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W-ALGEBRAS WITH INVOLUTION: POLYNOMIAL IDENTITIES AND ASYMPTOTIC GROWTH

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Abstract. Let W be an algebra with involution over a field of characteristic zero. We develop a theory of W-polynomial identities with involution for finite-dimensional $(W,*)$-algebras A, using the multiplier algebra with involution of A. We prove that the $(W,*)$-exponent of A exists and coincides with the ordinary $*$-exponent. We then consider a four-dimensional subalgebra M of the algebra of 4×4 upper triangular matrices, endowed with the reflection involution, and study two non-equivalent $(W,*)$-algebra structures on it. For both structures, we determine the corresponding T^*_W-ideals of W-polynomial identities with involution and compute the $(W,*)$-codimension sequences explicitly; for one of them, we also determine the complete $(W,*)$-cocharacter sequence. Finally, we prove that these two W-algebras with involution generate distinct varieties of almost polynomial growth.

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1. Introduction

Polynomial identities provide a powerful tool for studying algebras by means of the polynomial relations that hold under all evaluations. A natural extension is obtained by allowing the coefficients of the polynomials to belong to a fixed associative algebra W, rather than being restricted to the ground field. In this setting, elements of W may occur at arbitrary positions among the variables, reflecting their role as noncommutative coefficients. This leads to the notion of a *W-polynomial*.

The corresponding notion of a *W-polynomial identity* naturally extends the concept of a generalized polynomial identity, introduced by Amitsur in 1965 [3] and subsequently developed by several authors (see [31] and the references therein). For unital algebras, the two notions coincide, and W-polynomial identities are therefore often referred to as generalized polynomial identities.

The natural framework for W-polynomial identities is provided by the theory of W-algebras. Roughly speaking, a W-algebra is an associative algebra A endowed with a W-bimodule structure satisfying some additional W-associative conditions. This theory has been recently developed, proving to be particularly useful in the study of codimension growth and varieties of algebras with W-polynomial identities (see [23,24]).

The theory of generalized polynomial identities has a classical counterpart for algebras with involution. Following Amitsur's work on identities in rings with involution [4], Martindale studied generalized polynomial identities for prime rings with involution [22]. Rowen subsequently developed a structural framework for generalized identities in rings with involution [28–30], which was later incorporated into the systematic treatment of polynomial and generalized identities in [31]. In the present paper, we extend generalized polynomial identities with involution to the broader setting of W-algebras, focusing in particular on their codimensions, cocharacters, and asymptotic growth.

A fundamental role in our approach is played by multiplier algebras, which provide the natural setting for incorporating the action of a possibly non-unital algebra W on an associative algebra. The notion of a multiplier algebra of an associative algebra dates back to Hochschild [19] and was later developed in the context of C^*-algebras by Busby [6]; it has since found applications in various areas, including partial dynamical systems [7, 8, 10]. In our setting, an action of W on A is encoded by a homomorphism Φ from W into the multiplier algebra $\mathcal{M}(A)$ of A. When W and A are endowed with involutions, $\mathcal{M}(A)$ naturally

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inherits an involution, and Φ is a homomorphism of algebras with involution. This provides the structural framework for our study of W-algebras with involution.

A central problem in PI-theory is the asymptotic behavior of codimension sequences, which quantify the multilinear polynomial identities of an algebra. In particular, their exponential growth rate is captured by the PI-exponent in the ordinary setting [15, 16]. Analogous growth rates have been studied extensively for algebras endowed with additional structures; see, for instance, [1,2,9,11,12,17,18,20,21]. In the framework of W-algebras, the exponential growth of W-codimensions has recently been investigated for finite-dimensional unital W-algebras [23], finite-dimensional W-algebras not necessarily with unity [27], and finite-dimensional group-graded W-algebras [5]. For algebras with involution, the existence and integrality of the $*$-exponent were established in [14].

It is worth noting that the exponent associated with an additional algebraic structure need not coincide with the ordinary PI-exponent; this is already the case for algebras with involution [13]. On the other hand, for finite-dimensional algebras, the two exponents are known to coincide in some settings, including W-algebras, group-graded W-algebras, and algebras endowed with an action of a Lie algebra by derivations [26].

One of the main objectives of this paper is to show that, for a finite-dimensional W-algebra A with involution, the corresponding $(W,*)$-exponent $\exp^{W,*}(A)$ exists and coincides with the $*$-exponent of A. In particular, it is a non-negative integer. Thus, although the additional W-action may substantially affect the W-polynomial identities with involution and their codimension sequence, it does not alter their exponential rate of growth.

Since $\exp^{W,*}(A)$ is an integer, it is natural to investigate the boundary between polynomial and exponential growth, corresponding to the cases $\exp^{W,*}(A) = 1$ and $\exp^{W,*}(A) \geq 2$. To this end, we focus on the algebra with involution M, a subalgebra of the upper triangular matrices of order four endowed with the reflection involution introduced in [25].

We determine the only two W-algebra with involution structures on M, denoted by M^F and M^D, that generate distinct varieties of W-algebras with involution of almost polynomial growth. More precisely, $\exp^{W,*}(M^F) = \exp^{W,*}(M^D) = 2$, whereas every proper subvariety has polynomially bounded codimension sequence (equivalently $(W,*)$-exponent 1). For both algebras, we determine generators of the corresponding T^*_W-ideals of W-polynomial identities with involution and compute their codimensions explicitly. We also give a complete description of the generalized $*$-cocharacters of M^D.

The paper is organized as follows. Section 2 describes the structure of W-algebras with involution in terms of multiplier algebras. In Section 3, we introduce W-polynomial identities with involution, together with the corresponding codimensions and cocharacters. Section 4 is devoted to the $(W,*)$-exponent. In Section 5, we study the W-polynomial identities and explicitly determine the corresponding generalized $*$-codimensions of M^F and M^D, while Section 6 focuses on the generalized $*$-cocharacters of M^D. Finally, in Section 7, we investigate the growth of the varieties of W-algebras with involution generated by M^F and M^D.

2. W-algebras with involution

Let F be a field of characteristic zero and let A be an associative algebra over F. An involution on A is a linear map $*\colon A \to A$ such that, for any $a, b \in A$,

$$(a^*)^* = a, \qquad (ab)^* = b^*a^*.$$

If A is equipped with an involution $*$, then it is called a $*$-algebra. Since the field F has characteristic zero, a $*$-algebra A can be written as $A = A^+ \oplus A^-$, where

$$A^+ = \{a \in A : a^* = a\} \qquad \text{and} \qquad A^- = \{a \in A : a^* = -a\}$$

are the vector spaces of symmetric and skew-symmetric elements of A, respectively.

Now fix an F-algebra W endowed with an involution $\sharp$. We say that a $*$-algebra A is a $(W,*)$-*bimodule* if it is equipped with a structure of both left and right W-module and the following properties hold:

- the corresponding actions permute, i.e., for all $w_1, w_2 \in W$, $a \in A$, we have that

$$(w_1 a)w_2 = w_1(aw_2), \tag{2.1}$$

- the involutions on W and A satisfy the following compatibility relations: for all $a \in A$, $w \in W$,

$$(wa)^* = a^* w^{\#} \quad \text{and} \quad (aw)^* = w^{\#} a^*. \tag{2.2}$$

Moreover, if W has a unity element 1_W, we further assume that $1_W a = a 1_W = a$ for all $a \in A$.

Definition 2.1. We say that A is a $(W, *)$-*algebra*, or a W-*algebra with involution* $*$, if A is a $(W, *)$-bimodule satisfying the following associative relations: for all $w \in W$, $a_1, a_2 \in A$,

$$w(a_1 a_2) = (wa_1)a_2, \quad (a_1 a_2)w = a_1(a_2 w), \quad (a_1 w)a_2 = a_1(wa_2). \tag{2.3}$$

Remark 2.2. Given an algebra A, we can define the opposite algebra A^{op} as the underlying vector space of A endowed with the opposite product $\cdot^{\mathrm{op}}$, defined by $a \cdot^{\mathrm{op}} b = b \cdot a$, where $\cdot$ is the product of A.

Let $\mathrm{End}_F(A)$ be the algebra of the endomorphisms of A acting on the left of A. The multiplication in $\mathrm{End}_F(A)$ is given by the usual composition of endomorphisms, written by juxtaposition.

Definition 2.3. Define $\mathcal{M}(A)$ as the subset of $\mathrm{End}_F^{\mathrm{op}}(A) \times \mathrm{End}_F(A)$ of the pairs (R, L) satisfying, for all $a, b \in A$,

$$R(ab) = aR(b), \quad L(ab) = L(a)b, \quad R(a)b = aL(b). \tag{2.4}$$

We refer to R and L as a *right* and *left multiplier* of A, respectively, and to the elements of $\mathcal{M}(A)$ as *multipliers* of A.

An element $m \in A$ obviously determines the multiplier $(R_m, L_m) \in \mathcal{M}(A)$, where

$$R_m(a) = am \quad \text{and} \quad L_m(a) = ma$$

for all $a \in A$, called *inner multiplier* of A induced by the element m.

Our next goal is to establish a correspondence between multipliers and $(W, *)$-actions. To this end, we first record several useful properties of multipliers.

It is straightforward to verify that $\mathcal{M}(A)$ is a unital F-algebra under component-wise addition and multiplication, with unit element $(\mathrm{id}_A, \mathrm{id}_A)$.

Now suppose that A is an F-algebra equipped with an involution $*$. We can endow $\mathcal{M}(A)$ with a corresponding involution. To this end, if $T \in \mathrm{End}_F(A)$, we define the map $T^* \in \mathrm{End}_F(A)^{op}$ as

$$T^*(a) := (T(a^*))^*$$

for every $a \in A$. Note that if we take a map $S \in \mathrm{End}_F(A)^{op}$, one can define $S^* \in \mathrm{End}_F(A)$ in the same manner as before. Thus, we equip $\mathcal{M}(A)$ with an involution $*$ defined by

$$(R, L)^* := (L^*, R^*).$$

We refer to $\mathcal{M}(A)$ with this involution as the *multiplier* $*$-*algebra* of A. Note that this involution interchanges the right and left multipliers.

Recall that a $*$-ideal I of a $*$-algebra A is an ideal I of A such that $I^* = I$. Then, clearly, the set of all inner multipliers of A

$$\mathcal{IM}(A) = \{(R_m, L_m) : m \in A\}$$

is a two-sided $*$-ideal of $\mathcal{M}(A)$, and it is called the *inner multiplier* $*$-*ideal* of A.

Definition 2.4. A right multiplier R and a left multiplier L of A are *permutable* if

$$RL = LR. \tag{2.5}$$

In general, not all left and right multipliers are permutable, as shown by the following example.

Example 2.5. Let $A = Fe_{12} \oplus Fe_{13} \oplus Fe_{23}$, where the e_{ij}'s are the usual matrix units. Consider the multiplier $(R, L) \in \mathcal{M}(A)$ defined by

$$R(e_{12}) = e_{12}, \quad R(e_{13}) = 0, \quad R(e_{23}) = e_{13},$$
$$L(e_{12}) = e_{13}, \quad L(e_{13}) = 0, \quad L(e_{23}) = e_{23}.$$

Then R and L are not permutable. Indeed, $R(L(e_{12})) = R(e_{13}) = 0$, whereas $L(R(e_{12})) = L(e_{12}) = e_{13}$. Notice also that $(R, L) \notin \mathcal{IM}(A)$. Indeed, suppose there exists $m = \alpha e_{12} + \beta e_{13} + \gamma e_{23} \in A$ such that $(R, L) = (R_m, L_m)$, then applying the right multiplier to e_{12}, we obtain $e_{12} = \gamma e_{13}$, which is a contradiction.

It is straightforward to show that every multiplier permutes with any inner multiplier. Consequently, for unital algebras, we obtain the following result.

Proposition 2.6. *Let A be a unital $*$-algebra. Then $A \cong \mathcal{IM}(A) = \mathcal{M}(A)$ as $*$-algebras and $R_a L_b = L_b R_a$ for every $a, b \in A$.*

Proof. From [23, Corollary 2.5], we have that $\mathcal{IM}(A) = \mathcal{M}(A)$, and moreover, $A \cong \mathcal{IM}(A)$ as F-algebras via the isomorphism $\mu\colon A \to \mathcal{IM}(A)$ defined by $\mu(m) = (R_m, L_m)$, for all $m \in A$. Since $R_{m^*}(a) = L_m^*(a)$ and $L_{m^*}(a) = R_m^*(a)$ for all $a, m \in A$, it follows that μ is an isomorphism of $*$-algebras, which completes the proof. □

Now we can explain the connection between multiplier $*$-algebras and $(W,*)$-algebras. Recall that the right and left actions of W on A induce the following F-algebra homomorphisms

$$\rho\colon W \to \mathrm{End}_F(A)^{op} \quad \text{and} \quad \lambda\colon W \to \mathrm{End}_F(A)$$

defined by $\rho(w)(a) = aw$ and $\lambda(w)(a) = wa$, for all $w \in W, a \in A$, and vice versa. These are referred to as the right and left representations of W on A, respectively. From (2.3), it follows that $(\rho(w), \lambda(w)) \in \mathcal{M}(A)$, for all $w \in W$. Thus, the map

$$\begin{aligned} \Phi\colon W &\longrightarrow \mathcal{M}(A) \\ w &\mapsto (\rho(w), \lambda(w)) \end{aligned}$$

is a homomorphism of F-algebras. By (2.2), for all $w \in W$, we have that

$$\rho^*(w) = \lambda(w^{\#}) \quad \text{and} \quad \lambda^*(w) = \rho(w^{\#})$$

and, as a consequence, Φ is a homomorphism of $*$-algebras. Moreover, by (2.1), Φ satisfies the following additional condition: for all $w_1, w_2 \in W$

$$\rho(w_2)\lambda(w_1) = \lambda(w_1)\rho(w_2). \tag{2.6}$$

Furthermore, if W has unity, then $\Phi(1_W) = (\mathrm{id}_A, \mathrm{id}_A)$.

The converse is not generally true because not all $*$-homomorphisms from W to $\mathcal{M}(A)$ define a $(W,*)$-algebra structure on A; indeed, there may exist $*$-homomorphisms from W to $\mathcal{M}(A)$ that do not satisfy condition (2.6), since generally not all multipliers of A satisfy the permutability condition (2.5) (see Example 2.5).

However, if we consider a $*$-homomorphism $\Phi\colon W \to \mathcal{M}(A)$ such that $\Phi(W)$ satisfies the permutability condition (2.5), then A inherits a natural structure of $(W,*)$-algebra defined as follows: for any $w \in W$ and $a \in A$

$$wa := L(a) \quad \text{and} \quad aw := R(a),$$

where $\Phi(w) = (R, L) \in \mathcal{M}(A)$.

Thus, giving a $(W,*)$-algebra structure on A is equivalent to giving a $*$-homomorphism $\Phi\colon W \to \mathcal{M}(A)$ whose image $\Phi(W) \subseteq \mathcal{M}(A)$ satisfies the condition of permutability. Such a $*$-homomorphism is called the *acting $*$-homomorphism* of W on A.

An algebra A may admit several $(W,*)$-algebra structures corresponding to distinct acting $*$-homomorphisms $\Phi\colon W \to \mathcal{M}(A)$. When it is necessary to specify the $*$-homomorphism defining the action, we write A^{Φ} for the algebra A endowed with the $(W,*)$-action determined by Φ.

Notice that the action of W on A induces an action of the $*$-subalgebra $\overline{W} = \Phi(W)$ of $\mathcal{M}(A)$, so A can be regarded as a $(\overline{W},*)$-algebra. Thus, the action of W on A factors through $\overline{W}$. Moreover, distinct acting $*$-homomorphisms $\Phi\colon W \to \mathcal{M}(A)$ with the same image $\overline{W}$ induce the same $(\overline{W},*)$-algebra structure on A. Thus we make the following definition.

Definition 2.7. We say that $(W,*)$-algebra structures A^{Φ} and $A^{\Phi'}$ are *equivalent* if $\Phi(W) = \Phi'(W)$ as $*$-subalgebras of $\mathcal{M}(A)$.

This defines an equivalence relation on the set of $(W,*)$-algebra structures on A. Its equivalence classes are precisely determined by the $*$-subalgebras of $\mathcal{M}(A)$ that occur as images of acting $*$-homomorphisms. Accordingly, throughout the paper, we describe a $(W,*)$-action on A in terms of its image $\overline{W}$, and say that W acts on A via $\overline{W}$.

Whenever it is necessary to specify the image, we write $A^{\overline{W}}$ for A endowed with a fixed $(W,*)$-action whose image is $\overline{W}$. Thus, $A^{\overline{W}}$ always denotes a fixed representative of the corresponding equivalence class. The acting $*$-homomorphism Φ is consequently understood to be part of the data, even though it is not displayed in the notation $A^{\overline{W}}$. We will write Φ explicitly only when necessary to avoid ambiguity.

From the arguments above and Proposition 2.6, the following result holds.

Proposition 2.8. *If $A^{\overline{W}}$ is $(W,*)$-algebra with unit 1_A, then the action of W on A is induced by the left and right multiplication actions of a $*$-subalgebra B of A that is isomorphic to $\overline{W}$. Moreover, if W is unital, $1_A \in B$.*

In the last part of this section we will deal with finite dimensional $(W,*)$-algebras. Henceforth, from now on, A will denote a finite dimensional $*$-algebra and $J = J(A)$ the Jacobson radical of A.

Recall that A is called *semisimple* if $J = 0$ and $(*)$-*simple* if it has no-nontrivial $(*)$-ideals and $A^2 \neq 0$.

We next recall the Wedderburn–Malcev theorem for algebras with involution (see [17, Theorem 3.4.4]).

Theorem 2.9. *Let F be a field of characteristic 0. Any finite dimensional F-algebra A with involution $*$ has a decomposition of the form*

$$A = B + J \tag{2.7}$$

where the Jacobson radical $J = J(A)$ is a $*$-ideal and B is a maximal semisimple $*$-subalgebra of A. If the field F is algebraically closed, then

$$B = B_1 \oplus \cdots \oplus B_m,$$

where the B_is are $*$-simple subalgebras of A.

Given a finite dimensional $(W,*)$-algebra A and a subset $I \subseteq A$, we consider

$$WI = \{wa : w \in W,\ a \in I\} \quad \text{and} \quad IW = \{aw : a \in I,\ w \in W\}.$$

We say that I is W-invariant if $IW, WI \subseteq I$. A $*$-ideal ($*$-subalgebra) of I of A is called $(W,*)$-ideal ($(W,*)$-subalgebra) if it is W-invariant.

Notice that the Wedderburn-Malcev decomposition may not be W-invariant. Indeed, suppose that A admits a decomposition of the form (2.7) with $J \neq 0$ and the acting $*$-homomorphism Φ of W on A satisfies $\Phi(W) \subseteq \mathcal{IM}(J)$, i.e., W acts on A via the $*$-ideal J by left and right multiplications. In this case, we have BW, $WB \subseteq J$, and hence B is not W-invariant. Nevertheless, the following result holds.

Theorem 2.10. *Let W be a $\#$-algebra over a field F of characteristic zero and A be a finite dimensional $(W,*)$-algebra. Then the Jacobson radical J of A is a $(W,*)$-ideal of A. Moreover, if F is algebraically closed, then $A = B_1 \oplus \cdots \oplus B_m + J$, where B_i is a $*$-simple subalgebra such that WB_i, $B_iW \subseteq B_i + J$ for all $1 \leq i \leq m$.*

Proof. The result follows from Theorem 2.9 and by [23, Theorem 3.3]. □

We say that a $(W,*)$-algebra A is $(W,*)$-simple if $A^2 \neq 0$ and A has no non-zero $(W,*)$-ideals. As a consequence of Theorems 2.9 and 2.10, we obtain the following result, which will serve to describe the structure of $(W,*)$-simple algebras.

Corollary 2.11. *Let A be a finite dimensional $(W,*)$-algebra over an algebraically closed field F of characteristic zero. Then A is $(W,*)$-simple if and only if it is $*$-simple.*

Now, in order to describe simple $(W,*)$-algebras, we recall some notions from the theory of $*$-algebras.

Consider the matrix algebra $A = M_n(F)$. It is well known that we can define on A two involutions:

- the transpose involution t, that is the usual matrix transpose;
- the symplectic involution s defined, only when $n = 2k$ is even, as

$$\begin{pmatrix} A & B \\ C & D \end{pmatrix}^s = \begin{pmatrix} D^t & -B^t \\ -C^t & A^t \end{pmatrix}$$

  where $A, B, C, D \in M_k(F)$.

From Remark 2.2, given an algebra A we can consider the algebra A^{op}. Thus, the algebra $B = A \oplus A^{op}$ is a $*$-algebra endowed with the so-called exchange involution ex given by

$$(a, b)^{ex} = (b, a).$$

Finally, we can recall the characterization of $*$-simple algebras (see [17, 31]).

Theorem 2.12. *Let F be an algebraically closed field of characteristic zero. Then, up to isomorphisms, all finite dimensional $*$-simple algebras are the following ones*

- *$(M_n(F), t)$, the algebra of $n \times n$ matrices with transpose involution;*
- *$(M_{2k}(F), s)$, the algebra of $2k \times 2k$ matrices with symplectic involution;*
- *$(M_n(F) \oplus M_n(F)^{op}, ex)$, the direct sum of the algebra of $n \times n$ matrices and its opposite algebra with the exchange involution.*

From this characterization and Corollary 2.11 and Proposition 2.8, we have the following result.

Corollary 2.13. *If A is a finite dimensional $(W, *)$-simple algebra over an algebraically closed field F of characteristic zero, then A is isomorphic to one of the following $(W, *)$-algebras:*

- *$M_n(F)$ with transpose or symplectic involution,*
- *$M_n(F) \oplus M_n(F)^{op}$ with exchange involution $(a, b)^* = (b, a)$,*

with the W-action given by left and right multiplication of an appropriate unital $*$-subalgebra.

3. Free $(W, *)$-algebra and generalized $*$-identities

From now on, W will be a unital $*$-algebra endowed with an involution $\sharp$.

Let A and B be two $(W, *)$-algebras, endowed with the involutions $*_A$ and $*_B$, respectively. A $(W, *)$-homomorphism $\phi\colon A \to B$ must satisfy the conditions $\phi(a^{*_A}) = \phi(a)^{*_B}$ and $\phi(wav) = w\phi(a)v$, for all $a \in A$, $w, v \in W$.

We start by constructing the free $(W, *)$-algebra $W\langle X, *\rangle$. To this end, given two disjoint countable infinite sets of variables $X := \{x_1, x_2, \ldots\}$ and $X^* := \{x_1^*, x_2^*, \ldots\}$, we consider the free W-algebra $W\langle X \cup X^*\rangle$ freely generated by $X \cup X^*$ over F. Starting with the given involution $\#$ on W, we induce an involution $*$ on $W\langle X \cup X^*\rangle$ by defining, for all $w_1, w_2 \in W$ and $x_i \in X$,

$$(w_1 x_i w_2)^* := w_2^{\#} x_i^* w_1^{\#}.$$

In this way $W\langle X \cup X^*\rangle$ become the free $(W, *)$-algebra over X which we denote by $W\langle X, *\rangle$, and its elements are called *generalized $*$-polynomials* or *$(W, *)$-polynomials*. Thus, it is uniquely determined up to isomorphism by the following universal property: given a $(W, *)$-algebra A, any map $\varphi\colon X \to A$ can be uniquely extended to a homomorphism of $(W, *)$-algebras $\bar{\varphi}\colon W\langle X, *\rangle \to A$, called *evaluation* of $W\langle X, *\rangle$ in A. Notice that $\bar{\varphi}$ depends also on the acting $*$-homomorphism Φ of W on A, so we will denote the evaluation $\bar{\varphi}$ by $\bar{\varphi}_\Phi$, or simply φ_Φ, when necessary to clarify which action we are considering.

Since $W\langle X, *\rangle$ is, in particular, a $*$-algebra, then it decomposes as $W\langle X, *\rangle = W\langle X, *\rangle^+ \oplus W\langle X, *\rangle^-$ where the $(W, *)$-polynomials of $W\langle X, *\rangle^+$ are called symmetric, and those in $W\langle X, *\rangle^-$ are called skew-symmetric. Since W has unit 1_W, we set $1_W x_i 1_W = x_i$ and $1_W x_i^* 1_W = x_i^*$, $i \geq 1$. Thus, if we define

$$x_i^+ := x_i + x_i^* \quad \text{and} \quad x_i^- := x_i - x_i^*, \quad i \geq 1,$$

then $W\langle X, *\rangle$ may be regarded as the free $(W, *)$-algebra on the two sets $X^+ = \{x_1^+, x_2^+, \ldots\}$ and $X^- = \{x_1^-, x_2^-, \ldots\}$, where the variables from X^+ are symmetric, i.e., $x_i^+ \in W\langle X, *\rangle^+$, $i \geq 1$, and the variables from X^- are skew-symmetric, i.e., $x_i^- \in W\langle X, *\rangle^-$, $i \geq 1$.

Since W has unity, fix a basis $\mathcal{B}_W := \{w_i : i \geq 0\}$ of W, a vector-space basis of $W\langle X, *\rangle$ is

$$\mathcal{B}_{W\langle X,*\rangle} = \{w_{i_0} x_{j_1}^{\epsilon_1} w_{i_1} x_{j_2}^{\epsilon_2} w_{i_2} \cdots w_{i_{n-1}} x_{j_n}^{\epsilon_n} w_{i_n} \ : \ \epsilon_k \in \{+, -\},\ w_{i_l} \in \mathcal{B}_W,\ 1 \leq k \leq n,\ 0 \leq l \leq n,\ n \geq 1\}.$$

Given a $(W, *)$-algebra A, a generalized $*$-polynomial $f = f(x_1^+, \ldots, x_r^+, x_1^-, \ldots, x_s^-) \in W\langle X, *\rangle$ is a *$(W, *)$-identity*, or a *generalized $*$-identities* of A, and we write $f \equiv 0$, if $f(a_1^+, \ldots, a_r^+, a_1^-, \ldots, a_s^-) = 0$ for all choices of $a_1^+, \ldots, a_r^+ \in A^+$, $a_1^-, \ldots, a_s^- \in A^-$. We denote by $\mathrm{Id}^{W,*}(A)$ the set of generalized $*$-identities of A, and it forms a T_W^*-ideal of $W\langle X, *\rangle$, that is, a $(W, *)$-ideal invariant under all $(W, *)$-algebra endomorphisms of $W\langle X, *\rangle$. Given a set $S \subseteq W\langle X, *\rangle$, by $\langle S\rangle_{T_W^*}$ we denote the smallest T_W^*-ideal containing S.

For $n \geq 1$, we denote by $P_n^{W,*}$ the vector space of multilinear $(W,*)$-polynomials in the variables $x_1^+, x_1^-, \ldots, x_n^+, x_n^-$, that is

$$P_n^{W,*} := \mathrm{span}_F\{w_{i_0} x_{\sigma(1)}^{\epsilon_1} w_{i_1} x_{\sigma(2)}^{\epsilon_2} w_{i_2} \cdots w_{i_{n-1}} x_{\sigma(n)}^{\epsilon_n} w_{i_n} \,:\, \epsilon_k \in \{+,-\},\, w_{i_l} \in \mathcal{B}_W,\, 1 \leq k \leq n,\, 0 \leq l \leq n,\, \sigma \in S_n\}.$$

Remark 3.1. Since W is a $*$-algebra, we can consider a basis $\mathcal{B}_W$ of W which is the union $\mathcal{B}_W = \mathcal{B}_{W^+} \cup \mathcal{B}_{W^-}$ of bases of W^+ and W^-, respectively, where $\mathcal{B}_{W^+} := \{w_i^+ \,:\, i \geq 1\}$ and $\mathcal{B}_{W^-} := \{w_i^- \,:\, i \geq 1\}$. Then, for $n \geq 1$ we can write

$$P_n^{W,*} = \mathrm{span}_F\{w_{i_0}^{\eta_0} x_{\sigma(1)}^{\epsilon_1} w_{i_1}^{\eta_1} x_{\sigma(2)}^{\epsilon_2} w_{i_2}^{\eta_2} \cdots w_{i_{n-1}}^{\eta_{n-1}} x_{\sigma(n)}^{\epsilon_n} w_{i_n}^{\eta_n} \,:\, \epsilon_k, \eta_l \in \{+,-\},\, w_{i_l}^{\eta_l} \in \mathcal{B}_{W^{\eta_l}},\, 1 \leq k \leq n,\, 0 \leq l \leq n,\, \sigma \in S_n\ \}.$$

Since $\mathrm{char}F = 0$, a standard linearization argument shows that the T_W^*-ideal $\mathrm{Id}^{W,*}(A)$ is completely determined by its multilinear generalized $*$-polynomials. Thus, we can consider the quotient space

$$P_n^{W,*}(A) := \frac{P_n^{W,*}}{P_n^{W,*} \cap \mathrm{Id}^{W,*}(A)},$$

and the *n-th $(W,*)$-codimension*, or *generalized $*$-codimension*, of A is defined as

$$c_n^{W,*}(A) := \dim_F P_n^{W,*}(A).$$

Note that $c_n^{W,*}(A)$ is not necessarily finite. However, if at least one between W and A is finite dimensional, then $c_n^{W,*}(A)$ is finite for all $n \geq 1$.

Remark that when $W \cong F$ we are dealing with the ordinary $*$-identities. In this case we write $P_n^* := P_n^{F,*}$, $\mathrm{Id}^*(A) := \mathrm{Id}^{F,*}(A)$ and $c_n^*(A) := c_n^{F,*}(A)$. More generally, since W has unity, F embeds naturally as a subalgebra of W and it follows that $P_n^* \subseteq P_n^{W,*}$. Consequently, for any $(W,*)$-algebra A, the ordinary $*$-codimensions are bounded above by the generalized $*$-codimensions. That is,

$$c_n^*(A) \leq c_n^{W,*}(A), \quad n \geq 1. \tag{3.1}$$

The study of multilinear generalized $*$-polynomial can be reduced to the one of smaller spaces in the following way. For $n \geq 1$ and $0 \leq r \leq n$, define $P_{r,n-r}$ as the subspace of $P_n^{W,*}$ of multilinear $(W,*)$-polynomials in the variable $x_1^+, \ldots, x_r^+, x_{r+1}^-, \ldots, x_n^-$, i.e., the first r variables are symmetric and the last $n-r$ are skew-symmetric. It is clear that $P_n^{W,*}$ is isomorphic to the direct sum of $\binom{n}{r}$ copies of $P_{r,n-r}$ with $0 \leq r \leq n$ as well as that $P_n^{W,*}(A)$ inherits this decomposition. Thus, one sets

$$P_{r,n-r}(A) := \frac{P_{r,n-r}}{P_{r,n-r} \cap \mathrm{Id}^{W,*}(A)}$$

and

$$c_{r,n-r}(A) := \dim_F P_{r,n-r}(A),$$

and as a consequence, for all $n \geq 1$ we have that

$$P_n^{W,*}(A) \cong \bigoplus_{0 \leq r \leq n} \binom{n}{r} P_{r,n-r}(A) \tag{3.2}$$

and

$$c_n^{W,*}(A) = \sum_{r=0}^{n} \binom{n}{r} c_{r,n-r}(A). \tag{3.3}$$

We can attach another useful invariant to the spaces $P_{r,n-r}(A)$ in the following way. For $n \geq 1$ and $0 \leq r \leq n$ we consider the left permutation action of the group $S_r \times S_{n-r}$ on the space $P_{r,n-r}$ by letting S_r act on $x_1^+, \ldots, x_r^+$ and $S_{n-r} \equiv S_{\{r+1,\ldots,n\}}$ on $x_{r+1}^-, \ldots, x_n^-$. Since $\mathrm{Id}^{W,*}(A)$ is invariant under this $S_r \times S_{n-r}$ action, we have that $P_{r,n-r}(A)$ is a $S_r \times S_{n-r}$ left module. The corresponding character, denoted by $\chi_{r,n-r}^{W,*}(A)$ is called *$(r,n-r)$-th $(W,*)$-cocharacter*, or *generalized $*$-cocharacter*, of A.

Recall that the irreducible $(S_r \times S_{n-r})$-characters are the tensor products $\chi_\lambda \otimes \chi_\mu$ of the irreducible S_r-character χ_λ and S_{n-r}-character χ_μ, where $\lambda \vdash r$ and $\mu \vdash n-r$ are partitions. Since F is of characteristic zero, by complete reducibility we can write

$$\chi^{W,*}_{r,n-r}(A) = \sum_{(\lambda,\mu)\vdash(r,n-r)} m_{\lambda,\mu}(\chi_\lambda \otimes \chi_\mu) \tag{3.4}$$

where $\lambda \vdash r$, $\mu \vdash n-r$, and $m_{\lambda,\mu} \geq 0$ is the multiplicity corresponding to $\chi_\lambda \otimes \chi_\mu$.

Since there is a one-to-one correspondence between T^*_W-ideals and varieties of $(W,*)$-algebras, it is often convenient to translate all the objects we have introduced to the language of varieties. A variety $\mathcal{V}$ of $(W,*)$-algebras generated by a $(W,*)$-algebra A is denoted by $\mathrm{var}^{W,*}(A)$ and is called $(W,*)$-*variety*, or *generalized* $*$-*variety*. We define $\mathrm{Id}^{W,*}(\mathcal{V}) := \mathrm{Id}^{W,*}(A)$. Moreover, the growth of $\mathcal{V}$ is the growth of the sequence $c^{W,*}_n(\mathcal{V}) := c^{W,*}_n(A)$, $n \leq 1$, and $\chi^{W,*}_{r,n-r}(\mathcal{V}) := \chi^{W,*}_{r,n-r}(A)$, for all $n \geq 1$ and $0 \leq r \leq n$, is the $(r, n-r)$th $(W,*)$-cocharacter of $\mathcal{V}$.

We conclude this section with a result on the $(W,*)$-variety generated by finite-dimensional unital $(W,*)$-algebras, which will be useful in what follows.

Let $A^{\overline{W}}$ be a finite-dimensional $(W,*)$-algebra and let $\Phi\colon W \to \mathcal{M}(A)$ be the corresponding acting $*$-homomorphism of W on A, i.e., such that $\overline{W} = \Phi(W)$. Since $\overline{W}$ is also finite dimensional and F is of characteristic 0, the Wedderburn–Malcev decomposition for algebras with involution (Theorem 2.9) yields

$$\overline{W} = \overline{W}_{ss} + \bar{J}$$

where $\overline{W}_{ss}$ is a semisimple $*$-subalgebra of $\overline{W}$ isomorphic to $\overline{W}/\bar{J}$ and $\bar{J} = J(\overline{W})$ is the Jacobson radical of $\overline{W}$. If $\pi\colon \overline{W} \to \overline{W}/\bar{J}$ denotes the canonical epimorphism, then, via the above identification, we may assume that $\pi\colon \overline{W} \to \overline{W}_{ss}$.

Define a map $\Phi_{ss}\colon W \to \mathcal{M}(A)$ by $\Phi_{ss} := \pi\Phi$. Then Φ_{ss} is a homomorphism of $*$-algebras satisfying $\Phi_{ss}(W) = \overline{W}_{ss}$ and therefore determines another $(W,*)$-algebra structure on A. Therefore, $A^{\overline{W}_{ss}}$ is a $(W,*)$-algebra defined by the acting $*$-homomorphism Φ_{ss}.

We can now state the following result, whose proof follows from that of [23, Theorem 4.3] with mild and natural adaptations.

Theorem 3.2. *Let W be a unital algebra with involution over a field F of characteristic zero, and $A^{\overline{W}}$ be a finite-dimensional unital $(W,*)$-algebra. If $\bar{J} \subseteq \mathcal{IM}(J(A))$, where $\bar{J} = J(\overline{W})$ and $J(A)$ are the Jacobson radicals of $\overline{W}$ and A, respectively, then $A^{\overline{W}_{ss}} \in \mathrm{var}^{W,*}(A^{\overline{W}})$.*

4. The $(W,*)$-exponent

In [14] the authors proved that, for any $*$-algebra A that satisfies a non-trivial identity the limit

$$\exp^*(A) := \lim_{n\to\infty} \sqrt[n]{c^*_n(A)}$$

exists and is an integer, referred to as the $*$-exponent of A. Moreover, in [18] the same result was established for a broad class of algebras, which includes algebras with involution, under the additional assumption of finite dimensionality. By analogy, we define the generalized $(W,*)$-exponent of A by

$$\exp^{W,*}(A) := \lim_{n\to\infty} \sqrt[n]{c^{W,*}_n(A)}.$$

The main objective of this section is to show that, for any finite-dimensional $(W,*)$-algebra A, this limit exists and coincides with $\exp^*(A)$.

From this point on, we therefore assume that A is a finite-dimensional $(W,*)$-algebra. We begin by recalling the following definition.

Definition 4.1. Let A be a finite-dimensional $(W,*)$-algebra such that $A = B_1 \oplus \cdots \oplus B_m + J$, where $B_1, \ldots, B_m$ are $*$-simple algebras. We say that a $*$-subalgebra $B_{i_1} \oplus \cdots \oplus B_{i_k}$, where the B_{i_j}'s are distinct simple $*$-algebras, is *admissible* if for some permutation $(l_1, \ldots, l_k)$ of $(i_1, \ldots, i_k)$ we have that

$$B_{l_1} J B_{l_2} J \cdots J B_{l_k} \neq 0.$$

We start by proving the lower bound.

Lemma 4.2. *Let A be a finite dimensional $(W,*)$-algebra over an algebraically closed field F of characteristic zero. Then there exist constants $C>0$ and t such that*

$$c_n^{W,*}(A) \geq Cn^t d^n,$$

where d is the maximal dimension of an admissible $*$-subalgebra of A.

Proof. By [18, Theorem 5] and (3.1) there exist constants $C>0$ and t such that $c_n^{W,*}(A) \geq c_n^*(A) \geq Cn^t d^n$, where d is the maximal dimension of an admissible $*$-subalgebra of A, as required. □

In order to compute the upper bound we need the following construction. Let $A = A^+ \oplus A^-$ be a finite dimensional $(W,*)$-algebra. We fix a basis $\mathcal{B}_A$ of A which is the union $\mathcal{B}_A = \mathcal{B}^+ \cup \mathcal{B}^-$ of bases of A^+ and A^-, respectively, where $\mathcal{B}^+ = \{a_1, \ldots, a_s\}$ and $\mathcal{B}^- = \{b_1, \ldots, b_t\}$. Moreover, for a fixed $n \geq 1$ consider $n(s+t)$ commutative variables $\xi_{i,j}$, $1 \leq i \leq n$, $1 \leq j \leq s+t$. Then, for $1 \leq i \leq n$, we define the generic elements:

$$\xi_i^+ := \sum_{j=1}^{s} \xi_{i,j} \otimes a_j \quad \text{and} \quad \xi_i^- := \sum_{j=1}^{t} \xi_{i,s+j} \otimes b_j. \tag{4.1}$$

We denote by H the algebra generated by the ξ_i^+ and ξ_i^-, $1 \leq i \leq n$, over F. Then, the $(W,*)$-algebra structure of A induces naturally a corresponding $(W,*)$-algebra structure on H. The involution $*$ on H is defined by assuming that the elements $\xi_1^+, \ldots, \xi_n^+$ are symmetric and $\xi_1^-, \ldots, \xi_n^-$ are skew-symmetric. The action of W on H is given as follows: for $w, v \in W$ and $1 \leq i \leq n$,

$$w\xi_i^+ v = \sum_{j=1}^{s} \xi_{i,j} \otimes (wa_j v) \quad \text{and} \quad w\xi_i^- v = \sum_{j=1}^{t} \xi_{i,s+j} \otimes (wb_j v).$$

Moreover, let $F[\xi_{i,j}] := F[\xi_{i,j} | 1 \leq i \leq n, 1 \leq j \leq s+t]$ denote the polynomial ring over F in the indeterminates $\xi_{i,j}$, that is, the free commutative algebra over F in $n(s+t)$ variables. Then also the tensor product algebra $F[\xi_{i,j}] \otimes A$ inherits a natural $(W,*)$-algebra structure induced by the one of A, and we have that $H \subseteq F[\xi_{i,j}] \otimes A$.

Proposition 4.3. *Let $W\langle x_1^+, \ldots, x_n^+, x_1^-, \ldots, x_n^- \rangle$ be the free $(W,*)$-algebra in n symmetric variables and n skew-symmetric variables, and*

$$\psi \colon W\langle x_1^+, \ldots, x_n^+, x_1^-, \ldots, x_n^- \rangle \to H$$

be the $(W,*)$-homomorphism induced by $\psi(x_i^+) = \xi_i^+$ and $\psi(x_i^-) = \xi_i^-$, $1 \leq i \leq n$. Then

$$\ker \psi = \mathrm{Id}^{W,*}(A) \cap W\langle x_1^+, \ldots, x_n^+, x_1^-, \ldots, x_n^- \rangle.$$

Proof. We start by proving that $\mathrm{Id}^{W,*}(A) \cap W\langle x_1^+, \ldots, x_n^+, x_1^-, \ldots, x_n^- \rangle \subseteq \ker \psi$. To this end, let $f = f(x_1^+, \ldots, x_n^+, x_1^-, \ldots, x_n^-) \in W\langle x_1^+, \ldots, x_n^+, x_1^-, \ldots, x_n^- \rangle$ be a $(W,*)$-identity of A. Since the field F is of characteristic zero, we can assume that f is a multilinear $(W,*)$-polynomial. Then, by (4.1), $\psi(f) = f(\xi_1^+, \ldots, \xi_n^+, \xi_1^-, \ldots, \xi_n^-)$ is a sum of the expressions of the form

$$\bar{f} = f(\xi_{1,j_1} \otimes a_{j_1}, \ldots, \xi_{n,j_n} \otimes a_{j_n}, \xi_{1,l_1} \otimes b_{l_1}, \ldots, \xi_{n,l_n} \otimes b_{l_n}) = \xi_{1,p_1} \cdots \xi_{n,p_n} \otimes f(a_{j_1}, \ldots, a_{j_n}, b_{l_1}, \ldots, b_{l_n}),$$

where $p_i \in \{j_i, l_i\}$ for $1 \leq i \leq n$. Since $f \in \mathrm{Id}^{W,*}(A)$, we get that $\bar{f} = 0$, and as a consequence $f \in \ker \psi$, as required.

Suppose now that $f(x_1^+, \ldots, x_n^+, x_1^-, \ldots, x_n^-) \in \ker \psi$, i.e., $f(\xi_1^+, \ldots, \xi_n^+, \xi_1^-, \ldots, \xi_n^-) = 0$ in H, and let $c_1, \ldots, c_n$ be arbitrary symmetric elements of A and $d_1, \ldots, d_n$ be arbitrary skew-symmetric elements of A. Write each c_i as a linear combination of the basis $\mathcal{B}^+$ of A^+ and d_i as a linear combination of the basis $\mathcal{B}^-$ of A^-. Then, for $1 \leq i \leq n$, there exist $\alpha_{i,1}, \ldots, \alpha_{i,s}, \beta_{i,s+1}, \ldots, \beta_{i,s+t} \in F$ such that

$$c_i = \sum_{j=1}^{s} \alpha_{i,j} a_j \quad \text{and} \quad d_i = \sum_{j=1}^{t} \beta_{i,s+j} b_j.$$

By the universal property of the free commutative algebra $F[\xi_{i,j}]$ and of the tensor product, we can consider the $(W,*)$-homomorphism $\rho : F[\xi_{i,j}] \otimes A \to A$ which restricts to the identity on A and, for each $1 \leq i \leq n$, satisfies

$$\rho(\xi_{i,j}) = \begin{cases} \alpha_{i,j}, & \text{if } 1 \leq j \leq s \\ \beta_{i,j}, & \text{if } s+1 \leq j \leq s+t \end{cases}.$$

Since $\rho(\xi_i^+) = c_i$ and $\rho(\xi_i^-) = d_i$, $1 \leq i \leq n$, we obtain

$$0 = \rho(f(\xi_1^+, \ldots, \xi_n^+, \xi_1^-, \ldots, \xi_n^-)) = f(\rho(\xi_1^+) \ldots, \rho(\xi_n^+), \rho(\xi_1^-), \ldots, \rho(\xi_n^-)) = f(c_1, \ldots, c_n, d_1, \ldots, d_n).$$

Since $c_1, \ldots, c_n, d_1, \ldots, d_n$ are arbitrary elements of A, $f(x_1^+, \ldots, x_n^+, x_1^-, \ldots, x_n^-)$ is a $(W, *)$-identity of A. Consequently, $\ker \psi = \mathrm{Id}^{W,*}(A) \cap W\langle x_1^+, \ldots, x_n^+, x_1^-, \ldots, x_n^-\rangle$. □

As a consequence we have that H is the relatively free $(W, *)$-algebra in n symmetric generators and n skew-symmetric generators of the $(W, *)$-variety generated by A, and that a $(W, *)$-polynomial is a $(W, *)$-identity if and only if it vanishes on the generic elements $\xi_{i,j}$, $1 \leq i \leq n$, $1 \leq j \leq s + t$. We use this in the next lemma to determine the upper bound.

Lemma 4.4. *Let A be a finite dimensional $(W, *)$-algebra over an algebraically closed field F, where W is a unital F-algebra. Then there exist constants C, u such that*

$$c_n^{W,*}(A) \leq Cn^u d^n,$$

where d is the maximal dimension of a admissible $*$-subalgebra of A.

Proof. Consider the $(W, *)$-homomorphism $\psi\colon W\langle x_1^+, \ldots, x_n^+, x_1^-, \ldots, x_n^-\rangle \to H$ of the proposition above. Then $\psi(P_n^{W,*}) \cong P_n^{W,*}(A)$, and therefore, to estimate $c_n^{W,*}(A)$, it suffices to obtain an upper bound for $\dim_F \psi(P_n^{W,*})$. To this end, for each $0 \leq r \leq n$, set $\mathcal{P}_{r,n-r} := \psi(P_{r,n-r})$, that is, the vector space of multilinear $(W, *)$-polynomials in the variables $\xi_1^+, \ldots, \xi_r^+, \xi_{r+1}^-, \ldots, \xi_n^-$. We begin by determining an upper bound for $\dim_F \mathcal{P}_{r,n-r}$.

Consider a monomial $w_{i_0}\xi_{\sigma(1)}^{\epsilon_{\sigma(1)}} w_{i_1} \cdots w_{i_{n-1}}\xi_{\sigma(n)}^{\epsilon_{\sigma(n)}} w_{i_n} \in \mathcal{P}_{r,n-r}$, where $\sigma \in S_n$, $w_{i_l} \in \mathcal{B}_W$ for $1 \leq l \leq n$, $\epsilon_i = +$ for $1 \leq i \leq r$, and $\epsilon_i = -$ for $r+1 \leq i \leq n$. Using (4.1), such a monomial can be expressed as a linear combination of products of n elements from the fixed basis $\mathcal{B}_A = \mathcal{B}^+ \cup \mathcal{B}^-$ of A with coefficients that are polynomials in the variables ξ_{ij}. More precisely, when expanded with respect to the basis $\mathcal{B}_A$, it is a linear combination of elements of the form $c \otimes \xi_{1,j_1} \cdots \xi_{n,j_n}$, where $c \in \mathcal{B}_A$. Thus, we will estimate $\dim_F \mathcal{P}_{r,n-r}$ through an estimate of the number of possible monomials $\xi_{1,j_1} \cdots \xi_{n,j_n}$ of this last type which can appear as coefficients of $w_{i_0}\xi_{\sigma(1)}^{\epsilon_{\sigma(1)}} w_{i_1} \cdots w_{i_{n-1}}\xi_{\sigma(n)}^{\epsilon_{\sigma(n)}} w_{i_n}$.

Since F is algebraically closed of characteristic zero, by Theorem 2.10 we write $A = B_1 \oplus \cdots \oplus B_m + J$, where J is the Jacobson radical of A and each B_i is a simple $*$-algebra satisfying $WB_l, B_lW \subseteq B_l + J$ for $1 \leq l \leq m$. We may further assume that the chosen basis $\mathcal{B}_A = \mathcal{B}^+ \cup \mathcal{B}^-$ of A is adapted to this decomposition, in the sense that both $\mathcal{B}^+$ and $\mathcal{B}^-$ consist of elements drawn from J and of each of the B_i. Since each commutative variable $\xi_{i,j}$ in (4.1) is attached to an element of either $\mathcal{B}^+$ or $\mathcal{B}^-$, by a slight abuse of terminology, we therefore refer to $\xi_{i,j}$ as a symmetric radical, skew-symmetric radical, symmetric semisimple, or skew-symmetric semisimple variable.

Take a monomial $\xi_{1,j_1} \cdots \xi_{n,j_n}$ of degree n that contains i_1 symmetric radical variables and i_2 skew-symmetric radical variables. Since J is a nilpotent $(W, *)$-ideal of A, such a monomial is nonzero only if $i_1 + i_2 \leq u$, where u is such that $J^u \neq 0$ and $J^{u+1} = 0$. Now each such monomial involves $r - i_1$ symmetric semisimple variables and $n - r - i_2$ skew-symmetric semisimple variables which come from some distinct $*$-simple components $B_{l_1}, \ldots, B_{l_p}$. Since $WB_l, B_lW \subseteq B_l + J$, there exist distinct indices $q_1, \ldots, q_v \in \{l_1, \ldots, l_p\}$, with $1 \leq v \leq p$ such that $D := B_{q_1} \oplus \cdots \oplus B_{q_v}$ is an admissible $*$-subalgebra of A.

Now, fix an admissible $*$-subalgebra D of A. We recall that each monomial has r coefficients taken in the set $\{\xi_{ij} : 1 \leq i \leq r, 1 \leq j \leq t\}$ and $n - r$ taken in the set $\{\xi_{ij} : r + 1 \leq i \leq n, 1 \leq j \leq s\}$ and also the number of possible distributions of $i_1 \leq r$ symmetric radical variables and $i_2 \leq n - r$ skew-symmetric radical variables in a non-zero monomial is at most $\binom{r}{i_1}\binom{n-r}{i_2}(\dim_F J)^{i_1+i_2} \leq C_1 n^u$, where C_1 is a constant. So, the number of possible monomials with symmetric and skew-symmetric semisimple variables coming from D is bounded by $C_2 n^u (\dim D^+)^r (\dim D^-)^{n-r}$.

Now, let M be the number of maximal admissible $*$-subalgebras of A and let E be an admissible $*$-subalgebra of maximum dimension. Then, an upper bound for the number of possible nonzero monomials is $MC_2 n^u (\dim E^+)^r (\dim E^-)^{n-r}$. Since we rewrote any product of n basis elements of A as a linear combination of basis elements, we get that

$$\dim_F \mathcal{P}_{r,n-r} \leq C_3 n^u (\dim_F E^+)^r (\dim_F E^-)^{n-r},$$

with C_3 constant. Since, by Proposition 4.3, $\mathcal{P}_{r,n-r} \cong P_{r,n-r}(A)$, we have that $\dim_F \mathcal{P}_{r,n-r} = c_{r,n-r}(A)$. Thus, if $d = \dim_F E^+ + \dim_F E^- = \dim_F E$, by (3.3) we obtain

$$c_n^{W,*}(A) = \sum_{r=0}^{n} \binom{n}{r} \dim_F \mathcal{P}_{r,n-r} \leq C_3 n^u \sum_{r=0}^{n} \binom{n}{r} (\dim_F E^+)^r (\dim_F E^-)^{n-r} = C_3 n^u d^n,$$

as required. □

Finally, we can prove the main result of the section.

Theorem 4.5. *Let A be a finite dimensional $(W,*)$-algebra over a field F of characteristic zero, where W is a unital algebra with involution. Then the $(W,*)$-exponent $\exp^{W,*}(A)$ of A exists and is equal to $*$-exponent $\exp^*(A)$ of A.*

Proof. As in the ordinary case, a careful adaptation of the proof of [17, Theorem 4.1.9] shows that the codimensions remain unchanged under extension of the base field F, since F is infinite. Then by Lemmas 4.2 and 4.4 there exist constants C_1, C_2, t_1, t_2 such that $C_1 \geq 0$ and

$$C_1 n^{t_1} d^n \leq c_n^{W,*}(A) \leq C_2 n^{t_2} d^n,$$

where d is the maximal dimension of an admissible $*$-subalgebra of A. Thus, the $(W,*)$-exponent $\exp^{W,*}(A)$ of A exists and is equal to d. Moreover, by [18, Theorem 5], we also have $\exp^*(A) = d$, completing the proof. □

As a consequence of the previous theorem, we obtain that $\exp^{W,*}(A) = d$, where d denotes the maximal dimension of an admissible $*$-subalgebra of A. In particular, this yields the integrality of the $(W,*)$-exponent. We therefore obtain the following corollary.

Corollary 4.6. *If A is a finite dimensional $(W,*)$-algebra, then the sequence of $(W,*)$-codimensions $c_n^{W,*}(A)$, $n \geq 1$, either has exponential growth or is polynomially bounded.*

We say that the generalized $*$-variety $\mathcal{V}$ has *polynomial growth* if $c_n^{W,*}(\mathcal{V})$ is polynomially bounded and $\mathcal{V}$ has *almost polynomial growth* if $c_n^{W,*}(\mathcal{V})$ is not polynomially bounded but every proper $(W,*)$-subvariety $\mathcal{U}$ of $\mathcal{V}$ has polynomial growth. Our interest lies precisely in the study of this boundary case.

5. Generalized $*$-identities of M

In this section we introduce the $(W,*)$-algebra M, which is a subalgebra of the 4×4 upper triangular matrices over F, and we determine its generalized $*$-identities.

Let e_{ij} denote the usual matrix units. Then M is the $*$-algebra

$$M = F(e_{11} + e_{44}) + F(e_{22} + e_{33}) + Fe_{12} + Fe_{34},$$

endowed with the reflection involution $*$, i.e., the involution given by the formula: for $\alpha, \beta, \gamma, \delta \in F$

$$\big(\alpha(e_{11} + e_{44}) + \beta(e_{22} + e_{33}) + \gamma e_{12} + \delta e_{34}\big)^* = \alpha(e_{11} + e_{44}) + \beta(e_{22} + e_{33}) + \delta e_{12} + \gamma e_{34}.$$

As the field F has characteristic zero, M decomposes as the direct sum of its symmetric and skew-symmetric parts, $M = M^+ \oplus M^-$, where

$$M^+ = F(e_{11} + e_{44}) + F(e_{22} + e_{33}) + F(e_{12} + e_{34}) \quad \text{and} \quad M^- = F(e_{12} - e_{34}).$$

Our first goal in this section is to classify, up to equivalence, all possible $(W,*)$-algebra structures on M. To this end, let $\Phi\colon W \to \mathcal{M}(M)$ be an acting $*$-homomorphism of W on M and set $\overline{W} = \Phi(W)$. Since M is unital, Proposition 2.8 implies that the action of W on M is induced by the left and right multiplication actions of a $*$-subalgebra B of M isomorphic to $\overline{W}$ and containing the unit $1_M = e_{11} + e_{22} + e_{33} + e_{44}$.

Notice that, up to isomorphisms, the proper $*$-subalgebras of M containing 1_M are

$$F1_M \cong F, \qquad D := F(e_{11} + e_{44}) \oplus F(e_{22} + e_{33}),$$

$$C_1 := F1_M + Fe_{12} + Fe_{34}, \quad C_2 := F1_M + F(e_{12} + e_{34}), \quad C_3 := F1_M + F(e_{12} - e_{34}),$$

endowed with the corresponding induced involution.

Consequently, by Definition 2.7 and Proposition 2.8, M admits exactly six pairwise non-equivalent $(W,*)$-algebra structures according to which $*$-subalgebra of M induce the action by left and right multiplication. We denote these structures by

$$M^F, \qquad M^D, \qquad M^{C_i}\ (1 \le i \le 3), \qquad M,$$

according as $\overline{W}$ is isomorphic to F, D, C_i for some $1 \le i \le 3$, or to the full algebra M, respectively.

Notice that as a direct consequence of Theorem 4.5, we have that

$$\exp^{W,*}(M^F) = \exp^{W,*}(M^D) = \exp^{W,*}(M^{C_i}) = \exp^{W,*}(M) = 2,$$

where $1 \le i \le 3$. Moreover, by Theorems 2.10 and 3.2 it follows that

$$M^F \in \operatorname{var}^{W,*}(M^{C_i}),\ 1 \le i \le 3, \qquad \text{and} \qquad M^D \in \operatorname{var}^{W,*}(M).$$

Accordingly, since we aim to study the boundary case between polynomial and exponential growth, we proceed to determine generators for the T_W^*-ideals and to compute the exact values of the corresponding $(W,*)$-codimensions of M^F and M^D. We begin with the following technical lemma.

Lemma 5.1. *For any $k \ge 0$ we have $x_{k+1}^- x_1^+ \cdots x_k^+ x_{k+2}^- \in \langle x_1^- x_2^- \rangle_{T_W^*}$.*

Proof. If $k = 0$, there is nothing to prove, so assume $k \ge 1$. We proceed by induction on k.

First observe that $x_1^+ x_2^- + x_2^- x_1^+ \in W\langle X,* \rangle^-$. Then for the base case $k = 1$, substituting x_1^- with $x_1^+ x_2^- + x_2^- x_1^+$ in the $(W,*)$-polynomial $x_1^- x_2^-$, and then renaming the variables, yields $x_1^+ x_2^- x_3^- + x_2^- x_1^+ x_3^- \in \langle x_1^- x_2^- \rangle_{T_W^*}$. Hence $x_2^- x_1^+ x_3^- \in \langle x_1^- x_2^- \rangle_{T_W^*}$.

Now assume inductively that $x_k^- x_1^+ \cdots x_{k-1}^+ x_{k+1}^- \in \langle x_1^- x_2^- \rangle_{T_W^*}$. Replacing x_k^- with $x_k^+ x_{k+2}^- + x_{k+2}^- x_k^+$ in this $(W,*)$-polynomial and renaming variables, we obtain $x_{k+1}^- x_1^+ \cdots x_{k-1}^+ x_{k+2}^- x_k^+ + x_{k+1}^- x_1^+ \cdots x_k^+ x_{k+2}^- \in \langle x_1^- x_2^- \rangle_{T_W^*}$. Consequently, by the inductive hypothesis, $x_{k+1}^- x_1^+ \cdots x_k^+ x_{k+2}^- \in \langle x_1^- x_2^- \rangle_{T_W^*}$. □

Let $[x_1, x_2] := x_1 x_2 - x_2 x_1$ denote the commutator of x_1 and x_2. More generally, for $k \ge 2$, we write $[x_1, x_2, \ldots, x_k]$ for the left-normed commutator.

With this notation, we are ready to prove the main results of this section.

Theorem 5.2. *Let M^D be a $(W,*)$-algebra M where W acts on it as the algebra D by left and right multiplication. Then there exists a basis $\mathcal{B}_W = \mathcal{B}_{W^+} \cup \mathcal{B}_{W^-}$ of W, where $\mathcal{B}_{W^+} := \{1_W, w_i^+ : i \ge 1\}$ and $\mathcal{B}_{W^-} := \{w_i^- : i \ge 1\}$ are basis of W^+ and W^-, respectively, such that the T_W^*-ideal $\operatorname{Id}^{W,*}(M^D)$ of generalized $*$-identities of M^D is generated, as T_W^*-ideal, by the following $(W,*)$-polynomials:*

$$x_1^- x_2^-, \quad w_1^+ x^- + x^- w_1^+, \quad (w_1^+)^2 x^\epsilon - x^\epsilon, \quad w_i^+ x^\epsilon, \quad w_j^- x^\epsilon,$$

where $\epsilon \in \{+,-\}$, $i \ge 2$ and $j \ge 1$. Moreover $c_n^{W,*}(M^D) = n2^{n+1} + 2$.

Proof. Let Φ denote the acting homomorphism of W on M^D. Then $\Phi(W) \cong D$ and we can choose a basis $\mathcal{B}_W = \mathcal{B}_{W^+} \cup \mathcal{B}_{W^-}$ of $W = W^+ \oplus W^-$, where $\mathcal{B}_{W^+} := \{w_i^+ : i \ge 0\}$ and $\mathcal{B}_{W^-} := \{w_i^- : i \ge 1\}$ are basis of W^+ and W^-, respectively, such that $w_0^+ = 1_W$,

$$\Phi(w_1^+) = (R_{e_{11}+e_{44}-e_{22}-e_{33}}, L_{e_{11}+e_{44}-e_{22}-e_{33}}) \in \mathcal{M}(M) \qquad \text{and} \qquad w_i^+,\ w_j^- \in \ker\Phi \quad \text{for all } i \ge 2,\ j \ge 1.$$

Let I be the T_W^*-ideal generated by the $(W,*)$-polynomials in the statement. It is easy to check that $I \subseteq \operatorname{Id}^{W,*}(M^D)$.

To prove the opposite inclusion, let f be a multilinear generalized $*$-polynomial of degree n and assume, as we may, that $f \in P_{r,n-r}$ for some $0 \le r \le n$.

Let $\epsilon \in \{+,-\}$. Observe first that $x^\epsilon (w_1^+)^2 - x^\epsilon \in \langle (w_1^+)^2 x^\epsilon - x^\epsilon \rangle_{T_W^*}$. Moreover, for $i \ge 2$ we have $x^\epsilon w_i^+$, $w_1^+ w_i^+ x^\epsilon$, $x^\epsilon w_i^+ w_1^+ \in \langle w_i^+ x^\epsilon \rangle_{T_W^*}$ and for $j \ge 1$ we have $x^\epsilon w_j^-$, $w_1^+ w_j^- x^\epsilon$, $x^\epsilon w_j^- w_1^+ \in \langle w_j^- x^\epsilon \rangle_{T_W^*}$. Furthermore since the $(W,*)$-polynomial $w_1^+ x^\epsilon + x^\epsilon w_1^+$ is symmetric if $\epsilon = +$ and skew-symmetric if $\epsilon = -$, it follows that $x^\epsilon w_1^+ w_i^+$, $w_i^+ w_1^+ x^\epsilon \in \langle w_i^+ x^\epsilon \rangle_{T_W^*}$ for all $i \ge 2$, and $x^\epsilon w_1^+ w_j^-$, $w_j^- w_1^+ x^\epsilon \in \langle w_j^- x^\epsilon \rangle_{T_W^*}$ for all $j \ge 1$. Hence, since $(w_1^+)^2 x^\epsilon - x^\epsilon$, $w_i^+ x^\epsilon$, $w_j^- x^\epsilon \in I$, for $\epsilon \in \{+,-\}$, $i \ge 2$ and $j \ge 1$, we may suppose that, modulo I, only 1_W and w_1^+ appear in f.

Since $[x_1^+, w_1^+] \in W\langle X,* \rangle^-$, by Lemma 5.1 we get that for every $k \ge 0$

$$[x_{k+1}^+, w_1^+] x_1^+ \cdots x_k^+ x_{k+2}^-, \quad x_{k+2}^- x_1^+ \cdots x_k^+ [x_{k+1}^+, w_1^+] \ \in I.$$

Hence, using $w_1^+x^- + x^-w_1^+$, $(w_1^+)^2x^\epsilon - x^\epsilon$, $x^\epsilon(w_1^+)^2 - x^\epsilon \in I$, with $\epsilon \in \{+,-\}$, we obtain for every $k \geq 0$

$$v_0x_1^+v_1x_2^+ \cdots v_{k-1}x_k^+v_kx_{k+1}^-v_{k+1} \equiv x_1^+ \cdots x_k^+x_{k+1}^-v \pmod{I} \tag{5.1}$$

and

$$v_0x_{k+1}^-v_1x_1^+v_2x_2^+ \cdots v_kx_k^+v_{k+1} \equiv x_{k+1}^-x_1^+ \cdots x_k^+v \pmod{I} \tag{5.2}$$

where $v, v_0, \ldots, v_{k+1} \in \{1_W, w_1^+\}$.

Now, combining Lemma 5.1 with (5.1), we get for every $k \geq 0$

$$u_1x_{k+1}^-v_0x_1^+v_1x_2^+ \cdots v_{k-1}x_k^+v_kx_{k+2}^-u_2 \in I, \tag{5.3}$$

where $u_1, u_2, v_0, \ldots, v_{k+1} \in \{1_W, w_1^+\}$. Thus, it follows that, modulo I, every monomial appearing in f contains at most one variable of type x^-. In particular, either $f \in P_{n-1,1}$ or $f \in P_{n,0}$.

Let us first consider $f \in P_{n-1,1}$. From (5.1) and (5.2) it follows that we can move w_1^+ to the end of each monomial of f. Moreover, by Lemma 5.1 (see [25, Remark 2]), in every monomial $x_{i_1}^+ \cdots x_{i_k}^+x_n^-x_{j_1}^+ \cdots x_{j_{n-1-k}}^+$ we can reorder the variables to the left and to the right of x_n^-. Thus, we can write f, modulo I, as linear combination of the $(W,*)$-polynomials

$$x_{i_1}^+ \cdots x_{i_k}^+x_n^-x_{j_1}^+ \cdots x_{j_{n-k-1}}^+, \quad x_{i_1}^+ \cdots x_{i_k}^+x_n^-x_{j_1}^+ \cdots x_{j_{n-k-1}}^+w_1^+ \tag{5.4}$$

where $0 \leq k \leq n-1$, $i_1 < \ldots < i_k$ and $j_1 < \ldots < j_{n-k-1}$.

Next, we prove that $(W,*)$-polynomials in (5.4) are linearly independent modulo $\mathrm{Id}^{W,*}(M^D)$. To this end, let us assume that $f \in \mathrm{Id}^{W,*}(M^D)$ is a linear combination of the above $(W,*)$-polynomials, i.e.,

$$f = \sum_{\mathcal{I},\mathcal{J}} \alpha_{\mathcal{I},\mathcal{J}}x_{i_1}^+ \cdots x_{i_k}^+x_n^-x_{j_1}^+ \cdots x_{j_{n-k-1}}^+ + \sum_{\mathcal{I},\mathcal{J}} \beta_{\mathcal{I},\mathcal{J}}x_{i_1}^+ \cdots x_{i_k}^+x_n^-x_{j_1}^+ \cdots x_{j_{n-k-1}}^+w_1^+$$

where $\mathcal{I} = \{i_1, \ldots, i_k\}$ and $\mathcal{J} = \{j_1, \ldots, j_{n-k-1}\}$ are subsets of indices in $\{1, \ldots, n-1\}$ such that $\mathcal{I} \cap \mathcal{J} = \emptyset$ with $0 \leq k \leq n-1$, $i_1 < \cdots < i_k$, $j_1 < \cdots < j_{n-k-1}$, and $\alpha_{\mathcal{I},\mathcal{J}}, \beta_{\mathcal{I},\mathcal{J}} \in F$ for all $\mathcal{I}, \mathcal{J}$.

Let $\mathcal{I} = \{i_1, \ldots, i_k\}$ and $\mathcal{J} = \{j_1, \ldots, j_{n-k-1}\}$ be two fixed disjoint subsets of $\{1, \ldots, n-1\}$ such that $0 < k < n-1$, $i_1 < \cdots < i_k$, $j_1 < \cdots < j_{n-k-1}$. If we make the evaluation $x_{i_1}^+ = \cdots = x_{i_k}^+ = e_{11} + e_{44}$, $x_{j_1}^+ = \cdots = x_{j_{n-k-1}}^+ = e_{22} + e_{33}$, and $x_n^- = e_{12} - e_{34}$, we get $(\alpha_{\mathcal{I},\mathcal{J}} - \beta_{\mathcal{I},\mathcal{J}})e_{12} - (\alpha_{\mathcal{J},\mathcal{I}} + \beta_{\mathcal{J},\mathcal{I}})e_{34} = 0$, that is

$$\alpha_{\mathcal{I},\mathcal{J}} - \beta_{\mathcal{I},\mathcal{J}} = 0 \quad \text{and} \quad \alpha_{\mathcal{J},\mathcal{I}} + \beta_{\mathcal{J},\mathcal{I}} = 0. \tag{5.5}$$

Moreover, let us make the evaluation $x_{i_1}^+ = \cdots = x_{i_k}^+ = e_{22} + e_{33}$, $x_{j_1}^+ = \cdots = x_{j_{n-k-1}}^+ = e_{11} + e_{44}$, and $x_n^- = e_{12} - e_{34}$. In this case we get $-(\alpha_{\mathcal{I},\mathcal{J}} + \beta_{\mathcal{I},\mathcal{J}})e_{34} + (\alpha_{\mathcal{J},\mathcal{I}} - \beta_{\mathcal{J},\mathcal{I}})e_{12} = 0$, thus

$$\alpha_{\mathcal{I},\mathcal{J}} + \beta_{\mathcal{I},\mathcal{J}} = 0 \quad \text{and} \quad \alpha_{\mathcal{J},\mathcal{I}} - \beta_{\mathcal{J},\mathcal{I}} = 0. \tag{5.6}$$

Hence, by putting together (5.5) and (5.6) we obtain that $\alpha_{\mathcal{I},\mathcal{J}} = \beta_{\mathcal{I},\mathcal{J}} = \alpha_{\mathcal{J},\mathcal{I}} = \beta_{\mathcal{J},\mathcal{I}} = 0$.

Finally, fixed $\mathcal{I} = \{1, \ldots, n-1\}$, if we evaluate $x_1 = \cdots = x_{n-1} = e_{11} + e_{44}$ and $x_n^- = e_{12} - e_{34}$, then we get $(\alpha_{\mathcal{I},\emptyset} - \beta_{\mathcal{I},\emptyset})e_{12} - (\alpha_{\emptyset,\mathcal{I}} + \beta_{\emptyset,\mathcal{I}})e_{34} = 0$, thus

$$\alpha_{\mathcal{I},\emptyset} - \beta_{\mathcal{I},\emptyset} = 0 \quad \text{and} \quad \alpha_{\emptyset,\mathcal{I}} + \beta_{\emptyset,\mathcal{I}} = 0. \tag{5.7}$$

Moreover, from the evaluation $x_1 = \cdots = x_{n-1} = e_{22} + e_{33}$ and $x_n^- = e_{12} - e_{34}$, it follows that $-(\alpha_{\mathcal{I},\emptyset} + \beta_{\mathcal{I},\emptyset})e_{34} + (\alpha_{\emptyset,\mathcal{I}} - \beta_{\emptyset,\mathcal{I}})e_{12} = 0$, thus

$$\alpha_{\mathcal{I},\emptyset} + \beta_{\mathcal{I},\emptyset} = 0 \quad \text{and} \quad \alpha_{\emptyset,\mathcal{I}} - \beta_{\emptyset,\mathcal{I}} = 0. \tag{5.8}$$

Therefore, putting together (5.7) and (5.8) we get $\alpha_{\mathcal{I},\emptyset} = \beta_{\mathcal{I},\emptyset} = \alpha_{\emptyset,\mathcal{I}} = \beta_{\emptyset,\mathcal{I}} = 0$. As a consequence, all the scalars appearing in f are zero.

Thus, the polynomials in (5.4) are linearly independent modulo $\mathrm{Id}^{W,*}(M^D)$ and, since $P_{n-1,1} \cap I \subseteq P_{n-1,1} \cap \mathrm{Id}^{W,*}(M^D)$, this proves that the polynomials in (5.4) are a basis of $P_{n-1,1}$ modulo $P_{n-1,1} \cap \mathrm{Id}^{W,*}(M^D)$. Hence, by counting we get

$$c_{n-1,1}(M^D) = 2\sum_{k=0}^{n-1} \binom{n-1}{k} = 2^n. \tag{5.9}$$

Now, let us consider $f \in P_{n,0}$. Since $[x_1^+, x_2^+], [w_1^+, x_1^+] \in W\langle X, *\rangle^-$, from (5.1) we obtain for every $k \geq 0$

$$v_0 x_1^+ v_1 x_2^+ \cdots v_{k-1} x_k^+ v_k [x_{k+1}^+, x_{k+2}^+] v_{k+1} \equiv x_1^+ \cdots x_k^+ [x_{k+1}^+, x_{k+2}^+] v \pmod I, \tag{5.10}$$

$$v_0 x_1^+ v_1 x_2^+ \cdots v_{k-1} x_k^+ v_k [w_1^+, x_{k+1}^+] v_{k+1} \equiv x_1^+ \cdots x_k^+ [w_1^+, x_{k+1}^+] v \pmod I, \tag{5.11}$$

and from (5.2) we get for every $k \geq 0$

$$v_0 [x_{k+1}^+, x_{k+2}^+] v_1 x_1^+ v_2 x_2^+ \cdots v_k x_k^+ v_{k+1} \equiv [x_{k+1}^+, x_{k+2}^+] x_1^+ \cdots x_k^+ v \pmod I, \tag{5.12}$$

$$v_0 [w_1^+, x_{k+1}^+] v_1 x_1^+ v_2 x_2^+ \cdots v_k x_k^+ v_{k+1} \equiv [w_1^+, x_{k+1}^+] x_1^+ \cdots x_k^+ v \pmod I. \tag{5.13}$$

where $v, v_0, \ldots, v_{k+1} \in \{1_W, w_1^+\}$. As a consequence, by the Poincaré-Birkhoff-Witt Theorem, f can be written, modulo I, as a linear combination of polynomials of the type

$$x_{i_1}^+ \cdots x_{i_m}^+ c_1 c_2 \cdots c_s v, \tag{5.14}$$

where $i_1 < \cdots < i_m$, $c_1, \ldots, c_s$ are left-normed commutators containing symmetric variables and w_1^+, and $v \in \{1_W, w_1^+\}$.

Now, by (5.3) and again since $[x_1^+, x_2^+], [x_1^+, w_1^+] \in W\langle X, *\rangle^-$, we get for every $k \geq 0$

$$\begin{aligned}
u_1 [x_{k+1}^+, x_{k+2}^+] v_0 x_1^+ v_1 x_2^+ \cdots v_{k-1} x_k^+ v_k [x_{k+3}^+, x_{k+4}^+] u_2 &\in I, \\
u_1 [w_1^+, x_{k+1}^+] v_0 x_1^+ v_1 x_2^+ \cdots v_{k-1} x_k^+ v_k [x_{k+2}^+, x_{k+3}^+] u_2 &\in I, \\
u_1 [x_{k+1}^+, x_{k+2}^+] v_0 x_1^+ v_1 x_2^+ \cdots v_{k-1} x_k^+ v_k [w_1^+, x_{k+3}^+] u_2 &\in I, \\
u_1 [w_1^+, x_{k+1}^+] v_0 x_1^+ v_1 x_2^+ \cdots v_{k-1} x_k^+ v_k [w_1^+, x_{k+2}^+] u_2 &\in I,
\end{aligned}$$

where $u_1, u_2, v_0, \ldots, v_k \in \{1_W, w_1^+\}$. Thus it follows that in (5.14) only a left-normed commutator can appear, i.e., $s \in \{0, 1\}$. Moreover, by (5.10)–(5.13), we can assume that if w_1^+ appears in the commutator c_1, then it is in the first position. Hence, f can be written, modulo I, as a linear combination of $(W, *)$-monomials $x_1 \cdots x_n v$ and the $(W, *)$-polynomials of the type

$$x_{i_1}^+ \cdots x_{i_m}^+ [x_k^+, x_{j_1}^+, \ldots, x_{j_{n-m-1}}^+] v, \quad x_{i_1}^+ \cdots x_{i_m}^+ [w_1^+, x_{l_1}^+, \ldots, x_{l_{n-m}}^+] v$$

where $0 \leq m \leq n-2$, $i_1 < \cdots < i_m$ and $v \in \{1_W, w_1^+\}$. Since $[x_k^+, x_{j_1}^+], [w_1^+, x_{l_1}^+] \in W\langle X, *\rangle^-$, Lemma 5.1 implies that we can reorder the variables to the left and to the right of $[x_k^+, x_{j_1}^+]$ and $[w_1^+, x_{l_1}^+]$ (see [25, Remark 2]). Hence we may assume that $j_2 < \cdots < j_{n-m-1}$ and $l_2 < \cdots < l_{n-m}$. Moreover, by the Jacobi identity we may also suppose that $k > j_1 < j_2$. Additionally, applying the Jacobi identity, (5.10) and (5.12), we may suppose that $l_1 < l_2$. This implies that we can write f, modulo I, as linear combination of the $(W, *)$-polynomials

$$\begin{aligned}
&x_{i_1}^+ \cdots x_{i_m}^+ [x_k^+, x_{j_1}^+, \ldots, x_{j_{n-m-1}}^+], \quad x_{i_1}^+ \cdots x_{i_m}^+ [x_k^+, x_{j_1}^+, \ldots, x_{j_{n-m-1}}^+] w_1^+ \\
&x_{h_1}^+ \cdots x_{h_p}^+ [w_1^+, x_{l_1}^+, \ldots, x_{l_{n-p}}^+], \quad x_{h_1}^+ \cdots x_{h_p}^+ [w_1^+, x_{l_1}^+, \ldots, x_{l_{n-p}}^+] w_1^+,
\end{aligned} \tag{5.15}$$

where $0 \leq m \leq n$, $m \neq n-1$, $0 \leq p \leq n-1$, $i_1 < \cdots < i_m$, $k > j_1 < \cdots < j_{n-m-1}$, $h_1 < \cdots < h_p$, $l_1 < \cdots < l_{n-p}$.

Next, we prove that these $(W, *)$-polynomials are linearly independent modulo $\mathrm{Id}^{W,*}(M^D)$. To this end, let us assume $f \in \mathrm{Id}^{W,*}(M^D)$ is such that

$$\begin{aligned}
f = &\sum_{\mathcal{I},k} \alpha_{\mathcal{I},k} x_{i_1}^+ \cdots x_{i_m}^+ [x_k^+, x_{j_1}^+, \ldots, x_{j_{n-m-1}}^+] + \sum_{\mathcal{I},k} \beta_{\mathcal{I},k} x_{i_1}^+ \cdots x_{i_m}^+ [x_k^+, x_{j_1}^+, \ldots, x_{j_{n-m-1}}^+] w_1^+ \\
&+ \sum_{\mathcal{H}} \gamma_{\mathcal{H}} x_{h_1}^+ \cdots x_{h_p}^+ [w_1^+, x_{l_1}^+, \ldots, x_{l_{n-p}}^+] + \sum_{\mathcal{H}} \delta_{\mathcal{H}} x_{h_1}^+ \cdots x_{h_p}^+ [w_1^+, x_{l_1}^+, \ldots, x_{l_{n-p}}^+] w_1^+,
\end{aligned} \tag{5.16}$$

where $0 \leq m \leq n$, $m \neq n-1$, $0 \leq p \leq n-1$, $i_1 < \cdots < i_m$, $k > j_1 < \cdots < j_{n-m-1}$, $h_1 < \cdots < h_p$, $l_1 < \cdots < l_{n-p}$, $\mathcal{I} = \{i_1, \ldots, i_m\}$, $\mathcal{H} = \{h_1, \ldots, h_p\}$ and $\alpha_{\mathcal{I},k}, \beta_{\mathcal{I},k}, \gamma_{\mathcal{H}}, \delta_{\mathcal{H}} \in F$ for all $\mathcal{I}, \mathcal{H}$.

First, suppose that $\alpha_{\mathcal{I},0} \neq 0$ or $\beta_{\mathcal{I},0} \neq 0$ in (5.16) with $\mathcal{I} = \{1, \ldots, n\}$. From the evaluation $x_1^+ = \cdots = x_n^+ = e_{11} + e_{22} + e_{33} + e_{44}$ it follows that $\alpha_{\mathcal{I},0}(e_{11} + e_{22} + e_{33} + e_{44}) + \beta_{\mathcal{I},0}(e_{11} + e_{44} - e_{22} - e_{33}) = 0$, thus $\alpha_{\mathcal{I},0} = \beta_{\mathcal{I},0} = 0$, a contradiction. This says that in (5.16) only terms with one commutator can appear with a non-zero coefficient.

Now, let $k \in \{1, \dots, n\}$ and let $\mathcal{I} = \{i_1, \dots, i_m\}$ be a proper subset of $\{1, \dots, n\} \setminus \{k\}$ where $i_1 < \cdots < i_m$ and $0 \leq m \leq n-2$ is the maximum such that $\alpha_{\mathcal{I},k} \neq 0$ or $\beta_{\mathcal{I},k} \neq 0$ in (5.16). If we consider the evaluation $x_{i_1}^+ = \cdots = x_{i_m}^+ = e_{11} + e_{22} + e_{33} + e_{44}$, $x_k^+ = e_{12} + e_{34}$, and $x_{j_1}^+ = \cdots = x_{j_{n-m-1}}^+ = e_{22} + e_{33}$, then the third and fourth summations in (5.16) are evaluated to zero since $\Phi(w_1^+)(e_{22} + e_{33}) - (e_{22} + e_{33})\Phi(w_1^+) = 0$, and we get

$$(\alpha_{\mathcal{I},k} - \beta_{\mathcal{I},k})e_{12} + (-1)^{n-m-1}(\alpha_{\mathcal{I},k} + \beta_{\mathcal{I},k})e_{34} = 0.$$

Thus $\alpha_{\mathcal{I},k} - \beta_{\mathcal{I},k} = 0$ and $\alpha_{\mathcal{I},k} + \beta_{\mathcal{I},k} = 0$. As a consequence, it follows that $\alpha_{\mathcal{I},k} = \beta_{\mathcal{I},k} = 0$, a contradiction. By repeating this argument we obtain that $\alpha_{\mathcal{I},k} = \beta_{\mathcal{I},k} = 0$ for all $\mathcal{I}$ and k.

Now suppose that $\mathcal{H} = \{h_1, \dots, h_p\}$ is a proper subset of $\{1, \dots, n\}$ where $h_1 < \cdots < h_p$ and $0 \leq p \leq n-2$ is the maximum such that $\gamma_{\mathcal{H}} \neq 0$ or $\delta_{\mathcal{H}} \neq 0$ in (5.16), and consider the evaluation $x_{h_1}^+ = \cdots = x_{h_p}^+ = e_{11} + e_{22} + e_{33} + e_{44}$, $x_{l_1}^+ = e_{12} + e_{34}$, and $x_{l_2}^+ = \cdots = x_{l_{n-p}}^+ = e_{22} + e_{33}$. Under this evaluation, we obtain

$$2(\gamma_{\mathcal{H}} - \delta_{\mathcal{H}})e_{12} + 2(-1)^{n-p-1}(\gamma_{\mathcal{H}} + \delta_{\mathcal{H}})e_{34} = 0.$$

As a consequence, since $\operatorname{char} F = 0$, we get that $\gamma_{\mathcal{H}} = \delta_{\mathcal{H}} = 0$, a contradiction. As above, by repeating this argument we obtain that $\gamma_{\mathcal{H}} = \delta_{\mathcal{H}} = 0$ for all $\mathcal{H}$. As a consequence, all the scalars appearing in f are zero, i.e., $f = 0$.

Thus, the polynomials in (5.15) are linearly independent modulo $\mathrm{Id}^{W,*}(M^D)$ and, as in the previous case, this implies that the $(W,*)$-polynomials in (5.15) are a basis of $P_{n,0}$ modulo $P_{n,0} \cap \mathrm{Id}^{W,*}(M^D)$. Hence, by counting we get

$$c_{n,0}(M^D) = 2 + 2\sum_{m=2}^{n} \binom{n}{m}(m-1) + 2\sum_{m=0}^{n-1} \binom{n}{m} = 2\sum_{m=0}^{n} \binom{n}{m} m + 2 = n2^n + 2. \tag{5.17}$$

Since we have shown that $P_{r,n-r} \cap \mathrm{Id}^{W,*}(M^D) \supseteq P_{r,n-r} \cap I$ for all $n \in \mathbb{N}$ and $0 \leq r \leq n$, it follows that $\mathrm{Id}^{W,*}(M^D) = I$, as claimed.

Finally, since $P_{r,n-r} \equiv 0 \pmod{(P_{r,n-r} \cap \mathrm{Id}^{W,*}(M^D))}$ for $0 \leq r \leq n-2$, applying Formula (3.3) together with (5.9) and (5.17), we obtain

$$c_n^{W,*}(M^D) = \sum_{r=0}^{n} \binom{n}{r} c_{r,n-r}^{W,*}(M^D) = c_{n,0}^{W,*}(M^D) + n c_{n-1,1}^{W,*}(M^D) = n2^n + 2 + n2^n = n2^{n+1} + 2,$$

as desired. □

Theorem 5.3. *Let M^F be a $(W,*)$-algebra M where W acts on it as the algebra F by left and right multiplication. Then there exists a basis $\mathcal{B}_W = \mathcal{B}_{W^+} \cup \mathcal{B}_{W^-}$ of W, where $\mathcal{B}_{W^+} := \{1_W, w_i^+ : i \geq 1\}$ and $\mathcal{B}_{W^-} := \{w_i^- : i \geq 1\}$ are bases of W^+ and W^-, respectively, such that the T_W^*-ideal $\mathrm{Id}^{W,*}(M^F)$ of generalized $*$-identities of M^F is generated, as T_W^*-ideal, by the following polynomials:*

$$x_1^- x_2^-, \quad w_i^\eta x^\epsilon,$$

where $\epsilon, \eta \in \{+,-\}$ and $i \geq 1$. Moreover $c_n^{W,*}(M^F) = (n-1)2^n + 2$.

Proof. Let Φ denote the acting homomorphism of W on M^F. Then $\Phi(W) \cong F$ and we can choose a basis $\mathcal{B}_W = \mathcal{B}_{W^+} \cup \mathcal{B}_{W^-}$ of $W = W^+ \oplus W^-$, where $\mathcal{B}_{W^+} := \{w_i^+ : i \geq 0\}$ and $\mathcal{B}_{W^-} := \{w_i^- : i \geq 1\}$ are bases of W^+ and W^-, respectively, such that $w_0^+ = 1_W$, and $w_i^+, w_i^- \in \ker \Phi$ for all $i \geq 1$. Clearly, the $(W,*)$-identities of M^F correspond to the ordinary $*$-identities of M modulo the T_W^*-ideal $\langle w_i^\eta x^\epsilon : \epsilon, \eta \in \{+,-\},\ i \geq 1\rangle_{T_W^*}$. Thus by [25, Theorem 2] we obtain the required conclusion:

$$\mathrm{Id}^{W,*}(M^F) = \langle x_1^- x_2^-,\ w_i^\eta x^\epsilon : \epsilon, \eta \in \{+,-\},\ i \geq 1\rangle_{T_W^*}.$$

Moreover, by Lemma 5.1 and arguing along the same lines as in the proof of Theorem 5.2, with the necessary changes, we obtain the following.

The $(W,*)$-polynomials

$$x_{i_1}^+ \cdots x_{i_m}^+ [x_k^+, x_{j_1}^+, \dots, x_{j_{n-m-1}}^+],$$

where $0 \leq m \leq n$, $m \neq n-1$, $i_1 < \cdots < i_m$, $k > j_1 < \cdots < j_{n-m-1}$, and the $(W,*)$-polynomials

$$x_{i_1}^+ \cdots x_{i_k}^+ x_n^- x_{j_1}^+ \cdots x_{j_{n-k-1}}^+,$$

where $0 \leq k \leq n-1$, $i_1 < \cdots < i_k$, $j_1 < \cdots < j_{n-k-1}$, form a basis of $P_{r,n-r}$ modulo $P_{r,n-r} \cap \mathrm{Id}^{W,*}(M^F)$ for $r = n$ and $r = n-1$, respectively. Moreover, $P_{r,n-r} \equiv 0 \pmod{(P_{r,n-r} \cap \mathrm{Id}^{W,*}(M^F))}$ for $0 \leq r \leq n-2$. Consequently, a direct counting argument yields

$$c_{r,n-r}(M^F) = \begin{cases} (n-2)2^{n-1}+2, & \text{if } r = n, \\ 2^{n-1}, & \text{if } r = n-1, \\ 0, & \text{if } 0 \leq r \leq n-2. \end{cases}$$

Finally, by Formula (3.3), we conclude that $c_n^{W,*}(M^F) = (n-1)2^n + 2$, which completes the proof. □

Recall that M^D and M^F denote fixed representatives of two distinct equivalence classes in the sense of Definition (2.7). More precisely, there exist acting $*$-homomorphisms $\Phi_D \colon W \to \mathcal{M}(M)$ and $\Phi_F \colon W \to \mathcal{M}(M)$ with images $\Phi_D(W) \cong D$ and $\Phi_F(W) \cong F$ such that the $(W,*)$-algebras M^D and M^F are precisely M^{Φ_D} and M^{Φ_F}, respectively. Therefore, to prove that M^D and M^F generate distinct $(W,*)$-varieties, it suffices to show that, for every choice of acting $*$-homomorphisms Φ_D and Φ_F of W on M with $\Phi_D(W) \cong D$ and $\Phi_F(W) \cong F$, the $(W,*)$-algebras M^{Φ_D} and M^{Φ_F} generate distinct $(W,*)$-varieties.

Corollary 5.4. *The $(W,*)$-algebras M^F and M^D generates two distinct $(W,*)$-varieties.*

Proof. Given an acting $*$-homomorphism Φ_D of W on M such that $\Phi_D(W) \cong D$, by Theorem 5.2 there exists a symmetric element w_1^+ of W linearly independent with 1_W such that $(w_1^+)^2 x^+ - x^+ \in \mathrm{Id}^{W,*}(M^{\Phi_D})$. For any acting $*$-homomorphism Φ_F of W on M such that $\Phi_F(W) \cong F$, we have that $\Phi_F(w_1^+) = (0,0) \in \mathcal{M}(M)$. As a consequence, $(w_1^+)^2 x^+ - x^+ \notin \mathrm{Id}^{W,*}(M^F)$; indeed, $(w_1^+)^2 1_M - 1_M = -1_M \neq 0$. Therefore, $\mathrm{Id}^{W,*}(M^{\Phi_D}) \not\subset \mathrm{Id}^{W,*}(M^{\Phi_F})$ for any acting $*$-homomorphisms Φ_D and Φ_F of W on M with $\Phi_D(W) \cong D$ and $\Phi_F(W) \cong F$, and consequentially $M^F \notin \mathrm{var}^{W,*}(M^D)$.

Now, given an acting $*$-homomorphism Φ_F of W on M such that $\Phi_F(W) \cong F$, by Theorem 5.3 for any element w of W linearly independent with 1_W we have that $wx^+ \in \mathrm{Id}^{W,*}(M^{\Phi_F})$. For any acting $*$-homomorphism Φ_D of W on M such that $\Phi_D(W) \cong D$, there exists a symmetric element w_1^+ of W linearly independent with 1_W such that $\Phi_D(w_1^+) = (R_{e_{11}+e_{44}-e_{22}-e_{33}}, L_{e_{11}+e_{44}-e_{22}-e_{33}}) \in \mathcal{M}(M)$. As a consequence, $w_1^+ x^+ \notin \mathrm{Id}^{W,*}(M^{\Phi_D})$ since $w_1^+ 1_M = e_{11} + e_{44} - e_{22} - e_{33} \neq 0$. Thus, $\mathrm{Id}^{W,*}(M^{\Phi_D}) \not\subset \mathrm{Id}^{W,*}(M^{\Phi_F})$ for any acting $*$-homomorphisms Φ_D and Φ_F of W on M with $\Phi_D(W) \cong D$ and $\Phi_F(W) \cong F$, and as a consequence $M^D \notin \mathrm{var}^{W,*}(M^F)$. □

6. Generalized $*$-cocharacter of M^D

In this section, we compute the $(r, n-r)$th $(W,*)$-cocharacter of M^D, which will play a central role in the next section in studying the growth of the $(W,*)$-variety generated by M^D. More precisely, for $n \geq 1$ and $0 \leq r \leq n$, we determine the multiplicities $m_{\lambda,\mu}$ in the decomposition of the $(S_r \times S_{n-r})$-character

$$\chi_{r,n-r}^{W,*}(M^D) = \sum_{(\lambda,\mu)\vdash(r,n-r)} m_{\lambda,\mu}(\chi_\lambda \otimes \chi_\mu) \tag{6.1}$$

of $P_{r,n-r}(M^D)$ (see (3.4)). To this end recall that any irreducible left $(S_r \times S_{n-r})$-module $W_{\lambda,\mu} \subseteq P_{r,n-r}$ with character $\chi_\lambda \otimes \chi_\mu$ can be generated as an $(S_r \times S_{n-r})$-module by an element of the form $e_{T_\lambda} e_{T_\mu} f$, for some $f \in W_{\lambda,\mu}$ and some pair of Young tableaux (T_λ, T_μ) of shape $\lambda \vdash r$ and $\mu \vdash n-r$, respectively. Here $e_{T_\nu} = \sum_{\substack{\sigma \in R_{T_\nu} \\ \tau \in C_{T_\nu}}} (\operatorname{sgn}\tau)\sigma\tau$ stands for the quasi-idempotent corresponding to some Young tableau T_ν of shape $\nu \vdash t$, where R_{T_ν} and C_{T_ν} are the subgroups of S_t stabilizing the rows and columns of T_ν, respectively. Furthermore, recall that

$$c_{r,n-r}^{W,*}(M^D) = \sum_{(\lambda,\mu)\vdash(r,n-r)} m_{\lambda,\mu} d_\lambda d_\mu \tag{6.2}$$

where $d_\lambda = \chi_\lambda(1)$ and $d_\mu = \chi_\mu(1)$ are the degrees of χ_λ and χ_μ, i.e., the number of standard tableaux of shape λ and μ, respectively.

We start by proving the following lemmas.

Lemma 6.1. *If $r = n-1$, then in* (6.1) *we have*

$$m_{\lambda,\mu} = \begin{cases} 2(q+1) & \text{if } \lambda = (p+q,p), \mu = (1) \\ 2n & \text{if } \lambda = (n-1), \mu = (1) \\ 0 & \text{otherwise} \end{cases}.$$

Proof. Notice that, as a consequence of formula (6.2), it suffices to prove that:

(1) $m_{(p+q,p),(1)} \geq 2(q+1)$ for all $p \geq 1$ and $q \geq 0$ such that $2p+q = n-1$;

(2) $m_{(n-1),(1)} \geq 2n$.

Indeed, assume that $m_{(n-1),(1)} \geq 2n$ and $m_{(p+q,p),(1)} \geq 2(q+1)$ for all $p \geq 1$ and $q \geq 0$ such that $2p+q = n-1$. Then

$$c_{n-1,1}(M^D) \geq \sum_{\substack{1 \leq p \leq \lfloor \frac{n-1}{2} \rfloor \\ 0 \leq q \leq n-2p-1}} 2(q+1) d_{(p+q,p)} d_{(1)} + 2n d_{(n-1)} d_{(1)}. \tag{6.3}$$

Now, by the formula (4.8) proved in Theorem 4.4 of [24], we have

$$\sum_{\substack{1 \leq p \leq \lfloor \frac{n-1}{2} \rfloor \\ 0 \leq q \leq n-2p-1}} 2(q+1) d_{(p+q,p)} = 2(2^{n-1} - n) = 2^n - 2n.$$

Hence, recalling that $d_{(n-1)} = d_{(1)} = 1$, we get

$$\sum_{\substack{1 \leq p \leq \lfloor \frac{n-1}{2} \rfloor \\ 0 \leq q \leq n-2p-1}} 2(q+1) d_{(p+q,p)} d_{(1)} + 2n d_{(n-1)} d_{(1)} = 2^n - 2n + 2n = 2^n.$$

Since by (5.9) $c_{n-1,1}(M^D) = 2^n$, it follows that (6.3) is actually an equality. Thus the lemma follows once (1) and (2) are proved.

We compute the two different cases separately.

Case 1: We prove that $m_{(p+q,p),(1)} \geq 2(q+1)$ for all $p \geq 1$, $q \geq 0$ such that $2p+q = n-1$.

For $0 \leq i \leq q$, we consider the following two tableaux:

$$T^{(i)}_{(p+q,p)} = \begin{array}{|c|c|c|c|c|c|c|c|c|c|} \hline i+1 & i+2 & \dots & i+p & 1 & \dots & i & i+2p+2 & \dots & n \\ \hline i+p+2 & i+p+3 & \dots & i+2p+1 \\ \cline{1-4} \end{array}$$

$$T^{(i)}_{(1)} = \boxed{i+p+1}$$

and the associated polynomials

$$a^{(i)}_{p,q}(x_1^+, x_2^+, x^-) = (x_1^+)^i \underbrace{\overline{x_1^+} \cdots \widetilde{x_1^+}}_{p} x^- \underbrace{\overline{x_2^+} \cdots \widetilde{x_2^+}}_{p} (x_1^+)^{q-i}, \tag{6.4}$$

$$a^{(i)}_{p,q,w_1^+}(x_1^+, x_2^+, x^-) = (x_1^+)^i \underbrace{\overline{x_1^+} \cdots \widetilde{x_1^+}}_{p} x^- \underbrace{\overline{x_2^+} \cdots \widetilde{x_2^+}}_{p} (x_1^+)^{q-i} w_1^+, \tag{6.5}$$

where the symbol $\tilde{}$ or $\bar{}$ means alternation on the corresponding variables. These generalized $*$-polynomials are obtained from the quasi-idempotents corresponding to the pair of tableaux $\left(T^{(i)}_{(p+q,p)}, T^{(i)}_{(1)}\right)$ by identifying all the elements in each row of $(p+q,p)$.

It is easily checked that $a^{(i)}_{p,q}(x_1^+, x_2^+, x^-)$ and $a^{(i)}_{p,q,w_1^+}(x_1^+, x_2^+, x^-)$, for all $0 \leq i \leq q$, are not $(W,*)$-identities of M^D. We will prove that they are linearly independent modulo $\mathrm{Id}^{W,*}(M^D)$.

Suppose that

$$\sum_{i=0}^{q} \alpha_i a^{(i)}_{p,q}(x_1^+, x_2^+, x^-) + \sum_{i=0}^{q} \beta_i a^{(i)}_{p,q,w_1^+}(x_1^+, x_2^+, x^-) \equiv 0 \pmod{\mathrm{Id}^{W,*}(M^D)}.$$

If we evaluate $x_1^+ = k(e_{11}+e_{44})+e_{22}+e_{33}$, $x_2^+ = e_{11}+e_{44}$ and $x^- = e_{12}-e_{34}$, with $k \in F$, $k \neq 0$, we get

$$(-1)^p \left(\sum_{i=0}^{q} (\alpha_i - \beta_i) k^i \right) e_{12} - \left(\sum_{i=0}^{q} (\alpha_i + \beta_i) k^{q-i} \right) e_{34} = 0.$$

Hence, setting $t = q - i$, it follows that

$$\sum_{i=0}^{q} (\alpha_i - \beta_i) k^i = 0, \qquad \sum_{t=0}^{q} (\alpha_{q-t} + \beta_{q-t}) k^t = 0.$$

Now, for $0 \leq i \leq q$, let $\gamma_i^{(1)} := \alpha_i - \beta_i$ and $\gamma_i^{(2)} := \alpha_{q-i} + \beta_{q-i}$. Given $j \in \{1,2\}$, assume that there exists $\gamma_i^{(j)} \neq 0$ for some $0 \leq i \leq q$. Since F is an infinite field, we can choose $k_0, \ldots, k_q \in F$ such that $k_i \neq 0$ and $k_i \neq k_j$ for $0 \leq i \neq j \leq q$. Then, we get the following homogeneous linear system of $q+1$ equations in the $q+1$ variables $\gamma_0^{(j)}, \ldots, \gamma_q^{(j)}$

$$\sum_{i=0}^{q} k_s^i \gamma_i^{(j)} = 0 \qquad 0 \leq s \leq q.$$

Since the matrix associated to the above system is a Vandermonde matrix, it follows that $\gamma_i^{(j)} = 0$ for any $0 \leq i \leq q$, a contradiction. Hence, for all $0 \leq i \leq q$, $\alpha_i - \beta_i = \alpha_i + \beta_i = 0$ and so $\alpha_i = \beta_i = 0$.

Thus the $2(q+1)$ generalized $*$-polynomials $a_{p,q}^{(i)}(x_1^+, x_2^+, x^-)$, $a_{p,q,w_1^+}^{(i)}(x_1^+, x_2^+, x^-)$, $0 \leq i \leq q$, are linearly independent modulo $\mathrm{Id}^{W,*}(M^D)$.

Notice that for every $0 \leq i \leq q$, the complete linearization of $a_{p,q}^{(i)}(x_1^+, x_2^+, x^-)$ and $a_{p,q,w_1^+}^{(i)}(x_1^+, x_2^+, x^-)$ are, respectively, the $(W,*)$-polynomials

$$e_{T_{(p+q,p)}^{(i)}} e_{T_{(1)}^{(i)}}(x_1^+ \cdots x_{i+p}^+ x_{i+p+1}^- x_{i+p+2}^+ \cdots x_n^+) \quad \text{and} \quad e_{T_{(p+q,p)}^{(i)}} e_{T_{(1)}^{(i)}}(x_1^+ \cdots x_{i+p}^+ x_{i+p+1}^- x_{i+p+2}^+ \cdots x_n^+ w_1^+),$$

which are consequently linearly independent modulo $\mathrm{Id}^{W,*}(M^D)$. Hence, it follows that the multilinear $(W,*)$-polynomials

$$e_{(p+q,p),(1)}^{(i)}(x_1^+, \ldots, x_{n-1}^+, x_n^-) \quad \text{and} \quad e_{(p+q,p),(1)}^{(i),w_1^+}(x_1^+, \ldots, x_{n-1}^+, x_n^-),$$

obtained, respectively, from the expressions above by renaming the variables, are also linearly independent modulo $\mathrm{Id}^{W,*}(M^D)$, for $0 \leq i \leq q$, and they generate $2(q+1)$ distinct copies of the same irreducible module associated to the pair of partitions $\big((p+q,p),(1)\big)$ in the module decomposition of $P_{n-1,1}(M^D)$. So, we have that $m_{(p+q,p),(1)} \geq 2(q+1)$, as claimed.

Case 2: We prove that $m_{(n-1),(1)} \geq 2n$.

For $0 \leq i \leq n-1$ given the tableaux

$$T_{(n-1)}^{(i)} = \begin{array}{|c|c|c|c|c|c|} \hline 1 & \ldots & i & i+2 & \ldots & n \\ \hline \end{array}$$

and

$$T_{(1)}^{(i)} = \begin{array}{|c|} \hline i+1 \\ \hline \end{array},$$

the associated polynomials are respectively

$$b^{(i)}(x^+, x^-) = (x^+)^i x^- (x^+)^{n-i-1}, \tag{6.6}$$

$$b_{w_1^+}^{(i)}(x^+, x^-) = (x^+)^i x^- (x^+)^{n-i-1} w_1^+. \tag{6.7}$$

These polynomials are obtained from the quasi-idempotents corresponding to the pair of tableaux $\big(T_{(n-1)}, T_{(1)}\big)$ by identifying all the elements in $T_{(n-1)}$.

We will prove that the $2n$ $(W,*)$-polynomials $b^{(i)}(x^+, x^-)$, $b_{w_1^+}^{(i)}(x^+, x^-)$, $0 \leq i \leq n-1$, which are not $(W,*)$-identities of M^D, are linearly independent modulo $\mathrm{Id}^{W,*}(M^D)$.

Let

$$\sum_{0 \leq i \leq n-1} \alpha_i b^{(i)}(x^+, x^-) + \sum_{0 \leq i \leq n-1} \beta_i b_{w_1^+}^{(i)}(x^+, x^-) \equiv 0 \pmod{\mathrm{Id}^{W,*}(M^D)}.$$

By evaluating $x^+ = k(e_{11}+e_{44})+(e_{22}+e_{33})$, $x^- = e_{12}-e_{34}$, where $k \in F$, $k \neq 0$ we get

$$\left(\sum_{i=0}^{n-1}(\alpha_i-\beta_i)k^i\right)e_{12} - \left(\sum_{i=0}^{n-1}(\alpha_i+\beta_i)k^{n-i-1}\right)e_{34} = 0.$$

Hence, setting $t = n-i-1$, it follows

$$\sum_{i=0}^{n-1}(\alpha_i-\beta_i)k^i = 0, \qquad \sum_{t=0}^{n-1}(\alpha_{n-1-t}+\beta_{n-1-t})k^t = 0.$$

Now, from the same argument of the previous case, we obtain $\alpha_i = \beta_i = 0$ for all $0 \leq i \leq n-1$. Consequently, the $(W,*)$-polynomials $b^{(i)}(x^+,x^-)$ and $b^{(i)}_{w_1^+}(x^+,x^-)$, for $0 \leq i \leq n-1$, are linearly independent modulo $\mathrm{Id}^{W,*}(M^D)$. Applying a similar reasoning as in Case 1, this implies that $m_{(n-1),(1)} \geq 2n$, as claimed. □

Lemma 6.2. *If $r = n$, then in* (6.1) *we have*

$$m_{\lambda,\mu} = \begin{cases} 4(q+1) & \text{if } \lambda = (p+q,p), \mu = \emptyset \\ 2(q+1) & \text{if } \lambda = (p+q,p,1), \mu = \emptyset \\ 2n+2 & \text{if } \lambda = (n), \mu = \emptyset \\ 0 & \text{otherwise} \end{cases}.$$

Proof. We proceed as in the proof of the previous lemma. Observe that, in this case, as a consequence of formula (6.2), it is enough to prove that:

(1) $m_{(p+q,p),\emptyset} \geq 4(q+1)$ for all $p \geq 1$ and $q \geq 0$ such that $2p+q = n$;
(2) $m_{(p+q,p,1),\emptyset} \geq 2(q+1)$ for all $p \geq 1$ and $q \geq 0$ such that $2p+q = n-1$;
(3) $m_{(n),\emptyset} \geq 2n+2$.

Indeed, if these conditions hold, then

$$c_{n,0}(M^D) \geq \sum_{\substack{1\leq p\leq \lfloor \frac{n}{2}\rfloor \\ 0\leq q\leq n-2p}} 4(q+1)d_{(p+q,p)}d_\emptyset + \sum_{\substack{1\leq p\leq \lfloor \frac{n-1}{2}\rfloor \\ 0\leq q\leq n-2p-1}} 2(q+1)d_{(p+q,p,1)}d_\emptyset + (2n+2)d_{(n)}d_\emptyset. \tag{6.8}$$

Notice that by the formulas (4.8) and (4.9) proved in Theorem 4.4 of [24], we have

$$\sum_{\substack{1\leq p\leq \lfloor \frac{n}{2}\rfloor \\ 0\leq q\leq n-2p}} 4(q+1)d_{(p+q,p)} = 2^{n+2}-4n-4.$$

and

$$\sum_{\substack{1\leq p\leq \lfloor \frac{n-1}{2}\rfloor \\ 0\leq q\leq n-2p-1}} 2(q+1)d_{(p+q,p,1)} = (n-4)2^n+2n+4.$$

Hence, recalling that $d_{(n)} = d_\emptyset = 1$, we obtain

$$\sum_{\substack{1\leq p\leq \lfloor \frac{n}{2}\rfloor \\ 0\leq q\leq n-2p}} 4(q+1)d_{(p+q,p)}d_\emptyset + \sum_{\substack{1\leq p\leq \lfloor \frac{n-1}{2}\rfloor \\ 0\leq q\leq n-2p-1}} 2(q+1)d_{(p+q,p,1)}d_\emptyset + (2n+2)d_{(n)}d_\emptyset = n2^n+2.$$

Since, by (5.17), we have $c_{n,0}(M^D) = n2^n+2$, it follows that (6.8) is in fact an equality. Thus it remains to prove the three inequalities above.

We now compute the three cases separately.

Case 1: We prove that $m_{(p+q,p),\emptyset} \geq 4(q+1)$ for all $p \geq 1$ and $q \geq 0$ such that $2p+q = n$.

For $0 \leq i \leq q$, we consider the tableau

$T^{(i)}_{(p+q,p)} =$

$i+1$	$i+2$	$\dots$	$i+p-1$	$i+p$	1	$\dots$	i	$i+2p+1$	$\dots$	n
$i+p+2$	$i+p+3$	$\dots$	$i+2p$	$i+p+1$						

and its associated $(W,*)$-polynomials

$$f^{(i)}_{p,q}(x_1^+,x_2^+)=(x_1^+)^i\underbrace{\overline{x_1^+}\cdots\widetilde{x_1^+}}_{p-1}[x_1^+,x_2^+]\underbrace{\overline{x_2^+}\cdots\widetilde{x_2^+}}_{p-1}(x_1^+)^{q-i}, \tag{6.9}$$

$$f^{(i)}_{p,q,w_1^+}(x_1^+,x_2^+)=(x_1^+)^i\underbrace{\overline{x_1^+}\cdots\widetilde{x_1^+}}_{p-1}[x_1^+,x_2^+]\underbrace{\overline{x_2^+}\cdots\widetilde{x_2^+}}_{p-1}(x_1^+)^{q-i}w_1^+, \tag{6.10}$$

$$g^{(i)}_{p,q}(x_1^+,x_2^+)=(x_1^+)^i\underbrace{\overline{x_1^+}\cdots\widetilde{x_1^+}}_{p-1}(x_1^+w_1^+x_2^+-x_2^+w_1^+x_1^+)\underbrace{\overline{x_2^+}\cdots\widetilde{x_2^+}}_{p-1}(x_1^+)^{q-i}, \tag{6.11}$$

$$g^{(i)}_{p,q,w_1^+}(x_1^+,x_2^+)=(x_1^+)^i\underbrace{\overline{x_1^+}\cdots\widetilde{x_1^+}}_{p-1}(x_1^+w_1^+x_2^+-x_2^+w_1^+x_1^+)\underbrace{\overline{x_2^+}\cdots\widetilde{x_2^+}}_{p-1}(x_1^+)^{q-i}w_1^+, \tag{6.12}$$

where the symbol $\tilde{}$ or $\bar{}$ means alternation on the corresponding variables. For any $0\leq i\leq q$, these $(W,*)$-polynomials are obtained from the quasi-idempotents corresponding to the tableau $T^{(i)}_{(p+q,p)}$ by identifying all the elements in each row.

Clearly, the $4(q+1)$ polynomials $f^{(i)}_{p,q}(x_1^+,x_2^+)$, $f^{(i)}_{p,q,w_1^+}(x_1^+,x_2^+)$, $g^{(i)}_{p,q}(x_1^+,x_2^+)$, $g^{(i)}_{p,q,w_1^+}(x_1^+,x_2^+)$, for all $0\leq i\leq q$, do not belong to $\mathrm{Id}^{W,*}(M^D)$. Next, we will prove that they are linearly independent modulo $\mathrm{Id}^{W,*}(M^D)$.

Suppose that

$$\sum_{i=0}^{q}\alpha_i f^{(i)}_{p,q}(x_1^+,x_2^+)+\sum_{i=0}^{q}\beta_i f^{(i)}_{p,q,w_1^+}(x_1^+,x_2^+)+\sum_{i=0}^{q}\gamma_i g^{(i)}_{p,q}(x_1^+,x_2^+)+\sum_{i=0}^{q}\delta_i g^{(i)}_{p,q,w_1^+}(x_1^+,x_2^+)\equiv 0 \pmod{\mathrm{Id}^{W,*}(M^D)}.$$

If we evaluate $x_1^+=k(e_{11}+e_{44})+e_{22}+e_{33}+e_{12}+e_{34}$, with $k\in F$, $k\neq 0$, and $x_2^+=e_{11}+e_{44}$, we get

$$(-1)^{p-1}\left(\sum_{i=0}^{q}(-\alpha_i+\beta_i-\gamma_i+\delta_i)k^i\right)e_{12}+\left(\sum_{i=0}^{q}(\alpha_i+\beta_i+\gamma_i+\delta_i)k^{q-i}\right)e_{34}=0. \tag{6.13}$$

Also, by evaluating $x_1^+=e_{11}+e_{44}+k(e_{22}+e_{33})+e_{12}+e_{34}$, with $k\in F$, $k\neq 0$, and $x_2^+=e_{22}+e_{33}$, we obtain

$$\left(\sum_{i=0}^{q}(\alpha_i-\beta_i-\gamma_i+\delta_i)k^{q-i}\right)e_{12}+(-1)^{p-1}\left(\sum_{i=0}^{q}(-\alpha_i-\beta_i+\gamma_i+\delta_i)k^i\right)e_{34}=0. \tag{6.14}$$

Hence, setting $t=q-i$ and from (6.13), (6.14) it follows that

$$\sum_{i=0}^{q}k^i(-\alpha_i+\beta_i-\gamma_i+\delta_i)=0,\qquad \sum_{t=0}^{q}k^t(\alpha_{q-t}+\beta_{q-t}+\gamma_{q-t}+\delta_{q-t})=0,$$
$$\sum_{t=0}^{q}k^t(\alpha_{q-t}-\beta_{q-t}-\gamma_{q-t}+\delta_{q-t})=0,\qquad \sum_{i=0}^{q}k^i(-\alpha_i-\beta_i+\gamma_i+\delta_i)=0.$$

Now, for $0\leq i\leq q$, let

$$\begin{aligned}\eta_i^{(1)}&:=-\alpha_i+\beta_i-\gamma_i+\delta_i,\qquad \eta_i^{(2)}:=\alpha_{q-i}+\beta_{q-i}+\gamma_{q-i}+\delta_{q-i},\\ \eta_i^{(3)}&:=\alpha_{q-i}-\beta_{q-i}-\gamma_{q-i}+\delta_{q-i},\qquad \eta_i^{(4)}:=-\alpha_i-\beta_i+\gamma_i+\delta_i.\end{aligned} \tag{6.15}$$

Given $1\leq j\leq 4$, assume that there exists $\eta_i^{(j)}\neq 0$ for some $0\leq i\leq q$. Since F is an infinite field, we can choose $k_0,\ldots,k_q\in F$ such that $k_i\neq 0$ and $k_i\neq k_j$ for $0\leq i\neq j\leq q$. Then, we get the following homogeneous linear system of $q+1$ equation in the $q+1$ variables $\eta_0^{(j)},\ldots,\eta_q^{(j)}$

$$\sum_{i=0}^{q}k_s^i\eta_i^{(j)}=0,\qquad 0\leq s\leq q.$$

Since the matrix associated to the above system is a Vandermonde matrix, it follows that $\eta_i^{(j)}=0$ for any $0\leq i\leq q$, a contradiction. Thus, by (6.15) we get that $\alpha_i=\beta_i=\gamma_i=\delta_i=0$ for all $0\leq i\leq q$.

Therefore, the $4(q+1)$ generalized $*$-polynomials $f^{(i)}_{p,q}(x_1^+,x_2^+)$, $f^{(i)}_{p,q,w_1^+}(x_1^+,x_2^+)$, $g^{(i)}_{p,q}(x_1^+,x_2^+)$, $g^{(i)}_{p,q,w_1^+}(x_1^+,x_2^+)$ are linearly independent modulo $\mathrm{Id}^{W,*}(M^D)$.

Proceeding as in the previous lemma, we observe that for every $0 \le i \le q$, the complete linearizations of $f^{(i)}_{p,q}(x_1^+,x_2^+)$, $f^{(i)}_{p,q,w_1^+}(x_1^+,x_2^+)$, $g^{(i)}_{p,q}(x_1^+,x_2^+)$, and $g^{(i)}_{p,q,w_1^+}(x_1^+,x_2^+)$ are, respectively, the $(W,*)$-polynomials

$$e_{T^{(i)}_{(p+q,p)}}(x_1^+\cdots x_n^+), \quad e_{T^{(i)}_{(p+q,p)}}(x_1^+\cdots x_n^+w_1^+),$$
$$e_{T^{(i)}_{(p+q,p)}}(x_1^+\cdots x_{i+p}^+w_1^+x_{i+p+1}^+\cdots x_n^+), \quad e_{T^{(i)}_{(p+q,p)}}(x_1^+\cdots x_{i+p}^+w_1^+x_{i+p+1}^+\cdots x_n^+w_1^+).$$

Hence, these multilinear $(W,*)$-polynomials are also linearly independent modulo $\mathrm{Id}^{W,*}(M^D)$. Consequently, they generate $4(q+1)$ distinct copies of the same irreducible module corresponding to the pair of partitions $\big((p+q,p),\emptyset\big)$ in the module decomposition of $P_{n,0}(M^D)$. Therefore, we conclude that $m_{(p+q,p),\emptyset} \ge 4(q+1)$, as claimed.

Case 2: We prove that $m_{(p+q,p,1),\emptyset} \ge 2(q+1)$ for all $p \ge 1$ and $q \ge 0$ such that $2p+q = n-1$.

For $0 \le i \le q$, consider the tableau

$$T^{(i)}_{(p+q,p,1)} = \begin{array}{|c|c|c|c|c|c|c|c|c|c|}\hline i+p & i+1 & \dots & i+p-1 & 1 & \dots & i & i+2p+2 & \dots & n \\ \cline{1-4} i+p+1 & i+p+3 & \dots & i+2p+1 \\ \cline{1-1} i+p+2 \\ \cline{1-1}\end{array}$$

and its associated $(W,*)$-polynomials

$$h^{(i)}_{p,q}(x_1^+,x_2^+,x_3^+) = (x_1^+)^i \underbrace{\widehat{x_1^+}\cdots\widetilde{x_1^+}}_{p-1}\overline{x_1^+}\,\overline{x_2^+}\,\overline{x_3^+}\underbrace{\widehat{x_2^+}\cdots\widetilde{x_2^+}}_{p-1}(x_1^+)^{q-i}, \tag{6.16}$$

$$h^{(i)}_{p,q,w_1^+}(x_1^+,x_2^+,x_3^+) = (x_1^+)^i \underbrace{\widehat{x_1^+}\cdots\widetilde{x_1^+}}_{p-1}\overline{x_1^+}\,\overline{x_2^+}\,\overline{x_3^+}\underbrace{\widehat{x_2^+}\cdots\widetilde{x_2^+}}_{p-1}(x_1^+)^{q-i}w_1^+, \tag{6.17}$$

where the symbol $\widehat{}$ or $\widetilde{}$ or $\overline{}$ means alternation on the corresponding variables. For any $0 \le i \le q$, each of these generalized $*$-polynomials is obtained from the quasi-idempotents corresponding to the tableau $T^{(i)}_{(p+q,p,1)}$ by identifying all the elements in each row.

Clearly, $h^{(i)}_{p,q}(x_1^+,x_2^+,x_3^+)$, $h^{(i)}_{p,q,w_1^+}(x_1^+,x_2^+,x_3^+)$, for $0 \le i \le q$, do not belong to $\mathrm{Id}^{W,*}(M^D)$ and we will prove that they are linearly independent modulo $\mathrm{Id}^{W,*}(M^D)$.

We proceed as in the previous case. So, assume that

$$\sum_{i=0}^{q}\alpha_i h^{(i)}_{p,q}(x_1^+,x_2^+,x_3^+) + \sum_{i=0}^{q}\beta_i h^{(i)}_{p,q,w_1^+}(x_1^+,x_2^+,x_3^+) \equiv 0 \pmod{\mathrm{Id}^{W,*}(M^D)}.$$

With the same reasoning as before, if we evaluate first $x_1^+ = k(e_{11}+e_{44})+e_{22}+e_{33}+e_{12}+e_{34}$, $x_2^+ = e_{11}+e_{44}$, $x_3^+ = e_{22}+e_{33}$ with $0 \ne k \in F$, we get

$$(-1)^p\left(\sum_{i=0}^{q}(\alpha_i-\beta_i)k^i\right)e_{12} + \left(\sum_{i=0}^{q}(\alpha_i+\beta_i)k^{q-i}\right)e_{34} = 0,$$

and, as a consequence,

$$\sum_{i=0}^{q}(\alpha_i-\beta_i)k^i = 0, \qquad \sum_{i=0}^{q}(\alpha_i+\beta_i)k^{q-i} = 0.$$

Now, by applying the same argument as in Case 1 of the proof of Lemma 6.1, it follows that $\alpha_i = \beta_i = 0$ for all $0 \le i \le q$. Hence the $2(q+1)$ generalized $*$-polynomials $h^{(i)}_{p,q}(x_1^+,x_2^+,x_3^+)$ and $h^{(i)}_{p,q,w_1^+}(x_1^+,x_2^+,x_3^+)$ are linearly independent modulo $\mathrm{Id}^{W,*}(M^D)$.

Therefore, as in the previous case, we obtain $m_{(p+q,p,1),\emptyset} \ge 2(q+1)$, as claimed.

Case 3: We prove that $m_{(n),\emptyset} \ge 2n+2$.

Given the tableau

$$T_{(n)} = \begin{array}{|c|c|c|c|c|}\hline 1 & 2 & 3 & \dots & n \\ \cline{1-5}\end{array}$$

and its associated polynomials

$$t(x^+) = (x^+)^n, \tag{6.18}$$

$$t_{w_1^+}(x^+) = (x^+)^n w_1^+, \tag{6.19}$$

$$u^{(i)}(x^+) = (x^+)^i [x^+, w_1^+](x^+)^{n-i-1}, \qquad 0 \leq i \leq n-1, \tag{6.20}$$

$$u^{(i)}_{w_1^+}(x^+) = (x^+)^i [x^+, w_1^+](x^+)^{n-i-1} w_1^+, \qquad 0 \leq i \leq n-1, \tag{6.21}$$

These $(W,*)$-polynomials are obtained from the quasi-idempotents corresponding to the tableau $T_{(n)}$ by identifying all the elements.

We will prove that the $2n+2$ generalized $*$-polynomials $t(x^+)$, $t_{w_1^+}(x^+)$, $u^{(i)}(x^+)$, and $u^{(i)}_{w_1^+}(x^+)$, for $0 \leq i \leq n-1$, which are not $(W,*)$-identities of M^D, are linearly independent modulo $\mathrm{Id}^{W,*}(M^D)$.

Suppose that

$$\alpha t(x^+) + \beta t_{w_1^+}(x^+) + \sum_{0 \leq i \leq n-1} \gamma_i u^{(i)}(x^+) + \sum_{0 \leq i \leq n-1} \delta_i u^{(i)}_{w_1^+}(x^+) \equiv 0 \pmod{\mathrm{Id}^{W,*}(M^D)}.$$

First of all, note that if we evaluate them at $x^+ = e_{11} + e_{44} + e_{22} + e_{33}$, then the last two sums vanish and we get $(\alpha+\beta)(e_{11}+e_{44}) + (\alpha-\beta)(e_{22}+e_{33}) = 0$, so $\alpha = \beta = 0$.

Now, we want to prove that the last two sums are also linearly independent. By applying the same arguments as in the previous cases and evaluating them at $x^+ = k(e_{11}+e_{44}) + e_{22} + e_{33} + e_{12} + e_{34}$, with $0 \neq k \in F$, we get

$$-2\left(\sum_{i=0}^{n-1}(\gamma_i - \delta_i)k^i\right)e_{12} + 2\left(\sum_{i=0}^{n-1}(\gamma_i + \delta_i)k^{n-1-i}\right)e_{34} = 0$$

and, as a consequence, we have

$$\sum_{i=0}^{n-1}(\delta_i - \gamma_i)k^i = 0, \qquad \sum_{i=0}^{n-1}(\gamma_i + \delta_i)k^{n-1-i} = 0.$$

Again, from the same argument of Case 1 of the proof of Lemma 6.1, we obtain that $\gamma_i = \delta_i = 0$ for every $0 \leq i \leq n-1$.

Thus, the $(W,*)$-polynomials $t(x^+)$, $t_{w_1^+}(x^+)$, $u^{(i)}(x^+)$, and $u^{(i)}_{w_1^+}(x^+)$, for $0 \leq i \leq n-1$, are linearly independent, and as in the previous cases, we obtain $m_{(n),\emptyset} \geq 2n+2$, as desired. □

Now, we can finally prove the main result of this section.

Theorem 6.3. *Let M^D be the $(W,*)$-algebra M where W acts on it as the algebra D by left and right multiplication. For $n \geq 1$ and $0 \leq r \leq n$, consider*

$$\chi^{W,*}_{r,n-r}(M^D) = \sum_{(\lambda,\mu)\vdash(r,n-r)} m_{\lambda,\mu}(\chi_\lambda \otimes \chi_\mu),$$

the $(r,n-r)$-th $(W,*)$-cocharacter of M^D. Then:

(1) *for $r = n$,*

$$m_{\lambda,\mu} = \begin{cases} 4(q+1) & \text{if } \lambda = (p+q,p), \mu = \emptyset \\ 2(q+1) & \text{if } \lambda = (p+q,p,1), \mu = \emptyset \\ 2n+2 & \text{if } \lambda = (n), \mu = \emptyset \\ 0 & \text{otherwise} \end{cases};$$

(2) *for $r = n-1$,*

$$m_{\lambda,\mu} = \begin{cases} 2(q+1) & \text{if } \lambda = (p+q,p), \mu = (1) \\ 2n & \text{if } \lambda = (n-1), \mu = (1) \\ 0 & \text{otherwise} \end{cases};$$

(3) *for $0 \leq r \leq n-2$, we have $m_{\lambda,\mu} = 0$.*

Proof. Items (1) and (2) follow from Lemmas 6.2 and 6.1, respectively. Moreover, for $0 \leq r \leq n-2$, since $c_{r,n-r}(M^D) = 0$, it follows from (6.2) that $m_{\lambda,\mu} = 0$, which proves item (3). □

7. Growth of the generalized $*$-codimensions of M

In this section, we investigate the growth of $\mathrm{var}^{W,*}(M^D)$ and $\mathrm{var}^{W,*}(M^F)$. To this end, recall that a $(W,*)$-variety $\mathcal{V}$ is said to have almost polynomial growth if $c_n^{W,*}(\mathcal{V})$ is not polynomially bounded but every proper $(W,*)$-subvariety $\mathcal{U}$ of $\mathcal{V}$ has polynomial growth.

If we consider M^F, then $\Phi(W) \cong F$. Hence, we are dealing with the ordinary $*$-identities, and consequently, by [25, Theorem 3], we obtain the following result.

Theorem 7.1. $\mathrm{var}^{W,*}(M^F)$ *has almost polynomial growth.*

Next, we show that $\mathrm{var}^{W,*}(M^D)$ also has almost polynomial growth. If $\mathcal{V}$ is a $(W,*)$-variety, then for $n \geq 1$ and $0 \leq r \leq n$, by (3.4) we write

$$\chi_{r,n-r}^{W,*}(\mathcal{V}) = \sum_{(\lambda,\mu)\vdash(r,n-r)} m_{\lambda,\mu}^{\mathcal{V}}(\chi_\lambda \otimes \chi_\mu)$$

where $m_{\lambda,\mu}^{\mathcal{V}}$ denotes the multiplicity of $\chi_\lambda \otimes \chi_\mu$ in $\chi_{r,n-r}^{W,*}(\mathcal{V})$.

Lemma 7.2. *For any $\alpha, \beta \in F$ we have:*

$$([x^+,x^-](\alpha 1_W + \beta w_1^+))^* \equiv [x^+,x^-](\alpha 1_W - \beta w_1^+) \pmod{\mathrm{Id}^{W,*}(M^D)},$$

and

$$((x^+x^- + x^-x^+)(\alpha 1_W + \beta w_1^+))^* \equiv (x^+x^- + x^-x^+)(-\alpha 1_W + \beta w_1^+) \pmod{\mathrm{Id}^{W,*}(M^D)}.$$

Proof. Since $\alpha 1_W + \beta w_1^+$ is a symmetric element of W, $[x^+,x^-]$ is a symmetric polynomial, and $(x^+x^- + x^-x^+)$ is a skew-symmetric polynomial, we have

$$([x^+,x^-](\alpha 1_W + \beta w_1^+))^* = (\alpha 1_W + \beta w_1^+)[x^+,x^-],$$

and

$$((x^+x^- + x^-x^+)(\alpha 1_W + \beta w_1^+))^* = -(\alpha 1_W + \beta w_1^+)(x^+x^- + x^-x^+).$$

Then, since $[x^+, w_1^+]$ is skew-symmetric, $x_1^-x_2^-$, $w_1^+x^- + x^-w_1^+ \in \mathrm{Id}^{W,*}(M^D)$ and by Lemma 5.1, we obtain the desired conclusion. □

Lemma 7.3. *Let $\mathcal{U}$ be a proper subvariety of* $\mathrm{var}^{W,*}(M^D)$. *Then there exist constants $K \leq N$ such that*

$$(x_1^+)^K x_2^-(x_1^+)^{N-K} + \hat{\alpha}(x_1^+)^K x_2^-(x_1^+)^{N-K} w_1^+ + \sum_{i<K} \hat{\beta}_i (x_1^+)^i x_2^-(x_1^+)^{N-i} + \sum_{i<K} \hat{\gamma}_i (x_1^+)^i x_2^-(x_1^+)^{N-i} w_1^+ \in \mathrm{Id}^{W,*}(\mathcal{U}),$$

where $\hat{\alpha}, \hat{\beta}_i, \hat{\gamma}_i \in F$.

Proof. Since $\mathcal{U} \subsetneq \mathcal{V} = \mathrm{var}^{W,*}(M^D)$, there exist $n \geq 1$, $0 \leq r \leq n$, and partitions $\lambda \vdash r$ and $\mu \vdash n-r$ such that $m_{\lambda,\mu}^{\mathcal{U}} < m_{\lambda,\mu}^{\mathcal{V}}$. Then, by Theorem 6.3 we have that either

$$\sum_{i=0}^{q} \left(\alpha_i a_{p,q}^{(i)}(x_1^+, x_2^+, x^-) + \beta_i a_{p,q,w_1^+}^{(i)}(x_1^+, x_2^+, x^-) \right) \in \mathrm{Id}^{W,*}(\mathcal{U}) \tag{7.1}$$

with α_i, β_i not all zero, where the polynomials $a_{p,q}^{(i)}$ and $a_{p,q,w_1^+}^{(i)}$ are defined in (6.4) and (6.5), respectively, in the proof of Lemma 6.1, or

$$\sum_{i=0}^{n-1} \left(\gamma_i b^{(i)}(x^+, x^-) + \delta_i b_{w_1^+}^{(i)}(x^+, x^-) \right) \in \mathrm{Id}^{W,*}(\mathcal{U}) \tag{7.2}$$

with γ_i, δ_i not all zero, where the polynomials $b^{(i)}$ and $b_{w_1^+}^{(i)}$ are defined in (6.6) and (6.7), respectively, in the proof of Lemma 6.1, or

$$\sum_{i=0}^{q} \left(\varepsilon_i f_{p,q}^{(i)}(x_1^+, x_2^+) + \zeta_i f_{p,q,w_1^+}^{(i)}(x_1^+, x_2^+) + \eta_i g_{p,q}^{(i)}(x_1^+, x_2^+) + \theta_i g_{p,q,w_1^+}^{(i)}(x_1^+, x_2^+) \right) \in \mathrm{Id}^{W,*}(\mathcal{U}) \tag{7.3}$$

with $\varepsilon_i, \zeta_i, \eta_i, \theta_i$ not all zero, where the polynomials $f^{(i)}_{p,q}$, $f^{(i)}_{p,q,w_1^+}$, $g^{(i)}_{p,q}$ and $g^{(i)}_{p,q,w_1^+}$ are defined in (6.9), (6.10), (6.11), and (6.12), respectively, in the proof of Lemma 6.2, or

$$\sum_{i=0}^{q} \left(\nu_i h^{(i)}_{p,q}(x_1^+, x_2^+, x_3^+) + \pi_i h^{(i)}_{p,q,w_1^+}(x_1^+, x_2^+, x_3^+) \right) \in \mathrm{Id}^{W,*}(\mathcal{U}) \tag{7.4}$$

with ν_i, π_i not all zero, where the polynomials $h^{(i)}_{p,q}$ and $h^{(i)}_{p,q,w_1^+}$ are defined in (6.16) and (6.17), respectively, in the proof of Lemma 6.2, or

$$\rho t(x^+) + \sigma t_{w_1^+}(x^+) + \sum_{i=0}^{n-1} \left(\tau_i u^{(i)}(x^+) + \xi_i u^{(i)}_{w_1^+}(x^+) \right) \in \mathrm{Id}^{W,*}(\mathcal{U}) \tag{7.5}$$

with $\rho, \sigma, \tau_i, \xi_i$ not all zero, where the polynomials t, $t_{w_1^+}$, $u^{(i)}$, and $u^{(i)}_{w_1^+}$ are defined in (6.18), (6.19), (6.20), and (6.21), respectively, in the proof of Lemma 6.2.

Case 1: Suppose hold (7.3). Then

$$\begin{aligned}
&\sum_{i=0}^{q} (x_1^+)^i \underbrace{\overline{x_1^+} \cdots \widetilde{x_1^+}}_{p-1} [x_1^+, x_2^+] \underbrace{\overline{x_2^+} \cdots \widetilde{x_2^+}}_{p-1} (x_1^+)^{q-i} v_i \\
&+ \sum_{i=0}^{q} (x_1^+)^i \underbrace{\overline{x_1^+} \cdots \widetilde{x_1^+}}_{p-1} \left(x_1^+ w_1^+ x_2^+ - x_2^+ w_1^+ x_1^+ \right) \underbrace{\overline{x_2^+} \cdots \widetilde{x_2^+}}_{p-1} (x_1^+)^{q-i} v_i' \in \mathrm{Id}^{W,*}(\mathcal{U}),
\end{aligned}$$

where $v_i := \varepsilon_i 1_W + \zeta_i w_1^+ \in W$ and $v_i' := \eta_i 1_W + \theta_i w_1^+ \in W$ are not all zero. If we substitute x_2^+ with $x_3^+ + x_4^+$, we obtain

$$\begin{aligned}
&\sum_{i=0}^{q} (x_1^+)^i \underbrace{\overline{x_1^+} \cdots \widetilde{x_1^+}}_{p-1} [x_1^+, x_3^+ + x_4^+] \underbrace{\overline{(x_3^+ + x_4^+)} \cdots \widetilde{(x_3^+ + x_4^+)}}_{p-1} (x_1^+)^{q-i} v_i \\
&+ \sum_{i=0}^{q} (x_1^+)^i \underbrace{\overline{x_1^+} \cdots \widetilde{x_1^+}}_{p-1} (x_1^+ w_1^+ (x_3^+ + x_4^+) - (x_3^+ + x_4^+) w_1^+ x_1^+) \underbrace{\overline{(x_3^+ + x_4^+)} \cdots \widetilde{(x_3^+ + x_4^+)}}_{p-1} (x_1^+)^{q-i} v_i' \in \mathrm{Id}^{W,*}(\mathcal{U}).
\end{aligned}$$

Now, consider the homogeneous component c of degree 1 in x_4^+.

Since the $(W,*)$-polynomial $[x_2^-, w_1^+]$ is symmetric, we may substitute x_3^+ by $(x_1^+)^2$ and x_4^+ by $[x_2^-, w_1^+]$ in c. Therefore, since the $(W,*)$-polynomial $x_1^+ w_1^+ (x_1^+)^2 - (x_1^+)^2 w_1^+ x_1^+$ is skew-symmetric and $w_1^+ x^- + x^- w_1^+$, $x_1^- x_2^- \in \mathrm{Id}^{W,*}(M^D) \subset \mathrm{Id}^{W,*}(\mathcal{U})$, by Lemma 5.1 it follows that

$$\left(x_1^+ w_1^+ (x_1^+)^2 - (x_1^+)^2 w_1^+ x_1^+ \right) (x_1^+)^k [x_2^-, w_1^+],\ [x_2^-, w_1^+] (x_1^+)^k \left(x_1^+ w_1^+ (x_1^+)^2 - (x_1^+)^2 w_1^+ x_1^+ \right) \ \in \mathrm{Id}^{W,*}(\mathcal{U})$$

for all $k \geq 0$, and as a consequence, we have

$$\begin{aligned}
&\sum_{i=0}^{q} (x_1^+)^i \underbrace{\overline{x_1^+} \cdots \widetilde{x_1^+}}_{p-1} \left(x_1^+ [x_2^-, w_1^+] - [x_2^-, w_1^+] x_1^+ \right) \underbrace{\overline{(x_1^+)^2} \cdots \widetilde{(x_1^+)^2}}_{p-1} (x_1^+)^{q-i} v_i \\
&+ \sum_{i=0}^{q} (x_1^+)^i \underbrace{\overline{x_1^+} \cdots \widetilde{x_1^+}}_{p-1} (x_1^+ w_1^+ [x_2^-, w_1^+] - [x_2^-, w_1^+] w_1^+ x_1^+) \underbrace{\overline{(x_1^+)^2} \cdots \widetilde{(x_1^+)^2}}_{p-1} (x_1^+)^{q-i} v_i' \in \mathrm{Id}^{W,*}(\mathcal{U}).
\end{aligned}$$

Since $[x_2^-, w_1^+] \equiv 2x_2^- w_1^+ \pmod{\mathrm{Id}^{W,*}(M^D)}$ and $\mathrm{Id}^{W,*}(M^D) \subset \mathrm{Id}^{W,*}(\mathcal{U})$, also $[x_1^+, w_1^+]$ is skew-symmetric, $x_1^- x_2^- \in \mathrm{Id}^{W,*}(M^D) \subset \mathrm{Id}^{W,*}(\mathcal{U})$ and by Lemma 5.1, we obtain

$$\sum_{i=0}^{q} (x_1^+)^i \underbrace{\overline{x_1^+ \cdots \widetilde{x_1^+}}}_{p-1} \left(x_1^+ x_2^- - x_2^- x_1^+\right) \underbrace{\overline{(x_1^+)^2} \cdots \widetilde{(x_1^+)^2}}_{p-1} (x_1^+)^{q-i} v_i'' + \sum_{i=0}^{q} (x_1^+)^i \underbrace{\overline{x_1^+ \cdots \widetilde{x_1^+}}}_{p-1} (x_1^+ w_1^+ x_2^- w_1^+ - x_2^- (w_1^+)^2 x_1^+) \underbrace{\overline{(x_1^+)^2} \cdots \widetilde{(x_1^+)^2}}_{p-1} (x_1^+)^{q-i} v_i''' \in \mathrm{Id}^{W,*}(\mathcal{U}),$$

where $v_i'' := 2\zeta_i 1_W + 2\varepsilon_i w_1^+ \in W$ since $x^+(w_1^+)^2 - x^+ \in \mathrm{Id}^{W,*}(M^D) \subset \mathrm{Id}^{W,*}(\mathcal{U})$, $v_i''' := 2\eta_i 1_W + 2\theta_i w_1^+ \in W$. Now, since $(w_1^+)^2 x^+ - x^+$, $x^+(w_1^+)^2 - x^+$, $w_1^+ x_2^- + x_2^- w_1^+ \in \mathrm{Id}^{W,*}(M^D) \subset \mathrm{Id}^{W,*}(\mathcal{U})$, we get

$$x_1^+ w_1^+ x_2^- w_1^+ - x_2^- (w_1^+)^2 x_1^+ \equiv -x_1^+ x_2^- - x_2^- x_1^+ \pmod{\mathrm{Id}^{W,*}(\mathcal{U})},$$

and so it follows that

$$\sum_{i=0}^{q} (x_1^+)^i \underbrace{\overline{x_1^+ \cdots \widetilde{x_1^+}}}_{p-1} [x_1^+, x_2^-] \underbrace{\overline{(x_1^+)^2} \cdots \widetilde{(x_1^+)^2}}_{p-1} (x_1^+)^{q-i} v_i'' - \sum_{i=0}^{q} (x_1^+)^i \underbrace{\overline{x_1^+ \cdots \widetilde{x_1^+}}}_{p-1} (x_1^+ x_2^- + x_2^- x_1^+) \underbrace{\overline{(x_1^+)^2} \cdots \widetilde{(x_1^+)^2}}_{p-1} (x_1^+)^{q-i} v_i''' \in \mathrm{Id}^{W,*}(\mathcal{U}),$$

where $v_i'' = 2\zeta_i 1_W + 2\varepsilon_i w_1^+ \in W$ and $v_i''' = 2\eta_i 1_W + 2\theta_i w_1^+ \in W$ are not all zero.

Let t be the maximum i such that $v_i'' \neq 0$ or $v_i''' \neq 0$. Since $[x_1^+, w_1^+]$ is skew-symmetric and $x_1^- x_2^- \in \mathrm{Id}^{W,*}(M^D) \subset \mathrm{Id}^{W,*}(\mathcal{U})$, we can move v_i'', v_i''' and obtain

$$\begin{aligned} &(x_1^+)^t \underbrace{\overline{x_1^+ \cdots \widetilde{x_1^+}}}_{p-1} \left([x_1^+, x_2^-] v_t'' - (x_1^+ x_2^- + x_2^- x_1^+) v_t'''\right) \underbrace{\overline{(x_1^+)^2} \cdots \widetilde{(x_1^+)^2}}_{p-1} (x_1^+)^{q-t} \\ &+ \sum_{i<t} (x_1^+)^i \underbrace{\overline{x_1^+ \cdots \widetilde{x_1^+}}}_{p-1} \left([x_1^+, x_2^-] v_i'' - (x_1^+ x_2^- + x_2^- x_1^+) v_i'''\right) \underbrace{\overline{(x_1^+)^2} \cdots \widetilde{(x_1^+)^2}}_{p-1} (x_1^+)^{q-i} \in \mathrm{Id}^{W,*}(\mathcal{U}). \end{aligned} \tag{7.6}$$

First, observe that we may assume, without loss of generality, that $t = q - t$. Indeed, if $t < q - t$, we multiply (7.6) on the left by $(x_1^+)^{q-2t}$, thereby reducing to the case $t = q - t$. Similarly, if $t > q - t$, we can multiply (7.6) on the right by $(x_1^+)^{2t-q}$. Hence, (7.6) becomes

$$\begin{aligned} r(x_1^+, x_2^-) :=& (x_1^+)^t \underbrace{\overline{x_1^+ \cdots \widetilde{x_1^+}}}_{p-1} \left([x_1^+, x_2^-] v_t'' - (x_1^+ x_2^- + x_2^- x_1^+) v_t'''\right) \underbrace{\overline{(x_1^+)^2} \cdots \widetilde{(x_1^+)^2}}_{p-1} (x_1^+)^t \\ &+ \sum_{i<t} (x_1^+)^i \underbrace{\overline{x_1^+ \cdots \widetilde{x_1^+}}}_{p-1} \left([x_1^+, x_2^-] v_i'' - (x_1^+ x_2^- + x_2^- x_1^+) v_i'''\right) \underbrace{\overline{(x_1^+)^2} \cdots \widetilde{(x_1^+)^2}}_{p-1} (x_1^+)^{2t-i} \in \mathrm{Id}^{W,*}(\mathcal{U}), \end{aligned}$$

with $v_j'' = 2\zeta_j 1_W + 2\varepsilon_j w_1^+ \in W$ and $v_j''' = 2\eta_j 1_W + 2\theta_j w_1^+ \in W$. Since $(r(x_1^+, x_2^-))^* \in \mathrm{Id}^{W,*}(\mathcal{U})$, by Lemma 7.2 we have

$$\begin{aligned} r'(x_1^+, x_2^-) :=& (x_1^+)^t \underbrace{\overline{x_1^+ \cdots \widetilde{x_1^+}}}_{p-1} \left([x_1^+, x_2^-] u_t - (x_1^+ x_2^- + x_2^- x_1^+) u_t'\right) \underbrace{\overline{(x_1^+)^2} \cdots \widetilde{(x_1^+)^2}}_{p-1} (x_1^+)^t \\ &+ \sum_{i<t} (x_1^+)^i \underbrace{\overline{x_1^+ \cdots \widetilde{x_1^+}}}_{p-1} \left([x_1^+, x_2^-] u_i - (x_1^+ x_2^- + x_2^- x_1^+) u_i'\right) \underbrace{\overline{(x_1^+)^2} \cdots \widetilde{(x_1^+)^2}}_{p-1} (x_1^+)^{2t-i} \in \mathrm{Id}^{W,*}(\mathcal{U}), \end{aligned}$$

where $u_t = 2\zeta_t 1_W - 2\varepsilon_t w_1^+ \in W$, $u_t' = -2\eta_t 1_W + 2\theta_t w_1^+ \in W$, and for $i < t$, $u_i = 2\zeta_{2t-i} 1_W - 2\varepsilon_{2t-i} w_1^+ \in W$ and $u_i' = -2\eta_{2t-i} 1_W + 2\theta_{2t-i} w_1^+ \in W$. Moreover, notice that

$$v_t'' + u_t = 4\zeta_t 1_W, \quad v_t''' + u_t' = 4\theta_t w_1^+, \quad v_t'' - u_t = 4\varepsilon_t w_1^+, \quad v_t''' - u_t' = 4\eta_t 1_W.$$

Suppose first that $\zeta_t \neq 0$ or $\theta_t \neq 0$ and set $N = q + 3p - 2$. Since $r(x_1^+, x_2^-) + r'(x_1^+, x_2^-) \in \mathrm{Id}^{W,*}(\mathcal{U})$, we have

$$\zeta_t(x_1^+)^{t+2p-1}x_2^-(x_1^+)^{N-t-2p+1} - \theta_t(x_1^+)^{t+2p-1}x_2^-(x_1^+)^{N-t-2p+1}w_1^+ + \sum_{i<t+2p-1}\beta_i'(x_1^+)^i x_2^-(x_1^+)^{N-i} + \sum_{i<t+2p-1}\gamma_i'(x_1^+)^i x_2^-(x_1^+)^{N-i}w_1^+ \in \mathrm{Id}^{W,*}(\mathcal{U}),$$

for some $\beta_i', \gamma_i' \in F$. Since $\zeta_t \neq 0$ or $\theta_t \neq 0$ and and $x^\epsilon(w_1^+)^2 - x^\epsilon \in \mathrm{Id}^{W,*}(M^D) \subset \mathrm{Id}^{W,*}(\mathcal{U})$, $\epsilon \in \{+,-\}$, we have

$$(x_1^+)^{t_1+2p-1}x_2^-(x_1^+)^{N-t_1-2p+1} + \hat{\alpha}(x_1^+)^{t_1+2p-1}x_2^-(x_1^+)^{N-t_1-2p+1}w_1^+ + \sum_{i<t_1+2p-1}\hat{\beta}_i(x_1^+)^i x_2^-(x_1^+)^{N-i} + \sum_{i<t_1+2p-1}\hat{\gamma}_i(x_1^+)^i x_2^-(x_1^+)^{N-i}w_1^+ \in \mathrm{Id}^{W,*}(\mathcal{U}),$$

for some $\hat{\alpha}, \hat{\beta}_i, \hat{\gamma}_i \in F$. Hence, setting $K := t + 2p - 1$ gives the desired conclusion.

If $\zeta_t = \theta_t = 0$, then $\varepsilon_t \neq 0$ or $\eta_t \neq 0$. Thus, since $r(x_1^+, x_2^-) - r'(x_1^+, x_2^-) \in \mathrm{Id}^{W,*}(\mathcal{U})$, by proceeding as above we obtain the desired conclusion.

Case 2: Suppose hold (7.5). Then

$$(x^+)^n v + \sum_{i=0}^{n-1}(x^+)^i[x^+, w_1^+](x^+)^{n-i-1}v_i' \in \mathrm{Id}^{W,*}(\mathcal{U}), \tag{7.7}$$

where $v := \rho 1_W + \sigma w_1^+ \in W$ and $v_i' := \tau_i 1_W + \xi_i w_1^+ \in W$ are not all zero.

Let us substitute x^+ with $x_1^+ + x_2^+$ in (7.7), and consider the homogeneous component c of degree 1 in x_2^+. Since the $(W,*)$-polynomial $[x_2^-, w_1^+]$ is symmetric, we may substitute x_2^+ by $[x_2^-, w_1^+]$ in c. Therefore, since the $(W,*)$-polynomial $[x_2^+, w_1^+]$ is skew-symmetric and $w_1^+x^- + x^-w_1^+$, $x_1^-x_2^- \in \mathrm{Id}^{W,*}(M^D) \subset \mathrm{Id}^{W,*}(\mathcal{U})$, by Lemma 5.1 it follows that $[x_2^+, w_1^+](x_1^+)^k[x_2^-, w_1^+]$, $[x_2^-, w_1^+](x_1^+)^k[x_2^+, w_1^+] \in \mathrm{Id}^{W,*}(\mathcal{U})$ for all $k \geq 0$, and as a consequence, we have

$$\sum_{i=0}^{n-1}(x_1^+)^i[x_2^-, w_1^+](x_1^+)^{n-i-1}v + \sum_{i=0}^{n-1}(x_1^+)^i[[x_2^-, w_1^+], w_1^+](x_1^+)^{n-i-1}v_i' \in \mathrm{Id}^{W,*}(\mathcal{U}).$$

Now, since $[x_2^-, w_1^+] - 2x_2^-w_1^+$, $x_2^-(w_1^+)^2 - x_2^- \in \mathrm{Id}^{W,*}(M^D) \subset \mathrm{Id}^{W,*}(\mathcal{U})$ and since $[x_1^+, w_1^+]$ is a skew-symmetric element, by Lemma 5.1 we have

$$\sum_{i=0}^{n-1}(x_1^+)^i x_2^-(x_1^+)^{n-i-1}v' + \sum_{i=0}^{n-1}(x_1^+)^i x_2^-(x_1^+)^{n-i-1}v_i'' \in \mathrm{Id}^{W,*}(\mathcal{U}),$$

where $v' := 2\sigma 1_W + 2\rho w_1^+ \in W$ and $v_i'' := 4\tau_i 1_W + 4\xi_i w_1^+ \in W$ are not all zero. Therefore, by proceeding as in Case 1, we reach the desired conclusion.

Case 3: Suppose hold (7.1). Since

$$a_{p,q}^{(i)}(x_1^+, x_2^+, x^-) = (x_1^+)^i \underbrace{\overline{x_1^+}\cdots\widetilde{x_1^+}}_{p} x^- \underbrace{\overline{x_2^+}\cdots\widetilde{x_2^+}}_{p}(x_1^+)^{q-i},$$

$$a_{p,q,w_1^+}^{(i)}(x_1^+, x_2^+, x^-) = (x_1^+)^i \underbrace{\overline{x_1^+}\cdots\widetilde{x_1^+}}_{p} x^- \underbrace{\overline{x_2^+}\cdots\widetilde{x_2^+}}_{p}(x_1^+)^{q-i}w_1^+,$$

substituting x^- by $[x_1^+, x_2^+]$ in relation (7.1), we obtain relation (7.3). Hence, by Case 1, we get the desired conclusion.

Case 4: Suppose hold (7.2). Since $b^{(i)}(x^+, x^-) = (x^+)^i x^-(x^+)^{n-i-1}$ and $b_{w_1^+}^{(i)}(x^+, x^-) = (x^+)^i x^-(x^+)^{n-i-1}w_1^+$, as in the previous case by substituting x^- with $[x_1^+, x_2^+]$ in relation (7.2), we get relation (7.3). Therefore, by Case 1, we obtain the desired conclusion.

Case 5: Suppose hold (7.4). Since

$$h^{(i)}_{p,q}(x_1^+,x_2^+,x_3^+) = (x_1^+)^i \underbrace{\widehat{x_1^+}\cdots\widetilde{x_1^+}}_{p-1}\overline{x_1^+}\,\overline{x_2^+}\,\overline{x_3^+}\underbrace{\widehat{x_2^+}\cdots\widetilde{x_2^+}}_{p-1}(x_1^+)^{q-i},$$

$$h^{(i)}_{p,q,w_1^+}(x_1^+,x_2^+,x_3^+) = (x_1^+)^i \underbrace{\widehat{x_1^+}\cdots\widetilde{x_1^+}}_{p-1}\overline{x_1^+}\,\overline{x_2^+}\,\overline{x_3^+}\underbrace{\widehat{x_2^+}\cdots\widetilde{x_2^+}}_{p-1}(x_1^+)^{q-i}w_1^+,$$

by substituting x_3^+ with $(x_1^+)^2$ in (7.4) we get (7.3), and so by Case 1 we reach the desired conclusion. □

Lemma 7.4. *Let $\mathcal{U}$ be a proper subvariety of $\mathrm{var}^{W,*}(M^D)$. Then there exist constants $K' \le N'$ such that*

$$(x_1^+)^{K'}x_2^\epsilon(x_1^+)^{N'-K'} + \sum_{i<K'}\delta_i(x_1^+)^i x_2^\epsilon(x_1^+)^{N'-i} + \sum_{i<K'}\eta_i(x_1^+)^i x_2^\epsilon(x_1^+)^{N'-i}w_1^+ \in \mathrm{Id}^{W,*}(\mathcal{U}),$$

for some $\delta_i,\eta_i \in F$ and for any $\epsilon \in \{+,-\}$.

Proof. By Lemma 7.3, there exist constants $K \le N$ such that

$$(x_1^+)^K x_2^-(x_1^+)^{N-K} + \hat{\alpha}(x_1^+)^K x_2^-(x_1^+)^{N-K}w_1^+ + \sum_{i<K}\hat{\beta}_i(x_1^+)^i x_2^-(x_1^+)^{N-i} + \sum_{i<K}\hat{\gamma}_i(x_1^+)^i x_2^-(x_1^+)^{N-i}w_1^+ \in \mathrm{Id}^{W,*}(\mathcal{U}),$$

where $\hat{\alpha},\hat{\beta}_i,\hat{\gamma}_i \in F$. Since $[x_1^+,w_1^+]$ is skew-symmetric and $x_1^-x_2^- \in \mathrm{Id}^{W,*}(M^D) \subset \mathrm{Id}^{W,*}(\mathcal{U})$, we can move w_1^+ and obtain

$$(x_1^+)^K(x_2^- + \hat{\alpha}x_2^-w_1^+)(x_1^+)^{N-K} + \sum_{i<K}(x_1^+)^i(\hat{\beta}_i x_2^- + \hat{\gamma}_i x_2^-w_1^+)(x_1^+)^{N-i} \in \mathrm{Id}^{W,*}(\mathcal{U}). \tag{7.8}$$

First, observe that we may assume, without loss of generality, that $N = 2K$ in (7.8). Indeed, if $N > 2K$, we multiply (7.8) on the left by $(x_1^+)^{N-2K}$, thereby reducing to the case $N = 2K$. Similarly, if $N < 2k$, we multiply (7.8) on the right by $(x_1^+)^{2K-N}$, and again we reduce to the case $N = 2K$. Then, (7.8) can be written as

$$f := (x_1^+)^K(x_2^- + \hat{\alpha}x_2^-w_1^+)(x_1^+)^K + \sum_{i<K}(x_1^+)^i(\hat{\beta}_i x_2^- + \hat{\gamma}_i x_2^-w_1^+)(x_1^+)^{2K-i} \in \mathrm{Id}^{W,*}(\mathcal{U}).$$

Since $f^* \in \mathrm{Id}^{W,*}(\mathcal{U})$ and $(x_2^-w_1^+)^* - x_2^-w_1^+ \in \mathrm{Id}^{W,*}(M^D) \subset \mathrm{Id}^{W,*}(\mathcal{U})$, we have

$$g := (x_1^+)^K(-x_2^- + \hat{\alpha}x_2^-w_1^+)(x_1^+)^K + \sum_{i<K}(x_1^+)^i(-\hat{\beta}_{2K-i}x_2^- + \hat{\gamma}_{2K-i}x_2^-w_1^+)(x_1^+)^{2K-i} \in \mathrm{Id}^{W,*}(\mathcal{U}).$$

Therefore,

$$f - g = 2(x_1^+)^K x_2^-(x_1^+)^K + \sum_{i<K}(x_1^+)^i((\hat{\beta}_i + \hat{\beta}_{2K-i})x_2^- + (\hat{\gamma}_i - \hat{\gamma}_{2K-i})x_2^-w_1^+)(x_1^+)^{2K-i} \in \mathrm{Id}^{W,*}(\mathcal{U}).$$

As a consequence, we obtain

$$(x_1^+)^K x_2^-(x_1^+)^K + \sum_{i<K}\beta_i'(x_1^+)^i x_2^-(x_1^+)^{2K-i} + \sum_{i<K}\gamma_i'(x_1^+)^i x_2^-(x_1^+)^{2K-i}w_1^+ \in \mathrm{Id}^{W,*}(\mathcal{U}),$$

where $\beta_i' = 2^{-1}(\hat{\beta}_i + \hat{\beta}_{2K-i})$ and $\gamma_i' = 2^{-1}(\hat{\gamma}_i - \hat{\gamma}_{2K-i})$. Now, if we substitute x_2^- by $[x_1^+,x_2^+]$, we obtain

$$(x_1^+)^{K+1}x_2^+(x_1^+)^K + \sum_{i<K+1}\beta_i''(x_1^+)^i x_2^+(x_1^+)^{2K+1-i} + \sum_{i<K+1}\gamma_i''(x_1^+)^i x_2^+(x_1^+)^{2K+1-i}w_1^+ \in \mathrm{Id}^{W,*}(\mathcal{U}),$$

for some $\beta_i'',\gamma_i'' \in F$. Hence, if we set $K' = K+1$ and $N' = 2K+1$, then it follows that

$$(x_1^+)^{K'}x_2^\epsilon(x_1^+)^{N'-K'} + \sum_{i<K'}\delta_i(x_1^+)^i x_2^\epsilon(x_1^+)^{N'-i} + \sum_{i<K'}\eta_i(x_1^+)^i x_2^\epsilon(x_1^+)^{N'-i}w_1^+ \in \mathrm{Id}^{W,*}(\mathcal{U}),$$

for some $\delta_i,\eta_i \in F$, where $\epsilon \in \{+,-\}$, as desired. □

Proposition 7.5. *Let $\mathcal{U}$ be a proper subvariety of $\mathrm{var}^{W,*}(M^D)$. Then there exists a constant $\bar{N}$ such that $m^{\mathcal{U}}_{\lambda,\mu} \le \bar{N}$ for any $(\lambda,\mu) \vdash (r,n-r)$, $0 \le r \le n$, $n \ge 1$.*

Proof. By Lemma 7.4, there exists a constant N' such that

$$(x_1^+)^{K'}x_2^\epsilon(x_1^+)^{N'-K'} + \sum_{i<K'}\delta_i(x_1^+)^i x_2^\epsilon(x_1^+)^{N'-i} + \sum_{i<K'}\eta_i(x_1^+)^i x_2^\epsilon(x_1^+)^{N'-i}w_1^+ \in \mathrm{Id}^{W,*}(\mathcal{U}), \tag{7.9}$$

for some $\delta_i, \eta_i \in F$ and a suitable $K' \leq N'$, where $\epsilon \in \{+,-\}$. We will prove that $m_{\lambda,\mu}^{\mathcal{U}} \leq 4N'$ for any $(\lambda,\mu) \vdash (r, n-r)$, $0 \leq r \leq n$, $n \geq 1$.

By Theorem 6.3, we only have to consider the following cases:

- for $r = n$,
$$(\lambda,\mu) = \begin{cases} ((p+q,p),\emptyset) \\ ((p+q,p,1),\emptyset) \\ ((n),\emptyset) \end{cases};$$
- for $r = n-1$,
$$(\lambda,\mu) = \begin{cases} ((p+q,p),(1)) \\ ((n-1),(1)) \end{cases}.$$

We proceed with the same reasoning as in [23, Proposition 5.6] and [25, Theorem 3].

We prove the statement for $\lambda = (p+q,p)$ and $\mu = \emptyset$. The other cases will follow with similar arguments. If $q \leq N'-1$, then by Theorem 6.3 we have $m_{\lambda,\mu}^{\mathcal{U}} \leq 4N'$, as desired. So, let us assume that $q \geq N'$ and consider $(W,*)$-polynomials $f_{p,q}^{(i)}(x_1^+,x_2^+)$, $f_{p,q,w_1^+}^{(i)}(x_1^+,x_2^+)$, $g_{p,q}^{(i)}(x_1^+,x_2^+)$ and $g_{p,q,w_1^+}^{(i)}(x_1^+,x_2^+)$, $0 \leq i \leq q$, defined in (6.9), (6.10), (6.11) and (6.12), respectively, of Lemma 6.2. Since $\underbrace{\overline{x_1^+}\cdots\widetilde{x_1^+}}_{p-1}[x_1^+,x_2^+]\underbrace{\overline{x_2^+}\cdots\widetilde{x_2^+}}_{p-1}$ is a symmetric or a skew-symmetric element according as p is even or odd, respectively, from relation (7.9) it follows that

$$\begin{aligned}(x_1^+)^{K'}\underbrace{\overline{x_1^+}\cdots\widetilde{x_1^+}}_{p-1}[x_1^+,x_2^+]\underbrace{\overline{x_2^+}\cdots\widetilde{x_2^+}}_{p-1}(x_1^+)^{N'-K'} &\equiv \sum_{i<K'}\delta_i'(x_1^+)^i\underbrace{\overline{x_1^+}\cdots\widetilde{x_1^+}}_{p-1}[x_1^+,x_2^+]\underbrace{\overline{x_2^+}\cdots\widetilde{x_2^+}}_{p-1}(x_1^+)^{N'-i} \\ &\quad + \sum_{i<K'}\eta_i'(x_1^+)^i\underbrace{\overline{x_1^+}\cdots\widetilde{x_1^+}}_{p-1}[x_1^+,x_2^+]\underbrace{\overline{x_2^+}\cdots\widetilde{x_2^+}}_{p-1}(x_1^+)^{N'-i}w_1^+ \pmod{\mathrm{Id}^{W,*}(\mathcal{U})},\end{aligned}$$

and by acting on the right by w_1^+,

$$\begin{aligned}(x_1^+)^{K'}\underbrace{\overline{x_1^+}\cdots\widetilde{x_1^+}}_{p-1}[x_1^+,x_2^+]\underbrace{\overline{x_2^+}\cdots\widetilde{x_2^+}}_{p-1}(x_1^+)^{N'-K'}w_1^+ &\equiv \sum_{i<K'}\delta_i'(x_1^+)^i\underbrace{\overline{x_1^+}\cdots\widetilde{x_1^+}}_{p-1}[x_1^+,x_2^+]\underbrace{\overline{x_2^+}\cdots\widetilde{x_2^+}}_{p-1}(x_1^+)^{N'-i}w_1^+ \\ &\quad + \sum_{i<K'}\eta_i'(x_1^+)^i\underbrace{\overline{x_1^+}\cdots\widetilde{x_1^+}}_{p-1}[x_1^+,x_2^+]\underbrace{\overline{x_2^+}\cdots\widetilde{x_2^+}}_{p-1}(x_1^+)^{N'-i} \pmod{\mathrm{Id}^{W,*}(\mathcal{U})},\end{aligned}$$

where $\delta_i', \eta_i' \in F$. Hence, since $q \geq N'$, we can apply Relation (7.9) to any polynomials $f_{p,q}^{(i)}(x_1^+,x_2^+)$ and $f_{p,q,w_1^+}^{(i)}(x_1^+,x_2^+)$ such that $i \geq K'$ and, as a consequence, we get

$$f_{p,q}^{(i)}(x_1^+,x_2^+) \equiv \sum_{j<K'}\delta_j' f_{p,q}^{(j)}(x_1^+,x_2^+) + \sum_{j<K'}\eta_j' f_{p,q,w_1^+}^{(j)}(x_1^+,x_2^+) \pmod{\mathrm{Id}^{W,*}(\mathcal{U})}$$

and

$$f_{p,q,w_1^+}^{(i)}(x_1^+,x_2^+) \equiv \sum_{j<K'}\delta_j' f_{p,q,w_1^+}^{(j)}(x_1^+,x_2^+) + \sum_{j<K'}\eta_j' f_{p,q}^{(j)}(x_1^+,x_2^+) \pmod{\mathrm{Id}^{W,*}(\mathcal{U})}.$$

Similarly, since the polynomial $\underbrace{\overline{x_1^+}\cdots\widetilde{x_1^+}}_{p-1}\left(x_1^+w_1^+x_2^+ - x_2^+w_1^+x_1^+\right)\underbrace{\overline{x_2^+}\cdots\widetilde{x_2^+}}_{p-1}$ is symmetric when p is even and skew-symmetric when p is odd and $q \geq N'$, the relation (7.9) can be applied to the polynomials $g_{p,q}^{(i)}(x_1^+,x_2^+)$

and $g^{(i)}_{p,q,w_1^+}(x_1^+,x_2^+)$ for every $i \geq K'$. Consequently,

$$g^{(i)}_{p,q}(x_1^+,x_2^+) \equiv \sum_{j<K'} \delta'_j g^{(j)}_{p,q}(x_1^+,x_2^+) + \sum_{j<K'} \eta'_j g^{(j)}_{p,q,w_1^+}(x_1^+,x_2^+) \pmod{\mathrm{Id}^{W,*}(\mathcal{U})}$$

and

$$g^{(i)}_{p,q,w_1^+}(x_1^+,x_2^+) \equiv \sum_{j<K'} \delta'_j g^{(j)}_{p,q,w_1^+}(x_1^+,x_2^+) + \sum_{j<K'} \eta'_j g^{(j)}_{p,q}(x_1^+,x_2^+) \pmod{\mathrm{Id}^{W,*}(\mathcal{U})}.$$

Therefore, it follows that $m^{\mathcal{U}}_{\lambda,\mu} \leq 4K' \leq 4N'$, as desired. □

Now, we can finally prove the main theorem of this section.

Theorem 7.6. *The $(W,*)$-variety generated by M^D has almost polynomial growth.*

Proof. Let $\mathcal{U}$ be a proper subvariety of $\mathcal{V} = \mathrm{var}^{W,*}(M^D)$. We shall prove that $\mathcal{U}$ has polynomial growth of the $(W,*)$-codimensions. By Lemma 7.4, there exist constants $K' \leq N'$ such that

$$(x_1^+)^{K'} x_2^- (x_1^+)^{N'-K'} + \sum_{i<K'} \delta_i (x_1^+)^i x_2^- (x_1^+)^{N'-i} + \sum_{i<K'} \eta_i (x_1^+)^i x_2^- (x_1^+)^{N'-i} w_1^+ \in \mathrm{Id}^{W,*}(\mathcal{U}),$$

for some $\delta_i, \eta_i \in F$. By substituting x_2^- with $x^-_{N'+1}$ and applying the standard multilinearization process (see [17, Theorem 1.3.8]), we get

$$\sum_{\sigma\in S_{N'}} x^+_{\sigma(1)} \cdots x^+_{\sigma(K')} x^-_{N'+1} x^+_{\sigma(K'+1)} \cdots x^+_{\sigma(N')} + \sum_{i<K'} \sum_{\sigma\in S_{N'}} \delta_i x^+_{\sigma(1)} \cdots x^+_{\sigma(i)} x^-_{N'+1} x^+_{\sigma(i+1)} \cdots x^+_{\sigma(N')}$$
$$+ \sum_{i<K'} \sum_{\sigma\in S_{N'}} \eta_i x^+_{\sigma(1)} \cdots x^+_{\sigma(i)} x^-_{N'+1} x^+_{\sigma(i+1)} \cdots x^+_{\sigma(N')} w_1^+ \in \mathrm{Id}^{W,*}(\mathcal{U}).$$

Now, we multiply on the right by $y_1^+ \cdots y_{K'}^+$ where y_i^+ are new symmetric variables, and alternate x_i^+ with y_i^+ for all $1 \leq i \leq K'$. Since $[x_1^+, w_1^+]$ is skew-symmetric, $x_1^- x_2^- \in \mathrm{Id}^{W,*}(M^D) \subset \mathrm{Id}^{W,*}(\mathcal{U})$ and by Lemma 5.1, we can move w_1^+ to the right of any monomial in which it appears. Moreover, by Lemma 5.1 (see [25, Remark 2]), any symmetric variables to the right (and to the left) of $x^-_{N'+1}$ can be reordered modulo $\mathrm{Id}^{W,*}(M^D) \subset \mathrm{Id}^{W,*}(\mathcal{U})$. Hence, we get

$$\overline{x_1^+} \cdots \widetilde{x_{K'}^+} x^-_{N'+1} \overline{y_1^+} \cdots \widetilde{y_{K'}^+} x^+_{K'+1} \cdots x^+_{N'} \in \mathrm{Id}^{W,*}(\mathcal{U}).$$

If we multiply on the left by $y^+_{K'+1} \cdots y^+_{N'}$ and we alternate x_j^+ with y_j^+ for all $K'+1 \leq j \leq N'$, we obtain

$$\overline{x_1^+} \cdots \widetilde{x_{N'}^+} x^-_{N'+1} \overline{y_1^+} \cdots \widetilde{y_{N'}^+} \in \mathrm{Id}^{W,*}(\mathcal{U}), \tag{7.10}$$

and by acting on the left by w_1^+ we get

$$\overline{x_1^+} \cdots \widetilde{x_{N'}^+} x^-_{N'+1} \overline{y_1^+} \cdots \widetilde{y_{N'}^+} w_1^+ \in \mathrm{Id}^{W,*}(\mathcal{U}). \tag{7.11}$$

This proves that, for all $q \geq 0$ and $0 \leq i \leq q$, the $(W,*)$-polynomials $a^{(i)}_{p,q}$ and $a^{(i)}_{p,q,w_1^+}$, defined in (6.4) and (6.5), respectively, in Lemma 6.1, belong to $\mathrm{Id}^{W,*}(\mathcal{U})$ whenever $p \geq N'$. Consequently, $m^{\mathcal{U}}_{(q+p,p),(1)} = 0$ when $p \geq N'$.

Since $z_1^+ w_1^+ z_2^+ - z_2^+ w_1^+ z_1^+$ is skew-symmetric, if we substitute $x^-_{N'+1}$ in (7.10) and (7.11) with $[z_1^+, z_2^+]$ or with $z_1^+ w_1^+ z_2^+ - z_2^+ w_1^+ z_1^+$, where z_i^+ are new symmetric variables, we obtain

$$\overline{x_1^+} \cdots \widetilde{x_{N'}^+}[z_1^+, z_2^+] \overline{y_1^+} \cdots \widetilde{y_{N'}^+}, \quad \overline{x_1^+} \cdots \widetilde{x_{N'}^+}[z_1^+, z_2^+] \overline{y_1^+} \cdots \widetilde{y_{N'}^+} w_1^+ \in \mathrm{Id}^{W,*}(\mathcal{U}),$$

and

$$\overline{x_1^+} \cdots \widetilde{x_{N'}^+}(z_1^+ w_1^+ z_2^+ - z_2^+ w_1^+ z_1^+) \overline{y_1^+} \cdots \widetilde{y_{N'}^+}, \quad \overline{x_1^+} \cdots \widetilde{x_{N'}^+}(z_1^+ w_1^+ z_2^+ - z_2^+ w_1^+ z_1^+) \overline{y_1^+} \cdots \widetilde{y_{N'}^+} w_1^+ \in \mathrm{Id}^{W,*}(\mathcal{U}).$$

As above, for all $q \geq 0$ and $0 \leq i \leq q$, the $(W,*)$-polynomials $f^{(i)}_{p,q}$, $f^{(i)}_{p,q,w_1^+}$, $g^{(i)}_{p,q}$, and $g^{(i)}_{p,q,w_1^+}$, defined in (6.9)–(6.12), respectively, in Lemma 6.2, belong to $\mathrm{Id}^{W,*}(\mathcal{U})$ whenever $p \geq N'+1$. Consequently, $m^{\mathcal{U}}_{(q+p,p),\emptyset} = 0$ when $p \geq N'+1$.

If we substitute $x^-_{N'+1}$ in (7.10) and (7.11) with $\widehat{z_1^+ z_2^+ z_3^+}$, where z_i^+ are new symmetric variables, we obtain

$$\overline{x_1^+}\cdots\widetilde{x^+_{N'}}\widehat{z_1^+}\widehat{z_2^+}\widehat{z_3^+}\overline{y_1^+}\cdots\widetilde{y^+_{N'}},\quad \overline{x_1^+}\cdots\widetilde{x^+_{N'}}\widehat{z_1^+}\widehat{z_2^+}\widehat{z_3^+}\overline{y_1^+}\cdots\widetilde{y^+_{N'}}w_1^+ \ \in \mathrm{Id}^{W,*}(\mathcal{U}).$$

Thus, as in the previous cases, for all $q \geq 0$ and $0 \leq i \leq q$, the $(W,*)$-polynomials $h^{(i)}_{p,q}$ and $h^{(i)}_{p,q,w_1^+}$, defined in (6.16) and (6.17), respectively, in Lemma 6.2, belong to $\mathrm{Id}^{W,*}(\mathcal{U})$ whenever $p \geq N'+1$. Consequently, $m^{\mathcal{U}}_{(q+p,p,1),\emptyset} = 0$ when $p \geq N'+1$.

Finally, since $\mathcal{U}$ is a subvariety of $\mathrm{var}^{W,*}(M^D)$, by Theorem 6.3 we have that $m^{\mathcal{U}}_{\lambda,\mu} = 0$ for all $(\lambda,\mu) \vdash (r, n-r)$ and $0 \leq r \leq n-2$.

In conclusion, by Theorem 6.3 in all cases we have proved that $m^{\mathcal{U}}_{\lambda,\mu} = 0$ for all λ with more than $N'+2$ boxes below the first row or, equivalently, if $\chi_\lambda \otimes \chi_\mu$ appears with a non-zero multiplicity in the generalized $S_r \times S_{n-r}$-cocharacter of $\mathcal{U}$, then λ must contain at most $N'+1$ boxes below the first row. As a consequence, for $0 \leq r \leq n$ and $n \geq 0$ we have

$$\chi^{W,*}_{r,n-r}(\mathcal{U}) = \sum_{\substack{(\lambda,\mu)\vdash(r,n-r)\\ r-\lambda_1 \leq N'+1\\ n-r\leq 1}} m^{\mathcal{U}}_{\lambda,\mu}(\chi_\lambda \otimes \chi_\mu), \tag{7.12}$$

where λ_1 stands for the number of boxes of the first row of λ.

Since $r - \lambda_1 \leq N'+1$, then $\lambda_1 \geq r - (N'+1)$. By the hook formula

$$d_\lambda = \chi_\lambda(1) \leq \frac{r!}{(r-(N'+1))!} \leq r^{N'+1} \leq n^{N'+1}.$$

Also recall that $d_\mu = \chi_\mu(1) = 1$ whenever $\mu = \emptyset$ or $\mu = (1)$. Thus, since $c^{W,*}_{r,n-r}(\mathcal{U}) = \chi^{W,*}_{r,n-r}(\mathcal{U})(1)$, by Formulas (3.3) and (7.12), and Proposition 7.5, we get

$$\begin{aligned} c^{W,*}_n(\mathcal{U}) &= \sum_{r=0}^{n}\binom{n}{r}\chi^{W,*}_{r,n-r}(\mathcal{U})(1) = \sum_{r=0}^{n}\binom{n}{r}\sum_{\substack{(\lambda,\mu)\vdash(r,n-r)\\ r-\lambda_1\leq N'+1\\ n-r\leq 1}} m^{\mathcal{U}}_{\lambda,\mu}d_\lambda d_\mu \\ &= \sum_{\substack{(\lambda,\mu)\vdash(n,0)\\ n-\lambda_1\leq N'+1}} m^{\mathcal{U}}_{\lambda,\mu}d_\lambda d_\mu + n\sum_{\substack{(\lambda,\mu)\vdash(n-1,1)\\ n-1-\lambda_1\leq N'+1}} m^{\mathcal{U}}_{\lambda,\mu}d_\lambda d_\mu \\ &\leq \sum_{\substack{(\lambda,\mu)\vdash(n,0)\\ n-\lambda_1\leq N'+1}} \bar{N}n^{N'+1} + n\sum_{\substack{(\lambda,\mu)\vdash(n-1,1)\\ n-1-\lambda_1\leq N'+1}} \bar{N}n^{N'+1} \leq (N'+1)^2\bar{N}(n+1)n^{N'+1}, \end{aligned}$$

since the number of partitions such that $r - \lambda_1 \leq N'+1$ is bounded by $(N'+1)^2$. Therefore, $c^{W,*}_n(\mathcal{U})$ is polynomially bounded and $\mathrm{var}^{W,*}(M^D)$ has almost polynomial growth, as required. □

As a direct consequence of Corollary 5.4, Theorem 7.1 and Theorem 7.6, we have the following result.

Corollary 7.7. *The $(W,*)$-algebras M^F and M^D generate two distinct $(W,*)$-varieties of almost polynomial growth.*

Notice that the same cannot be said for $\mathrm{var}^{W,*}(M^{C_i})$, for $1 \leq i \leq 3$, and $\mathrm{var}^{W,*}(M)$ since by Theorems 2.10 and 3.2 they contain either M^F or M^D. Thus, we conclude with the following result.

Corollary 7.8. *The $(W,*)$-variety $\mathrm{var}^{W,*}(M^\Phi)$ has almost polynomial growth if and only if $\Phi(W)$ is either isomorphic to F or D.*

References

[1] E. Aljadeff, A. Giambruno, *Multialternating graded polynomials and growth of polynomial identities*, Proc. Amer. Math. Soc. **141** (2013), 3055–3065.

[2] E. Aljadeff, A. Giambruno, D. La Mattina, *Graded polynomial identities and exponential growth*, J. Reine Angew. Math. **650** (2011), 83–100.

[3] S. A. Amitsur, *Generalized polynomial identities and pivotal monomials*, Trans. Amer. Math. Soc. **114** (1965), 210–226.

[4] S. A. Amitsur, *Identities in rings with involutions*, Israel Journal of Mathematics **7** (1969), 63–68.

[5] G. Busalacchi, F. Martino, C. Rizzo, *Varieties of graded W-algebras and asymptotic behavior of codimension growth*, Linear Algebra Appl. **739** (2026), 124–153.
[6] R. C. Busby, *Double centralizers and extensions of C^*-algebras*, Trans. Amer. Math. Soc. **132** (1968), 79–99.
[7] M. Dokuchaev, A. del Río, J. J. Simón, *Globalizations of partial actions on nonunital rings*, Proc. Amer. Math. Soc. **135** (2007), no. 2, 343–352.
[8] M. Dokuchaev, R. Exel, *Associativity of crossed products by partial actions, enveloping actions and partial representations*, Trans. Amer. Math. Soc. **357** (2005), no. 5, 1931–1952.
[9] R. B. dos Santos, $*$-*Superalgebras and exponential growth*, J. Algebra **473** (2017), 283–306.
[10] R. Exel, *Partial Dynamical Systems, Fell Bundles and Applications*, Mathematical Surveys and Monographs, vol. 224, American Mathematical Society, Providence, RI, 2017.
[11] A. Giambruno, A. Ioppolo, D. La Mattina, *Trace codimensions of algebras and their exponential growth*, Israel J. Math. **254** (2023), no. 1, 431–459.
[12] A. Giambruno, D. La Mattina, *Graded polynomial identities and codimensions: Computing the exponential growth*, Adv. Math. **225** (2010), 859–881.
[13] A. Giambruno, S. Mishchenko, *Growth of the $*$-codimensions and Young diagrams*, Comm. Algebra Comm. Algebra **29** (2001), 277–284.
[14] A. Giambruno, C. Polcino Milies, A. Valenti, *Star-polynomial identities: computing the exponential growth of the codimensions*, J. Algebra **469** (2017), 302–322.
[15] A. Giambruno, M. Zaicev, *On codimension growth of finitely generated associative algebras*, Adv. Math. **140** (1998), no. 2, 145–155.
[16] A. Giambruno, M. Zaicev, *Exponential codimension growth of PI algebras: an exact estimate*, Adv. Math. **142** (1999), no. 2, 221–243.
[17] A. Giambruno, M. Zaicev, *Polynomial Identities and Asymptotic Methods*, Mathematical Surveys and Monographs, vol. 122, American Mathematical Society, Providence, RI, 2005.
[18] A. Gordienko, *Amitsur's conjecture for associative algebras with a generalized Hopf action*, J. Pure Appl. Algebra **217** (2013), no. 8, 1395–1411.
[19] G. Hochschild, *Cohomology and representations of associative algebras*, Duke Math. J. **14** (1947), 921–948.
[20] A. Ioppolo, *The exponent for superalgebras with superinvolution*, Linear Algebra Appl. **555** (2018), 1–20.
[21] A. Ioppolo, *A characterization of superalgebras with pseudoinvolution of exponent 2*, Algebr. Represent. Theory **24** (2021), no. 6, 1415–1429.
[22] W.S. Martindale 3rd, *Prime rings with involution and generalized polynomial identities*, J. Algebra **22** (1972), 502–516.
[23] F. Martino, C. Rizzo, *Multipliers, W-algebras and the growth of generalized polynomial identities*, Bull. Lond. Math. Soc. (2026), accepted.
[24] F. Martino, C. Rizzo, *The 2×2 upper triangular matrix algebra and its generalized polynomial identities*, J. Algebra **666** (2025), 308–330.
[25] S. Mishchenko, A. Valenti, *A star-variety with almost polynomial growth*, J. Algebra **223** (2000), no. 1, 66–84.
[26] C. Rizzo, *Differential codimensions and exponential growth*, Linear Algebra Appl. **675** (2023), 294–311.
[27] C. Rizzo, *On W-algebras and their polynomial identities: recent developments and perspectives*, Polynomial Identities and Related Topics, Springer INdAM Series, (2026) accepted.
[28] L. H. Rowen, *Generalized polynomial identities*, J. Algebra **34** (1975), 458–480.
[29] L. H. Rowen, *General polynomial identities. II*, J. Algebra **38** (1976), no. 2, 380–392.
[30] L. H. Rowen, *Generalized polynomial identities, III*, J. Algebra **46** (1977), no. 2, 305–314.
[31] L. H. Rowen, *Polynomial Identities in Ring Theory*, Pure and Applied Mathematics, vol. 84, Academic Press, New York, 1980.

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