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ON W-ALGEBRAS AND THEIR POLYNOMIAL IDENTITIES: RECENT DEVELOPMENTS AND PERSPECTIVES

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Abstract. This paper surveys recent results on W-polynomial identities and presents some new contributions to the theory of W-algebras. Using the multiplier algebra as a fundamental tool, we develop a general and self-contained framework for W-actions and their structural properties. We investigate the asymptotic behavior of the W-codimension sequence, proving the existence of the W-exponent for finite-dimensional (not necessarily unital) W-algebras and showing that it coincides with the ordinary PI-exponent. We then characterize the generalized varieties of almost polynomial growth generated by finite-dimensional W-algebras. Furthermore, we present explicit examples of W-polynomial identities for the algebra $M_k(F)$ of $k \times k$ matrices, the algebra UT_2 of 2×2 upper triangular matrices, and the infinite-dimensional Grassmann algebra G.

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1. Introduction

Let A and W be associative algebras over a field F of characteristic zero. We say that A is a W-associative algebra, or simply a W-algebra, if it is endowed with a W-bimodule structure satisfying some additional W-associative conditions. This notion recovers several familiar situations. For example, A is naturally an A-algebra when the bimodule structure is given by left and right multiplication. Moreover, when $W = F$, the definition reduces to the classical notion of an associative algebra over F.

Within the framework of associative algebras equipped with additional structure, W-algebras emerge as a particularly natural choice. Indeed, studying W-algebras is equivalent to considering internal object actions in the category of associative algebras. The notion of internal object action was introduced in category theory by Borceux, Janelidze, and Kelly [4] as a unifying categorical framework for various classical algebraic actions, such as the action of a group G on another group H, or the action of a Lie algebra $\mathfrak{g}$ on another Lie algebra $\mathfrak{h}$.

Let $W\langle X\rangle$ denote the free W-algebra generated by a countable set $X = \{x_1, x_2, \ldots\}$. Its elements, called W-polynomials, may be viewed informally as non-commutative polynomials in the variables from X, where elements of W play the role of some sort of coefficients that are allowed to appear also between variables. A W-polynomial is an identity of a W-algebra A if it vanishes under all substitutions of its variables by elements of A. The notion W-polynomial identities naturally extends the concept of generalized polynomial identities, introduced by Amitsur in 1965 [1] and subsequently developed by several authors (see [3] and the references therein). In the unital case, the two notions coincide, and for this reason, W-polynomial identities are often referred to as generalized polynomial identities.

In this paper, we focus on W-polynomial identities, placing particular emphasis on the asymptotic behavior of their associated W-codimension sequences.

The codimension sequence $c_n(A)$ was introduced by Regev in [28] as the dimension of the space of multilinear polynomials of degree n modulo the polynomial identities of A. Since the exact computation of the codimensions of a given algebra is a highly non-trivial problem and known only in few cases, research has naturally focused on their asymptotic behavior. In 1999, Giambruno and Zaicev [14] proved that if A is an algebra satisfying a non-trivial polynomial identity, then the limit $\exp(A) := \lim_{n\to\infty} \sqrt[n]{c_n(A)}$ exists and is always a non-negative integer, known as the PI-exponent of A. Since then, the study of exponents has been extensively developed also in the broader context of algebras endowed with additional structure.

Similarly to the classical setting, to every W-algebra A one can associate the sequence of W-codimension $c_n^W(A)$. In [25], the authors determined the exponential growth rate of $c_n^W(A)$ for finite-dimensional

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unital W-algebras, confirming Amitsur's conjecture in this context and introducing the notion of the W-exponent of A. They also showed that the W-exponent coincides with the ordinary PI-exponent of A. Since, in the unital case, W-polynomial identities correspond to generalized polynomial identities, these results naturally extend [16, Theorem 3]. In the present paper, we extend these results to the broader class of finite-dimensional W-algebras not necessarily unital.

It is worth emphasizing that the exponent of an algebra with additional structure generally differs from its ordinary PI-exponent, as seen in algebras with involution [11] or those graded by a finite abelian group [31]. To date, the only other known case where the two exponents coincide is that of algebras equipped with an action of a Lie algebra by derivations [30].

One of the main advantages of the exponent is that it provides an integer scale for measuring the growth of any non-trivial variety of algebras. This makes it natural to investigate varieties with the same exponent and to identify those exhibiting particularly distinguished properties. In this context, a celebrated theorem of Kemer (see [20]) states that $\exp(A) \leq 1$ if and only if G and UT_2 do not belong to the variety generated by A, where G is the infinite-dimensional Grassmann algebra and UT_2 the algebra of 2×2 upper triangular matrices. Consequently, the varieties generated by G and UT_2 are precisely those of almost polynomial growth, that is, varieties whose codimension sequences are not polynomially bounded, while every proper subvariety has polynomially bounded codimensions.

Motivated by these developments, the authors of [24] studied the W-variety generated by UT_2 under different W-actions, showing that this variety has almost polynomial growth depending on the chosen W-action. As in the case of algebras with a Lie algebra acting by derivations [12, 29], this highlights the strong influence of the W-action on the asymptotic behavior of W-codimensions.

Subsequently, in [25], they extended this analysis to the W-variety $\mathrm{var}^W(M_k)$ generated by $M_k(F)$ with the natural action W of $M_k(F)$ by left and right multiplication, thereby extending [3, Proposition 6.5]. They also proved a notable and somewhat unexpected PI result: unlike the ordinary case, the W-variety $\mathrm{var}^W(M_k)$ has almost polynomial growth for all $k \geq 2$. The only other known example of such behavior for full matrix algebras occurs for those with a Lie algebra action by derivations [5].

In the same work, using the multiplier algebra to avoid fixing W *a priori*, they classified all W-varieties of almost polynomial growth generated by finite-dimensional algebras.

In the infinite-dimensional setting, they determined the W-polynomial identities of the Grassmann algebra G under various W-actions, providing an example of a W-algebra with infinite W-codimensions.

This paper aims to provide a survey of the aforementioned recent results on W-polynomial identities and their codimension growth, while also presenting some new contributions. In particular, we develop a comprehensive, self-contained, and as general as possible theory of W-associative algebras.

For completeness, we mention that [2, 6] further extend the theory of W-polynomial identities by introducing the W-cocharacter theory and W-associative algebras graded by a finite group, respectively.

The paper is organized as follows. Section 2 introduces the multiplier algebra $\mathcal{M}(A)$ and its main properties, which constitute the primary tool in the study of W-associative algebras. Section 3 is devoted to W-associative algebras, establishing in particular the duality between W-actions and $\mathcal{M}(A)$-actions satisfying an additional compatibility condition, and presenting some of the fundamental structural properties of this class of algebras. In Section 4, we recall the necessary background on W-polynomial identities and prove the existence of the W-exponent, showing that it coincides with the ordinary PI-exponent for finite-dimensional (not necessarily unital) W-algebras. Section 5 surveys the results of [24] and [25] on the W-varieties generated by UT_2 and $M_k(F)$, extending the analysis to all W-actions. It also presents a classification of W-varieties of almost polynomial growth generated by finite-dimensional algebras, described in particular in terms of the W-exponent. Finally, Section 6 deals with the W-polynomial identities of the infinite-dimensional Grassmann algebra and their asymptotic growth, providing in particular a counterexample to the Specht property in characteristic zero.

Throughout this paper, F denotes a field of characteristic zero, and all algebras are assumed to be associative F-algebras.

2. Multiplier algebra

The notion of a multiplier algebra of an associative algebra goes back to Hochschild [17] in his work on cohomology and extensions of associative algebras, to Mac Lane [22] in his research on extension of rings, and to Johnson [19] in his study of centralizers in topological algebra. Dauns initiated the first pure algebraic investigation of this notion in [8]. Moreover, Busby in [7] applied the multiplier algebra in the study of extension of C^*-algebras, which are particular Banach algebras with an involution, and afterwards it was deeply used in functional analysis.

In this section, we will recall the construction of a multiplier algebra and introduce the necessary results and notation needed in the following.

Let A be an F-algebra. We denote by $\mathrm{End}_F(A)$ the algebra of F-module endomorphisms of A acting on the left. The multiplication in $\mathrm{End}_F(A)$ is given by the usual composition of endomorphisms, written by juxtaposition. We further denote by $\mathrm{End}_F(A)^{\mathrm{op}}$ the *opposite algebra* of $\mathrm{End}_F(A)$, namely the vector space $\mathrm{End}_F(A)$ endowed with the opposite product $\cdot^{\mathrm{op}}$ defined by $\psi_1 \cdot^{\mathrm{op}} \psi_2 := \psi_2\psi_1$ for all $\psi_1, \psi_2 \in \mathrm{End}_F(A)$. With this multiplication, $\mathrm{End}_F(A)^{\mathrm{op}}$ is the algebra of F-module endomorphisms of A acting on the right. A pair $(R, L) \in \mathrm{End}_F(A)^{\mathrm{op}} \times \mathrm{End}_F(A)$ is a *bi-multiplier*, or simply a *multiplier*, of A if for all $a, b \in A$,

$$R(ab) = aR(b), \quad L(ab) = L(a)b, \quad R(a)b = aL(b). \tag{2.1}$$

We refer to R and L as a *right* and a *left multiplier* of A, respectively.

Clearly, the set

$$\mathcal{M}(A) = \{(R, L) \in End_F(A)^{\mathrm{op}} \times \mathrm{End}_F(A) \ : \ (R, L) \text{ is a multiplier of } A\}$$

is a unital F-algebra under componentwise addition and multiplication, with unit element $(\mathrm{id}_A, \mathrm{id}_A)$, which is called *multiplier algebra* of A.

For $m \in A$, we denote by R_m and L_m the endomorphism of A defined by left and right multiplication by m, respectively, namely,

$$R_m(a) := am, \qquad L_m(a) := ma,$$

for all $a \in A$. The associativity of A implies that the pair (R_m, L_m) is a multiplier of A, which is referred to as an *inner multiplier* of A induced by m.

Define the map

$$\begin{array}{rccc} \mu : & A & \to & \mathcal{M}(A) \\ & m & \mapsto & (R_m, L_m) \end{array}. \tag{2.2}$$

It is immediate that μ is a homomorphism of F-algebras. Moreover, the kernel of μ coincides with the annihilator of A, that is,

$$\ker \mu = \{a \in A \ : \ ab = ba = 0 \text{ for all } b \in A\}.$$

Furthermore, since for any $m \in A$ and any $(R, L) \in \mathcal{M}(A)$ one has

$$(R_m, L_m)(R, L) = (R_{R(m)}, L_{R(m)}) \quad \text{ and } \quad (R, L)(R_m, L_m) = (R_{L(m)}, L_{L(m)}),$$

the image $\mathcal{IM}(A) := \mu(A)$ of μ is a two-sided ideal of $\mathcal{M}(A)$, called the *inner multiplier ideal* of A.

Example 2.1. Let $A = Fe_{11} \oplus Fe_{12}$, where the e_{ij}'s are the usual matrix units. Clearly, the inner multiplier ideal $\mathcal{IM}(A)$ of A is two-dimensional, with vector space basis

$$\mathcal{B}_{\mathcal{IM}(A)} := \{(R_{e_{11}}, L_{e_{11}}), (R_{e_{12}}, L_{e_{12}})\}.$$

Note also that $(\mathrm{id}_A, \mathrm{id}_A) \in \mathcal{M}(A)$ is not an inner multiplier. Indeed, suppose there exist scalars $\alpha, \beta \in F$ such that $(\mathrm{id}_A, \mathrm{id}_A) = (\alpha R_{e_{11}} + \beta R_{e_{12}}, \alpha L_{e_{11}} + \beta L_{e_{12}})$. Applying the right multiplier to e_{12}, we obtain $e_{12} = \alpha e_{12}e_{11} + \beta e_{12}e_{12} = 0$, which is a contradiction. Hence

$$(\mathrm{id}_A, \mathrm{id}_A) \notin \mathcal{IM}(A).$$

We now determine the multiplier algebra $\mathcal{M}(A)$. Let $(R, L) \in \mathcal{M}(A)$. Since $L(e_{11}) = L(e_{11}e_{11}) = L(e_{11})e_{11}$, there exists $\alpha \in F$ such that $L(e_{11}) = \alpha e_{11}$. As a consequence, since $L(e_{12}) = L(e_{11}e_{12}) = L(e_{11})e_{12}$, we obtain $L(e_{12}) = \alpha e_{12}$. Moreover, since $R(e_{12})e_{11} = e_{12}L(e_{11}) = 0$, it follows that $R(e_{12}) = \beta e_{12}$, for some $\beta \in F$. Finally, from $R(e_{11})e_{11} = e_{11}L(e_{11}) = \alpha e_{11}$, we deduce that $R(e_{11}) = \alpha e_{11} + \gamma e_{12}$, for some $\gamma \in F$. Thus every multiplier (R, L) of A is determined by three scalars α, β, γ and therefore $\dim_F \mathcal{M}(A) \leq 3$. Thus, since the elements $(\mathrm{id}_A, \mathrm{id}_A), (R_{e_{11}}, L_{e_{11}}), (R_{e_{12}}, L_{e_{12}}) \in \mathcal{M}(A)$ are linearly independent, they form a basis of $\mathcal{M}(A)$. In particular,

$$\mathcal{M}(A) = \mathrm{span}_F\{(\mathrm{id}_A, \mathrm{id}_A),\ (R_{e_{11}}, L_{e_{11}}),\ (R_{e_{12}}, L_{e_{12}})\}.$$

From the above example, we can deduce that, in general, not all multipliers are inner. Nevertheless, for unital F-algebras we have the following.

Proposition 2.2. *$A \cong \mathcal{M}(A) = \mathcal{IM}(A)$ if and only if A has unity.*

Proof. It is clear that if $A \cong \mathcal{M}(A)$, then A must be unital.

Conversely, suppose that A has unity 1_A. Let $\mu : A \to \mathcal{M}(A)$ be the homomorphism defined in (2.2). Then $\mu(1_A) = (\mathrm{id}_A, \mathrm{id}_A)$. Therefore, $\mu(A)$ contains the identity of $\mathcal{M}(A)$, and since $\mathcal{IM}(A) = \mu(A)$ is a two-sided ideal of $\mathcal{M}(A)$, it follows that $\mathcal{IM}(A) = \mathcal{M}(A)$. Moreover, if $a \in \ker \mu$, then in particular $a = a1_A = 0$, showing that μ is injective. Hence, μ is an isomorphism, and $A \cong \mathcal{M}(A) = \mathcal{IM}(A)$ as desired. □

A right multiplier R and a left multiplier L of A are *permutable* if

$$RL = LR. \tag{2.3}$$

We say that a subalgebra M of the multiplier algebra $\mathcal{M}(A)$ is *permutable* if any two elements of M are permutable, that is, $R_1L_2 = L_2R_1$ for all $(R_1, L_1), (R_2, L_2) \in M$.

Example 2.3. Given an F-algebra A, every multiplier permutes with any inner multiplier. Indeed, given $(R, L) \in \mathcal{M}(A)$ and $(R_m, L_m) \in \mathcal{IM}(A)$, by (2.1) for all $a \in A$, one has

$$R(L_m(a)) = R(ma) = mR(a) = L_m(R(a)) \quad \text{and} \quad L(R_m(a)) = L(am) = L(a)m = R_m(L(a)).$$

As a consequence, $\mathcal{IM}(A)$ is permutable.

Example 2.4. The multiplier algebra $\mathcal{M}(A)$ of a unital F-algebra is permutable. Indeed, Proposition 2.2 shows that $\mathcal{M}(A) = \mathcal{IM}(A)$ in the unital case, and by the above example the conclusion follows.

Unital algebras are not the only type of algebras for which the multiplier algebra satisfies the permutability property.

Example 2.5. The F-algebra $A = Fe_{11} \oplus Fe_{12}$ considered in Example 2.1 is non-unital. Nevertheless, the associated multiplier algebra is clearly permutable.

In general, not all multipliers of a non-unital algebra are permutable, as shown by the following example.

Example 2.6. Let $A = Fe_{12} \oplus Fe_{13} \oplus Fe_{23}$, where the e_{ij}'s are the usual matrix units. Define a pair $(R, L) \in End_F(A)^{\mathrm{op}} \times \mathrm{End}_F(A)$ by

$$R(e_{12}) = e_{12}, \quad R(e_{13}) = 0, \quad R(e_{23}) = e_{13},$$
$$L(e_{12}) = e_{13}, \quad L(e_{13}) = 0, \quad L(e_{23}) = e_{23}.$$

A straightforward computation shows that $(R, L) \in \mathcal{M}(A)$. However, R and L are not permutable. Indeed, $R(L(e_{12})) = R(e_{13}) = 0$, whereas $L(R(e_{12})) = L(e_{12}) = e_{13}$. Moreover, as a consequence and by Example 2.3, $(R, L) \notin \mathcal{IM}(A)$.

Recall that A is said *non-degenerate algebra* if its multiplication is non-degenerate as a bilinear form. This means that there is no nonzero element $a \in A$ such that $ab = ba = 0$, for all $b \in A$, or equivalently the homomorphism μ defined in (2.2) is injective.

Example 2.7. Every unital F-algebra is non-degenerate.

Example 2.8. The F-algebra $A = Fe_{11} \oplus Fe_{12}$ considered in Example 2.1 is non-degenerate.

Example 2.9. The F-algebra $A = Fe_{12} \oplus Fe_{13} \oplus Fe_{23}$ of the Example 2.6 is not non-degenerate.

Proposition 2.10 [10, Proposition 7.9]**.** *Let A be either non-degenerate or idempotent, i.e., $A^2 = A$. Then $\mathcal{M}(A)$ is permutable.*

3. W-algebras

In this section we introduce the main object of study of the paper and describe its relationship with multiplier algebras. We also collect several properties that will be used later.

Now fix an F-algebra W. Recall that a F-algebra A is a W-bimodule if it is endowed with both right and left W-module structures with the compatibility relation, for all $w_1, w_2 \in W$, $a \in A$,

$$w_1(aw_2) = (w_1a)w_2. \tag{3.1}$$

Moreover, if W has a unity element 1_W, we further assume that $1_Wa = a1_W = a$ for all $a \in A$.

Notice that if A is a right (resp. left) W-module, the corresponding action induces an F-algebra homomorphism

$$\rho : W \to \mathrm{End}_F(A)^{\mathrm{op}} \qquad \big(\text{resp. } \lambda : W \to \mathrm{End}_F(A)\big),$$

where for all $w \in W$ and $a \in A$,

$$\rho(w)(a) = aw \qquad \big(\text{resp. } \lambda(w)(a) = wa\big).$$

Such a homomorphism is called the *right* (resp. *left*) *representation* of W on A. Conversely, any F-algebra homomorphism $\rho : W \to \mathrm{End}_F(A)^{\mathrm{op}}$ (resp. $\lambda : W \to \mathrm{End}_F(A)$) uniquely determines a right (resp. left) W-module structure on A.

Remark also that if A is a W-bimodule, then the compatibility condition (3.1) implies that the right and left representations commute, namely $\rho(w_1)\lambda(w_2) = \lambda(w_2)\rho(w_1)$ for all $w_1, w_2 \in W$.

Definition 3.1. We say that A is a *W-associative algebra*, or simply *W-algebra*, if A is a W-bimodule satisfying the following associative relations: for all $w \in W$, $a_1, a_2 \in A$,

$$w(a_1a_2) = (wa_1)a_2, \quad (a_1a_2)w = a_1(a_2w), \quad (a_1w)a_2 = a_1(wa_2). \tag{3.2}$$

Example 3.2. If $W \cong F$, then A is just an F-algebra.

Example 3.3. The F-algebra W has itself the structure of W-algebra by taking the left and right W-actions as the usual left and right multiplications of W.

Example 3.4. Let A be an F-algebra and $\psi : W \to A$ a homomorphism. Then A inherits a W-algebra structure by defining the left and right W-actions via composition with ψ and the usual right and left multiplication in A.

Example 3.5. Let A be F-algebra, and M be a permutable subalgebra of the multiplier algebra $\mathcal{M}(A)$. We define the following right and left action of M on A: for any $(R, L) \in M$ and $a \in A$

$$a(R, L) = R(a) \quad \text{and} \quad (R, L)a = L(a).$$

Thus, these actions endow A with an M-algebra structure. Indeed, since $(R, L) \in \mathrm{End}_F(A)^{\mathrm{op}} \times \mathrm{End}_F(A)$, A is both a right and a left M-module. Also, since M is permutable, the compatibility relation (3.1) is satisfied, and consequently A is an M-bimodule. Finally, from the condition (2.1) follows that A satisfies the associative relations (3.2), thus A is actually an M-algebra.

Note that this is not generally true if we consider instead of M the full multiplier algebra $\mathcal{M}(A)$ because there may be a multiplier of A that does not satisfy the condition (2.3) of permutability (see Example 2.6). However, if, for instance, A is either non-degenerate or idempotent, then by Proposition 2.10 A is an $\mathcal{M}(A)$-algebra.

Let A be a W-algebra. Since A is both a left and a right W-module, it affords the corresponding right and left representations ρ and λ of W on A. By (3.2), for every $w \in W$ the pair $\big(\rho(w), \lambda(w)\big)$ belongs to $\mathcal{M}(A)$. Thus, defining

$$\begin{array}{rccc} \Phi: & W & \longrightarrow & \mathcal{M}(A) \\ & w & \longmapsto & \big(\rho(w), \lambda(w)\big), \end{array}$$

we obtain an F-algebra homomorphism which, by (3.1), satisfies

$$\rho(w_2)\lambda(w_1) = \lambda(w_1)\rho(w_2) \quad \text{for all } w_1, w_2 \in W. \tag{3.3}$$

In general, not all homomorphisms from W to $\mathcal{M}(A)$ define a W-algebra structure on A; indeed, there may exist homomorphisms from W to $\mathcal{M}(A)$ that do not satisfy condition (3.3), since generally not all multipliers of A satisfy the permutability condition (2.3) (see Example 2.6). However, if we consider a homomorphism $\Phi : W \to \mathcal{M}(A)$ such that the image $\Phi(W) \subseteq \mathcal{M}(A)$ is pemutable, then, as shown in Example 3.5, the algebra A inherits a natural structure of W-algebra by setting, for all $w \in W$, $a \in A$,

$$aw := a\Phi(w) \quad \text{and} \quad wa := \Phi(w)a.$$

Thus, giving a W-algebra structure on A is equivalent to giving a homomorphism $\Phi\colon W \to \mathcal{M}(A)$ whose image is a permutable subalgebra of $\mathcal{M}(A)$. Such a homomorphism is called the *acting homomorphism* of W on A. We denote its image by $\overline{W} = \Phi(W) \subseteq \mathcal{M}(A)$.

Note that the action of W on A induces a natural action of the subalgebra $\overline{W}$ of $\mathcal{M}(A)$ on A, so that $a\Phi(w) = aw$ and $\Phi(w)a = wa$ for all $w \in W$ and $a \in A$. Hence, A may equivalently be viewed as a $\overline{W}$-algebra. Since the action of W on A factors through the image $\overline{W}$, we describe, throughout the paper, a W-action on A via the corresponding image $\overline{W}$. In this sense, we say that W acts on A as the algebra $\overline{W}$.

The algebra A may carry distinct W-algebra structures arising from different acting homomorphisms Φ, and consequently from different images $\overline{W} \subseteq \mathcal{M}(A)$. When it is necessary to specify the chosen action, we denote the W-algebra A by A^{Φ} or equivalently by $A^{\Phi(W)} = A^{\overline{W}}$.

Now, we state the following result on the structure of unital W-algebras. Its proof is an immediate consequence of the previous arguments, Proposition 2.2, Example 2.4, and the observation that, when W is a unital algebra, Φ is a homomorphism of unital algebras, in particular satisfying $\Phi(1_W) = (\mathrm{id}_A, \mathrm{id}_A)$.

Proposition 3.6. *Let W be an F-algebra. If A is a W-algebra with unity 1_A, then the action of W on A is equivalent to the action of a suitable subalgebra B of A by left and right multiplication. Moreover, if W is a unital algebra, then $1_A \in B$.*

Given that the above proposition fully characterizes unital W-algebras, we now examine how non-unital W-algebras relate to unital ones. To this end, we first recall that a subalgebra (respectively, an ideal) B of a W-algebra A is called a *W-subalgebra* (respectively, a *W-ideal*) if it is invariant under the action of W; that is,

$$WB \subseteq B \quad \text{and} \quad BW \subseteq B$$

where

$$WB = \{wb \ : \ w \in W,\ b \in B\} \quad \text{and} \quad BW = \{bw \ : \ b \in B,\ w \in W\}.$$

Given a (not necessarily unital) W-algebra A, we define a multiplication on the direct sum of vector spaces $W \oplus A$ by

$$(w_1, a_1)(w_2, a_2) := (w_1 w_2, w_1 a_2 + a_1 w_2 + a_1 a_2).$$

We call the vector space $W \oplus A$, endowed with this multiplication, the *semi-direct product* of W and A, and we denote it by $W \ltimes A$.

This construction gives rise to the following diagram:

$$0 \longrightarrow A \xrightarrow{\iota_2} W \ltimes A \underset{\iota_1}{\overset{\pi_1}{\rightleftarrows}} W \longrightarrow 0 \tag{3.4}$$

where $\pi_1(w, a) = w$, $\iota_1(w) = (w, 0)$ and $\iota_2(a) = (0, a)$, respectively. Since $\pi_1 \circ \iota_1 = \mathrm{id}_W$ and $\iota_2(A) \subseteq \ker \pi_1$, the diagram (3.4) defines a split extension of F-algebras. Conversely, every split extension arises in this way. Hence, the notions of split extensions of associative F-algebras and W-associative algebras are equivalent. For a detailed discussion of this equivalence, see [4, 18].

The F-algebra $W \ltimes A$ naturally inherits a W-algebra structure from those of W and A. Moreover, under the homomorphism ι_2 in (3.4), the subalgebra $\iota_2(A)$ is a W-subalgebra of $W \ltimes A$ that is isomorphic to A as W-algebra. Likewise, the subalgebra $\iota_1(W)$ is a W-subalgebra of $W \ltimes A$ that is isomorphic to W as W-algebra. If W is a unital, then $W \ltimes A$ is also a unital, with unit $(1_W, 0)$.

These observations lead to the following result, which relates (not necessarily unital) W-algerbas to unital W-algebras whose structure is described by the semi-direct product construction.

Proposition 3.7. *Let A be a W-algebra. Then the semi-direct product $W \ltimes A$ is a W-algebra containing W-subalgebras canonically isomorphic to A and W. Moreover, if W is a unital, then $W \ltimes A$ is unital as well.*

In the remainder of this section, we restrict our attention to finite-dimensional W-algebras. From now on, let A denote a finite-dimensional algebra over F, and let $J = J(A)$ be its Jacobson radical. Recall that A is *semisimple* if and only if $J = 0$, and *simple* if it has no nontrivial ideals and $A^2 \neq 0$.

Since the base field F has characteristic 0, the Wedderburn–Malcev Theorem for associative algebras (see [15, Theorem 3.4.3]) guarantees the existence of a maximal semisimple subalgebra $B \subseteq A$, unique up to isomorphism, such that

$$A = B \oplus J \tag{3.5}$$

as a direct sum of vector spaces.

Notice that the Wedderburn-Malcev decomposition may not be W-invariant. In fact, for instance, if $J \neq 0$ and W acts on A via left and right multiplication by elements of J, then $WB + BW \subseteq J$, showing that B is not W-invariant. Nevertheless, the following result holds.

Theorem 3.8 [25, Theorem 3.3]**.** *Let W be an algebra over a field F of characteristic zero and A be a finite dimensional W-algebra. Then the Jacobson radical J of A is a W-ideal of A. Moreover, if F is algebraically closed, then $A = B_1 \oplus \cdots \oplus B_k + J$, where B_i is a simple algebra such that $WB_i, B_iW \subseteq B_i + J$ for all $1 \leq i \leq k$.*

As a consequence and by Proposition 3.6, we get the following characterization of W-simple algebras.

Corollary 3.9 [25, Corolary 3.5]**.** *If A is a finite dimensional W-simple algebra over an algebraically closed field F of characteristic 0, then A is isomorphic to the W-algebra $M_k(F)$ of matrices of order k over F for some $k \geq 1$ with the W-action given by left and right multiplication of an appropriate subalgebra $B \subseteq M_k(F)$ containing $1_{M_k(F)}$.*

4. W-identities and their codimensions growth

In this section, we first construct the free W-algebra. Then, we define the W-codimension sequence and present some fundamental results on the growth of such a sequence.

Let from now on W be a fixed unital F-algebra. Since the class of W-algebras is equational, it forms a variety in the sense of universal algebra. This variety is non-trivial, as it contains W itself. A homomorphism $\varphi : A \to B$ between W-algebras A e B is required to satisfy $\varphi(wav) = w\varphi(a)v$ for all $a \in A$, $w, v \in W$, and is called *W-homomorphism*.

The variety of W-algebras contains the *free (non-unital associative) W-algebra* $W\langle X\rangle$ freely generated by the countably infinite set of variables $X := \{x_1, x_2, \ldots\}$, i.e., $W\langle X\rangle$ is uniquely determined up to an isomorphism by the following universal property: given a W-algebra A, any map $\varphi : X \to A$ can be uniquely extended to a W-homomorphism $\bar{\varphi} : W\langle X\rangle \to A$, which we call *evaluation* of $W\langle X\rangle$ in A.

Notice that an evaluation $\bar{\varphi}$ of $W\langle X\rangle$ in the W-algebra A depends not only on the map φ but also on the right and left actions of W on A, and consequently on on the acting homomorphism Φ of W on A. Thus, since A can be a W-algebra with respect to different acting homomorphisms, we denote the evaluation $\bar{\varphi}$ by $\bar{\varphi}_\Phi$, or simply φ_Φ, when necessary to clarify which action we are considering.

We can construct $W\langle X\rangle$ as follows: given the free vector space V_X spanned by X, we consider the tensor product $V_X \otimes W$ and its corresponding non-unital tensor algebra, i.e., $T(V_X \otimes W) = \bigoplus_{n=1}^{\infty} \big(V_X \otimes W\big)^{\otimes n}$, then the free W-algebra is the tensor product

$$W\langle X\rangle = W \otimes T(V_X \otimes W) \cong \bigoplus_{n=1}^{\infty} W \otimes \big(V_X \otimes W\big)^{\otimes n}, \tag{4.1}$$

where W acts on it by right and left multiplication on the corresponding right or left component W, respectively. Notice that the algebra W is not contained in the free algebra $W\langle X\rangle$ as in the summation (4.1) the index begins at $n = 1$.

From now on, we write $wxv := w \otimes x \otimes v$ and $x := 1_W \otimes x \otimes 1_W$ for $x \in X$ and $w, v, 1_W \in W$. Then, given a vector space basis $\mathcal{B}_W := \{w_i : i \geq 0\}$ of W, a vector space basis of $W\langle X\rangle$ is the following

$$\mathcal{B}_{W\langle X\rangle} := \left\{ w_{i_0} x_{j_1} w_{i_1} x_{j_2} \cdots w_{i_{n-1}} x_{j_n} w_{i_n} \ : \ j_1, \ldots, j_n \geq 1,\ w_{i_0}, \ldots, w_{i_n} \in \mathcal{B}_W,\ n \geq 1 \right\}.$$

Indeed, if $\mathcal{B}_W$ contains 1_W, then it is immediate that between any two variables there always appears an element of the basis. Otherwise, we can express 1_W as a linear combination of some elements of $\mathcal{B}_W$, say $1_W = \alpha_1 w_{i_1} + \cdots + \alpha_s w_{i_s}$. It follows that any monomial containing two or more consecutive variables can be rewritten as a linear combination of monomials in which an element of $\mathcal{B}_W$ is inserted between each pair of adjacent variables. For example,

$$w_j x_1 x_2 w_k = w_j x_1 1_W x_2 w_k = w_j x_1 (\alpha_1 w_{i_1} + \cdots + \alpha_s w_{i_s}) x_2 w_k = \alpha_1 w_j x_1 w_{i_1} x_2 w_k + \cdots + \alpha_s w_j x_1 w_{i_s} x_2 w_k.$$

The elements of $W\langle X\rangle$ are called *W-polynomials*, or *generalized polynomials* when there is no ambiguity about the role of W. Note that we consider polynomials without constant terms since in the summation (4.1) the index begins at $n = 1$.

A *T_W-ideal* of the free W-algebra is an W-ideal that is also invariant under all W-algebra endomorphisms of $W\langle X\rangle$, i.e., under the endomorphisms that we call *substitutions*, which send variables of $x_i \in X$ to elements of $W\langle X\rangle$. Given a set $S \subseteq W\langle X\rangle$, by $\langle S\rangle_{T_W}$ we denote the T_W-ideal generated by S, that is, the smallest T_W-ideal containing S.

Given a W-algebra A^Φ, a W-polynomial $f(x_1, \ldots, x_n) \in W\langle X\rangle$ is a *W-identity* of A, or *generalized identity* when the role of W is clear, and we write $f \equiv 0$, if $f(a_1, \ldots, a_n) = 0$ for any $a_1, \ldots, a_n \in A$. We denote by $\mathrm{Id}^W(A)$ the set of generalized identities of A, which is a T_W-ideal of the free W-algebra $W\langle X\rangle$, called T_W-ideal of A. Note that $\mathrm{Id}^W(A)$ is the intersection of all kernels of evaluations of $W\langle X\rangle$ from A.

Consider the T_W-ideal

$$K_\Phi := \langle wx,\ xw \ : \ w \in \ker \Phi\rangle_{T_W}.$$

It is contained in the T_W-ideal $\mathrm{Id}^W(A)$ of A, and its elements are called *Φ-trivial* W-polynomials.

Observe that the subalgebra $\overline{W} = \Phi(W)$ of $\mathcal{M}(A)$ carries a natural structure of W-algebra given by left and right multiplication. More precisely, for all $w \in W$ and $\bar{w} \in \overline{W}$, we define

$$w\bar{w} := \Phi(w)\bar{w} \quad \text{and} \quad \bar{w}w := \bar{w}\Phi(w). \tag{4.2}$$

As a consequence, the free $\overline{W}$-algebra $\overline{W}\langle X\rangle$ inherits a natural structure of W-algebra. Moreover, since A^Φ is also a $\overline{W}$-algebra, then we may consider the $T_{\overline{W}}$-ideal $\mathrm{Id}^{\overline{W}}(A)$ consisting of all $\overline{W}$-identities of A. The following result relates the W-identities of A with its $\overline{W}$-identities.

Proposition 4.1. *Let A^Φ be a W-algebra. Then* $\dfrac{\mathrm{Id}^W(A)}{K_\Phi} \cong_W \mathrm{Id}^{\overline{W}}(A)$.

Proof. Since $W\langle X\rangle$ and $\overline{W}\langle X\rangle$ are both W-algebras, we can define a W-homomorphism $\Psi : W\langle X\rangle \to \overline{W}\langle X\rangle$ by setting $\Psi(wxv) = \Phi(w)x\Phi(v)$, for all $w, v \in W$ and $x \in X$, where, by abuse of notation, Φ also denotes its corestriction to the image $\overline{W}$. Because Φ is surjective, the map Ψ is also surjective. Moreover, it is straightforward to verify that $\ker\Psi = K_\Phi$. Hence $W\langle X\rangle/K_\Phi \cong_W \overline{W}\langle X\rangle$. By definition $\Psi(\mathrm{Id}^W(A)) = \mathrm{Id}^{\overline{W}}(A)$ and $K_\Phi \subseteq \mathrm{Id}^W(A)$. Hence by the isomorphism theorems we get $\mathrm{Id}^W(A)/K_\Phi \cong_W \mathrm{Id}^{\overline{W}}(A)$, as desired. □

For $n \geq 1$, we denote by P_n^W the vector space of *multilinear W-polynomials* in the variables $x_1, \ldots, x_n$, that is

$$P_n^W := \mathrm{span}_F\{w_{i_0}x_{\sigma(1)}w_{i_1}x_{\sigma(2)}\cdots w_{i_{n-1}}x_{\sigma(n)}w_{i_n} \ : \ \sigma \in S_n,\ w_{i_0}, \ldots, w_{i_n} \in \mathcal{B}_W\},$$

where S_n denotes the symmetric group acting on $\{1, \ldots, n\}$. We now have the following useful result.

Corollary 4.2. *Let A^Φ be a W-algebra. Then* $\mathrm{Id}^W(\overline{W}) \cap P_1^W = K_\Phi \cap P_1^W$.

Proof. Since K_Φ is a W-ideal of $\mathrm{Id}^W(\overline{W})$, we clearly have $K_\Phi \cap P_1^W \subseteq \mathrm{Id}^W(\overline{W}) \cap P_1^W$.

Conversely, let $f \in \mathrm{Id}^W(\overline{W}) \cap P_1^W$. From the definition of the W-homomorphism Ψ in the proof of Proposition 4.1 and (4.2), it follows that $\Psi(f) \in \mathrm{Id}^{\overline{W}}(\overline{W}) \cap P_1^{\overline{W}}$. By [24, Corollary 2.3], the polynomial $\Psi(f)$ is $\overline{W}$-trivial, that is, $\Psi(f) = 0$. Therefore $f \in \ker\Psi = K_\Phi$, as claimed. □

Given a W-algebra A^Φ. Since F has characteristic zero, a standard argument shows that the T_W-ideal $\mathrm{Id}^W(A)$ is completely determined by its multilinear generalized polynomials (see [15, Theorem 1.3.8]). Thus, we can consider the quotient space

$$P_n^W(A) := \frac{P_n^W}{P_n^W \cap \mathrm{Id}^W(A)},$$

and the nth *W-codimension*, or *generalized codimension*, of A is defined as

$$c_n^W(A) := \dim_F P_n^W(A), \quad n \geq 1.$$

Note that $c_n^W(A)$ is not necessarily finite, indeed we will show a concrete example of infinite generalized codimensions in Theorem 6.6. However, if at least one between $\Phi(W)$ and A is finite dimensional, then $c_n^W(A)$ is finite for all $n \geq 1$.

When $W \cong F$, we are dealing with the ordinary polynomial identities. In this case we write $P_n := P_n^F$, $\mathrm{Id}(A) := \mathrm{Id}^F(A)$ and $c_n(A) := c_n^F(A)$. More generally, since W has unity, F embeds naturally as a subalgebra of W and it follows that $P_n \subseteq P_n^W$. Consequently, for any W-algebra A, the ordinary codimensions are bounded above by the generalized codimensions. That is,

$$c_n(A) \leq c_n^W(A), \quad n \geq 1.$$

In [13,14], the authors proved that if A is a PI-algebra, i.e., if A is an algebra that satisfies a non-trivial identity, then the limit

$$\exp(A) := \lim_{n\to\infty} \sqrt[n]{c_n(A)}$$

exists and is an integer called *(ordinary) exponent of A*. In particular, if A is a finite dimensional F-algebra, then, after extending the base field to its algebraic closure, we have

$$\exp(A) = \max\{\dim_F \left(B_{i_1} \oplus B_{i_2} \oplus \cdots \oplus B_{i_r}\right) \ : \ B_{i_1}JB_{i_2}J\cdots JB_{i_r} \neq 0,\ 1 \leq r \leq k,\ i_p \neq i_s,\ 1 \leq p, s \leq r\},$$

where $A = B_1 \oplus \cdots \oplus B_k + J$ with $B_1, \ldots, B_k$ simple algebras and $J = J(A)$ the Jacobson radical.

Analogously, we define the generalized W-exponent of A by

$$\exp^W(A) := \lim_{n\to\infty} \sqrt[n]{c_n^W(A)},$$

and we wonder whether this limit exists. In [25], the authors determined the exponential growth of the generalized codimension sequence of a finite-dimensional unital W-algebra A. More precisely, they proved that $\exp^W(A)$ exists and $\exp^W(A) = \exp(A)$. Now, we generalize this result to arbitrary finite dimensional W-algebras, not necessarily unital.

Theorem 4.3. *Let W be a unital algebra over a field F of characteristic zero. If A is a finite dimensional W-algebra, then* $\exp^W(A)$ *exists and* $\exp^W(A) = \exp(A)$.

Proof. The claim follows immediately by viewing A as a G-graded W-algebra with trivial grading and applying [6, Theorem 4.5]. □

Corollary 4.4. *If A is a finite dimensional W-algebra, then the sequence of generalized codimensions $c_n^W(A)$, $n \geq 1$, either grows exponentially or is polynomially bounded.*

Since there is a one-to-one correspondence between T_W-ideals and varieties of W-algebras, it is often convenient to translate all the objects we have introduced to the language of varieties. A variety $\mathcal{V}$ of W-algebras generated by a W-algebra A is denoted by $\mathrm{var}^W(A)$ and is called *W-variety*, or *generalized variety*, and $\mathrm{Id}^W(\mathcal{V}) := \mathrm{Id}^W(A)$. The growth of $\mathcal{V} = \mathrm{var}^W(A)$ is the growth of the sequence $c_n^W(\mathcal{V}) := c_n^W(A)$, $n \leq 1$. We say that the generalized variety $\mathcal{V}$ has *polynomial growth* if $c_n^W(\mathcal{V})$ is polynomially bounded and $\mathcal{V}$ has *almost polynomial growth* if $c_n^W(\mathcal{V})$ is not polynomially bounded but every proper W-subvariety $\mathcal{U}$ of $\mathcal{V}$ has polynomial growth. Moreover, if we are dealing with the ordinary polynomial identities of A, i.e., $W \cong F$, then we write $\mathrm{var}(A) := \mathrm{var}^F(A)$.

Next, we present an interesting result on the W-variety generated by finite-dimensional unital W-algebras.

Let $A^{\overline{W}}$ be a finite-dimensional W-algebra. Since $\overline{W} \subseteq \mathcal{M}(A)$ is finite dimensional and F is of characteristic 0, the Wedderburn–Malcev decomposition for F-algebras yields and

$$\overline{W} = \overline{W}_{ss} \oplus \bar{J},$$

where $\overline{W}_{ss}$ is a semisimple subalgebra of $\overline{W}$ isomorphic to $\overline{W}/\bar{J}$ and $\bar{J} = J(\overline{W})$ is the Jacobson radical of $\overline{W}$. If $\pi\colon \overline{W} \to \overline{W}/\bar{J}$ denotes the canonical epimorphism, then, via the above identification, we may assume that $\pi\colon \overline{W} \to \overline{W}_{ss}$.

Define a map $\Phi_{ss}\colon W \to \mathcal{M}(A)$ by $\Phi_{ss} := \pi\Phi$. Then Φ_{ss} is a homomorphism of W-algebras satisfying $\Phi_{ss}(W) = \overline{W}_{ss}$ and therefore determines another W-algebra structure on A, denote by $A^{\overline{W}_{ss}}$.

We now have the following result.

Theorem 4.5 [25, Theorem 4.3]**.** *Let W be a unital algebra over a field F of characteristic* 0*, and let $A^{\overline{W}}$ be a finite-dimensional unital W-algebra. Denote by J the Jacobson radical of A. If the Jacobson radical $\bar{J}$ of $\overline{W}$ satisfies $\bar{J} \subseteq \mathcal{IM}(J)$, then*

$$A^{\overline{W}_{ss}} \in \mathrm{var}^W(A^{\overline{W}}),$$

where $\overline{W}_{ss} \cong \overline{W}/\bar{J}$.

We conclude this section by clarifying the relationship between the classical notion of generalized polynomial identities and the W-identities introduced in this paper.

Historically, generalized polynomials were defined as elements of the free product $W *_F F\langle X\rangle$ that is, the coproduct in the category of associative F-algebras of the F-algebra W and the classical free algebra $F\langle X\rangle$. More precisely, the free product $W *_F F\langle X\rangle$ is the F-algebra uniquely characterized by the following universal property: given any F–algebra A, a homomorphism from $W *_F F\langle X\rangle$ to A is uniquely determined by assigning arbitrarily a homomorphism from W to A together with a map from X to A. Accordingly, if one considers the category of *W-generalized algebras*, namely F-algebras A endowed with a homomorphism $\iota : W \to A$, then, in this category, the free product $W *_F F\langle X\rangle$ plays the role of the free object.

Generalized polynomial identities have been studied primarily in the case $W = A$ and $\iota = \mathrm{id}_W$. This focus stems from the original motivation for their introduction: in the study of polynomial identities of matrix algebras, one is often interested in evaluations where certain elements are fixed in advance.

Notice that, in the unital case, the notions of W-associative algebras and W-generalized algebras coincide. Indeed, via the homomorphism μ defined in (2.2), an F-algebra A with unit is W-associative if and only if it is W-generalized.

More in general, if A is a W-associative algebra in the sense of Definition (3.1), then the semidirect product $W \ltimes A$ is a unital associative algebra, and the canonical inclusion $\iota_1 : W \to W \ltimes A$ (see the diagram (3.4)) endows $W \ltimes A$ with a structure of W-generalized algebra. Consequently, the free object in the category of W-generalized algebras can also be interpreted as the free W-algebra with constant term, $W\langle X\rangle^+ = W \ltimes W\langle X\rangle$.

A key distinction between the free W-algebra $W\langle X\rangle$ and the free product $W *_F F\langle X\rangle$ is that W is not embedded in $W\langle X\rangle$, whereas it is canonically embedded in $W *_F F\langle X\rangle$. This difference has significant consequences, leading to different notions of T-ideals in the two settings (see Remark 6.7).

Therefore, the framework of W-associative algebras is broader than the classical one, as it arises naturally from the more general theory of internal actions in the category of associative algebras. The classical theory of generalized polynomial identities appears as a special case of this approach, but with an essential conceptual shift: in the present setting, the action of W is encoded as a unary operation, whereas in the classical framework it is treated as a nullary operation.

5. On almost polynomial growth and matrix algebras

In this section, we present the results in [24, 25] on W-identities of the algebra UT_2 of 2×2 upper triangular matrices and the algebra $M_k(F)$ of $k \times k$ matrices for $k \geq 2$. We then exhibit a characterization of the W-varieties of almost polynomial growth.

We begin by analyzing the W-identities of UT_2. Since UT_2 is unital, Proposition 3.6 implies that the action of W on UT_2 is equivalent to the action induced by left and right multiplication by a suitable subalgebra B of UT_2 containing the unit 1_{UT_2} of UT_2.

Let e_{ij} denote the usual matrix units. Observe that, up to isomorphisms, the proper subalgebras of UT_2 containing $1_{UT_2} = e_{11} + e_{22}$ are

$$F1_{UT_2} \cong F, \qquad C := F1_{UT_2} + Fe_{12}, \qquad D := Fe_{11} \oplus Fe_{22}.$$

Consequently, UT_2 admits exactly four pairwise non-equivalent structures of a W-algebra according to which subalgebra of UT_2 acts by left and right multiplication. We denote these structures by UT_2^F, UT_2^C, UT_2^D, and simply UT_2 depending on whether UT_2 is endowed with the action of W induced by left and right multiplication by the subalgebra isomorphic to F, C, D, and the full algebra UT_2, respectively.

Notice that as a direct consequence of Theorem 4.3, we have that

$$\exp^W(UT_2^F) = \exp^W(UT_2^C) = \exp^W(UT_2^D) = \exp^W(UT_2) = 2.$$

Moreover, by Theorems 3.8 and 4.5 it follows that

$$UT_2^F \in \mathrm{var}^W(UT_2^C), \quad \text{and} \quad UT_2^D \in \mathrm{var}^W(UT_2).$$

We now introduce the notation used in the subsequent results on UT_2. Let Φ denote the acting homomorphism of W on UT_2. We choose a generating set $\mathcal{W} := \{w_i : i \geq 0\}$ of W as an algebra, with $w_0 = 1_W$, such that:

- for UT_2^F, $w_i \in \ker\Phi$ for all $i \geq 1$;
- for UT_2^C, $\Phi(w_1) = (R_{e_{12}}, L_{e_{12}})$ and $w_i \in \ker\Phi$ for all $i \geq 2$;
- for UT_2^D, $\Phi(w_1) = (R_{e_{22}}, L_{e_{22}})$ and $w_i \in \ker\Phi$ for all $i \geq 2$;
- for UT_2, $\Phi(w_1) = (R_{e_{22}}, L_{e_{22}})$, $\Phi(w_2) = (R_{e_{12}}, L_{e_{12}})$ and $w_i \in \ker\Phi$ for all $i \geq 3$.

Let $[x_1, x_2] := x_1x_2 - x_2x_1$ denote the commutator of x_1 and x_2. More generally, for $k \geq 2$, we write $[x_1, x_2, \ldots, x_k]$ for the left-normed commutator.

With this notation, we have the following results.

Theorem 5.1. *Let UT_2^F be the W-algebra UT_2 where W acts on it as the algebra F by left and right multiplication. Then $\mathrm{Id}^W(UT_2^F)$ is generated as T_W-ideal by the polynomials:*

$$[x_1, x_2][x_3, x_4], \quad w_ix, \quad xw_i,$$

for all $i \geq 1$. Moreover, $c_n^W(UT_2^F) = (n-2)2^{n-1} + 2$.

Proof. Clearly, by Proposition 4.1 and by the definition of the acting homomorphism Φ of W on UT_2^F, the W-identities of UT_2^F correspond to the ordinary identities of UT_2 modulo the T_W-ideal $K_\Phi := \langle w_ix,\, xw_i : i \geq 1\rangle_{T_W}$. Thus, by [23] we obtain the required conclusion. □

Theorem 5.2. *Let UT_2^C be the W-algebra UT_2 where W acts on it as the algebra C by left and right multiplication. Then $\mathrm{Id}^W(UT_2^C)$ is generated as T_W-ideal by the polynomials:*

$$[x_1, x_2][x_3, x_4], \quad w_1[x_1, x_2], \quad [x_1, x_2]w_1, \quad w_1^2x, \quad xw_1^2, \quad w_ix, \quad xw_i,$$

for all $i \geq 2$. Moreover, $c_n^W(UT_2^C) = n2^{n-1} + 2$.

Proof. Consider the T_W-ideal $K_\Phi = \langle wx,\, xw \,:\, w \in \ker\Phi\rangle_{T_W}$. By the definition of the acting homomorphism Φ of W on UT_2^C, it follows that $\ker\Phi$ is generated as a two-sided ideal by the set $\{w_1^2,\ w_i \ : i \geq 2\}$, and consequently $K_\Phi = \langle w_1^2x,\, xw_1^2,\, w_ix,\, xw_i \,:\, i \geq 2\rangle_{T_W}$. By Proposition 4.1 the W-identities of UT_2^C correspond, modulo K_Φ, to the $\overline{W}$-identities of UT_2^C, and, in particular, $c_n^W(UT_2^C) = c_n^{\overline{W}}(UT_2^C)$. For convenience, we set $\Phi(w_1) = \bar{w}_1$. Hence, it is enough to prove that $\mathrm{Id}^{\overline{W}}(UT_2^C) = \langle [x_1,x_2][x_3,x_4],\, \bar{w}_1[x_1,x_2],\, [x_1,x_2]\bar{w}_1\rangle_{T_{\overline{W}}}$ and determine its corresponding $\overline{W}$-codimensions to conclude the proof.

Let I be the $T_{\overline{W}}$-ideal generated by the $\overline{W}$-polynomials $[x_1,x_2][x_3,x_4]$, $\bar{w}_1[x_1,x_2]$ and $[x_1,x_2]\bar{w}_1$. A straightforward computation shows that $I \subseteq \mathrm{Id}^{\overline{W}}(UT_2^C)$. Thus in order to prove the opposite inclusion, let $f \in \mathrm{Id}^{\overline{W}}(UT_2^C)$ be a multilinear $\overline{W}$-polynomial of degree n.

Observe that $w_1^2x,\, xw_1^2 \in K_\Phi$, and by Corollary 4.2 we also have $w_1xw_1 \in K_\Phi$. Consequently, in $\overline{W}\langle X\rangle$ it holds that $\bar{w}_1^2x = 0$, $x\bar{w}_1^2 = 0$ and $\bar{w}_1x\bar{w}_1 = 0$. Hence, we may assume that in each monomial of f appears at most one $\bar{w}_1$. Moreover, since $\bar{w}_1[x_1,x_2],\, [x_1,x_2]\bar{w}_1 \in I$, any variables to the left or right of $\bar{w}_1$ may be reordered modulo I.

If a monomial of f does not involve $\bar{w}_1$, then, because $[x_1,x_2][x_3,x_4] \in I$, standard arguments and the Poincaré–Birkhoff–Witt theorem imply that it can be written modulo I as a linear combination of polynomials of the form

$$x_{i_1}\cdots x_{i_r}[x_k, x_{j_1},\ldots,x_{j_{n-r-1}}],$$

where $0 \leq r \leq n$, $r \neq n-1$, $i_1 < \cdots < i_r$, $k > j_1 < \cdots < j_{n-r-1}$.

Collecting these observations, we conclude that $P_n^{\overline{W}}$ is generated modulo I by polynomials of the types

$$x_{i_1}\cdots x_{i_r}[x_k, x_{j_1},\ldots,x_{j_{n-r-1}}], \qquad x_{h_1}\cdots x_{h_p}\bar{w}_1x_{l_1}\ldots x_{l_{n-p}}, \tag{5.1}$$

where $0 \leq r \leq n$, $r \neq n-1$, $0 \leq p \leq n$, $i_1 < \cdots < i_r$, $k > j_1 < \cdots < j_{n-r-1}$, $h_1 < \cdots < h_p$, $l_1 < \cdots < l_{n-p}$.

Next, we prove that these $\overline{W}$-polynomials are linearly independent modulo $\mathrm{Id}^{\overline{W}}(UT_2^C)$. Suppose $f \in \mathrm{Id}^{\overline{W}}(UT_2^C)$ is a linear combination of the polynomials in (5.1), i.e.,

$$f = \sum_{\mathcal{I},\mathcal{J}} \alpha_{\mathcal{I},\mathcal{J}} x_{i_1}\cdots x_{i_r}[x_k, x_{j_1},\ldots,x_{j_{n-r-1}}] + \sum_{\mathcal{H}} \beta_{\mathcal{H}} x_{h_1}\cdots x_{h_p}\bar{w}_1x_{l_1}\ldots x_{l_{n-p}} \in \mathrm{Id}^{\overline{W}}(UT_2^C),$$

where $\mathcal{I} = \{i_1,\ldots,i_r\}$, $\mathcal{J} = \{j_1,\ldots,j_{n-r-1}\}$, $\mathcal{H} = \{h_1,\ldots,h_p\}$ and $\alpha_{\mathcal{I},\mathcal{J}}, \beta_{\mathcal{H}} \in F$ for all $\mathcal{I},\mathcal{J},\mathcal{H}$ satisfying the conditions of (5.1).

First, if we evaluate $x_1 = \cdots = x_n = e_{11}$, then we get $\alpha_{\{1,\ldots,n\},\emptyset}e_{11} + \beta_{\{1,\ldots,n\}}e_{12} = 0$, thus $\alpha_{\{1,\ldots,n\},\emptyset} = \beta_{\{1,\ldots,n\}} = 0$. Now let us make the evaluation $x_1 = \cdots = x_n = e_{22}$. In this case we get $\beta_\emptyset e_{12} = 0$, so $\beta_\emptyset = 0$. For a fixed $\mathcal{H} = \{h_1,\ldots,h_p\}$, we set $x_{h_1} = \cdots = x_{h_p} = e_{11}$ and $x_{l_1} = \cdots = x_{l_{n-p}} = e_{22}$, so we obtain $\beta_{\mathcal{H}}e_{12} = 0$ and consequently $\beta_{\mathcal{H}} = 0$. Finally, for fixed $\mathcal{I} = \{i_1,\ldots,i_r\}$ and $\mathcal{J} = \{j_1,\ldots,j_{n-r-1}\}$, from the evaluation $x_{i_1} = \cdots = x_{i_r} = e_{11} + e_{22}$, $x_k = e_{12}$ and $x_{j_1} = \cdots = x_{j_{n-r-1}} = e_{22}$ we get $\alpha_{\mathcal{I},\mathcal{J}}e_{12} = 0$, thus $\alpha_{\mathcal{I},\mathcal{J}} = 0$. Therefore, all the scalars appearing in f are zero and $f = 0$, proving that the polynomials in (5.1) are linearly independent modulo $\mathrm{Id}^{\overline{W}}(UT_2^C)$. Since $P_n^{\overline{W}} \cap I \subseteq P_n^{\overline{W}} \cap \mathrm{Id}^{\overline{W}}(UT_2^C)$, this proves that $\mathrm{Id}^{\overline{W}}(UT_2^C) = I$ and the polynomials in (5.1) are a basis of $P_n^{\overline{W}}$ modulo $P_n^{\overline{W}} \cap \mathrm{Id}^{\overline{W}}(UT_2^C)$. Hence, by counting we get

$$c_n^{\overline{W}}(UT_2^C) = 1 + \sum_{i=2}^{n}\binom{n}{i}(i-1) + \sum_{i=0}^{n}\binom{n}{i} = n2^{n-1} + 2.$$

□

Theorem 5.3. *Let UT_2^D be the W-algebra UT_2 where W acts on it as the algebra D by left and right multiplication. Then $\mathrm{Id}^W(UT_2^D)$ is generated as T_W-ideal by the polynomials:*

$$[x_1,x_2] - [x_1,x_2,w_1],\quad w_1^2x - w_1x,\quad xw_1^2 - xw_1,\quad w_ix,\quad xw_i,$$

for all $i \geq 2$. Moreover, $c_n^W(UT_2^D) = n2^{n-1} + 2$.

Proof. Consider the T_W-ideal $K_\Phi = \langle wx,\, xw \,:\, w \in \ker\Phi\rangle_{T_W}$. Since $\Phi(W) \cong D$, it follows that $\ker\Phi$ is generated as a two-sided ideal by the set $\{w_1^2 - w_1,\ w_i \ : i \geq 2\}$, and consequently $K_\Phi = \langle w_1^2x - w_1x,\, xw_1^2 - xw_1,\, w_ix,\, xw_i \,:\, i \geq 2\rangle_{T_W}$. Moreover, by Proposition 4.1 the W-identities of UT_2^D correspond, modulo K_Φ, precisely to the D-identities of UT_2^D. Therefore, the desired conclusion follows from [24, Theorem 5.1]. □

Theorem 5.4. *Let UT_2 be the W-algebra UT_2 where W acts on it as UT_2 itself by left and right multiplication. Then $\mathrm{Id}^W(UT_2)$ is generated as T_W-ideal by the polynomials:*

$$[x_1,x_2]-[x_1,x_2,w_1],\quad w_1^2x-w_1x,\quad xw_1^2-xw_1,\quad w_2^2x,\quad xw_2^2,$$
$$w_1w_2x,\quad xw_1w_2,\quad w_2w_1x-w_2x,\quad xw_2w_1-xw_2,\quad w_ix,\quad xw_i,$$

for all $i\geq 3$. Moreover, $c_n^W(UT_2)=(n+2)2^{n-1}+2$.

Proof. Consider the T_W-ideal $K_\Phi=\langle wx, xw \,:\, w\in\ker\Phi\rangle_{T_W}$. By the definition of the acting homomorphism Φ of W in UT_2, we find that $\ker\Phi$ is generated as a two-sided ideal by the set $\{w_1^2-w_1,\, w_2^2,\, w_1w_2,\, w_2w_1-w_2,\, w_i \,:\, i\geq 3\}$. Consequently, K_Φ is generated as T_W-ideal by the following W-polynomials:

$$w_1^2x-w_1x,\ xw_1^2-xw_1,\ w_2^2x,\ xw_2^2, w_1w_2x,\ xw_1w_2,\ w_2w_1x-w_2x,\ xw_2w_1-xw_2,\ w_ix,\ xw_i,$$

for $i\geq 3$. Moreover, by Proposition 4.1 the W-identities of UT_2 correspond, modulo K_Φ, to the UT_2-identities of UT_2. Hence, the desired conclusion follows from [24, Theorem 3.2]. □

As consequence of Theorems 5.1 and 5.3, we have that $\mathrm{Id}^W(UT_2^F)\not\subseteq\mathrm{Id}^W(UT_2^D)$ and $\mathrm{Id}^W(UT_2^D)\not\subseteq\mathrm{Id}^W(UT_2^F)$. Thus, by Proposition 4.1, Theorem 5.7 of [24] and the results in [20], we have the following.

Theorem 5.5. *The W-algebras UT_2^F and UT_2^D generate two distinct W-varieties of almost polynomial growth.*

Notice that the same cannot be said for $\mathrm{var}^W(UT_2^C)$ and $\mathrm{var}^W(UT_2)$ since they contain UT_2^F and UT_2^D, respectively. Thus, we have the following result.

Corollary 5.6. *The W-variety $\mathrm{var}^W(UT_2^B)$ has almost polynomial growth if and only if B is either isomorphic to F or D.*

Matrix algebras are central in mathematics and its applications, and their identities have been studied since the origins of polynomial identity theory. Nevertheless, despite more than seventy years of research, a complete characterization of the polynomial identities of $M_k(F)$ for all $k\geq 2$ is still unknown and is considered one of the central open problems in the area. To date, the problem has been fully resolved only in the case $k=2$ (see [9,27]). By contrast, generalized polynomial identities are completely described for all $k\geq 2$ when $M_k(F)$ is viewed as an $M_k(F)$-generalized algebra (see, e.g., Proposition 6.5.5 in [3]). More recently, in [25], these results were extended to the setting of W-associative algebras where W acts by left and right multiplication of $M_k(F)$ itself, which we present in the following.

Since $M_k(F)$ is unital, by Proposition 3.6 it follows that the action of W on $M_k(F)$ is equivalent to the action induced by left and right multiplication by a suitable unital subalgebra $B\subseteq M_k(F)$. We denote by M_k^B the W-algebra $M_k(F)$ with the W-action given by left and right multiplication by B. When $B=M_k(F)$, we simply write M_k.

Notice that, as a direct consequence of Theorem 4.3, for any unital subalgebra $B\subseteq M_k(F)$, we have

$$\exp^W\left(M_k^B\right)=k^2.$$

We first consider M_k with $k\geq 2$. Since $\Phi_k(W)\cong M_k(F)$, where Φ_k is the acting homomorphism of W on $M_k(F)$, we may choose a generating set $\mathcal{W}_k:=\{w_{ij}^{(k)} \,:\, i,j\geq 1\}$ of W as an algebra such that:

- $\Phi_k\left(w_{ij}^{(k)}\right)=(R_{e_{ij}},L_{e_{ij}})$ for all $1\leq i,j\leq k$;
- $w_{pq}^{(k)}\in\ker\Phi_k$ if at least one p or q greater than k.

Note that the identity element of W is not contained in $\mathcal{W}$.

With this notation, by Theorem 5.2 of [25], we have the following result.

Theorem 5.7. *Let M_k, with $k\geq 2$, be the W-algebra $M_k(F)$ of $k\times k$ matrices where W acts on it as $M_k(F)$ itself by left and right multiplication. Then $\mathrm{Id}^W(M_k)$ is generated as T_W-ideal by the following W-polynomials:*

$$\left[w_{11}^{(k)}x_1w_{11}^{(k)}, w_{11}^{(k)}x_2w_{11}^{(k)}\right],\quad w_{ij}^{(k)}w_{ml}^{(k)}x-\delta_{jm}w_{il}^{(k)}x,\quad xw_{ij}^{(k)}w_{ml}^{(k)}-\delta_{jm}xw_{il}^{(k)},\quad w_{pq}^{(k)}x,\quad xw_{pq}^{(k)},$$

where $i,j,l,m\in\{1,\dots,k\}$, δ_{jm} denotes the Kronecker delta and $p,q\geq 1$ with at least one p or q greater than k. Moreover, $c_n^W(M_k)=k^{2(n+1)}$ for all $n\geq 1$.

By Theorem 5.4 of [25], we obtain the following surprising result.

Theorem 5.8. *For each $k \geq 2$, the W-algebra M_k generates pairwise distinct W-varieties of almost polynomial growth.*

Proof. From [25, Theorem 5.4], it follows that $\mathrm{var}^W(M_k)$ has almost polynomial growth for every $k \geq 2$. Moreover, combining this with the fact that $\exp^W(M_k) = k^2 \neq h^2 = \exp^W(M_h)$ whenever $k \neq h$, we conclude that the corresponding W-varieties $\mathrm{var}^W(M_k)$ are pairwise distinct. □

Now we focus on M_k^B, where B is a proper unital subalgebra of $M_k(F)$. To this end, recall that, for an ordered set of positive integers $(t_1, \ldots, t_m)$, the upper block triangular subalgebra of the matrix algebra $M_{t_1+\cdots+t_m}(F)$ is defined as

$$UT(t_1, \ldots, t_m) = \begin{pmatrix} U_{11} & U_{12} & \cdots & U_{1m} \\ & U_{22} & \cdots & U_{2m} \\ & & \ddots & \vdots \\ & & & U_{mm} \end{pmatrix}$$

where $U_{ii} = M_{t_i}(F)$ for $1 \leq i \leq m$, U_{ij} is a rectangular $t_i \times t_j$ matrix over F for $1 \leq i < j \leq m$, and all the others entries are 0. We refer to U_{ii}, $1 \leq i \leq m$, as diagonal blocks.

In particular, if $m = 1$, we have just one diagonal block and $UT(t_1) = M_{t_1}(F)$. On the other hand, if $t_i = 1$ for all $1 \leq i \leq m$, then we have exactly m one-dimensional diagonal block and $UT(t_1, \ldots, t_m) = UT_m$, the algebra of $m \times m$ upper block triangular matrices.

Observe that $UT(t_1, \ldots, t_m)$ decompose as

$$UT(t_1, \ldots, t_m) \cong M_{t_1}(F) \oplus \cdots \oplus M_{t_m}(F) + J,$$

where $J \cong \oplus_{1 \leq i < j \leq k} U_{ij}$ is the Jacobson radical. Consequently, as an application of the block triangularization theorem [26, Theorem 1.5.1], we obtain the following.

Proposition 5.9. *Let B be a unital subalgebra of the algebra $M_k(F)$ of matrices of order $k > 1$ over F. Then there exist positive integers $(t_1, \ldots, t_m)$ such that $\sum_{i=1}^m t_i = k$ and B is isomorphic to a unital subalgebra of $UT(t_1, \ldots, t_m)$ with the diagonal blocks being full matrix algebras of the corresponding dimensions.*

Next, we recall a useful result established in [25].

Proposition 5.10 [25, Proposition 7.1]**.** *Let F be a field of characteristic zero and let us consider the W-algebra UT_k of upper triangular matrices of order k over F for some $k \geq 2$. Then either $UT_2^F \in \mathrm{var}^W(UT_k)$ or $UT_2^D \in \mathrm{var}^W(UT_k)$.*

We are now ready to prove the following result on the W-varieties generated by M_k^B, where $k \geq 2$ and $B \subsetneq M_k(F)$.

Theorem 5.11. *Let M_k^B, with $k \geq 2$, be the W-algebra $M_k(F)$ with the W-action given by left and right multiplication by a proper unital subalgebra B of $M_k(F)$. Then either $UT_2^F \in \mathrm{var}^W(M_k^B)$, or $UT_2^D \in \mathrm{var}^W(M_k^B)$, or $M_h \in \mathrm{var}^W(M_k^B)$ for some $h < k$.*

Proof. By Proposition 5.9, B is isomorphic to a unital subalgebra of $UT(t_1, \ldots, t_m)$ whose diagonal blocks are full matrix algebras of the corresponding sizes, for some positive integers $(t_1, \ldots, t_m)$ with $\sum_{i=1}^m t_i = k$. Moreover, since $B \subsetneq M_k(F)$, we necessarily have $m > 1$.

Suppose first that $m = k$. Then $t_i = 1$ for all $1 \leq i \leq k$, and hence $B \subseteq UT_k$. Consequently, the W-algebra UT_k^B of $k \times k$ upper triangular matrices, endowed with the W-action given by left and right multiplication by B, is a W-subalgebra of M_k^B. Therefore, $\mathrm{var}^W(UT_k^B) \subseteq \mathrm{var}^W(M_k^B)$, and by Proposition 5.10, it follows that $\mathrm{var}^W(M_k^B)$ contains at least one of the algebras in $\{UT_2^F, UT_2^D\}$, completing this case.

If $k = 2$, the proof is completed, so assume $k \geq 3$ and $2 \leq m \leq k-1$. Then there exists $1 \leq i \leq m$ such that $t_i \geq 2$, and clearly $t_i \leq k-1$. We equip B with a natural W-algebra structure by letting W act via left and right multiplication by B itself. In this way, $B \in \mathrm{var}^W(M_k^B)$. Moreover, by Theorem 4.5, we may assume without loss of generality that only the semisimple component $B_{ss} \cong M_{t_1}(F) \oplus \cdots \oplus M_{t_m}(F)$ acts on B via left and right multiplication. Since each $M_{t_i}(F)$ is a minimal ideal of B_{ss}, it is stable under this action. Hence, the W-algebra M_{t_i}, that is, $M_{t_i}(F)$ endowed with the W-action induced by left and right multiplication by $M_{t_i}(F)$, is a W-subalgebra of B. Thus, it follows that $M_{t_i} \in \mathrm{var}^W(B) \subseteq \mathrm{var}^W(M_k^B)$, with $t_i < k$, as required. □

As a consequence of Theorems 5.8 and 5.11, we have the following.

Corollary 5.12. *For any $k \geq 2$, the W-variety $\mathrm{var}^W(M_k^B)$ has almost polynomial growth if and only if $B = M_k(F)$.*

Note that, in the finite-dimensional setting, Corollaries 5.6 and 5.12, together with Theorem 5.11, suggest that UT_2^F, UT_2^D, and M_k for $k \geq 2$ are the only plausible candidates for generating W-varieties of almost polynomial growth. This was indeed confirmed in [25]. More precisely, as a consequence of [25, Theorem 7.5], Theorem 4.3, and the results of [20] (see also [15, Theorem 7.2.5]), we obtain the following characterization of finite-dimensional W-algebras of polynomial growth.

Theorem 5.13. *Let W be a unital algebra over a field F of characteristic zero and A be a finite dimensional W-algebra. Then the following conditions are equivalent:*

(1) $c_n^W(A)$, $n \geq 1$, *is polynomially bounded;*
(2) $\exp^W(A) \leq 1$*;*
(3) $UT_2^F, UT_2^D, M_k \notin \mathrm{var}^W(A)$ *for all $k \geq 2$;*
(4) $c_n(A)$, $n \geq 1$, *is polynomially bounded;*
(5) $\exp(A) \leq 1$*;*
(6) $UT_2 \notin \mathrm{var}(A)$.

As a result, we have the following characterization of W-varieties of almost polynomial growth.

Corollary 5.14. $\mathrm{var}^W(UT_2^F)$, $\mathrm{var}^W(UT_2^D)$ *and* $\mathrm{var}^W(M_k)$, *for all $k \geq 2$, are the only W-varieties of almost polynomial growth generated by finite dimensional W-algebras.*

6. W-identities of the Grassmann algebra and their codimensions growth

In this section, we present an example of infinite dimensional W-algebra with different W-actions.

Let G be the infinite dimensional unital Grassmann algebra. Recall that G is the algebra generated over F by 1_G and a countable set of elements $e_1, e_2, \ldots$ subject to the condition $e_i e_j = -e_j e_i$ for all $i, j \geq 0$. It is clear that by construction G is not finitely generated.

Since G is unital, Proposition 3.6 shows that any action of W on G is equivalent to the action induced by left and right multiplication by a suitable subalgebra $B \subseteq G$ containing the unit 1_G.

For each finite $k \geq 0$, let G_k be the the subalgebra of G generated by the finite set 1_G, $e_1, e_2, \ldots, e_k$. Then $\dim_F G_k = 2^k$, and so G_k is the finite-dimensional Grassmann algebra on k generators. Notice that, in particular, when $k = 0$ we have $G_0 = F1_G$. Extending this notation to the infinite case, we may also denote the full algebra G by G_∞.

We thus consider the W-algebras $G^{(k)}$, for finite $k \geq 0$, as well as $G^{(\infty)}$, depending on whether G is endowed with the action of W induced by left and right multiplication by the subalgebra G_k, for finite $k \geq 0$, or by the whole algebra G_∞, respectively. Notice that the last case makes sense only if W is not finitely generated.

Similarly to the previous section, we adopt the following notation. If Φ denotes the acting homomorphism of W on G, then we choose a generating set $\mathcal{W} := \{w_i : i \geq 0\}$ of W as an algebra, with $w_0 = 1_W$, such that:

- for $G^{(0)}$, $w_i \in \ker \Phi$ for all $i \geq 1$;
- for $G^{(k)}$, for finite $k \geq 1$, $\Phi(w_i) = (R_{e_i}, L_{e_i})$ for all $1 \leq i \leq k$, and $w_j \in \ker \Phi$ for all $j \geq k+1$;
- for $G^{(\infty)}$, $\Phi(w_i) = (R_{e_i}, L_{e_i})$ for all $i \geq 1$.

With this notation, we have the following results.

Theorem 6.1. *Let $G^{(0)}$ be the W-algebra G where W acts on it as the algebra $G_0 \cong F$ by left and right multiplication. Then $\mathrm{Id}^W(G^{(0)})$ is generated, as T_W-ideal, by the W-polynomials*

$$[x_1, x_2, x_3], \quad w_i x, \quad x w_i$$

for all $i \geq 1$. Moreover, $c_n^W(G^{(0)}) = 2^{n-1}$ for all $n \geq 1$.

Proof. Since the W-identities of $G^{(0)}$ correspond to the ordinary identities of G modulo the T_W-ideal $K_\Phi = \langle w_i x,\, x w_i : i \geq 1 \rangle_{T_W}$. Thus, by [21] we obtain the required conclusion. □

Corollary 6.2. *The limit $\exp^W(G^{(0)}) := \lim_{n\to\infty} \sqrt[n]{c_n^W(G^{(0)})}$ exists, and precisely $\exp^W(G^{(0)}) = 2$.*

As a consequence of Proposition 4.1 and the results in [20], we also have the following.

Theorem 6.3. *The W-variety $\mathrm{var}^W(G^{(0)})$ has almost polynomial growth.*

Theorem 6.4. *Let $G^{(k)}$, for finite $k \geq 1$, be the W-algebra G where W acts on it as the algebra G_k by left and right multiplication. Then* $\mathrm{Id}^W(G^{(k)})$ *is generated, as T_W-ideal, by the polynomials*

$$[x_1, x_2, x_3], \quad [w_i, x_1, x_2], \quad w_i w_j x + w_j w_i x, \quad x w_i w_j + x w_j w_i, \quad w_l x, \quad x w_l,$$

for all $1 \leq i, j \leq k$ and for all $l \geq k+1$. Moreover, $c_n^W(G^{(k)}) = 2^{n-1}(2^{k+1}-1)$ for all $n \geq 1$.

Proof. Consider the T_W-ideal $K_\Phi = \langle wx,\, xw \,:\, w \in \ker \Phi \rangle_{T_W}$. Since $\Phi(W) \cong G_k$, it follows that $\ker \Phi$ is generated as a two-sided ideal by the set $\{w_i w_j + w_j w_i,\, w_l \,:\, 1 \leq i, j \leq k,\, l \geq k+1\}$, and consequently $K_\Phi = \langle w_i w_j x + w_j w_i x,\, x w_i w_j + x w_j w_i,\, w_l x,\, x w_l, \,:\, 1 \leq i, j \leq k,\, l \geq k+1 \rangle_{T_W}$. Moreover, by Proposition 4.1 the W-identities of $G^{(k)}$ correspond, modulo K_Φ, precisely to the G_k-identities of $G^{(k)}$. Therefore, the desired conclusion follows from [25, Theorem 8.3]. □

Corollary 6.5. *For any finite $k \geq 1$, the limit* $\exp^W(G^{(k)}) := \lim_{n\to\infty} \sqrt[n]{c_n^W(G^{(k)})}$ *exists, and precisely* $\exp^W(G^{(k)}) = 2$.

Theorem 6.6. *Let W be a unital F-algebra not finitely generated, and $G^{(\infty)}$ be the W-algebra G where W acts on it as G itself by left and right multiplication. Then* $\mathrm{Id}^W(G^{(\infty)})$ *is generated, as T_W-ideal, by the polynomials*

$$[x_1, x_2, x_3], \quad [w_i, x_1, x_2], \quad w_i w_j x + w_j w_i x, \quad x w_i w_j + x w_j w_i$$

for all $i, j \geq 1$. Moreover, $c_n^W(G^{(\infty)}) = +\infty$ for all $n \geq 1$.

Proof. Consider the T_W-ideal $K_\Phi = \langle wx,\, xw \,:\, w \in \ker \Phi \rangle_{T_W}$. By the definition of the acting homomorphism Φ of W in G, the kernel $\ker \Phi$ is generated, as a two-sided ideal, by the elements $\{w_i w_j + w_j w_i \,:\, i, j \geq 1\}$. Consequently $K_\Phi = \langle w_i w_j x + w_j w_i x,\, x w_i w_j + x w_j w_i \,:\, i, j \geq 1 \rangle_{T_W}$. Moreover, by Proposition 4.1 the W-identities of $G^{(\infty)}$ correspond, modulo K_Φ, to the G-identities of $G^{(\infty)}$. The desired conclusion therefore follows from [25, Theorem 8.5]. □

Remark 6.7. Since G is unital, it may also be regarded as a W-generalized algebra instead of a W-associative algebra. In this context, for every $i \geq 1$, the polynomials $[w_i, x_1, x_2]$ are consequence of $[x_1, x_2, x_3]$. Consequently, the generators of the T_W-ideal of identity in Theorems 6.4 and 6.6 change substantially, depending on whether one works within the framework of W-generalized or W-associative algebras.

Now, observe that Theorems 6.1, 6.4, and 6.6 indicate that the key element governing the behavior of the codimensions is not the dimension of W nor that of A, but rather whether the action of W on A is finite, namely, whether $\Phi(W)$ has finite dimension. In addition, Corollaries 6.2 and 6.5 show that, for every finite $k \geq 0$, the W-exponent of $G^{(k)}$ coincides with the ordinary exponent of G. Motivated by this, the authors of [25] formulated the following conjecture.

Conjecture 6.8. *Let W be a unital F-algebra over a field of characteristic zero. If A is a W-algebra such that the action of W on A is finite, then the limit*

$$\exp^W(A) := \lim_{n\to\infty} \sqrt[n]{c_n^W(A)}$$

exists and is a non-negative integer. Furthermore, $\exp^W(A) = \exp(A)$.

By Theorems 6.1, 6.4, and 6.6, we obtain the following infinite descending chain of T_W-ideals:

$$\mathrm{Id}^W(G^{(0)}) \supset \mathrm{Id}^W(G^{(1)}) \supset \cdots \supset \mathrm{Id}^W(G^{(k)}) \supset \cdots \supset \mathrm{Id}^W(G^{(\infty)}). \tag{6.1}$$

Consequently, for every $k \geq 1$, the W-variety $\mathrm{var}^W(G^{(k)})$ does not have almost polynomial growth, and by Theorem 6.3 we have the following.

Corollary 6.9. $\mathrm{var}^W(G^{(k)})$ *has almost polynomial growth if and only if $k = 0$.*

Recall that, for a W-algebra A, the T_W-ideal of identities $\mathrm{Id}^W(A)$ is said to have the Specht property if every T_W-ideal J containing $\mathrm{Id}^W(A)$ admits a finite basis, that is, J is finitely generated as a T_W-ideal. In particular, this requires $\mathrm{Id}^W(A)$ itself to be finitely generated as a T_W-ideal. Thus, by (6.1) we have the following counterexample to the Specht property in characteristic zero.

Corollary 6.10. *The T_W-ideal* $\mathrm{Id}^W(G^{(\infty)})$ *does not satisfy the Specht property.*

Theorems 6.1, 6.4 and 6.6 highlight an interesting issue concerning the number of generators of a T_W-ideal associated with an infinite-dimensional algebra. Indeed, the T_W-ideals $\mathrm{Id}^W(G^{(k)})$, for finite $k \geq 0$, are finitely generated provided that W itself is finitely generated. In addition, Theorems 5.1–5.4 establish that the same phenomenon occurs for the T_W-ideals arising from the finite-dimensional algebra UT_2. These results suggest that the Specht property may be governed more by the number of generators of W than by the specific nature of its action on A. Motivated by this observation, the authors in [25] proposed the following conjecture.

Conjecture 6.11. *Let A be a W-algebra over a field of characteristic zero. If W is finitely generated, then $\mathrm{Id}^W(A)$ has the Specht property.*

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