_{i_r}[x_{l_1}^{u_{12}},x_{l_2},\dots,x_{l_{v_1}}][x_{l'_1}^{u_{23}},x_{l'_2},\dots,x_{l'_{v_2}}],
\end{aligned} \tag{4.11}$$

where $r,s,t,w_1,w_2 \geq 0$, with $s,t,w_1,w_2 \neq 1$, and $q_1,q_2,v_1,v_2 \geq 1$ such that $i_1 < \cdots < i_r$, $j_1 > j_2 < \cdots < j_s$, $k_1 > k_2 < \cdots < k_t$, $h_2 < \cdots < h_{q_1}$, $m_1 > m_2 < \cdots < m_{w_1}$, $h'_2 < \cdots < h'_{q_2}$, $m'_1 > m'_2 < \cdots < m'_{w_2}$, $l_1 < \cdots < l_{v_1}$, $l'_1 < \cdots < l'_{v_2}$. Additionally, if $w_1 \geq 2$, then $h_1 < h_2$, and if $w_2 \geq 2$, then $h'_1 < h'_2$.

Step 4 is as in the proof of Theorem 4.12: since UT_3 has a unit, we may, without loss of generality, disregard the tail $x_{i_1}\cdots x_{i_r}$. Hence, if $f \in \mathrm{Id}^L(UT_3^{\varepsilon_1-\varepsilon_2})$ is a linear combination of the polynomials in (4.11), we may suppose that

$$\begin{aligned}
f = &\sum_{\mathcal{J}_{j_1},\mathcal{K}_{k_1}} \alpha_{\mathcal{J}_{j_1},\mathcal{K}_{k_1}}[x_{j_1},x_{j_2},\dots,x_{j_s}][x_{k_1},x_{k_2},\dots,x_{k_t}]\\
&+\sum_{\mathcal{H}_{h_1},\mathcal{M}_{m_1}} \beta_{\mathcal{H}_{h_1},\mathcal{M}_{m_1}}[x_{h_1}^{u_{12}},x_{h_2},\dots,x_{h_{q_1}}][x_{m_1},x_{m_2},\dots,x_{m_{w_1}}]\\
&+\sum_{\mathcal{M}'_{m'_1},\mathcal{H}'_{h'_1}} \gamma_{\mathcal{M}'_{m'_1},\mathcal{H}'_{h'_1}}[x_{m'_1},x_{m'_2},\dots,x_{m'_{w_2}}][x_{h'_1}^{u_{23}},x_{h'_2},\dots,x_{h'_{q_2}}]\\
&+\sum_{\mathcal{L},\mathcal{L}'} \lambda_{\mathcal{L},\mathcal{L}'}[x_{l_1}^{u_{12}},x_{l_2},\dots,x_{l_{v_1}}][x_{l'_1}^{u_{23}},x_{l'_2},\dots,x_{l'_{v_2}}],
\end{aligned}$$

where where $\mathcal{J}_{j_1} = \{j_1,\dots,j_s\}$, $\mathcal{K}_{k_1} = \{k_1,\dots,k_t\}$, $\mathcal{H}_{h_1} = \{h_1,\dots,h_{q_1}\}$, $\mathcal{M}_{m_1} = \{m_1,\dots,m_{w_1}\}$, $\mathcal{H}'_{h'_1} = \{h'_1,\dots,h'_{q_2}\}$, $\mathcal{M}'_{m'_1} = \{m'_1,\dots,m'_{w_2}\}$, $\mathcal{L} = \{l_1,\dots,l_{v_1}\}$ and $\mathcal{L}' = \{l'_1,\dots,l'_{v_2}\}$ are subsets of indices in $\{1,\dots,n\}$ such that $\mathcal{J}_{j_1} \cap \mathcal{K}_{k_1} = \mathcal{H}_{h_1} \cap \mathcal{M}_{m_1} = \mathcal{H}'_{h'_1} \cap \mathcal{M}'_{m'_1} = \mathcal{L} \cap \mathcal{L}' = \emptyset$, with $s \geq 2$, $t,w_1,w_2 \geq 0$, $t,w_1,w_2 \neq 1$, $q_1,q_2,v_1,v_2 \geq 1$, $s+t = q_1+w_1 = w_2+q_2 = v_1+v_2 = n$, and subjected to the conditions $j_1 > j_2 < \cdots < j_s$, $k_1 > k_2 < \cdots < k_t$, $m_1 > m_2 < \cdots < m_{w_1}$, $m'_1 > m'_2 < \cdots < m'_{w_2}$, $l_1 < \cdots < l_{v_1}$, $l'_1 < \cdots < l'_{v_2}$, $h_2 < \cdots < h_{q_1}$, $h'_2 < \cdots < h'_{q_2}$. Additionally, if $w_1 \geq 2$, then $h_1 < h_2$, and if $w_2 \geq 2$, then $h'_1 < h'_2$.

Step 5. We show that the L-polynomials (4.11) are linearly independent modulo $\mathrm{Id}^L(UT_3^{\varepsilon_1-\varepsilon_2})$; by (4.9) and (4.10), we make suitable evaluations to prove that all the coefficients of f are zero.

Let $\mathcal{J}_{j_1} = \{j_1,\dots,j_n\}$ be a fixed set such that $j_1 > j_2 < \cdots < j_n$. If we evaluate $x_{j_1} = e_{13}$ and $x_{j_2} = \cdots = x_{j_n} = e_{33}$, then we get $\alpha_{\mathcal{J}_{j_1},\emptyset}\, e_{13} = 0$, and hence $\alpha_{\mathcal{J}_{j_1},\emptyset} = 0$.

Now, let $\mathcal{H}_{h_1} = \{h_1,\dots,h_n\}$ be a fixed set such that $h_2 < \cdots < h_n$. By evaluating $x_{h_1} = e_{12}$ and $x_{h_2} = \cdots = x_{h_n} = e_{22}$, we get $\beta_{\mathcal{H}_{h_1},\emptyset}\, e_{12} = 0$, and so $\beta_{\mathcal{H}_{h_1},\emptyset} = 0$. Also, by making the evaluation $x_{h_1} = e_{23}$ and $x_{h_2} = \cdots = x_{h_n} = e_{33}$, we get $\gamma_{\emptyset,\mathcal{H}_{h_1}}\, e_{23} = 0$, and hence $\gamma_{\emptyset,\mathcal{H}_{h_1}} = 0$.

Next, let $\mathcal{J}_{j_1} = \{j_1,\dots,j_s\}$ and $\mathcal{K}_{k_1} = \{k_1,\dots,k_t\}$ be two fixed disjoint subsets of $\{1,\dots,n\}$ such that $s,t \geq 2$, $s+t = n$ and $j_1 > j_2 < \cdots < j_s$, $k_1 > k_2 < \cdots < k_t$. If we evaluate $x_{j_1} = e_{12}$, $x_{j_2} = \cdots = x_{j_s} = e_{11}$, $x_{k_1} = e_{23}$ and $x_{k_2} = \cdots = x_{k_t} = e_{33}$, then we get $\alpha_{\mathcal{J}_{j_1},\mathcal{K}_{k_1}}\, e_{13} = 0$ and hence $\alpha_{\mathcal{J}_{j_1},\mathcal{K}_{k_1}} = 0$. Here the polynomials of the form $[x_{h_1}^{u_{12}},x_{h_2},\dots,x_{h_s}][x_{k_1},x_{k_2},\dots,x_{k_t}]$ evaluate to zero since $j_1 > j_2 < \cdots < j_s$ whereas $h_1 < h_2 < \cdots < h_s$ and $e_{11}^{\nu_{12}} = 0$. Similarly, the polynomials of the form $[x_{j_1},x_{j_2},\dots,x_{j_s}][x_{h'_1}^{u_{23}},x_{h'_2},\dots,x_{h'_t}]$ evaluate to zero since $k_1 > k_2 < \cdots < k_t$ whereas $h'_1 < h'_2 < \cdots < h'_t$ and $e_{33}^{\nu_{23}} = 0$. Also, with the same reasoning, all the polynomials of the type $[x_{l_1}^{u_{12}},x_{l_2},\dots,x_{l_{v_1}}][x_{l'_1}^{u_{23}},x_{l'_2},\dots,x_{l'_{v_2}}]$ evaluate to zero.

Let $\mathcal{H}_{h_1} = \{h_1,\dots,h_{q_1}\}$ and $\mathcal{M}_{m_1} = \{m_1,\dots,m_{w_1}\}$ be two fixed disjoint subsets of $\{1,\dots,n\}$ with $q_1 \geq 1$, $w_1 \geq 2$, $q_1+w_1 = n$ and subject to the conditions $h_1 < h_2 < \cdots < h_{q_1}$, $m_1 > m_2 < \cdots < m_{w_1}$. If we make the evaluation $x_{h_1} = e_{12}$, $x_{h_2} = \cdots = x_{h_{q_1}} = e_{11}$, $x_{m_1} = e_{23}$ and $x_{m_2} = \cdots = x_{m_{w_1}} = e_{33}$, we get $\beta_{\mathcal{H}_{h_1},\mathcal{M}_{m_1}}\, e_{13} = 0$, that is $\beta_{\mathcal{H}_{h_1},\mathcal{M}_{m_1}} = 0$.

Similarly, if we evaluate $x_{h_1} = e_{23}$, $x_{h_2} = \cdots = x_{h_{q_1}} = e_{33}$, $x_{m_1} = e_{12}$ and $x_{m_2} = \cdots = x_{m_{w_1}} = e_{11}$, we get $\gamma_{\mathcal{M}_{m_1},\mathcal{H}_{h_1}}\, e_{13} = 0$, and so $\gamma_{\mathcal{M}_{m_1},\mathcal{H}_{h_1}} = 0$.

Finally, fixed $\mathcal{L}=\{l_1,\dots,l_{v_1}\}$ and $\mathcal{L}'=\{l'_1,\dots,l'_{v_2}\}$ disjoint subsets of $\{1,\dots,n\}$ such that $v_1,v_2\geq 1$, $v_1+v_2=n$ and $l_1<\cdots<l_{v_1}$, $l'_1<\cdots<l'_{v_2}$, then from the evaluation $x_{l_1}=e_{12}$, $x_{l_2}=\cdots=x_{l_{v_1}}=e_{11}$, $x_{l'_1}=e_{23}$ and $x_{l'_2}=\cdots=x_{l'_{v_2}}=e_{33}$, it follows that $\lambda_{\mathcal{L},\mathcal{L}'}\,e_{13}=0$ and consequently $\lambda_{\mathcal{L},\mathcal{L}'}=0$.

Therefore, all the scalars appearing in f are zero, and consequently, the L-polynomials (4.11) are linearly independent modulo $\mathrm{Id}^L(UT_3^{\varepsilon_1-\varepsilon_2})$. This proves that $\mathrm{Id}^L(UT_3^{\varepsilon_1-\varepsilon_2})=I$ and the L-polynomials (4.11) are a basis of P_n^L modulo $P_n^L\cap\mathrm{Id}^L(UT_3^{\varepsilon_1-\varepsilon_2})$.

Next we determine the L-codimensions of $UT_3^{\varepsilon_1-\varepsilon_2}$ by counting the L-polynomials (4.11). Hence, by Theorems 4.8 and 4.12, we get

$$\begin{aligned}c_n^L(UT_3^{\varepsilon_1-\varepsilon_2})=&2c_n^L(UT_3^{\varepsilon_1})-c_n^L(UT_3)+\sum_{r=0}^{n-2}\binom{n}{r}\left(\sum_{v=1}^{n-r-1}\binom{n-r}{v}\right)\\ =&2(n^2-4n)3^{n-2}+2(3n-2)2^{n-1}+4-(n^2-7n+9)3^{n-2}-(3n-6)2^{n-1}-3\\ &+3^n-2^{n+1}+1\\ =&(n^2-n)3^{n-2}+(3n-2)2^{n-1}+2,\end{aligned}$$

as claimed. □

Next, consider the case $\beta=1$. In this case, L acts on UT_3 as the one-dimensional Lie algebra generated by $\delta:=\varepsilon_1+\varepsilon_2$. Moreover, if we set $\nu_2:=2\delta-\delta^2$ and $\nu_{13}:=2^{-1}(\delta^2-\delta)$, we get

$$(e_{12})^{\nu_2}=e_{12},\quad (e_{23})^{\nu_2}=e_{23},\quad (e_{11})^{\nu_2}=(e_{22})^{\nu_2}=(e_{33})^{\nu_2}=(e_{13})^{\nu_2}=0, \tag{4.12}$$

and

$$(e_{13})^{\nu_{13}}=e_{13},\quad (e_{11})^{\nu_{13}}=(e_{22})^{\nu_{13}}=(e_{33})^{\nu_{13}}=(e_{12})^{\nu_{13}}=(e_{23})^{\nu_{13}}=0. \tag{4.13}$$

Our next goal is to determine the T_L-ideal of the differential identities of the L-algebra $UT_3^{\varepsilon_1+\varepsilon_2}$. We begin with several technical lemmas.

Lemma 4.26. *Let $d\in L$. If we denote $u_{13}:=2^{-1}(d^2-d)$, then $x_1^{u_{13}}x_2^{u_{13}}\in\langle x^{d^3}-3x^{d^2}+2x^d\rangle_{T_L}$.*

Proof. By the Leibniz Rule, we have

$$\begin{aligned}(x_1x_2)^{d^3}-3(x_1x_2)^{d^2}+2(x_1x_2)^d=&x_1^{d^3}x_2+x_1x_2^{d^3}+3x_1^{d^2}x_2^d+3x_1^dx_2^{d^2}\\ &-3x_1^{d^2}x_2-3x_1x_2^{d^2}-6x_1^dx_2^d+2x_1^dx_2+2x_1x_2^d\end{aligned}$$

and

$$\begin{aligned}(x_1x_2)^{d^4}-3(x_1x_2)^{d^3}+2(x_1x_2)^{d^2}=&x_1^{d^4}x_2+x_1x_2^{d^4}+4x_1^{d^3}x_2^d+4x_1^dx_2^{d^3}+6x_1^{d^2}x_2^{d^2}-3x_1^{d^3}x_2\\ &-3x_1x_2^{d^3}-9x_1^{d^2}x_2^d-9x_1^dx_2^{d^2}+2x_1^{d^2}x_2+2x_1x_2^{d^2}+4x_1^dx_2^d.\end{aligned}$$

As a consequence, it follows that

$$x_1^{d^2}x_2^d+x_1^dx_2^{d^2}-2x_1^dx_2^d,\ x_1^{d^2}x_2^{d^2}-x_1^dx_2^d\in\langle x^{d^3}-3x^{d^2}+2x^d\rangle_{T_L}.$$

Now, since $u_{13}=2^{-1}(d^2-d)$, we have

$$x_1^{u_{13}}x_2^{u_{13}}=4^{-1}(x_1^{d^2}x_2^{d^2}-x_1^{d^2}x_2^d-x_1^dx_2^{d^2}+x_1^dx_2^d)\in\langle x^{d^3}-3x^{d^2}+2x^d\rangle_{T_L},$$

as desired. □

As a direct consequence of the above lemma, we obtain the following result.

Lemma 4.27. *Let $d\in L$. If we denote $u_2:=2d-d^2$ and $u_{13}:=2^{-1}(d^2-d)$, then:*

(1) $x_1^{u_{13}}[x_2,x_3]\in\langle x^{d^3}-3x^{d^2}+2x^d,\ x_1^{u_{13}}x_2^{u_2},\ [x_1,x_2]^{u_{13}}+[x_1,x_2]^{u_2}-[x_1,x_2]\rangle_{T_L}$;

(2) $[x_1,x_2]x_3^{u_{13}}\in\langle x^{d^3}-3x^{d^2}+2x^d,\ x_1^{u_2}x_2^{u_{13}},\ [x_1,x_2]^{u_{13}}+[x_1,x_2]^{u_2}-[x_1,x_2]\rangle_{T_L}$.

Lemma 4.28. *Let $d\in L$. If we denote $u_2:=2d-d^2$ and $u_{13}:=2^{-1}(d^2-d)$, then*

(1) $x_1^{u_2}x_2^{u_2}[x_3,x_4],\ x_1^{u_2}[x_2,x_3][x_4,x_5]\in\langle x_1^{u_2}x_2^{u_{13}},\ x_1^{u_2}x_2^{u_2}x_3^{u_2},\ [x_1,x_2]^{u_{13}}+[x_1,x_2]^{u_2}-[x_1,x_2]\rangle_{T_L}$;

(2) $[x_1,x_2]x_3^{u_2}x_4^{u_2},\ x_1^{u_2}[x_2,x_3]x_4^{u_2}\in\langle x_1^{u_{13}}x_2^{u_2},\ x_1^{u_2}x_2^{u_2}x_3^{u_2},\ [x_1,x_2]^{u_{13}}+[x_1,x_2]^{u_2}-[x_1,x_2]\rangle_{T_L}$.

Proof. We have

$$x_1^{u_2}x_2^{u_2}[x_3,x_4]^{u_{13}} + x_1^{u_2}x_2^{u_2}[x_3,x_4]^{u_2} - x_1^{u_2}x_2^{u_2}[x_3,x_4] \equiv -x_1^{u_2}x_2^{u_2}[x_3,x_4] \pmod{\langle x_1^{u_2}x_2^{u_{13}},\, x_1^{u_2}x_2^{u_2}x_3^{u_2}\rangle_{T_L}}$$

and, as a consequence,

$$x_1^{u_2}[x_2,x_3]^{u_{13}}[x_4,x_5] + x_1^{u_2}[x_2,x_3]^{u_2}[x_4,x_5] - x_1^{u_2}[x_2,x_3][x_4,x_5] \equiv -x_1^{u_2}[x_2,x_3][x_4,x_5] \pmod{\langle x_1^{u_2}x_2^{u_{13}},\, x_1^{u_2}x_2^{u_2}x_3^{u_2}\rangle_{T_L}}.$$

This proves (1). The proof of (2) is analogous. □

Lemma 4.29. *Let $d \in L$. If we denote $u_2 := 2d - d^2$ and $u_{13} := 2^{-1}(d^2 - d)$, then*

$$[x_1,x_2][x_3,x_4][x_5,x_6]$$

is a consequence of the L-polynomials in

$$\mathcal{S} := \{x^{d^3} - 3x^{d^2} + 2x^d,\ x_1^{u_2}x_2^{u_{13}},\ x_1^{u_{13}}x_2^{u_2},\ x_1^{u_2}x_2^{u_2}x_3^{u_2},\ [x_1,x_2]^{u_{13}} + [x_1,x_2]^{u_2} - [x_1,x_2]\}.$$

Proof. From item (1) of Lemmas 4.27 and 4.28, it follows that

$$[x_1,x_2]^{u_{13}}[x_3,x_4][x_5,x_6] + [x_1,x_2]^{u_2}[x_3,x_4][x_5,x_6] - [x_1,x_2][x_3,x_4][x_5,x_6] \equiv -[x_1,x_2][x_3,x_4][x_5,x_6] \pmod{\langle \mathcal{S}\rangle_{T_L}}.$$

This proves the desired conclusion. □

Lemma 4.30. *Let $d \in L$. If we denote $u_2 := 2d - d^2$ and $u_{13} := 2^{-1}(d^2 - d)$, then*

$$(x_1x_2)^{u_2} - x_1^{u_2}x_2 - x_1x_2^{u_2} + 2x_1^{u_2}x_2^{u_2} \quad \text{and} \quad (x_1x_2)^{u_{13}} - x_1^{u_{13}}x_2 - x_1x_2^{u_{13}} + x_1^{u_2}x_2^{u_2}$$

are consequences of the L-polynomials in $\{x^{d^3} - 3x^{d^2} + 2x^d,\ x_1^{u_2}x_2^{u_{13}},\ x_1^{u_{13}}x_2^{u_2}\}$.

Proof. Since $d = u_2 + 2u_{13}$, it follows that $x_1^d x_2^d = x_1^{u_2}x_2^{u_2} + 2x_1^{u_2}x_2^{u_{13}} + 2x_1^{u_{13}}x_2^{u_2} + 4x_1^{u_{13}}x_2^{u_{13}}$. Then, by Lemma 4.26, we get

$$x_1^d x_2^d - x_1^{u_2}x_2^{u_2} \in \langle x^{d^3} - 3x^{d^2} + 2x^d,\, x_1^{u_2}x_2^{u_{13}},\, x_1^{u_{13}}x_2^{u_2}\rangle_{T_L}.$$

As a consequence, since $u_2 = 2d - d^2$ and $u_{13} = 2^{-1}(d^2 - d)$ and by the Leibniz rule, we obtain

$$(x_1x_2)^{u_2} \equiv x_1^{u_2}x_2 + x_1x_2^{u_2} - 2x_1^{u_2}x_2^{u_2} \pmod{\langle x^{d^3} - 3x^{d^2} + 2x^d,\, x_1^{u_2}x_2^{u_{13}},\, x_1^{u_{13}}x_2^{u_2}\rangle_{T_L}},$$

and

$$(x_1x_2)^{u_{13}} \equiv x_1^{u_{13}}x_2 + x_1x_2^{u_{13}} - x_1^{u_2}x_2^{u_2} \pmod{\langle x^{d^3} - 3x^{d^2} + 2x^d,\, x_1^{u_2}x_2^{u_{13}},\, x_1^{u_{13}}x_2^{u_2}\rangle_{T_L}},$$

as desired. □

As a direct consequence of Lemma 4.30 we obtain the following.

Lemma 4.31. *Let $d \in L$. If we denote $u_2 := 2d - d^2$ and $u_{13} := 2^{-1}(d^2 - d)$, then:*

(1) $x_1^{u_2}x_2x_3^{u_{13}},\ x_1^{u_{13}}x_2x_3^{u_2},\ x_1^{u_{13}}x_2x_3^{u_{13}} \in \langle x^{d^3} - 3x^{d^2} + 2x^d,\, x_1^{u_2}x_2^{u_{13}},\, x_1^{u_{13}}x_2^{u_2}\rangle_{T_L}$;

(2) $x_1^{u_2}x_2x_3^{u_2}x_4^{u_2},\ x_1^{u_2}x_2^{u_2}x_3x_4^{u_2} \in \langle x^{d^3} - 3x^{d^2} + 2x^d,\, x_1^{u_2}x_2^{u_{13}},\, x_1^{u_{13}}x_2^{u_2},\, x_1^{u_2}x_2^{u_2}x_3^{u_2}\rangle_{T_L}$.;

(3) $x_1^{u_2}x_2x_3^{u_2}[x_4,x_5] \in \langle x^{d^3} - 3x^{d^2} + 2x^d,\ x_1^{u_2}x_2^{u_{13}},\, x_1^{u_2}x_2^{u_2}x_3^{u_2},\, [x_1,x_2]^{u_{13}} + [x_1,x_2]^{u_2} - [x_1,x_2]\rangle_{T_L}$;

(4) $[x_1,x_2]x_3^{u_2}x_4x_5^{u_2} \in \langle x^{d^3} - 3x^{d^2} + 2x^d,\ x_1^{u_2}x_2^{u_{13}},\, x_1^{u_2}x_2^{u_2}x_3^{u_2},\, [x_1,x_2]^{u_{13}} + [x_1,x_2]^{u_2} - [x_1,x_2]\rangle_{T_L}$.

Lemma 4.32. *Let $d \in L$. If we denote $u_2 := 2d - d^2$ and $u_{13} := 2^{-1}(d^2 - d)$, then:*

$$\begin{aligned}
&[x_1^{u_2},x_2]x_3^{u_2} - [x_2^{u_2},x_1]x_3^{u_2} - [x_1,x_2]x_3^{u_2},\\
&x_1^{u_2}[x_2^{u_2},x_3] - x_1^{u_2}[x_3^{u_2},x_2] - x_1^{u_2}[x_2,x_3],\\
&[x_1^{u_2},x_2][x_3,x_4] - [x_2^{u_2},x_1][x_3,x_4] - [x_1,x_2][x_3,x_4],\\
&[x_1,x_2][x_3^{u_2},x_4] - [x_1,x_2][x_4^{u_2},x_3] - [x_1,x_2][x_3,x_4],
\end{aligned}$$

are consequences of the L-polynomials in

$$\mathcal{S} := \Big\{x^{d^3} - 3x^{d^2} + 2x^d,\ x_1^{u_2}x_2^{u_{13}},\ x_1^{u_{13}}x_2^{u_2},\ x_1^{u_2}x_2^{u_2}x_3^{u_2},\ [x_1,x_2]^{u_{13}} + [x_1,x_2]^{u_2} - [x_1,x_2]\Big\}.$$

Proof. By Lemma 4.30,

$$[x_1,x_2]^{u_2} \equiv [x_1^{u_2},x_2]-[x_2^{u_2},x_1]-2[x_1^{u_2},x_2^{u_2}] \pmod{\langle x^{d^3}-3x^{d^2}+2x^d,\ x_1^{u_2}x_2^{u_{13}},\ x_1^{u_{13}}x_2^{u_2}\rangle_{T_L}}. \quad (4.14)$$

Hence,

$$[x_1^{u_2},x_2]x_3^{u_2}-[x_2^{u_2},x_1]x_3^{u_2}-[x_1,x_2]x_3^{u_2},\ x_1^{u_2}[x_2^{u_2},x_3]-x_1^{u_2}[x_3^{u_2},x_2]-x_1^{u_2}[x_2,x_3]\in\langle\mathcal{S}\rangle_{T_L}.$$

Combining (4.14) with Lemmas 4.27 and 4.28, we obtain

$$[x_1^{u_2},x_2][x_3,x_4]-[x_2^{u_2},x_1][x_3,x_4]-[x_1,x_2][x_3,x_4],\ [x_1,x_2][x_3^{u_2},x_4]-[x_1,x_2][x_4^{u_2},x_3]-[x_1,x_2][x_3,x_4]\in\langle\mathcal{S}\rangle_{T_L}.$$

□

As a direct consequence of Lemma 4.30 we also obtain the following.

Lemma 4.33. *Let $d\in L$. If we denote $u_2:=2d-d^2$ and $u_{13}:=2^{-1}(d^2-d)$, then:*

$$[x_1^{u_{13}},x_2]-[x_2^{u_{13}},x_2]+[x_1^{u_2},x_2]-[x_2^{u_2},x_1]-3[x_1^{u_2},x_2^{u_2}]-[x_1,x_2]$$

is a consequence of the L-polynomials in

$$\Big\{x^{d^3}-3x^{d^2}+2x^d,\ x_1^{u_2}x_2^{u_{13}},\ x_1^{u_{13}}x_2^{u_2},\ [x_1,x_2]^{u_{13}}+[x_1,x_2]^{u_2}-[x_1,x_2]\Big\}.$$

Again, by following a strategy similar to that of Theorems 4.12 and 4.25, and carrying out some additional computations, we can compute the T_L-ideal of differential identities of $UT_3^{\varepsilon_1+\varepsilon_2}$.

Theorem 4.34. *Let L be a Lie algebra and let $UT_3^{\varepsilon_1+\varepsilon_2}$ be the L-algebra UT_3 where L acts via the one-dimensional Lie subalgebra of $\mathrm{Der}(UT_3)$ spanned by the inner derivation $\varepsilon_1+\varepsilon_2=\mathrm{ad}_{(e_{33}-e_{11})}$. Then there exists a basis $\mathcal{B}_L=\{d_i\mid i\geq 1\}$ of L over F such that the T_L-ideal of L-identities of $UT_3^{\varepsilon_1+\varepsilon_2}$ is generated by the following L-polynomials*

$$x^{d_i},\quad x^{d_1^3}-3x^{d_1^2}+2x^{d_1},\quad x_1^{u_2}x_2^{u_{13}},\quad x_1^{u_{13}}x_2^{u_2},\quad x_1^{u_2}x_2^{u_2}x_3^{u_2},\quad [x_1,x_2]^{u_{13}}+[x_1,x_2]^{u_2}-[x_1,x_2],$$

for all $i\geq 2$, where $u_2:=2d_1-d_1^2$ and $u_{13}:=2^{-1}(d_1^2-d_1)$.

Moreover, $c_n^L(UT_3^{\varepsilon_1+\varepsilon_2})=(n^2-n)3^{n-2}+n2^n+1$.

Proof. We follow the scheme of Remark 4.16, with $\theta(d_1)=\varepsilon_1+\varepsilon_2$; recall that $\varepsilon_1+\varepsilon_2$ acts on the matrix units as in (4.12) and (4.13), via $u_2:=2d_1-d_1^2$ and $u_{13}:=2^{-1}(d_1^2-d_1)$, whose action annihilates every matrix unit except e_{12} and e_{23} (for u_2) and e_{13} (for u_{13}).

Step 2 is immediate from (4.12) and (4.13).

Step 3. Let $f\in P_n^L$. As in the proof of Theorem 4.12, since x^{d_i}, $x^{d_1^3}-3x^{d_1^2}+2x^{d_1}\in I$ for $i\geq 2$, the exponents occurring modulo I reduce to $\{1_{U(L)},u_2,u_{13}\}$; at most two left-normed commutators occur (Lemma 4.29); and, by Lemma 4.26 and items (1)–(2) of Lemma 4.31, a monomial may involve at most two distinct variables whose exponent differs from $1_{U(L)}$.

If no variable occurs with an exponent different from $1_{U(L)}$, the argument is exactly as in Theorem 4.8, giving the ordinary family

$$x_{i_1}\dots x_{i_r}[x_{j_1},x_{j_2},\dots,x_{j_s}][x_{k_1},x_{k_2},\dots,x_{k_t}],$$

where $r,s,t\geq 0$ with $s,t\neq 1$, and $i_1<\dots<i_r$, $j_1>j_2<\dots<j_s$, $k_1>k_2<\dots<k_t$.

Now suppose that exactly one variable appears with an exponent different from $1_{U(L)}$, namely either u_2 or u_{13}.

If this exponent is u_{13}, following the argument as in the proof of Theorem 4.12 for the single exponent d_1: by Lemma 4.27 and items (1)–(2) of Remark 4.24, the variable must occur, modulo I, either in the tail or in the first entry of the (unique) commutator, and (4.8) (applied to u_{13}) moves a tail occurrence into a commutator.

If this exponent is u_2, the argument mirrors the single-exponent case of Theorem 4.25 (applied twice, once with the variable of exponent u_2 occurring in the first commutator and once in the second): by Lemma 4.29 the variable must occur, modulo I, inside a commutator, and (4.8) (applied to u_2) moves a tail occurrence into a commutator, which by Lemma 4.3 and the Jacobi identity may be assumed to occur in its first entry. Therefore, this gives the families

$$x_{i_1}\dots x_{i_r}[x_{h_1}^{u_2},x_{h_2},\dots,x_{h_{q_1}}][x_{m_1},x_{m_2},\dots,x_{m_{w_1}}], \quad (4.15)$$

$$x_{i_1}\dots x_{i_r}[x_{m_1'},x_{m_2'},\dots,x_{m_{w_2}'}][x_{h_1'}^{u_2},x_{h_2'},\dots,x_{h_{q_2}'}], \quad (4.16)$$

$$x_{i_1}\dots x_{i_r}[x_{p_1}^{u_{13}},x_{p_2},\dots,x_{p_z}], \quad (4.17)$$

where $r, w_1 \geq 0$, $w_1 \neq 1$, $q_1, q_2, z \geq 1$, $w_2 \geq 2$ and $i_1 < \cdots < i_r$. By Lemma 4.3 and the Jacobi identity, we may assume, modulo I, that $m_1 > m_2 < \cdots < m_{w_1}$ in (4.15) and $m'_1 > m'_2 < \cdots < m'_{w_2}$ in (4.16). In addition, by Lemma 4.7, we may assume, modulo I, that $h_2 < \cdots < h_{q_1}$, $h'_2 < \cdots < h'_{q_2}$ and $p_2 < \cdots < p_z$ in (4.15), (4.16) and (4.17), respectively. Moreover, by Lemma 4.32, we may assume, modulo I, that $h'_1 < h'_2$ in (4.16), and also if $w_1 \geq 2$ in (4.15), then $h_1 < h_2$. Furthermore, by Lemma 4.33, we may also assume, modulo I, that $p_1 < p_2$ in (4.17).

Finally, suppose that a monomial g contains exactly two distinct variables whose exponents differ from $1_{U(L)}$; by Lemma 4.26 and item (1) of Lemma 4.31, both must equal u_2. We examine in turn the cases where the number of commutators b is equal to $2, 1, 0$.

Case $b = 2$. By Lemma 4.28, items (3)–(4) of Lemma 4.31, and the Leibniz rule for the commutator, every monomial with an occurrence of u_2 outside the first entry of c_1 or c_2 lies in I; as in Theorem 4.25, this forces the two occurrences of u_2 into the first entries of c_1 and c_2, respectively.

Case $b = 1$. As in the proof of Theorem 4.25, Lemma 4.28, items (3)–(4) of Lemma 4.31, and the Leibniz rule for the commutator imply that the two occurrences of u_2 must be distributed as follows: one occurs in the single commutator, while the other may initially occur either in the tail or in the commutator. In the former case, the occurrence of u_2 in the tail can be rewritten using (4.8), yielding a linear combination of terms involving two separate commutators, each containing one occurrence of u_2. In the latter case, both occurrences of u_2 are forced into the first two positions of the commutator by Lemma 4.28. Lemma 4.6 then decomposes it into two separate commutators, each containing one of the two occurrences of u_2.

Case $b = 0$. As in Theorem 4.25, applying (4.8) to each of the two tail occurrences in turn yields the family $x_{i_1} \cdots x_{i_r}[x_{k_1}^{u_2}, x_{k_2}, \ldots, x_{k_m}][x_{j_1}^{u_2}, x_{j_2}, \ldots, x_{j_{n-r-m}}]$.

In each of the above sub-case, modulo I, both occurrences of u_2 lie in the first entries of two separate commutators; ordering the remaining entries via Lemma 4.7 and fixing $l_1 < l_2$, $l'_1 < l'_2$ via Lemma 4.32, we obtain, modulo I, the family

$$x_{i_1} \ldots x_{i_r}[x_{l_1}^{u_2}, x_{l_2}, \ldots, x_{l_{v_1}}][x_{l'_1}^{u_2}, x_{l'_2}, \ldots, x_{l'_{v_2}}], \tag{4.18}$$

where $0 \leq r \leq n-2$ and $v_1, v_2 \geq 1$.

Combining the observations above, we conclude that the following L-polynomials generate P_n^L modulo $P_n^L \cap I$:

$$\begin{aligned}
&x_{i_1} \ldots x_{i_r}[x_{j_1}, x_{j_2}, \ldots, x_{j_s}][x_{k_1}, x_{k_2}, \ldots, x_{k_t}],\\
&x_{i_1} \ldots x_{i_r}[x_{h_1}^{u_2}, x_{h_2}, \ldots, x_{h_{q_1}}][x_{m_1}, x_{m_2}, \ldots, x_{m_{w_1}}],\\
&x_{i_1} \ldots x_{i_r}[x_{m'_1}, x_{m'_2}, \ldots, x_{m'_{w_2}}][x_{h'_1}^{u_2}, x_{h'_2}, \ldots, x_{h'_{q_2}}],\\
&x_{i_1} \ldots x_{i_r}[x_{l_1}^{u_2}, x_{l_2}, \ldots, x_{l_{v_1}}][x_{l'_1}^{u_2}, x_{l'_2}, \ldots, x_{l'_{v_2}}],\\
&x_{i_1} \ldots x_{i_r}[x_{p_1}^{u_{13}}, x_{p_2}, \ldots, x_{p_z}],
\end{aligned} \tag{4.19}$$

where $r, s, t, w_1 \geq 0$, with $s, t, w_1 \neq 1$, $q_1, q_2, v_1, v_2, z \geq 1$ and $w_2 \geq 2$ such that $i_1 < \cdots < i_r$, $j_1 > j_2 < \cdots < j_s$, $k_1 > k_2 < \cdots < k_t$, $h_2 < \cdots < h_{q_1}$, $m_1 > m_2 < \cdots < m_{w_1}$, $m'_1 > m'_2 < \cdots < m'_{w_2}$, $h'_1 < \cdots < h'_{q_2}$, $l_1 < \cdots < l_{v_1}$, $l'_1 < \cdots < l'_{v_2}$, $p_1 < \cdots < p_z$. Additionally, if $w_1 \geq 2$, then $h_1 < h_2$.

Step 4 is as in the proof of Theorem 4.12: since UT_3 has a unit element, we may, without loss of generality, disregard the tail $x_{i_1} \ldots x_{i_r}$, so we may suppose that every variable $x_1, \ldots, x_n$ that appears in a polynomial $f \in \mathrm{Id}^L(UT_3^{\varepsilon_1+\varepsilon_2})$ with exponent $1_{U(L)}$ occurs exclusively within a commutator. Thus, we may write

$$\begin{aligned}
f = &\sum_{\mathcal{J}_{j_1}, \mathcal{K}_{k_1}} \alpha_{\mathcal{J}_{j_1}, \mathcal{K}_{k_1}}[x_{j_1}, x_{j_2}, \ldots, x_{j_s}][x_{k_1}, x_{k_2}, \ldots, x_{k_t}]\\
&+ \sum_{\mathcal{H}_{h_1}, \mathcal{M}_{m_1}} \beta_{\mathcal{H}_{h_1}, \mathcal{M}_{m_1}}[x_{h_1}^{u_2}, x_{h_2}, \ldots, x_{h_{q_1}}][x_{m_1}, x_{m_2}, \ldots, x_{m_{w_1}}]\\
&+ \sum_{\mathcal{M}'_{m'_1}, \mathcal{H}'_{h'_1}} \gamma_{\mathcal{M}'_{m'_1}, \mathcal{H}'_{h'_1}}[x_{m'_1}, x_{m'_2}, \ldots, x_{m'_{w_2}}][x_{h'_1}^{u_2}, x_{h'_2}, \ldots, x_{h'_{q_2}}]\\
&+ \sum_{\mathcal{L}, \mathcal{L}'} \lambda_{\mathcal{L}, \mathcal{L}'}[x_{l_1}^{u_2}, x_{l_2}, \ldots, x_{l_{v_1}}][x_{l'_1}^{u_2}, x_{l'_2}, \ldots, x_{l'_{v_2}}] + \mu[x_1^{u_{13}}, x_2, \ldots, x_n],
\end{aligned}$$

where $\mathcal{J}_{j_1} = \{j_1, \ldots, j_s\}$, $\mathcal{K}_{k_1} = \{k_1, \ldots, k_t\}$, $\mathcal{H}_{h_1} = \{h_1, \ldots, h_{q_1}\}$, $\mathcal{M}_{m_1} = \{m_1, \ldots, m_{w_1}\}$, $\mathcal{H}'_{h'_1} = \{h'_1, \ldots, h'_{q_2}\}$, $\mathcal{M}'_{m'_1} = \{m'_1, \ldots, m'_{w_2}\}$, $\mathcal{L} = \{l_1, \ldots, l_{v_1}\}$ and $\mathcal{L}' = \{l'_1, \ldots, l'_{v_2}\}$ are subsets of $\{1, \ldots, n\}$

satisfying $\mathcal{J}_{j_1}\cap\mathcal{K}_{k_1}=\mathcal{H}_{h_1}\cap\mathcal{M}_{m_1}=\mathcal{H}'_{h'_1}\cap\mathcal{M}'_{m'_1}=\mathcal{L}\cap\mathcal{L}'=\emptyset$. Moreover, the indices fulfill the conditions $s,w_2\geq 2$, $t,w_1\geq 0$, $t,w_1\neq 1$, $q_1,q_2,v_1,v_2\geq 1$, $s+t=q_1+w_1=w_2+q_2=v_1+v_2=n$, together with the ordering constraints $j_1>j_2<\cdots<j_s$, $k_1>k_2<\cdots<k_t$, $m_1>m_2<\cdots<m_{w_1}$, $h_2<\cdots<h_{q_1}$, $m'_1>m'_2<\cdots<m'_{w_2}$, $h'_1<\cdots<h'_{q_2}$, $l_1<\cdots<l_{v_1}$, $l'_1<\cdots<l'_{v_2}$. In addition, if $w_1\geq 2$ then $h_1<h_2$.

Step 5. We show that the L-polynomials appearing in the decomposition of f above are linearly independent modulo $\mathrm{Id}^L(UT_3^{\varepsilon_1+\varepsilon_2})$.

Let $\mathcal{J}_{j_1}=\{j_1,\ldots,j_n\}$ be a fixed set such that $j_1>j_2<\cdots<j_n$. If we evaluate $x_{j_1}=e_{13}$ and $x_{j_2}=\cdots=x_{j_n}=e_{33}$, then we get $\alpha_{\mathcal{J}_{j_1},\emptyset}e_{13}=0$ since $j_1\neq 1$ and $e_{33}^{\nu_{13}}=0$. So, it follows that $\alpha_{\mathcal{J}_{j_1},\emptyset}=0$. Moreover, evaluating $x_1=e_{13}$ and $x_2=\cdots=x_n=e_{33}$, we get $\mu e_{13}=0$, and hence $\mu=0$.

Now, let $\mathcal{H}_{h_1}=\{h_1,\ldots,h_n\}$ be a fixed set such that $h_2<\cdots<h_n$. By evaluating $x_{h_1}=e_{12}$ and $x_{h_2}=\cdots=x_{h_n}=e_{22}$, we get $\beta_{\mathcal{H}_{h_1},\emptyset}e_{12}=0$, so $\beta_{\mathcal{H}_{h_1},\emptyset}=0$.

Next, let $\mathcal{J}_{j_1}=\{j_1,\ldots,j_s\}$ and $\mathcal{K}_{k_1}=\{k_1,\ldots,k_t\}$ be two fixed disjoint subsets of $\{1,\ldots,n\}$ such that $s,t\geq 2$, $s+t=n$ and $j_1>j_2<\cdots<j_s$, $k_1>k_2<\cdots<k_t$. If we evaluate $x_{j_1}=e_{12}$, $x_{j_2}=\cdots=x_{j_s}=e_{11}$, $x_{k_1}=e_{23}$ and $x_{k_2}=\cdots=x_{k_t}=e_{33}$, then we get $\alpha_{\mathcal{J}_{j_1},\mathcal{K}_{k_1}}e_{13}=0$, thus $\alpha_{\mathcal{J}_{j_1},\mathcal{K}_{k_1}}=0$. Here the polynomials $[x_{h_1}^{u_2},x_{h_2},\ldots,x_{h_s}][x_{k_1},x_{k_2},\ldots,x_{k_t}]$ evaluate to zero since $j_1>j_2<\cdots<j_s$ whereas $h_1<h_2<\cdots<h_s$ and $e_{11}^{\nu_2}=0$. Similarly, the polynomials $[x_{j_1},x_{j_2},\ldots,x_{j_s}][x_{h'_1}^{u_2},x_{h'_2},\ldots,x_{h'_t}]$ evaluate to zero since $k_1>k_2<\cdots<k_t$ whereas $h'_1<h'_2<\cdots<h'_t$ and $e_{33}^{\nu_2}=0$. Also, with the same reasoning all the polynomials of the type $[x_{l_1}^{u_2},x_{l_2},\ldots,x_{l_{v_1}}][x_{l'_1}^{u_2},x_{l'_2},\ldots,x_{l'_{v_2}}]$ evaluate to zero.

Let $\mathcal{H}_{h_1}=\{h_1,\ldots,h_{q_1}\}$ and $\mathcal{M}_{m_1}=\{m_1,\ldots,m_{w_1}\}$ be two fixed disjoint subsets of $\{1,\ldots,n\}$ with $q_1\geq 1$, $w_1\geq 2$, $q_1+w_1=n$ and subject to the conditions $h_1<h_2<\cdots<h_{q_1}$, $m_1>m_2<\cdots<m_{w_1}$. If we make the evaluation $x_{h_1}=e_{12}$, $x_{h_2}=\cdots=x_{h_{q_1}}=e_{11}$, $x_{m_1}=e_{23}$ and $x_{m_2}=\cdots=x_{m_{w_1}}=e_{33}$, we get $\beta_{\mathcal{H}_{h_1},\mathcal{M}_{m_1}}e_{13}=0$, that is $\beta_{\mathcal{H}_{h_1},\mathcal{M}_{m_1}}=0$. Notice that, as above, with this evaluation all the polynomials of the type $[x_{m'_1},x_{m'_2},\ldots,x_{m'_{w_2}}][x_{h'_1}^{u_2},x_{h'_2},\ldots,x_{h'_{q_2}}]$ and of the type $[x_{l_1}^{u_2},x_{l_2},\ldots,x_{l_{v_1}}][x_{l'_1}^{u_2},x_{l'_2},\ldots,x_{l'_{v_2}}]$ evaluate to zero. Similarly, if we evaluate $x_{h_1}=e_{23}$, $x_{h_2}=\cdots=x_{h_{q_1}}=e_{33}$, $x_{m_1}=e_{12}$ and $x_{m_2}=\cdots=x_{m_{w_1}}=e_{11}$, we get $\gamma_{\mathcal{M}_{m_1},\mathcal{H}_{h_1}}e_{13}=0$, and so $\gamma_{\mathcal{M}_{m_1},\mathcal{H}_{h_1}}=0$.

Finally, for fixed $\mathcal{L}=\{l_1,\ldots,l_{v_1}\}$ and $\mathcal{L}'=\{l'_1,\ldots,l'_{v_2}\}$ disjoint subsets of $\{1,\ldots,n\}$ such that $v_1,v_2\geq 1$, $v_1+v_2=n$ and $l_1<\cdots<l_{v_1}$, $l'_1<\cdots<l'_{v_2}$, then from the evaluation $x_{l_1}=e_{12}$, $x_{l_2}=\cdots=x_{l_{v_1}}=e_{11}$, $x_{l'_1}=e_{23}$ and $x_{l'_2}=\cdots=x_{l'_{v_2}}=e_{33}$, it follows that $\lambda_{\mathcal{L},\mathcal{L}'}e_{13}=0$ and consequently $\lambda_{\mathcal{L},\mathcal{L}'}=0$.

Therefore, all the scalars appearing in f are zero, and as a consequence the L-polynomials (4.19) are linearly independent modulo $\mathrm{Id}^L(UT_3^{\varepsilon_1+\varepsilon_2})$. Since $P_n^L\cap I\subseteq P_n^L\cap\mathrm{Id}^L(UT_3^{\varepsilon_1+\varepsilon_2})$, it follows that $\mathrm{Id}^L(UT_3^{\varepsilon_1+\varepsilon_2})=I$. Consequently, the L-polynomials (4.19) form a basis of P_n^L modulo $P_n^L\cap\mathrm{Id}^L(UT_3^{\varepsilon_1+\varepsilon_2})$.

Finally, by counting these L-polynomials, we determine the L-codimensions of $UT_3^{\varepsilon_1+\varepsilon_2}$. Hence, by Theorem 4.25, we get

$$c_n^L(UT_3^{\varepsilon_1+\varepsilon_2})=c_n^L(UT_3^{\varepsilon_1-\varepsilon_2})-\sum_{m=1}^{n}\binom{n}{m}m+\sum_{r=0}^{n-1}\binom{n}{r}=(n^2-n)3^{n-2}+n2^n+1,$$

as claimed. □

Finally, consider the case where $\beta\neq 0,\pm 1$. In this situation, L acts on UT_3 as the one-dimensional Lie algebra generated by $\delta:=\varepsilon_1+\beta\,\varepsilon_2$. Moreover, defining

$$\nu_{12}:=\frac{\delta^3-(2\beta+1)\delta^2+\beta(\beta+1)\delta}{\beta(\beta-1)},\ \nu_{23}:=\frac{-\delta^3+(\beta+2)\delta^2-(\beta+1)\delta}{\beta(\beta-1)},\ \nu_{13}:=\frac{\delta^3-(\beta+1)\delta^2+\beta\delta}{\beta(\beta+1)},$$

we obtain

$$(e_{12})^{\nu_{12}}=e_{12},\quad (e_{11})^{\nu_{12}}=(e_{22})^{\nu_{12}}=(e_{33})^{\nu_{12}}=(e_{13})^{\nu_{12}}=(e_{23})^{\nu_{12}}=0, \tag{4.20}$$

$$(e_{23})^{\nu_{23}}=e_{23},\quad (e_{11})^{\nu_{23}}=(e_{22})^{\nu_{23}}=(e_{33})^{\nu_{23}}=(e_{12})^{\nu_{23}}=(e_{13})^{\nu_{23}}=0, \tag{4.21}$$

$$(e_{13})^{\nu_{13}}=e_{13},\quad (e_{11})^{\nu_{13}}=(e_{22})^{\nu_{13}}=(e_{33})^{\nu_{13}}=(e_{12})^{\nu_{13}}=(e_{23})^{\nu_{13}}=0. \tag{4.22}$$

We now proceed to determine the T_L-ideal of differential identities of the L-algebra $UT_3^{\varepsilon_1+\beta\,\varepsilon_2}$. To this end, we first establish several technical lemmas.

Lemma 4.35. *Let $d \in L$ and $\beta \in F$ such that $\beta \neq 0, \pm 1$. If we denote $u_{12} := \frac{d^3-(2\beta+1)d^2+\beta(\beta+1)d}{\beta(\beta-1)}$, $u_{23} := \frac{-d^3+(\beta+2)d^2-(\beta+1)d}{\beta(\beta-1)}$ and $u_{13} := \frac{d^3-(\beta+1)d^2+\beta d}{\beta(\beta+1)}$, then*

$$(x_1x_2)^{u_{12}} - x_1^{u_{12}}x_2 - x_1x_2^{u_{12}} + x_1^{u_{12}}x_2^{u_{23}},$$
$$(x_1x_2)^{u_{23}} - x_1^{u_{23}}x_2 - x_1x_2^{u_{23}} + x_1^{u_{12}}x_2^{u_{23}},$$
$$(x_1x_2)^{u_{13}} - x_1^{u_{13}}x_2 - x_1x_2^{u_{13}} - x_1^{u_{12}}x_2^{u_{23}},$$

are consequences of the L-polynomials in

$$\mathcal{S}_1 := \left\{x_1^{u_{ij}}x_2^{u_{kl}} \ : \ j \neq k\right\}.$$

Proof. Since $d = u_{12} + \beta u_{23} + (\beta+1)u_{13}$ and $d^2 = u_{12} + \beta^2 u_{23} + (\beta+1)^2 u_{13}$, it follows that

$$x_1^d x_2^d - \beta x_1^{u_{12}}x_2^{u_{23}},\ x_1^{d^2}x_2^d - \beta x_1^{u_{12}}x_2^{u_{23}},\ x_1^d x_2^{d^2} - \beta^2 x_1^{u_{12}}x_2^{u_{23}} \in \langle \mathcal{S}_1\rangle_{T_L}.$$

As a consequence, since $u_{12} = \frac{d^3-(2\beta+1)d^2+\beta(\beta+1)d}{\beta(\beta-1)}$ and by applying the Leibniz rule, we obtain

$$\begin{aligned}(x_1x_2)^{u_{12}} &= x_1^{u_{12}}x_2 + x_1x_2^{u_{12}} + \frac{3}{\beta(\beta-1)}(x_1^{d^2}x_2^d + x_1^d x_2^{d^2}) - \frac{4\beta+2}{\beta(\beta-1)}x_1^d x_2^d \\ &\equiv x_1^{u_{12}}x_2 + x_1x_2^{u_{12}} - x_1^{u_{12}}x_2^{u_{23}} \pmod{\langle \mathcal{S}_1\rangle_{T_L}}.\end{aligned}$$

Similarly, we can prove the others and complete the proof. □

As a direct consequence of the above lemma, we have the following result.

Lemma 4.36. *Let $d \in L$ and $\beta \in F$ such that $\beta \neq 0, \pm 1$. If we denote $u_{12} := \frac{d^3-(2\beta+1)d^2+\beta(\beta+1)d}{\beta(\beta-1)}$, $u_{23} := \frac{-d^3+(\beta+2)d^2-(\beta+1)d}{\beta(\beta-1)}$ and $u_{13} := \frac{d^3-(\beta+1)d^2+\beta d}{\beta(\beta+1)}$, then*

$$x_1^{u_{ij}}x_2x_3^{u_{kl}}, \quad j \neq k,$$

are consequences of the L-polynomials in

$$\mathcal{S}_1 := \left\{x_1^{u_{ij}}x_2^{u_{kl}} \ : \ j \neq k\right\}.$$

By straightforward computation, we have the following.

Lemma 4.37. *Let $d \in L$ and $\beta \in F$ such that $\beta \neq 0, \pm 1$. If we denote $u_{12} := \frac{d^3-(2\beta+1)d^2+\beta(\beta+1)d}{\beta(\beta-1)}$, $u_{23} := \frac{-d^3+(\beta+2)d^2-(\beta+1)d}{\beta(\beta-1)}$ and $u_{13} := \frac{d^3-(\beta+1)d^2+\beta d}{\beta(\beta+1)}$, then:*

(1) $x_1^{u_{13}}[x_2,x_3] \in \langle x_1^{u_{13}}x_2^{u_{13}},\ x_1^{u_{13}}x_2^{u_{12}},\ x_1^{u_{13}}x_2^{u_{23}},\ [x_1,x_2]^{u_{13}} + [x_1,x_2]^{u_{12}} + [x_1,x_2]^{u_{23}} - [x_1,x_2]\rangle_{T_L}$;

(2) $[x_1,x_2]x_3^{u_{13}} \in \langle x_1^{u_{13}}x_2^{u_{13}},\ x_1^{u_{12}}x_2^{u_{13}},\ x_1^{u_{23}}x_2^{u_{13}},\ [x_1,x_2]^{u_{13}} + [x_1,x_2]^{u_{12}} + [x_1,x_2]^{u_{23}} - [x_1,x_2]\rangle_{T_L}$;

(3) $[x_1,x_2]x_3^{u_{12}} \in \langle x_1^{u_{13}}x_2^{u_{12}},\ x_1^{u_{12}}x_2^{u_{12}},\ x_1^{u_{23}}x_2^{u_{12}},\ [x_1,x_2]^{u_{13}} + [x_1,x_2]^{u_{12}} + [x_1,x_2]^{u_{23}} - [x_1,x_2]\rangle_{T_L}$;

(4) $x_1^{u_{23}}[x_2,x_3] \in \langle x_1^{u_{23}}x_2^{u_{13}},\ x_1^{u_{23}}x_2^{u_{12}},\ x_1^{u_{23}}x_2^{u_{23}},\ [x_1,x_2]^{u_{13}} + [x_1,x_2]^{u_{12}} + [x_1,x_2]^{u_{23}} - [x_1,x_2]\rangle_{T_L}$;

(5) $x_1^{u_{12}}[x_2,x_3]x_4^{u_{23}} \in \langle x_1^{u_{12}}x_2^{u_{13}},\ x_1^{u_{12}}x_2^{u_{12}},\ x_1^{u_{23}}x_2^{u_{23}},\ [x_1,x_2]^{u_{13}} + [x_1,x_2]^{u_{12}} + [x_1,x_2]^{u_{23}} - [x_1,x_2]\rangle_{T_L}$.

Lemma 4.38. *Let $d \in L$ and $\beta \in F$ such that $\beta \neq 0, \pm 1$. If we denote $u_{12} := \frac{d^3-(2\beta+1)d^2+\beta(\beta+1)d}{\beta(\beta-1)}$, $u_{23} := \frac{-d^3+(\beta+2)d^2-(\beta+1)d}{\beta(\beta-1)}$ and $u_{13} := \frac{d^3-(\beta+1)d^2+\beta d}{\beta(\beta+1)}$, then*

$$[x_1,x_2][x_3,x_4][x_5,x_6]$$

is a consequence of the L-polynomials in

$$\mathcal{S}_2 := \left\{[x_1,x_2]^{u_{13}} + [x_1,x_2]^{u_{12}} + [x_1,x_2]^{u_{23}} - [x_1,x_2],\ x_1^{u_{ij}}x_2^{u_{kl}} \ : \ j \neq k\right\}.$$

Proof. From items (1) and (4) of Lemma 4.37, we deduce that $\left([x_1,x_2]^{u_{12}} - [x_1,x_2]\right)[x_3,x_4] \in \langle \mathcal{S}_2\rangle_{T_L}$. Consequently, applying item (3) of Lemma 4.37 together with Lemma 4.11, we arrive at the desired conclusion. □

Lemma 4.39. *Let $d \in L$ and $\beta \in F$ such that $\beta \neq 0, \pm 1$. If we denote $u_{12} := \frac{d^3-(2\beta+1)d^2+\beta(\beta+1)d}{\beta(\beta-1)}$, $u_{23} := \frac{-d^3+(\beta+2)d^2-(\beta+1)d}{\beta(\beta-1)}$ and $u_{13} := \frac{d^3-(\beta+1)d^2+\beta d}{\beta(\beta+1)}$, then*

$$[x_1^{u_{12}},x_2][x_3,x_4] - [x_2^{u_{12}},x_1][x_3,x_4] - [x_1,x_2][x_3,x_4],$$
$$[x_1,x_2][x_3^{u_{23}},x_4] - [x_1,x_2][x_4^{u_{23}},x_3] - [x_1,x_2][x_3,x_4],$$

are consequences of the L-polynomials in

$$\mathcal{S}_2 := \left\{[x_1,x_2]^{u_{13}} + [x_1,x_2]^{u_{12}} + [x_1,x_2]^{u_{23}} - [x_1,x_2],\ x_1^{u_{ij}}x_2^{u_{kl}} \ : \ j \neq k\right\}.$$

Proof. From Lemma 4.35 we obtain

$$[x_1,x_2]^{u_{12}} \equiv [x_1^{u_{12}},x_2] - [x_2^{u_{12}},x_1] - x_1^{u_{12}}x_2^{u_{23}} + x_2^{u_{12}}x_1^{u_{23}} \pmod{\langle \mathcal{S}_2\rangle_{T_L}}.$$

and

$$[x_1,x_2]^{u_{23}} \equiv [x_1^{u_{23}},x_2] - [x_2^{u_{23}},x_1] - x_1^{u_{12}}x_2^{u_{23}} + x_2^{u_{12}}x_1^{u_{23}} \pmod{\langle \mathcal{S}_2\rangle_{T_L}}.$$

Consequently, by items (1)–(4) of Lemma 4.37, we obtain the desired conclusion. □

Similarly, as a direct consequence of Lemma 4.35, we also obtain the following two results.

Lemma 4.40. *Let $d \in L$ and $\beta \in F$ such that $\beta \neq 0, \pm 1$. If we denote $u_{12} := \frac{d^3-(2\beta+1)d^2+\beta(\beta+1)d}{\beta(\beta-1)}$, $u_{23} := \frac{-d^3+(\beta+2)d^2-(\beta+1)d}{\beta(\beta-1)}$ and $u_{13} := \frac{d^3-(\beta+1)d^2+\beta d}{\beta(\beta+1)}$, then*

$$[x_1^{u_{12}},x_2]x_3^{u_{23}} - [x_2^{u_{12}},x_1]x_3^{u_{23}} - [x_1,x_2]x_3^{u_{23}},$$
$$x_1^{u_{12}}[x_2^{u_{23}},x_3] - x_1^{u_{12}}[x_3^{u_{23}},x_2] - x_1^{u_{12}}[x_2,x_3],$$

are consequences of the L-polynomials in

$$\mathcal{S}_2 := \left\{ [x_1,x_2]^{u_{13}} + [x_1,x_2]^{u_{12}} + [x_1,x_2]^{u_{23}} - [x_1,x_2],\ x_1^{u_{ij}}x_2^{u_{kl}} \ :\ j \neq k \right\}.$$

Lemma 4.41. *Let $d \in L$ and $\beta \in F$ such that $\beta \neq 0, \pm 1$. If we denote $u_{12} := \frac{d^3-(2\beta+1)d^2+\beta(\beta+1)d}{\beta(\beta-1)}$, $u_{23} := \frac{-d^3+(\beta+2)d^2-(\beta+1)d}{\beta(\beta-1)}$ and $u_{13} := \frac{d^3-(\beta+1)d^2+\beta d}{\beta(\beta+1)}$, then*

$$[x_1^{u_{13}},x_2] - [x_2^{u_{13}},x_1] + [x_1^{u_{12}},x_2] - [x_2^{u_{12}},x_1] + [x_1^{u_{23}},x_2] - [x_2^{u_{23}},x_1] - [x_1,x_2] - x_1^{u_{12}}x_2^{u_{23}} + x_2^{u_{12}}x_1^{u_{23}}$$

is a consequence of the L-polynomials in

$$\mathcal{S}_2 := \left\{ [x_1,x_2]^{u_{13}} + [x_1,x_2]^{u_{12}} + [x_1,x_2]^{u_{23}} - [x_1,x_2],\ x_1^{u_{ij}}x_2^{u_{kl}} \ :\ j \neq k \right\}.$$

Once again, by adopting an approach analogous to that used in Theorems 4.12, 4.25, and 4.34, and performing some additional computations, we determine the T_L-ideal of differential identities of $UT_3^{\varepsilon_1+\beta\,\varepsilon_2}$.

Theorem 4.42. *Let L be a Lie algebra and $UT_3^{\varepsilon_1+\beta\,\varepsilon_2}$, $\beta \neq 0, \pm 1$, be the L-algebra UT_3 where L acts via the one-dimensional Lie subalgebra of $\mathrm{Der}(UT_3)$ spanned by the inner derivation $\varepsilon_1+\beta\,\varepsilon_2 = \mathrm{ad}_{(-e_{11}+\beta e_{33})}$. Then there exists a basis $\mathcal{B}_L = \{d_i \mid i \geq 1\}$ of L over F such that the T_L-ideal of L-identities of $UT_3^{\varepsilon_1+\beta\,\varepsilon_2}$ is generated by the following L-polynomials*

$$x^{d_i}, \quad x^{d_1^4} - 2(\beta+1)x^{d_1^3} + (\beta^2+3\beta+1)x^{d_1^2} - \beta(\beta+1)x^{d_1},$$
$$x_1^{u_{jk}}x_2^{u_{lm}}, \quad [x_1,x_2]^{u_{13}} + [x_1,x_2]^{u_{12}} + [x_1,x_2]^{u_{23}} - [x_1,x_2],$$

for all $i \geq 2$, $1 \leq j < k \leq 3$, $1 \leq l < m \leq 3$, $k \neq l$, where $u_{12} = \frac{d_1^3-(2\beta+1)d_1^2+\beta(\beta+1)d_1}{\beta(\beta-1)}$, $u_{23} = \frac{-d_1^3+(\beta+2)d_1^2-(\beta+1)d_1}{\beta(\beta-1)}$ and $u_{13} = \frac{d_1^3-(\beta+1)d_1^2+\beta d_1}{\beta(\beta+1)}$.

Moreover, $c_n^L(UT_3^{\varepsilon_1+\beta\,\varepsilon_2}) = (n^2-n)3^{n-2} + 3n2^{n-1} + 1$.

Proof. **Step 1** is as in Remark 4.16, with $\theta(d_1) = \varepsilon_1 + \beta\,\varepsilon_2$.

Step 2 is immediate from (4.20), (4.21) and (4.22).

Step 3. Let $f \in P_n^L$. Since $x^{d_i} \in I$ for $i \geq 2$ and $x^{d_1^4} - 2(\beta+1)x^{d_1^3} + (\beta^2+3\beta+1)x^{d_1^2} - \beta(\beta+1)x^{d_1} \in I$, the exponents occurring modulo I reduce to $\{1_{U(L)}, d_1, d_1^2, d_1^3\}$; hence, since $\beta \neq 0, \pm 1$, without loss of generality we may assume that they lie in $\{1_{U(L)}, u_{12}, u_{23}, u_{13}\}$. By Lemma 4.38, at most two left-normed commutators occur; and, since $x_1^{u_{ij}}x_2^{u_{kl}} \in I$ and, by Lemma 4.36, also $x_1^{u_{ij}}x_2x_3^{u_{kl}} \in I$ for $j \neq k$, a monomial may involve at most two distinct variables whose exponent differs from $1_{U(L)}$.

If no variable occurs with an exponent different from $1_{U(L)}$, then, as in the previous theorems, the argument is exactly as in Theorem 4.8, giving the ordinary family

$$x_{i_1}\dots x_{i_r}[x_{j_1},x_{j_2},\dots,x_{j_s}][x_{k_1},x_{k_2},\dots,x_{k_t}],$$

where $r,s,t \geq 0$ with $s,t \neq 1$, and the indices satisfy $i_1 < \dots < i_r$, $j_1 > j_2 < \dots < j_s$, $k_1 > k_2 < \dots < k_t$.

Now suppose that exactly one variable appears with an exponent different from $1_{U(L)}$, namely either u_{12} or u_{23} or u_{13}.

Assume first that this exponent is u_{13}. Following the argument as in the proof of Theorem 4.34: by Lemma 4.37 and items (1)–(2) of Remark 4.24, the variable must occur, modulo I, either in the tail or in the first entry of the (unique) commutator, and (4.8) (applied to u_{13}) moves a tail occurrence into a commutator.

Assume now that exactly one variable occurs with exponent u_{12} (resp. u_{23}). Following the argument in the proof of Theorem 4.25 and using item (3) (resp. item (4)) of Lemma 4.37 instead of the corresponding lemmas, we may assume, modulo I, that this variable occurs in the first entry of the first commutator (resp. in the first entry of either the only commutator or the second commutator).

Therefore, this gives the following families

$$x_{i_1}\dots x_{i_r}[x_{h_1}^{u_{12}}, x_{h_2}, \dots, x_{h_{q_1}}][x_{m_1}, x_{m_2}, \dots, x_{m_{w_1}}], \tag{4.23}$$

$$x_{i_1}\dots x_{i_r}[x_{m'_1}, x_{m'_2}, \dots, x_{m'_{w_2}}][x_{h'_1}^{u_{23}}, x_{h'_2}, \dots, x_{h'_{q_2}}], \tag{4.24}$$

$$x_{i_1}\dots x_{i_r}[x_{p_1}^{u_{13}}, x_{p_2}, \dots, x_{p_z}], \tag{4.25}$$

where $r, w_1, w_2 \geq 0$, $w_1, w_2 \neq 1$,, $q_1, q_2, z \geq 1$ and $i_1 < \dots < i_r$. By Lemma 4.3 and the Jacobi identity, we may assume, modulo I, that $m_1 > m_2 < \dots < m_{w_1}$ in (4.23) and $m'_1 > m'_2 < \dots < m'_{w_2}$ in (4.24). Moreover, by Lemma 4.7, we may assume, modulo I, that $h_2 < \dots < h_{q_1}$, $h'_2 < \dots < h'_{q_2}$ and $p_2 < \dots < p_z$ in (4.23), (4.24) and (4.25), respectively. In addition, by Lemma 4.39, we may further assume, modulo I, that if $w_1 \geq 2$ in (4.23), then $h_1 < h_2$; similarly, if $w_2 \geq 2$ in (4.24), then $h'_1 < h'_2$. Finally, by Lemma 4.41, we may also assume, modulo I, that $p_1 < p_2$ in (4.25).

Finally, suppose that an L-polynomial g contains exactly two distinct variables whose exponents differ from $1_{U(L)}$. Since $x_1^{u_{ij}}x_2^{u_{kl}} \in I$ for $j \neq k$ and, by Lemma 4.36, also $x_1^{u_{ij}}x_2x_3^{u_{kl}} \in I$ for $j \neq k$, it follows that, modulo I, the only possibility is that the exponents different from $1_{U(L)}$ are u_{12} and u_{23}, with the first variable carrying exponent u_{12} and the second carrying exponent u_{23}.

Therefore, taking into account items (3)–(5) of Lemma 4.37 together with Remark 4.24, and proceeding exactly as in Theorem 4.25, we deduce that if a polynomial contains exactly two variables whose exponents differ from $1_{U(L)}$, then, modulo I, it can be written as a linear combination of polynomials of the type

$$x_{i_1}\dots x_{i_r}[x_{l_1}^{u_{12}}, x_{l_2}, \dots, x_{l_{v_1}}][x_{l'_1}^{u_{23}}, x_{l'_2}, \dots, x_{l'_{v_2}}], \tag{4.26}$$

where $0 \leq r \leq n-2$, $v_1, v_2 \geq 1$ and $i_1 < \dots < i_r$. Moreover, by Lemma 4.7, we may assume, modulo I, that $l_2 < \dots < l_{v_1}$ and $l'_2 < \dots < l'_{v_2}$ in (4.26). In addition, by Lemma 4.40, we may further assume, modulo I, that $l_1 < l_2$ and $l'_1 < l'_2$ in (4.26).

By combining the arguments above, we conclude that the following L-polynomials generate P_n^L modulo $P_n^L \cap I$:

$$\begin{aligned}
&x_{i_1}\dots x_{i_r}[x_{j_1}, x_{j_2}, \dots, x_{j_s}][x_{k_1}, x_{k_2}, \dots, x_{k_t}],\\
&x_{i_1}\dots x_{i_r}[x_{h_1}^{u_{12}}, x_{h_2}, \dots, x_{h_{q_1}}][x_{m_1}, x_{m_2}, \dots, x_{m_{w_1}}],\\
&x_{i_1}\dots x_{i_r}[x_{m'_1}, x_{m'_2}, \dots, x_{m'_{w_2}}][x_{h'_1}^{u_{23}}, x_{h'_2}, \dots, x_{h'_{q_2}}],\\
&x_{i_1}\dots x_{i_r}[x_{l_1}^{u_{12}}, x_{l_2}, \dots, x_{l_{v_1}}][x_{l'_1}^{u_{23}}, x_{l'_2}, \dots, x_{l'_{v_2}}],\\
&x_{i_1}\dots x_{i_r}[x_{p_1}^{u_{13}}, x_{p_2}, \dots, x_{p_z}],
\end{aligned} \tag{4.27}$$

where $r, s, t, w_1, w_2 \geq 0$, with $s, t, w_1, w_2 \neq 1$, and $q_1, q_2, v_1, v_2, z \geq 1$ such that $i_1 < \dots < i_r$, $j_1 > j_2 < \dots < j_s$, $k_1 > k_2 < \dots < k_t$, $m_1 > m_2 < \dots < m_{w_1}$, $h_2 < \dots < h_{q_1}$, $m'_1 > m'_2 < \dots < m'_{w_2}$, $h'_2 < \dots < h'_{q_2}$, $l_1 < \dots < l_{v_1}$, $l'_1 < \dots < l'_{v_2}$, $p_1 < \dots < p_z$. Additionally, if $w_1 \geq 2$, then $h_1 < h_2$, and if $w_2 \geq 2$, then $h'_1 < h'_2$.

Step 4 is as in Remark 4.16: since UT_3 has a unit element, we may, without loss of generality, disregard the tail $x_{i_1}\dots x_{i_r}$. Hence, if $f \in I^L(UT_3^{\varepsilon_1 + \beta\,\varepsilon_2})$, we may write

$$\begin{aligned}
f = &\sum_{\mathcal{J}_{j_1}, \mathcal{K}_{k_1}} \alpha_{\mathcal{J}_{j_1}, \mathcal{K}_{k_1}}[x_{j_1}, x_{j_2}, \dots, x_{j_s}][x_{k_1}, x_{k_2}, \dots, x_{k_t}]\\
&+ \sum_{\mathcal{H}_{h_1}, \mathcal{M}_{m_1}} \beta_{\mathcal{H}_{h_1}, \mathcal{M}_{m_1}}[x_{h_1}^{u_{12}}, x_{h_2}, \dots, x_{h_{q_1}}][x_{m_1}, x_{m_2}, \dots, x_{m_{w_1}}]\\
&+ \sum_{\mathcal{M}'_{m'_1}, \mathcal{H}'_{h'_1}} \gamma_{\mathcal{M}'_{m'_1}, \mathcal{H}'_{h'_1}}[x_{m'_1}, x_{m'_2}, \dots, x_{m'_{w_2}}][x_{h'_1}^{u_{23}}, x_{h'_2}, \dots, x_{h'_{q_2}}]\\
&+ \sum_{\mathcal{L}, \mathcal{L}'} \lambda_{\mathcal{L}, \mathcal{L}'}[x_{l_1}^{u_{12}}, x_{l_2}, \dots, x_{l_{v_1}}][x_{l'_1}^{u_{23}}, x_{l'_2}, \dots, x_{l'_{v_2}}] + \mu[x_1^{u_{13}}, x_2, \dots, x_n],
\end{aligned}$$

where $\mathcal{J}_{j_1} = \{j_1, \dots, j_s\}$, $\mathcal{K}_{k_1} = \{k_1, \dots, k_t\}$, $\mathcal{H}_{h_1} = \{h_1, \dots, h_{q_1}\}$, $\mathcal{M}_{m_1} = \{m_1, \dots, m_{w_1}\}$, $\mathcal{H}'_{h'_1} = \{h'_1, \dots, h'_{q_2}\}$, $\mathcal{M}'_{m'_1} = \{m'_1, \dots, m'_{w_2}\}$, $\mathcal{L} = \{l_1, \dots, l_{v_1}\}$ and $\mathcal{L}' = \{l'_1, \dots, l'_{v_2}\}$ are subsets of $\{1, \dots, n\}$ satisfying $\mathcal{J}_{j_1} \cap \mathcal{K}_{k_1} = \mathcal{H}_{h_1} \cap \mathcal{M}_{m_1} = \mathcal{H}'_{h'_1} \cap \mathcal{M}'_{m'_1} = \mathcal{L} \cap \mathcal{L}' = \emptyset$. Moreover, the indices fulfill the

conditions $s \geq 2$, $t, w_1, w_2 \geq 0$, $t, w_1, w_2 \neq 1$, $q_1, q_2, v_1, v_2 \geq 1$, $s+t = q_1+w_1 = w_2+q_2 = v_1+v_2 = n$, together with the ordering constraints $j_1 > j_2 < \cdots < j_s$, $k_1 > k_2 < \cdots < k_t$, $m_1 > m_2 < \cdots < m_{w_1}$, $m'_1 > m'_2 < \cdots < m'_{w_2}$, $l_1 < \cdots < l_{v_1}$, $l'_1 < \cdots < l'_{v_2}$, $h_2 < \cdots < h_{q_1}$, $h'_2 < \cdots < h'_{q_2}$. In addition, if $w_1 \geq 2$ then $h_1 < h_2$, and if $w_2 \geq 2$ then $h'_1 < h'_2$.

Step 5. We show that the L-polynomials appearing in the decomposition of f above are linearly independent modulo $\mathrm{Id}^L(UT_3^{\varepsilon_1+\beta\varepsilon_2})$. We carry out essentially the same evaluations as in Theorem 4.25, with only a slight modification. We begin in the same manner by fixing a set $\mathcal{J}_{j_1} = \{j_1, \ldots, j_n\}$ such that $j_1 > j_2 < \cdots < j_n$. Evaluating at $x_{j_1} = e_{13}$, $x_{j_2} = \cdots = x_{j_n} = e_{33}$, we obtain $\alpha_{\mathcal{J}_{j_1},\emptyset} e_{13} = 0$ since $j_1 \neq 1$ and $e_{33}^{\nu_{13}} = 0$. So, it follows that $\alpha_{\mathcal{J}_{j_1},\emptyset} = 0$. Next, evaluating $x_1 = e_{13}$ and $x_2 = \cdots = x_n = e_{33}$, we get $\mu e_{13} = 0$, and hence $\mu = 0$. Then, proceeding exactly as in Theorem 4.25, we also obtain that $\alpha_{\mathcal{J}_{j_1},\mathcal{K}_{k_1}} = \beta_{\mathcal{H}_{h_1},\mathcal{M}_{m_1}} = \gamma_{\mathcal{M}'_{m'_1},\mathcal{H}'_{h'_1}} = \lambda_{\mathcal{L},\mathcal{L}'} = 0$ for all $\mathcal{J}_{j_1}, \mathcal{K}_{k_1}, \mathcal{H}_{h_1}, \mathcal{M}_{m_1}, \mathcal{H}'_{h'_1}, \mathcal{M}'_{m'_1}, \mathcal{L}, \mathcal{L}'$.

Therefore, we conclude that all coefficients appearing in f vanish. As a consequence, it follows that the L-polynomials (4.27) are linearly independent modulo $\mathrm{Id}^L(UT_3^{\varepsilon_1+\beta\varepsilon_2})$ and $\mathrm{Id}^L(UT_3^{\varepsilon_1+\beta\varepsilon_2}) = I$. Consequently, the L-polynomials (4.27) form a basis of P_n^L modulo $P_n^L \cap \mathrm{Id}^L(UT_3^{\varepsilon_1+\beta\varepsilon_2})$.

Finally, by counting these L-polynomials, we determine the L-codimensions of $UT_3^{\varepsilon_1+\beta\varepsilon_2}$, we get

$$c_n^L(UT_3^{\varepsilon_1+\beta\varepsilon_2}) = c_n^L(UT_3^{\varepsilon_1-\varepsilon_2}) + \sum_{r=0}^{n-1}\binom{n}{r} = (n^2-n)3^{n-2} + 3n2^{n-1} + 1,$$

as claimed. □

Finally, we determine the T_L-ideal of the differential identities of $UT_3^{\varepsilon_1,\varepsilon_2}$. To this end, we establish the following technical lemmas.

Lemma 4.43. *Let $d_1, d_2 \in L$. Then $x_1^{d_1}x_2^{d_1d_2}$, $x_1^{d_1}x_2x_3^{d_1d_2} \in \langle x^{d_1^2} - x^{d_1},\ x_1^{d_2}x_2^{d_1}\rangle_{T_L}$.*

Proof. By Lemma 4.9, we have $x_1^{d_1}x_2^{d_1} \in \langle x^{d_1^2} - x^{d_1}\rangle_{T_L}$. It then follows, by applying the Leibniz rule, the desired conclusion. □

Lemma 4.44. *Let $d_1, d_2 \in L$. Then:*

(1) $[x_1,x_2]x_3^{d_1},\ [x_1,x_2]x_3^{d_1d_2} \in \langle x^{d_1^2} - x^{d_1},\ x_1^{d_2}x_2^{d_1},\ [x_1,x_2]^{d_1d_2} - [x_1,x_2]^{d_1} - [x_1,x_2]^{d_2} + [x_1,x_2]\rangle_{T_L}$;

(2) $x_1^{d_2}[x_2,x_3],\ x_1^{d_1d_2}[x_2,x_3] \in \langle x^{d_2^2} - x^{d_2},\ x_1^{d_2}x_2^{d_1},\ [x_1,x_2]^{d_1d_2} - [x_1,x_2]^{d_1} - [x_1,x_2]^{d_2} + [x_1,x_2]\rangle_{T_L}$;

(3) $x_1^{d_1}[x_2,x_3]x_4^{d_2} \in \langle x^{d_1^2} - x^{d_1},\ x^{d_2^2} - x^{d_2},\ x_1^{d_2}x_2^{d_1},\ [x_1,x_2]^{d_1d_2} - [x_1,x_2]^{d_1} - [x_1,x_2]^{d_2} + [x_1,x_2]\rangle_{T_L}$.

Proof. Let $\mathcal{S}_1 := \{x^{d_1^2} - x^{d_1},\ x_1^{d_2}x_2^{d_1},\ [x_1,x_2]^{d_1d_2} - [x_1,x_2]^{d_1} - [x_1,x_2]^{d_2} + [x_1,x_2]\}$. Since by Lemma 4.9 we have $x_1^{d_1}x_2^{d_1} \in \langle x^{d_1^2} - x^{d_1}\rangle_{T_L}$, it follows that

$$[x_1,x_2]^{d_1d_2}x_3^{d_1} - [x_1,x_2]^{d_1}x_3^{d_1} - [x_1,x_2]^{d_2}x_3^{d_1} + [x_1,x_2]x_3^{d_1} \equiv [x_1,x_2]x_3^{d_1} \pmod{\langle x^{d_1^2} - x^{d_1},\ x_1^{d_2}x_2^{d_1}\rangle_{T_L}},$$

and, as a consequence, $[x_1,x_2]x_3^{d_1} \in \langle \mathcal{S}_1\rangle_{T_L}$.

Now, since $\big([x_1,x_2]x_3^{d_1}\big)^{d_2} = [x_1,x_2]^{d_2}x_3^{d_1} + [x_1,x_2]x_3^{d_1d_2}$, it follows, by the above, that $[x_1,x_2]x_3^{d_1d_2} \in \langle \mathcal{S}_1\rangle_{T_L}$. This completes the proof of (1). Similarly, we can also prove (2) and (3). □

As a direct consequence of the above Lemma, we obtain the following result.

Lemma 4.45. *Let $d_1, d_2 \in L$. Then:*

(1) $[x_1,x_2]\big([x_3,x_4]^{d_2} - [x_3,x_4]\big) \in \langle x^{d_1^2} - x^{d_1},\ x_1^{d_2}x_2^{d_1},\ [x_1,x_2]^{d_1d_2} - [x_1,x_2]^{d_1} - [x_1,x_2]^{d_2} + [x_1,x_2]\rangle_{T_L}$;

(2) $\big([x_1,x_2]^{d_1} - [x_1,x_2]\big)[x_3,x_4] \in \langle x^{d_2^2} - x^{d_2},\ x_1^{d_2}x_2^{d_1},\ [x_1,x_2]^{d_1d_2} - [x_1,x_2]^{d_1} - [x_1,x_2]^{d_2} + [x_1,x_2]\rangle_{T_L}$.

Lemma 4.46. *Let $d_1, d_2 \in L$. Then:*

(1) $x_1^{d_1}\big([x_2,x_3]^{d_2} - [x_2,x_3]\big) \in \langle x^{d_1^2} - x^{d_1},\ x_1^{d_2}x_2^{d_1},\ [x_1,x_2]^{d_1d_2} - [x_1,x_2]^{d_1} - [x_1,x_2]^{d_2} + [x_1,x_2]\rangle_{T_L}$;

(2) $\big([x_1,x_2]^{d_1} - [x_1,x_2]\big)x_3^{d_2} \in \langle x^{d_2^2} - x^{d_2},\ x_1^{d_2}x_2^{d_1},\ [x_1,x_2]^{d_1d_2} - [x_1,x_2]^{d_1} - [x_1,x_2]^{d_2} + [x_1,x_2]\rangle_{T_L}$.

Proof. Since, by Lemma 4.9, we have $x_1^{d_1}x_2^{d_1} \in \langle x^{d_1^2} - x^{d_1}\rangle_{T_L}$ and, by Lemma 4.43, also $x_1^{d_1}x_2^{d_1d_2} \in \langle x^{d_1^2} - x^{d_1},\ x_1^{d_2}x_2^{d_1}\rangle_{T_L}$, we get

$$x_1^{d_1}[x_2,x_3]^{d_1d_2} - x_1^{d_1}[x_2,x_3]^{d_1} - x_1^{d_1}[x_2,x_3]^{d_2} + x_1^{d_1}[x_2,x_3] \equiv x_1^{d_1}\big([x_2,x_3]^{d_2} - [x_2,x_3]\big) \pmod{\langle x^{d_1^2} - x^{d_1},\ x_1^{d_2}x_2^{d_1}\rangle_{T_L}},$$

and statement (1) follows. The proof of (2) is analogous. □

Lemma 4.47. *Let $d_1, d_2 \in L$. Then*

$$[x_1^{d_1d_2}, x_2] - [x_2^{d_1d_2}, x_1] - [x_1^{d_1}, x_2] + [x_2^{d_1}, x_1] - [x_1^{d_2}, x_2] + [x_2^{d_2}, x_1] + [x_1, x_2] + x_1^{d_1}x_2^{d_2} - x_2^{d_1}x_1^{d_2}$$

is a consequence of the L-polynomials in

$$\left\{x_1^{d_2}x_2^{d_1},\ [x_1,x_2]^{d_1d_2} - [x_1,x_2]^{d_1} - [x_1,x_2]^{d_2} + [x_1,x_2]\right\}.$$

Proof. Since $[x_1,x_2]^{d_1d_2} \equiv [x_1^{d_1d_2}, x_2] - [x_2^{d_1d_2}, x_1] + x_1^{d_1}x_2^{d_2} - x_2^{d_1}x_1^{d_2} \pmod{\langle x_1^{d_2}x_2^{d_1}\rangle_{T_L}}$, the desired conclusion follows. □

Theorem 4.48. *Let L be a Lie algebra and let $UT_3^{\varepsilon_1,\varepsilon_2}$ be the L-algebra UT_3 where L acts via the two-dimensional abelian Lie subalgebra of $\mathrm{Der}(UT_3)$ with ordered basis $\{\varepsilon_1 = \mathrm{ad}_{(-e_{11})}, \varepsilon_2 = \mathrm{ad}_{e_{33}}\}$. Then there exists a basis $\mathcal{B}_L = \{d_i \mid i \geq 1\}$ of L over F such that the T_L-ideal of differential identities of $UT_3^{\varepsilon_1,\varepsilon_2}$ is generated by the following L-polynomials*

$$x^{d_i}, \quad x^{d_1^2} - x^{d_1}, \quad x^{d_2^2} - x^{d_2}, \quad x^{d_1d_2} - x^{d_2d_1}, \quad x_1^{d_2}x_2^{d_1}, \quad [x_1,x_2]^{d_1d_2} - [x_1,x_2]^{d_1} - [x_1,x_2]^{d_2} + [x_1,x_2],$$

for all $i \geq 3$. Moreover, $c_n^L(UT_3^{\varepsilon_1,\varepsilon_2}) = (n^2-n)3^{n-2} + 3n2^{n-1} + 1$.

Proof. **Step 1** is as in Remark 4.16, with $\theta(d_1) = \varepsilon_1$ and $\theta(d_2) = \varepsilon_2$.

Step 2 is immediate from (4.1) and (4.7).

Step 3. Let $f \in P_n^L$. At most two left-normed commutators occur modulo I (item (1) of Lemma 4.44, item (2) of Lemma 4.45 and Lemma 4.11).

Now, since $x^{d_i},\ x^{d_1^2} - x^{d_1},\ x^{d_2^2} - x^{d_2} \in I$, the exponents occurring modulo I reduce to $\{1_{U(L)}, d_1, d_2, d_1d_2\}$. Moreover, from Lemma 4.9 it follows that $x_1^{d_1}x_2^{d_1}$, $x_1^{d_2}x_2^{d_2}$, $x_1^{d_1d_2}x_2^{d_1d_2} \in I$, and consequentially we also get $x_1^{d_1}x_2x_3^{d_1}$, $x_1^{d_2}x_2x_3^{d_2}$, $x_1^{d_1d_2}x_2x_3^{d_1d_2} \in I$. In addition, the same lemma also yields $x_1^{d_1d_2}x_2^{d_2}$, $x_1^{d_2}x_2^{d_1d_2} \in I$ and $x_1^{d_1d_2}x_2x_3^{d_2}$, $x_1^{d_2}x_2x_3^{d_1d_2} \in I$. Also, by Lemma 4.43 we get $x_1^{d_1}x_2^{d_1d_2},\ x_1^{d_1}x_2x_3^{d_1d_2} \in I$. Additionally, since $x_1^{d_2}x_2^{d_1} \in I$, clearly we also have $x_1^{d_2}x_2x_3^{d_1} \in I$ and also $x_1^{d_1d_2}x_2^{d_1}$, $x_1^{d_1d_2}x_2x_3^{d_1} \in I$. Therefore, modulo I, an L-polynomial may contain either at most one variable with exponent d_1 or d_2 or d_1d_2, or two distinct variables with exponent the first one d_1 and the other one d_2.

Therefore, taking into account Lemmas 4.43–4.47, together with Remark 4.24, Lemmas 4.3, 4.6, and 4.7, as well as the Jacobi identity, and arguing exactly as in Theorem 4.42, we conclude that f can be expressed, modulo I, as a linear combination of L-polynomials:

$$\begin{aligned}
&x_{i_1}\dots x_{i_r}[x_{j_1}, x_{j_2}, \dots, x_{j_s}][x_{k_1}, x_{k_2}, \dots, x_{k_t}],\\
&x_{i_1}\dots x_{i_r}[x_{h_1}^{d_1}, x_{h_2}, \dots, x_{h_{q_1}}][x_{m_1}, x_{m_2}, \dots, x_{m_{w_1}}],\\
&x_{i_1}\dots x_{i_r}[x_{m_1'}, x_{m_2'}, \dots, x_{m_{w_2}'}][x_{h_1'}^{d_2}, x_{h_2'}, \dots, x_{h_{q_2}'}],\\
&x_{i_1}\dots x_{i_r}[x_{l_1}^{d_1}, x_{l_2}, \dots, x_{l_{v_1}}][x_{l_1'}^{d_2}, x_{l_2'}, \dots, x_{l_{v_2}'}],\\
&x_{i_1}\dots x_{i_r}[x_{p_1}^{d_1d_2}, x_{p_2}, \dots, x_{p_z}],
\end{aligned} \tag{4.28}$$

where $r, s, t, w_1, w_2 \geq 0$, with $s, t, w_1, w_2 \neq 1$, and $q_1, q_2, v_1, v_2, z \geq 1$ such that $i_1 < \dots < i_r$, $j_1 > j_2 < \dots < j_s$, $k_1 > k_2 < \dots < k_t$, $m_1 > m_2 < \dots < m_{w_1}$, $h_2 < \dots < h_{q_1}$, $m_1' > m_2' < \dots < m_{w_2}'$, $h_2' < \dots < h_{q_2}'$, $l_1 < \dots < l_{v_1}$, $l_1' < \dots < l_{v_2}'$, $p_1 < \dots < p_z$. Additionally, if $w_1 \geq 2$, then $h_1 < h_2$, and if $w_2 \geq 2$, then $h_1' < h_2'$.

Step 4 is as in Remark 4.16: since UT_3 has a unit element, we may, without loss of generality, disregard the tail $x_{i_1}\dots x_{i_r}$. Hence, if $f \in \mathrm{Id}^L(UT_3^{\varepsilon_1,\varepsilon_2})$, we may write

$$\begin{aligned}
f = &\sum_{\mathcal{J}_{j_1},\mathcal{K}_{k_1}} \alpha_{\mathcal{J}_{j_1},\mathcal{K}_{k_1}}[x_{j_1}, x_{j_2}, \dots, x_{j_s}][x_{k_1}, x_{k_2}, \dots, x_{k_t}]\\
&+ \sum_{\mathcal{H}_{h_1},\mathcal{M}_{m_1}} \beta_{\mathcal{H}_{h_1},\mathcal{M}_{m_1}}[x_{h_1}^{d_1}, x_{h_2}, \dots, x_{h_{q_1}}][x_{m_1}, x_{m_2}, \dots, x_{m_{w_1}}]\\
&+ \sum_{\mathcal{M}'_{m_1'},\mathcal{H}'_{h_1'}} \gamma_{\mathcal{M}'_{m_1'},\mathcal{H}'_{h_1'}}[x_{m_1'}, x_{m_2'}, \dots, x_{m_{w_2}'}][x_{h_1'}^{d_2}, x_{h_2'}, \dots, x_{h_{q_2}'}]\\
&+ \sum_{\mathcal{L},\mathcal{L}'} \lambda_{\mathcal{L},\mathcal{L}'}[x_{l_1}^{d_1}, x_{l_2}, \dots, x_{l_{v_1}}][x_{l_1'}^{d_2}, x_{l_2'}, \dots, x_{l_{v_2}'}] + \mu[x_1^{d_1d_2}, x_2, \dots, x_n],
\end{aligned}$$

where $\mathcal{J}_{j_1} = \{j_1, \dots, j_s\}$, $\mathcal{K}_{k_1} = \{k_1, \dots, k_t\}$, $\mathcal{H}_{h_1} = \{h_1, \dots, h_{q_1}\}$, $\mathcal{M}_{m_1} = \{m_1, \dots, m_{w_1}\}$, $\mathcal{H}'_{h'_1} = \{h'_1, \dots, h'_{q_2}\}$, $\mathcal{M}'_{m'_1} = \{m'_1, \dots, m'_{w_2}\}$, $\mathcal{L} = \{l_1, \dots, l_{v_1}\}$ and $\mathcal{L}' = \{l'_1, \dots, l'_{v_2}\}$ are subsets of indices in $\{1, \dots, n\}$ such that $\mathcal{J}_{j_1} \cap \mathcal{K}_{k_1} = \mathcal{H}_{h_1} \cap \mathcal{M}_{m_1} = \mathcal{H}'_{h'_1} \cap \mathcal{M}'_{m'_1} = \mathcal{L} \cap \mathcal{L}' = \emptyset$, with $s \geq 2$, $t, w_1, w_2 \geq 0$, $t, w_1, w_2 \neq 1$, $q_1, q_2, v_1, v_2 \geq 1$, $s + t = q_1 + w_1 = w_2 + q_2 = v_1 + v_2 = n$, and subjected to the conditions $j_1 > j_2 < \cdots < j_s$, $k_1 > k_2 < \cdots < k_t$, $h_2 < \cdots < h_{q_1}$, $m_1 > m_2 < \cdots < m_{w_1}$, $m'_1 > m'_2 < \cdots < m'_{w_2}$, $h'_2 < \cdots < h'_{q_2}$, $l_1 < \cdots < l_{v_1}$, $l'_1 < \cdots < l'_{v_2}$. Additionally, if $w_1 \geq 2$, then $h_1 < h_2$, and if $w_2 \geq 2$, then $h'_1 < h'_2$.

Step 5. We show that the L-polynomials appearing in the decomposition of f above are linearly independent modulo $\mathrm{Id}^L(UT_3^{\varepsilon_1,\varepsilon_2})$.

Let $\mathcal{J}_{j_1} = \{j_1, \dots, j_n\}$ be a fixed set such that $j_1 > j_2 < \cdots < j_n$. If we evaluate $x_{j_1} = e_{12}$ and $x_{j_2} = \cdots = x_{j_n} = e_{22}$, then we get $(\alpha_{\mathcal{J}_{j_1},\emptyset} + \beta_{\mathcal{J}_{j_1},\emptyset})e_{12} = 0$, thus

$$\alpha_{\mathcal{J}_{j_1},\emptyset} + \beta_{\mathcal{J}_{j_1},\emptyset} = 0. \tag{4.29}$$

If we make the evaluation $x_{j_1} = e_{23}$ and $x_{j_2} = \cdots = x_{j_n} = e_{33}$, then we obtain $(\alpha_{\mathcal{J}_{j_1},\emptyset} + \gamma_{\emptyset,\mathcal{J}_{j_1}})e_{23} = 0$, thus

$$\alpha_{\mathcal{J}_{j_1},\emptyset} + \gamma_{\emptyset,\mathcal{J}_{j_1}} = 0. \tag{4.30}$$

Moreover, from the evaluation $x_{j_1} = e_{13}$ and $x_{j_2} = \cdots = x_{j_n} = e_{33}$, it follows that $(\alpha_{\mathcal{J}_{j_1},\emptyset} + \beta_{\mathcal{J}_{j_1},\emptyset} + \gamma_{\emptyset,\mathcal{J}_{j_1}})e_{13} = 0$, thus

$$\alpha_{\mathcal{J}_{j_1},\emptyset} + \beta_{\mathcal{J}_{j_1},\emptyset} + \gamma_{\emptyset,\mathcal{J}_{j_1}} = 0. \tag{4.31}$$

Therefore, putting together (4.29), (4.30) and (4.31) we get $\alpha_{\mathcal{J}_{j_1},\emptyset} = \beta_{\mathcal{J}_{j_1},\emptyset} = \gamma_{\mathcal{J}_{j_1},\emptyset} = 0$.

Now, let us make the evaluation $x_1 = e_{12}$ and $x_2 = \cdots = x_n = e_{22}$. In this case we get $\beta_{\mathcal{H}_1,\emptyset}e_{12} = 0$, where $\mathcal{H}_1 = \{1, \dots, n\}$, so $\beta_{\mathcal{H}_1,\emptyset} = 0$. Also, by making the evaluation $x_1 = e_{23}$ and $x_2 = \cdots = x_n = e_{33}$, we get $\gamma_{\emptyset,\mathcal{H}'_1}e_{23} = 0$, where $\mathcal{H}'_1 = \{1, \dots, n\}$, then $\gamma_{\emptyset,\mathcal{H}'_1} = 0$. Furthermore, from the evaluation $x_1 = e_{13}$ and $x_2 = \cdots = x_n = e_{33}$, we get $\mu e_{13} = 0$, thus $\mu = 0$.

Next, let $\mathcal{J}_{j_1} = \{j_1, \dots, j_s\}$ and $\mathcal{K}_{k_1} = \{k_1, \dots, k_t\}$ be two fixed disjoint subsets of $\{1, \dots, n\}$ such that $s, t \geq 2$, $s + t = n$ and $j_1 > j_2 < \cdots < j_s$, $k_1 > k_2 < \cdots < k_t$. If we evaluate $x_{j_1} = e_{12}$, $x_{j_2} = \cdots = x_{j_s} = e_{11}$, $x_{k_1} = e_{23}$ and $x_{k_2} = \cdots = x_{k_t} = e_{33}$, then we get $\alpha_{\mathcal{J}_{j_1},\mathcal{K}_{k_1}}e_{13} = 0$, thus $\alpha_{\mathcal{J}_{j_1},\mathcal{K}_{k_1}} = 0$. Here the polynomials $[x_{h_1}^{d_1}, x_{h_2}, \dots, x_{h_s}][x_{k_1}, x_{k_2}, \dots, x_{k_t}]$ evaluate to zero since $j_1 > j_2 < \cdots < j_s$ whereas $h_1 < h_2 < \cdots < h_s$ and $e_{11}^{\varepsilon_1} = 0$. Similarly, the polynomials $[x_{j_1}, x_{j_2}, \dots, x_{j_s}][x_{h'_1}^{d_2}, x_{h'_2}, \dots, x_{h'_t}]$ evaluate to zero since $k_1 > k_2 < \cdots < k_t$ whereas $h'_1 < h'_2 < \cdots < h'_t$ and $e_{33}^{\varepsilon_2} = 0$. Also, with the same reasoning all the polynomials of the type $[x_{l_1}^{d_1}, x_{l_2}, \dots, x_{l_{v_1}}][x_{l'_1}^{d_2}, x_{l'_2}, \dots, x_{l'_{v_2}}]$ evaluate to zero.

Let $\mathcal{H}_{h_1} = \{h_1, \dots, h_{q_1}\}$ and $\mathcal{M}_{m_1} = \{m_1, \dots, m_{w_1}\}$ be two fixed disjoint subsets of $\{1, \dots, n\}$ with $q_1 \geq 1$, $w_1 \geq 2$, $q_1 + w_1 = n$ and subject to the conditions $h_1 < h_2 < \cdots < h_{q_1}$, $m_1 > m_2 < \cdots < m_{w_1}$. If we make the evaluation $x_{h_1} = e_{12}$, $x_{h_2} = \cdots = x_{h_{q_1}} = e_{11}$, $x_{m_1} = e_{23}$ and $x_{m_2} = \cdots = x_{m_{w_1}} = e_{33}$, we get $\beta_{\mathcal{H}_{h_1},\mathcal{M}_{m_1}}e_{13} = 0$, that is $\beta_{\mathcal{H}_{h_1},\mathcal{M}_{m_1}} = 0$. Notice that as above with this evaluation all the polynomials of the type $[x_{m'_1}, x_{m'_2}, \dots, x_{m'_{w_2}}][x_{h'_1}^{d_2}, x_{h'_2}, \dots, x_{h'_{q_2}}]$ and of the type $[x_{l_1}^{d_1}, x_{l_2}, \dots, x_{l_{v_1}}][x_{l'_1}^{d_2}, x_{l'_2}, \dots, x_{l'_{v_2}}]$ evaluate to zero. Similarly, if we evaluate $x_{h_1} = e_{23}$, $x_{h_2} = \cdots = x_{h_{q_1}} = e_{33}$, $x_{m_1} = e_{12}$ and $x_{m_2} = \cdots = x_{m_{w_1}} = e_{11}$, we get $\gamma_{\mathcal{M}_{m_1},\mathcal{H}_{h_1}}e_{13} = 0$, and so $\gamma_{\mathcal{M}_{m_1},\mathcal{H}_{h_1}} = 0$.

Finally, fixed $\mathcal{L} = \{l_1, \dots, l_{v_1}\}$ and $\mathcal{L}' = \{l'_1, \dots, l'_{v_2}\}$ disjoint subset of $\{1, \dots, n\}$ such that $v_1, v_2 \geq 1$, $v_1 + v_2 = n$ and $l_1 < \cdots < l_{v_1}$, $l'_1 < \cdots < l'_{v_2}$, then from the evaluation $x_{l_1} = e_{12}$, $x_{l_2} = \cdots = x_{l_{v_1}} = e_{11}$, $x_{l'_1} = e_{23}$ and $x_{l'_2} = \cdots = x_{l'_{v_2}} = e_{33}$, it follows that $\lambda_{\mathcal{L},\mathcal{L}'}e_{13} = 0$ and consequently $\lambda_{\mathcal{L},\mathcal{L}'} = 0$.

Therefore, all the scalars appearing in f are zero, and as a consequence, the L-polynomials (4.28) are linearly independent modulo $\mathrm{Id}^L(UT_3^{\varepsilon_1,\varepsilon_2})$. This proves that $\mathrm{Id}^L(UT_3^{\varepsilon_1,\varepsilon_2}) = I$ and the polynomials (4.28) are a basis of P_n^L modulo $P_n^L \cap \mathrm{Id}^L(UT_3^{\varepsilon_1,\varepsilon_2})$. Hence, as in Theorem 4.42, by counting we get $c_n^L(UT_3^{\varepsilon_1,\varepsilon_2}) = (n^2 - n)3^{n-2} + 3n2^{n-1} + 1$, as claimed. □

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