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PIÑEIRO MIXED ORTHOGONAL POLYNOMIALS AND BIDIAGONAL FACTORIZATION

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Abstract. We study mixed multiple orthogonal polynomials associated with rank-one matrices of measures and their recurrence relations. On the step line we identify the corresponding banded recurrence matrix through the Gauss–Borel factorization of the moment matrix. Elementary Christoffel transformations lead to a factorization of this recurrence matrix into lower and upper bidiagonal factors. For the mixed Piñeiro system we derive contour-integral and generalized hypergeometric representations and formulas for the recurrence and factorization coefficients.

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1. Introduction

Mixed multiple orthogonal polynomials [2, 18, 26] constitute a natural extension of the classical theory of multiple orthogonal polynomials. They arise in several areas of mathematics and mathematical physics. In probability theory, for example, they appear in the analysis of stochastic models such as Brownian bridges and systems of non-intersecting Brownian motions with p starting points and q endpoints [18]. They are also closely connected with multicomponent Toda lattices [1, 2]. Their relevance extends to number theory, where mixed Hermite-Padé approximation has proved to be a powerful tool. A celebrated example is Apéry's proof of the irrationality of $\zeta(3)$, which relies on a mixed Hermite-Padé approximation involving three functions [3]. Similar approximation techniques have subsequently been employed to show that infinitely many odd values of the Riemann zeta function are irrational [5], as well as to establish that at least one among $\zeta(5), \zeta(7), \zeta(9)$, and $\zeta(11)$ is irrational [29].

Bidiagonal factorizations are central tools in the analysis of structured matrices; see, for example, [19, 24, 11]. They are also one of the fundamental ingredients in spectral Favard theorems for banded operators [14, 15, 16]. Beyond this setting, bidiagonal factorizations appear naturally in Hermite–Padé approximation [4], integrable systems and Darboux transformations [6], and Christoffel transformations [12]. From the probabilistic viewpoint, they correspond to stochastic factorizations of transition matrices of Markov chains [8]. Explicit factorizations for recurrence matrices of several families of multiple orthogonal

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polynomials have been obtained in [4, 7, 13, 22]; further connections with branched continued fractions and combinatorics appear in [21, 25, 9].

Reference [15] establishes a simultaneous Gaussian quadrature formula for mixed orthogonality. More recently, the authors of [23] developed a method for computing the nodes and weights of simultaneous n-point Gaussian quadrature rules. Their approach computes the relevant eigenvalues and eigenvectors in a numerically stable way.

We study recurrence relations satisfied by mixed orthogonal linear forms and their type-I and type-II polynomial components. For step-line multi-indices, we refine the algorithm of [17] for the bidiagonal factorization of the recurrence matrix. In our normalization, the diagonal entries of the lower bidiagonal factors and the superdiagonal entries of the upper bidiagonal factors equal 1. As a case study, we consider mixed Piñeiro polynomials. We derive contour-integral and hypergeometric representations for both types of mixed linear forms, using Mellin-transform techniques related to [10, 27, 28]. These representations yield recurrence coefficients, from which the factorization coefficients on the step line are obtained exactly.

We begin by recalling the required definitions.

1.1. **Generalized hypergeometric series.** For $n \in \mathbb{N}_0$, define the Pochhammer symbol algebraically by

$$(z)_0 = 1, \qquad (z)_n = \prod_{r=0}^{n-1} (z+r), \qquad n \geq 1.$$

Whenever the quotient is defined, this agrees with $(z)_n = \Gamma(z+n)/\Gamma(z)$; elsewhere the algebraic definition, or its meromorphic continuation after cancellation, is used. For later compact step-line formulas we also use

$$(z)_{-n} := \frac{1}{(z-n)_n}, \qquad n \in \mathbb{N}, \tag{1}$$

away from the poles of the right-hand side.

The generalized hypergeometric series is

$${}_pF_q\left[\begin{matrix} a_1, \dots, a_p \\ \alpha_1, \dots, \alpha_q \end{matrix}; x\right] := \sum_{l=0}^{\infty} \frac{(a_1)_l \cdots (a_p)_l}{(\alpha_1)_l \cdots (\alpha_q)_l} \frac{x^l}{l!}, \tag{2}$$

provided that no denominator Pochhammer vanishes before termination. All hypergeometric series used below terminate because one upper parameter is a nonpositive integer; thus no additional convergence condition is needed in the formulas of this paper. Here Euler's gamma function is

$$\Gamma(z) := \int_0^{\infty} t^{z-1} \, \mathrm{e}^{-t} \, \mathrm{d}\, t, \qquad \operatorname{Re} z > 0.$$

Remark 1.1. If $n \in \mathbb{N}_0$, then

$$(-n)_m = \begin{cases} 0, & \text{if } m > n, \\ (-1)^m \frac{n!}{(n-m)!}, & \text{if } m \leqslant n. \end{cases}$$

This implies that if any of the $a_1, \dots, a_p$ is a negative integer, then the corresponding generalized hypergeometric series (2) is finite. Moreover, if any of the $a_1, \dots, a_p$ is zero, then the corresponding function (2) equals 1.

1.2. **Mixed orthogonal polynomials.** We consider a matrix of measures

$$\mathrm{d}\,\mu = \begin{bmatrix} \mathrm{d}\,\mu_{1,1} & \cdots\cdots & \mathrm{d}\,\mu_{1,p} \\ \vdots & & \vdots \\ \mathrm{d}\,\mu_{q,1} & \cdots\cdots & \mathrm{d}\,\mu_{q,p} \end{bmatrix},$$

where the measures $\mu_{i,j}$ are supported on the interval $\Delta \subseteq \mathbb{R}$.

Let $\vec{n} = (n_1, \dots, n_p)$ and $\vec{m} = (m_1, \dots, m_q)$ be multi-indices, and write $|\vec{n}| = n_1 + \cdots + n_p$ and $|\vec{m}| = m_1 + \cdots + m_q$. We use the standard convention that $n_i = 0$ implies $A^{(i)} \equiv 0$, and $m_j = 0$ implies $B^{(j)} \equiv 0$. Accordingly, formulas containing $(n_i - 1)!$ or $(m_j - 1)!$ are asserted only when $n_i \geq 1$ or $m_j \geq 1$, respectively.

If $|\vec{n}| = |\vec{m}| + 1$, there is a vector polynomial

$$(A^{(1)}_{\vec{n},\vec{m}}, \dots, A^{(p)}_{\vec{n},\vec{m}}), \qquad \deg A^{(i)}_{\vec{n},\vec{m}} \leq n_i - 1,$$

satisfying the mixed orthogonality conditions

$$\sum_{i=1}^{p} \int_{\Delta} A^{(i)}_{\vec{n},\vec{m}}(x) x^k \, \mathrm{d}\mu_{j,i}(x) = 0, \qquad 0 \leq k \leq m_j - 1, \qquad 1 \leq j \leq q. \tag{3}$$

Analogously, if $|\vec{m}| = |\vec{n}| + 1$, there is a vector polynomial

$$(B^{(1)}_{\vec{n},\vec{m}}, \dots, B^{(q)}_{\vec{n},\vec{m}}), \qquad \deg B^{(j)}_{\vec{n},\vec{m}} \leq m_j - 1,$$

satisfying

$$\sum_{j=1}^{q} \int_{\Delta} B^{(j)}_{\vec{n},\vec{m}}(x) x^k \, \mathrm{d}\mu_{j,i}(x) = 0, \qquad 0 \leq k \leq n_i - 1, \qquad 1 \leq i \leq p. \tag{4}$$

1.3. **Mixed orthogonal polynomials for rank-one matrix measures**. Let $\mathrm{d}w$ be a positive Borel measure on Δ, and consider positive measurable weight functions

$$u_1, \dots, u_p : \Delta \subseteq \mathbb{R} \to \mathbb{R}_+, \qquad v_1, \dots, v_q : \Delta \subseteq \mathbb{R} \to \mathbb{R}_+.$$

We assume that all moments used below are finite; explicitly,

$$\int_{\Delta} |x|^r u_i(x) v_j(x) \, \mathrm{d}w(x) < \infty, \qquad r \in \mathbb{N}_0, \qquad 1 \leq i \leq p, \qquad 1 \leq j \leq q.$$

Define the vectors and the rank-one matrix of weights by $u := (u_1, \dots, u_p)$, $v := (v_1, \dots, v_q)$,

$$\mathrm{d}\mu := v^{\top} u \, \mathrm{d}w = \begin{bmatrix} v_1 u_1 & \cdots & v_1 u_p \\ \vdots & & \vdots \\ v_q u_1 & \cdots & v_q u_p \end{bmatrix} \mathrm{d}w.$$

Taking

$$A_{\vec{n},\vec{m}}(x) := \sum_{i=1}^{p} A^{(i)}_{\vec{n},\vec{m}}(x) u_i(x), \qquad B_{\vec{n},\vec{m}}(x) := \sum_{j=1}^{q} B^{(j)}_{\vec{n},\vec{m}}(x) v_j(x)$$

we can rewrite (3) and (4) as:

$$\int_{\Delta} x^k \sum_{i=1}^{p} A^{(i)}_{\vec{n},\vec{m}}(x) u_i(x) v_j(x) \, \mathrm{d}w(x) = 0, \qquad j \in \{1, \dots, q\}, \qquad k \in \{0, \dots, m_j - 1\}, \tag{5}$$

$$\int_{\Delta} x^k \sum_{j=1}^{q} B^{(j)}_{\vec{n},\vec{m}}(x) v_j(x) u_i(x) \, \mathrm{d}w(x) = 0, \qquad i \in \{1, \dots, p\}, \qquad k \in \{0, \dots, n_i - 1\}. \tag{6}$$

For $d \in \{p, q\}$, let

$$\vec{e}^{\,(d)}_k := (\underbrace{0, \dots, 0}_{k-1}, 1, \underbrace{0, \dots, 0}_{d-k}) \in \mathbb{R}^d, \qquad 1 \leq k \leq d.$$

When the dimension is determined by the surrounding multi-index, we write simply $\vec{e}_k$.

If $u = (u_1, \dots, u_p)$ and $v = (v_1, \dots, v_q)$ are AT systems on Δ, every admissible multi-index is normal [20]. Thus the corresponding mixed forms are unique up to a nonzero factor, and every component with positive upper degree has maximal degree:

$$\deg A^{(i)}_{\vec{n},\vec{m}} = n_i - 1, \qquad \deg B^{(j)}_{\vec{n},\vec{m}} = m_j - 1$$

whenever $n_i > 0$ and $m_j > 0$, respectively.

The next proposition records the biorthogonality relation used below to isolate the recurrence coefficients.

Proposition 1.2 (Biorthogonality). *Assume that $u = (u_1, \dots, u_p)$ and $v = (v_1, \dots, v_q)$ are AT systems on Δ. Let $A_{\vec{n},\vec{m}}$ be a type-I form, so $|\vec{n}| = |\vec{m}| + 1$, and let $B_{\vec{\mathfrak{n}},\vec{\mathfrak{m}}}$ be a type-II form, so $|\vec{\mathfrak{m}}| = |\vec{\mathfrak{n}}| + 1$. Then*

$$\int_\Delta A_{\vec{n},\vec{m}}(x) B_{\vec{\mathfrak{n}},\vec{\mathfrak{m}}}(x)\ \mathrm{d}\, w(x) = 0 \qquad \textit{whenever} \qquad \vec{n} \le \vec{\mathfrak{n}} \qquad \textit{or} \qquad \vec{\mathfrak{m}} \le \vec{m}. \tag{7}$$

Moreover, this integral is nonzero in either adjacent case

$$\vec{\mathfrak{m}} = \vec{m} + \vec{e}_{j_0} \qquad \textit{or} \qquad \vec{n} = \vec{\mathfrak{n}} + \vec{e}_{i_0}. \tag{8}$$

Proof. Write

$$A^{(i)}_{\vec{n},\vec{m}}(x) = \sum_{\ell=0}^{n_i-1} A^{(i)}_{\vec{n},\vec{m}}[\ell] x^\ell.$$

If $\vec{n} \le \vec{\mathfrak{n}}$, then

$$\int_\Delta A_{\vec{n},\vec{m}}(x) B_{\vec{\mathfrak{n}},\vec{\mathfrak{m}}}(x)\ \mathrm{d}\, w(x) = \sum_{i=1}^{p} \sum_{\ell=0}^{n_i-1} A^{(i)}_{\vec{n},\vec{m}}[\ell] \int_\Delta x^\ell u_i(x) B_{\vec{\mathfrak{n}},\vec{\mathfrak{m}}}(x)\ \mathrm{d}\, w(x) = 0$$

by the type-II conditions. Likewise, expanding

$$B^{(j)}_{\vec{\mathfrak{n}},\vec{\mathfrak{m}}}(x) = \sum_{\ell=0}^{\mathfrak{m}_j-1} B^{(j)}_{\vec{\mathfrak{n}},\vec{\mathfrak{m}}}[\ell] x^\ell$$

and using the type-I conditions gives

$$\int_\Delta A_{\vec{n},\vec{m}}(x) B_{\vec{\mathfrak{n}},\vec{\mathfrak{m}}}(x)\ \mathrm{d}\, w(x) = 0 \qquad \text{if} \qquad \vec{\mathfrak{m}} \le \vec{m}.$$

Suppose now that $\vec{\mathfrak{m}} = \vec{m} + \vec{e}_{j_0}$. The type-I conditions annihilate every term except the leading term of the j_0-component:

$$\int_\Delta A_{\vec{n},\vec{m}}(x) B_{\vec{\mathfrak{n}},\vec{\mathfrak{m}}}(x)\ \mathrm{d}\, w(x) = B^{(j_0)}_{\vec{\mathfrak{n}},\vec{\mathfrak{m}}}[m_{j_0}] \int_\Delta x^{m_{j_0}} v_{j_0}(x) A_{\vec{n},\vec{m}}(x)\ \mathrm{d}\, w(x).$$

The coefficient $B^{(j_0)}_{\vec{\mathfrak{n}},\vec{\mathfrak{m}}}[m_{j_0}]$ is nonzero by normality. Choose i_0 with $n_{i_0} > 0$. If the remaining moment vanished, then $A_{\vec{n},\vec{m}}$, regarded as an element of the larger polynomial space corresponding to $(\vec{n}+\vec{e}_{i_0}, \vec{m}+\vec{e}_{j_0})$, would satisfy all the type-I conditions for this enlarged index. This contradicts normality: its i_0-component has degree at most $n_{i_0}-1$, whereas every nonzero form for the enlarged index has i_0-component of degree n_{i_0}.

Finally, suppose that $\vec{n} = \vec{\mathfrak{n}} + \vec{e}_{i_0}$. The type-II conditions now give

$$\int_\Delta A_{\vec{n},\vec{m}}(x) B_{\vec{\mathfrak{n}},\vec{\mathfrak{m}}}(x)\ \mathrm{d}\, w(x) = A^{(i_0)}_{\vec{n},\vec{m}}[\mathfrak{n}_{i_0}] \int_\Delta x^{\mathfrak{n}_{i_0}} u_{i_0}(x) B_{\vec{\mathfrak{n}},\vec{\mathfrak{m}}}(x)\ \mathrm{d}\, w(x).$$

The displayed coefficient is nonzero by normality. If the remaining moment vanished, then, for any j_0, the original form $B_{\vec{\mathfrak{n}},\vec{\mathfrak{m}}}$ would satisfy all the type-II conditions for $(\vec{\mathfrak{n}}+\vec{e}_{i_0}, \vec{\mathfrak{m}}+\vec{e}_{j_0})$, while its j_0-component would still have degree at most $\mathfrak{m}_{j_0} - 1$. Normality of the enlarged index requires that degree to be $\mathfrak{m}_{j_0}$, which is again a contradiction. □

2. Recurrence relations

Consider the vectors $\vec{1}_p := (1,\dots,1) \in \mathbb{R}^p$, $\vec{0}_p := (0,\dots,0) \in \mathbb{R}^p$. Let π_s and $\pi_{\mathfrak{s}}$ be permutations of $\{1,\dots,p\}$, and let π_t and $\pi_{\mathfrak{t}}$ be permutations of $\{1,\dots,q\}$. Define

$$\vec{s}_{rp+i} := r\vec{1}_p + \sum_{k=1}^{i} \vec{e}_{\pi_s(k)}, \qquad \vec{\mathfrak{s}}_{rp+i} := r\vec{1}_p + \sum_{k=1}^{i} \vec{e}_{\pi_{\mathfrak{s}}(k)}, \qquad r \in \mathbb{N}_0, \qquad i \in \{0,\dots,p-1\},$$

$$\vec{t}_{rq+j} := r\vec{1}_q + \sum_{k=1}^{j} \vec{e}_{\pi_t(k)}, \qquad \vec{\mathfrak{t}}_{rq+j} := r\vec{1}_q + \sum_{k=1}^{j} \vec{e}_{\pi_{\mathfrak{t}}(k)}, \qquad r \in \mathbb{N}_0, \qquad j \in \{0,\dots,q-1\}.$$

The vectors $\vec{s}_i, \vec{\mathfrak{s}}_i$ belong to $\mathbb{N}_0^p$, whereas $\vec{t}_j, \vec{\mathfrak{t}}_j$ belong to $\mathbb{N}_0^q$. They define chains starting at $\vec{s}_0 = \vec{\mathfrak{s}}_0 = \vec{0}_p$, $\vec{t}_0 = \vec{\mathfrak{t}}_0 = \vec{0}_q$. At each step the modulus increases by one, with the component order determined by π_s, $\pi_{\mathfrak{s}}$, π_t, and $\pi_{\mathfrak{t}}$.

The multi-index $\vec{n} \pm \vec{s}_{rp+i}$ is obtained by first adding or subtracting $r\vec{1}_p$ and then changing by 1 the i entries $\{n_{\pi_s(1)}, n_{\pi_s(2)}, \dots, n_{\pi_s(i)}\}$. Similarly, for $\vec{n} \pm \vec{\mathfrak{s}}_{rp+i}$ with $\pi_{\mathfrak{s}}$. Consequently, $|\vec{n} \pm \vec{s}_i| = |\vec{n} \pm \vec{\mathfrak{s}}_i| = |\vec{n}| \pm i$. In the same way, $\vec{m} \pm \vec{t}_{rq+j}$ is obtained by changing the j entries $\{m_{\pi_t(1)}, m_{\pi_t(2)}, \dots, m_{\pi_t(j)}\}$. Furthermore, for $\vec{m} \pm \vec{\mathfrak{t}}_{rq+j}$ with $\pi_{\mathfrak{t}}$. Consequently, $|\vec{m} \pm \vec{t}_j| = |\vec{m} \pm \vec{\mathfrak{t}}_j| = |\vec{m}| \pm j$.

The four admissible index paths introduced above lead to the following mixed recurrence relations.

Proposition 2.1. *Assume that $u = (u_1,\dots,u_p)$ and $v = (v_1,\dots,v_q)$ are AT systems on Δ. Assume in this proposition that all components of $\vec{n}$ and $\vec{m}$ are positive and that every shifted multi-index displayed below is nonnegative. Fix an arbitrary nonzero normalization for every form on the four chains, then the linear forms* (6) *and* (5) *satisfy respective recurrence relations*

$$xB_{\vec{n},\vec{m}}(x) = \sum_{j=1}^{q} b^{j}_{\vec{n},\vec{m}}\, B_{\vec{n}+\vec{\mathfrak{s}}_j,\vec{m}+\vec{t}_j}(x) + b^{0}_{\vec{n},\vec{m}}\, B_{\vec{n},\vec{m}}(x) + \sum_{i=1}^{p} b^{-i}_{\vec{n},\vec{m}}\, B_{\vec{n}-\vec{s}_i,\vec{m}-\vec{\mathfrak{t}}_i}(x), \tag{9}$$

$$xA_{\vec{n},\vec{m}}(x) = \sum_{j=1}^{q} a^{-j}_{\vec{n},\vec{m}}\, A_{\vec{n}-\vec{\mathfrak{s}}_j,\vec{m}-\vec{t}_j}(x) + a^{0}_{\vec{n},\vec{m}}\, A_{\vec{n},\vec{m}}(x) + \sum_{i=1}^{p} a^{i}_{\vec{n},\vec{m}}\, A_{\vec{n}+\vec{s}_i,\vec{m}+\vec{\mathfrak{t}}_i}(x), \tag{10}$$

where

$$\begin{cases} a^{-j}_{\vec{n},\vec{m}} = b^{j}_{\vec{n}-\vec{\mathfrak{s}}_{j+1},\vec{m}-\vec{t}_{j-1}} \dfrac{\int_\Delta B_{\vec{n}-\vec{\mathfrak{s}}_{j+1}+\vec{\mathfrak{s}}_j,\vec{m}-\vec{t}_{j-1}+\vec{t}_j}(x) A_{\vec{n},\vec{m}}(x)\,\mathrm{d}\,w(x)}{\int_\Delta B_{\vec{n}-\vec{\mathfrak{s}}_{j+1},\vec{m}-\vec{t}_{j-1}}(x) A_{\vec{n}-\vec{\mathfrak{s}}_j,\vec{m}-\vec{t}_j}(x)\,\mathrm{d}\,w(x)}, & j \in \{1,\dots,q\}, \\ a^{0}_{\vec{n},\vec{m}} = b^{0}_{\vec{n}-\vec{\mathfrak{s}}_1,\vec{m}+\vec{\mathfrak{t}}_1}, \\ a^{i}_{\vec{n},\vec{m}} = b^{-i}_{\vec{n}+\vec{s}_{i-1},\vec{m}+\vec{\mathfrak{t}}_{i+1}} \dfrac{\int_\Delta B_{\vec{n}-\vec{s}_i+\vec{s}_{i-1},\vec{m}-\vec{\mathfrak{t}}_i+\vec{\mathfrak{t}}_{i+1}}(x) A_{\vec{n},\vec{m}}(x)\,\mathrm{d}\,w(x)}{\int_\Delta B_{\vec{n}+\vec{s}_{i-1},\vec{m}+\vec{\mathfrak{t}}_{i+1}}(x) A_{\vec{n}+\vec{s}_i,\vec{m}+\vec{\mathfrak{t}}_i}(x)\,\mathrm{d}\,w(x)}, & i \in \{1,\dots,p\}, \end{cases} \tag{11}$$

and

$$\begin{cases} b^{j}_{\vec{n},\vec{m}} = \dfrac{\int_\Delta x\, A_{\vec{n}+\vec{\mathfrak{s}}_{j+1},\vec{m}+\vec{t}_{j-1}}(x) B_{\vec{n},\vec{m}}(x)\,\mathrm{d}\,w(x)}{\int_\Delta A_{\vec{n}+\vec{\mathfrak{s}}_{j+1},\vec{m}+\vec{t}_{j-1}}(x) B_{\vec{n}+\vec{\mathfrak{s}}_j,\vec{m}+\vec{t}_j}(x)\,\mathrm{d}\,w(x)}, & j \in \{1,\dots,q\}, \\ b^{0}_{\vec{n},\vec{m}} = \dfrac{\int_\Delta x\, A_{\vec{n}+\vec{\mathfrak{s}}_1,\vec{m}-\vec{\mathfrak{t}}_1}(x) B_{\vec{n},\vec{m}}(x)\,\mathrm{d}\,w(x)}{\int_\Delta A_{\vec{n}+\vec{\mathfrak{s}}_1,\vec{m}-\vec{\mathfrak{t}}_1}(x) B_{\vec{n},\vec{m}}(x)\,\mathrm{d}\,w(x)}, \\ b^{-i}_{\vec{n},\vec{m}} = \dfrac{\int_\Delta x\, A_{\vec{n}-\vec{s}_{i-1},\vec{m}-\vec{\mathfrak{t}}_{i+1}}(x) B_{\vec{n},\vec{m}}(x)\,\mathrm{d}\,w(x)}{\int_\Delta A_{\vec{n}-\vec{s}_{i-1},\vec{m}-\vec{\mathfrak{t}}_{i+1}}(x) B_{\vec{n}-\vec{s}_i,\vec{m}-\vec{\mathfrak{t}}_i}(x)\,\mathrm{d}\,w(x)}, & i \in \{1,\dots,p\}. \end{cases} \tag{12}$$

Proof. Existence and uniqueness. Let $\mathcal{V}$ be the space of forms $F = \sum_{j=1}^{q} P_j v_j$ with $\deg P_j \le m_j$ that satisfy

$$\int_\Delta x^k F(x) u_i(x)\, \mathrm{d}w(x) = 0, \qquad 0 \le k \le n_i - 2, \quad 1 \le i \le p,$$

with an empty range when $n_i = 1$. Write $P_j(x) = \sum_{\ell=0}^{m_j} c_{j,\ell} x^\ell$. The coefficient matrix of the displayed conditions is

$$\left[\int_\Delta x^{k+\ell} u_i(x) v_j(x)\, \mathrm{d}w(x) \right]_{\substack{1\le i\le p,\ 0\le k\le n_i-2 \\ 1\le j\le q,\ 0\le \ell\le m_j}}.$$

The perfectness theorem for the two AT systems gives full row rank for this matrix [20]. Hence the $|\vec{n}| - p$ displayed conditions are independent. The coefficient space before imposing them has dimension

$$\sum_{j=1}^{q} (m_j + 1) = |\vec{m}| + q = |\vec{n}| + q + 1.$$

Consequently,

$$\dim \mathcal{V} = (|\vec{n}| + q + 1) - (|\vec{n}| - p) = p + q + 1.$$

Consider the following $p + q + 1$ forms:

$$\mathcal{B}_{\vec{n},\vec{m}} := \left\{ B_{\vec{n}+\vec{\mathfrak{s}}_j, \vec{m}+\vec{t}_j} : 1 \le j \le q \right\} \cup \{ B_{\vec{n},\vec{m}} \} \cup \left\{ B_{\vec{n}-\vec{s}_i, \vec{m}-\vec{\mathfrak{t}}_i} : 1 \le i \le p \right\}.$$

Every form in this collection belongs to $\mathcal{V}$. For a forward form, the degree increase in $\vec{m} + \vec{t}_j$ is at most one in each component, while its type-II orthogonality conditions contain those defining $\mathcal{V}$. The central form satisfies stronger degree bounds, and the backward forms satisfy the required moment conditions because the corresponding components of $\vec{n} - \vec{s}_i$ have decreased by at most one.

The collection is linearly independent. Indeed, pair a linear combination of its elements successively with $A_{\vec{n}+\vec{\mathfrak{s}}_{j+1}, \vec{m}+\vec{t}_{j-1}}$, $1 \le j \le q$, then with $A_{\vec{n}+\vec{\mathfrak{s}}_1, \vec{m}-\vec{\mathfrak{t}}_1}$, and finally with $A_{\vec{n}-\vec{s}_{i-1}, \vec{m}-\vec{\mathfrak{t}}_{i+1}}$, $1 \le i \le p$. The biorthogonality relations annihilate all terms except the corresponding forward, central, or backward pivot. The precise vanishing relations are (14)–(17) below, and every pivot is nonzero by Proposition 1.2. Thus $\mathcal{B}_{\vec{n},\vec{m}}$ is a basis of $\mathcal{V}$.

Finally, $x B_{\vec{n},\vec{m}} \in \mathcal{V}$: its component degrees are at most m_j, and multiplication by x shifts every type-II moment condition by one. Therefore it has a unique expansion in the preceding basis,

$$x B_{\vec{n},\vec{m}}(x) = \sum_{j=1}^{q} b^{j}_{\vec{n},\vec{m}}\, B_{\vec{n}+\vec{\mathfrak{s}}_j, \vec{m}+\vec{t}_j}(x) b^{0}_{\vec{n},\vec{m}} B_{\vec{n},\vec{m}}(x) + \sum_{i=1}^{p} b^{-i}_{\vec{n},\vec{m}}\, B_{\vec{n}-\vec{s}_i, \vec{m}-\vec{\mathfrak{t}}_i}(x). \tag{13}$$

This proves existence and uniqueness of the type-II recurrence. The type-I recurrence (10) follows by the same argument with the two families interchanged.

Forward coefficients. Fix $j_0 \in \{1, \dots, q\}$. We have

$$\int_\Delta A_{\vec{n}+\vec{\mathfrak{s}}_{j_0+1}, \vec{m}+\vec{t}_{j_0-1}}(x) B_{\vec{n}+\vec{\mathfrak{s}}_j, \vec{m}+\vec{t}_j}(x)\, \mathrm{d}w(x) = 0, \qquad j \ne j_0,$$

taking into account

$$\vec{n} + \vec{\mathfrak{s}}_{j_0+1} \le \vec{n} + \vec{\mathfrak{s}}_j, \qquad j \in \{j_0 + 1, \dots, q\}, \qquad \vec{m} + \vec{t}_j \le \vec{m} + \vec{t}_{j_0-1}, \qquad j \in \{1, \dots, j_0 - 1\}, \tag{14}$$

and the biorthogonal relations (7). Noticing that $\vec{m} - \vec{\mathfrak{t}}_i \le \vec{m} + \vec{t}_{j_0-1}$ and again using the biorthogonal relations (7), it holds

$$\int_\Delta A_{\vec{n}+\vec{\mathfrak{s}}_{j_0+1}, \vec{m}+\vec{t}_{j_0-1}}(x) B_{\vec{n}-\vec{s}_i, \vec{m}-\vec{\mathfrak{t}}_i}(x)\, \mathrm{d}w(x) = 0, \qquad i > 0. \tag{15}$$

Moreover, using (7)

$$\int_{\Delta} A_{\vec{n}+\vec{s}_{j_0+1},\vec{m}+\vec{t}_{j_0-1}}(x) B_{\vec{n}+\vec{s}_{j_0},\vec{m}+\vec{t}_{j_0}}(x)\,\mathrm{d}\,w(x) \neq 0.$$

Multiplying (13) by $A_{\vec{n}+\vec{s}_{j_0+1},\vec{m}+\vec{t}_{j_0-1}}$, integrating, and using (14) and (15) gives the formula for $b^{j_0}_{\vec{n},\vec{m}}$.

Diagonal coefficient. For the diagonal coefficient, pair (13) with $A_{\vec{n}+\vec{s}_1,\vec{m}-\vec{t}_1}$. Part (a) of Proposition 1.2 annihilates every forward term, and part (b) annihilates every backward term. The remaining pairing with $B_{\vec{n},\vec{m}}$ is nonzero by (8). Division by this pairing gives the second line of (12).

Backward coefficients. Fix $i_0 \in \{1,\dots,p\}$. We have

$$\int_{\Delta} A_{\vec{n}-\vec{s}_{i_0-1},\vec{m}-\vec{t}_{i_0+1}}(x) B_{\vec{n}-\vec{s}_i,\vec{m}-\vec{t}_i}(x)\,\mathrm{d}\,w(x) = 0, \qquad i \neq i_0,$$

taking also into account

$$\vec{n}-\vec{s}_{i_0-1} \leq \vec{n}-\vec{s}_i, \qquad i \in \{1,\dots,i_0-1\}, \qquad \vec{m}-\vec{t}_i \leq \vec{m}-\vec{t}_{i_0+1}, \qquad i \geq i_0+1. \tag{16}$$

and the biorthogonal relations (7). Noticing that $\vec{n}-\vec{s}_{i_0-1} \leq \vec{n}+\vec{s}_j$ and again using the biorthogonal relations (7), it holds

$$\int_{\Delta} A_{\vec{n}-\vec{s}_{i_0-1},\vec{m}-\vec{t}_{i_0+1}}(x) B_{\vec{n}+\vec{s}_j,\vec{m}+\vec{t}_j}(x)\,\mathrm{d}\,w(x) = 0, \qquad j \in \{1,\dots,q\} \tag{17}$$

Moreover,

$$\int_{\Delta} A_{\vec{n}-\vec{s}_{i_0-1},\vec{m}-\vec{t}_{i_0+1}}(x) B_{\vec{n}-\vec{s}_{i_0},\vec{m}-\vec{t}_{i_0}}(x)\,\mathrm{d}\,w(x) \neq 0.$$

Multiplying (13) by $A_{\vec{n}-\vec{s}_{i_0-1},\vec{m}-\vec{t}_{i_0+1}}$, integrating, and using (16) and (17) gives the formula for $b^{-i_0}_{\vec{n},\vec{m}}$ in (12).

The type-I coefficients follow by duality. Substituting the shifted type-II coefficient formulas gives (11). □

Write the polynomial components of the mixed forms as

$$A^{(i)}_{\vec{n},\vec{m}}(x) = \sum_{l=0}^{n_i-1} A^{(i)}_{\vec{n},\vec{m}}[l]\,x^l, \quad i \in \{1,\dots,p\}, \qquad B^{(j)}_{\vec{\mathfrak{n}},\vec{\mathfrak{m}}}(x) = \sum_{l=0}^{\mathfrak{m}_j-1} B^{(j)}_{\vec{\mathfrak{n}},\vec{\mathfrak{m}}}[l]\,x^l, \quad j \in \{1,\dots,q\}.$$

For later boundary formulas, set $A^{(i)}_{\vec{n},\vec{m}}[l] = 0$ and $B^{(j)}_{\vec{n},\vec{m}}[l] = 0$ whenever l lies outside the indicated coefficient range.

Using these coefficients and the orthogonality conditions (6)–(5), we can rewrite the recurrence coefficients in (12) as follows:

$$
\begin{aligned}
b_{\vec{n},\vec{m}}^{j} &= \frac{\int_\Delta x\, A_{\vec{n}+\vec{\mathfrak{s}}_{j+1},\vec{m}+\vec{t}_{j-1}}(x) B_{\vec{n},\vec{m}}(x)\,\mathrm{d}\,w(x)}{\int_\Delta A_{\vec{n}+\vec{\mathfrak{s}}_{j+1},\vec{m}+\vec{t}_{j-1}}(x) B_{\vec{n}+\vec{\mathfrak{s}}_{j},\vec{m}+\vec{t}_{j}}(x)\,\mathrm{d}\,w(x)} \\
&= \sum_{k=j}^{q} \frac{B_{\vec{n},\vec{m}}^{(\pi_t(k))}[m_{\pi_t(k)}-1]\int_\Delta x^{m_{\pi_t(k)}} A_{\vec{n}+\vec{\mathfrak{s}}_{j+1},\vec{m}+\vec{t}_{j-1}}(x) v_{\pi_t(k)}(x)\,\mathrm{d}\,w(x)}{B_{\vec{n}+\vec{\mathfrak{s}}_{j},\vec{m}+\vec{t}_{j}}^{(\pi_t(j))}[m_{\pi_t(j)}]\int_\Delta x^{m_{\pi_t(j)}} A_{\vec{n}+\vec{\mathfrak{s}}_{j+1},\vec{m}+\vec{t}_{j-1}}(x) v_{\pi_t(j)}(x)\,\mathrm{d}\,w(x)},
\end{aligned}
\tag{18}
$$

$$
\begin{aligned}
b_{\vec{n},\vec{m}}^{0} &= \frac{\int_\Delta x\, A_{\vec{n}+\vec{\mathfrak{s}}_{1},\vec{m}-\vec{\mathfrak{t}}_{1}}(x) B_{\vec{n},\vec{m}}(x)\,\mathrm{d}\,w(x)}{\int_\Delta A_{\vec{n}+\vec{\mathfrak{s}}_{1},\vec{m}-\vec{\mathfrak{t}}_{1}}(x) B_{\vec{n},\vec{m}}(x)\,\mathrm{d}\,w(x)} \\
&= \sum_{k=1}^{q} \frac{B_{\vec{n},\vec{m}}^{(k)}[m_k-1]}{B_{\vec{n},\vec{m}}^{(\pi_{\mathfrak{t}}(1))}[m_{\pi_{\mathfrak{t}}(1)}-1]} \frac{\int_\Delta x^{m_k} A_{\vec{n}+\vec{\mathfrak{s}}_{1},\vec{m}-\vec{\mathfrak{t}}_{1}}(x) v_k(x)\,\mathrm{d}\,w(x)}{\int_\Delta x^{m_{\pi_{\mathfrak{t}}(1)}-1} A_{\vec{n}+\vec{\mathfrak{s}}_{1},\vec{m}-\vec{\mathfrak{t}}_{1}}(x) v_{\pi_{\mathfrak{t}}(1)}(x)\,\mathrm{d}\,w(x)} + \frac{B_{\vec{n},\vec{m}}^{(\pi_{\mathfrak{t}}(1))}[m_{\pi_{\mathfrak{t}}(1)}-2]}{B_{\vec{n},\vec{m}}^{(\pi_{\mathfrak{t}}(1))}[m_{\pi_{\mathfrak{t}}(1)}-1]} \\
&= \sum_{k=1}^{p} \frac{A_{\vec{n}+\vec{\mathfrak{s}}_{1},\vec{m}-\vec{\mathfrak{t}}_{1}}^{(k)}[n_k-1]}{A_{\vec{n}+\vec{\mathfrak{s}}_{1},\vec{m}-\vec{\mathfrak{t}}_{1}}^{(\pi_{\mathfrak{s}}(1))}[n_{\pi_{\mathfrak{s}}(1)}]} \frac{\int_\Delta x^{n_k} B_{\vec{n},\vec{m}}(x) u_k(x)\,\mathrm{d}\,w(x)}{\int_\Delta x^{n_{\pi_{\mathfrak{s}}(1)}} B_{\vec{n},\vec{m}}(x) u_{\pi_{\mathfrak{s}}(1)}(x)\,\mathrm{d}\,w(x)} + \frac{\int_\Delta x^{n_{\pi_{\mathfrak{s}}(1)}+1} B_{\vec{n},\vec{m}}(x) u_{\pi_{\mathfrak{s}}(1)}(x)\,\mathrm{d}\,w(x)}{\int_\Delta x^{n_{\pi_{\mathfrak{s}}(1)}} B_{\vec{n},\vec{m}}(x) u_{\pi_{\mathfrak{s}}(1)}(x)\,\mathrm{d}\,w(x)},
\end{aligned}
\tag{19}
$$

$$
\begin{aligned}
b_{\vec{n},\vec{m}}^{-i} &= \frac{\int_\Delta x\, A_{\vec{n}-\vec{s}_{i-1},\vec{m}-\vec{\mathfrak{t}}_{i+1}}(x) B_{\vec{n},\vec{m}}(x)\,\mathrm{d}\,w(x)}{\int_\Delta A_{\vec{n}-\vec{s}_{i-1},\vec{m}-\vec{\mathfrak{t}}_{i+1}}(x) B_{\vec{n}-\vec{s}_{i},\vec{m}-\vec{\mathfrak{t}}_{i}}(x)\,\mathrm{d}\,w(x)} \\
&= \sum_{k=i}^{p} \frac{A_{\vec{n}-\vec{s}_{i-1},\vec{m}-\vec{\mathfrak{t}}_{i+1}}^{(\pi_s(k))}[n_{\pi_s(k)}-1]\int_\Delta x^{n_{\pi_s(k)}} B_{\vec{n},\vec{m}}(x) u_{\pi_s(k)}(x)\,\mathrm{d}\,w(x)}{A_{\vec{n}-\vec{s}_{i-1},\vec{m}-\vec{\mathfrak{t}}_{i+1}}^{(\pi_s(i))}[n_{\pi_s(i)}-1]\int_\Delta x^{n_{\pi_s(i)}-1} B_{\vec{n}-\vec{s}_{i},\vec{m}-\vec{\mathfrak{t}}_{i}}(x) u_{\pi_s(i)}(x)\,\mathrm{d}\,w(x)}.
\end{aligned}
\tag{20}
$$

3. The step line

We say that a pair of multi-indices $(\vec{n},\vec{m})$ lies on the step line if it has the form

$$
\begin{aligned}
\vec{n} &= (\underbrace{\mathfrak{r}+1,\dots,\mathfrak{r}+1}_{\mathfrak{l}\text{ times}},\underbrace{\mathfrak{r},\dots,\mathfrak{r}}_{p-\mathfrak{l}\text{ times}}), && \mathfrak{r}\in\mathbb{N}_0, && \mathfrak{l}\in\{0,\dots,p-1\}, \\
\vec{m} &= (\underbrace{r+1,\dots,r+1}_{l\text{ times}},\underbrace{r,\dots,r}_{q-l\text{ times}}), && r\in\mathbb{N}_0, && l\in\{0,\dots,q-1\}.
\end{aligned}
\tag{21}
$$

Thus $|\vec{n}| = p\mathfrak{r}+\mathfrak{l}$ and $|\vec{m}| = qr+l$. Equivalently, their components are

$$
n_i = \left\lceil \frac{|\vec{n}|+1-i}{p} \right\rceil \qquad \text{and} \qquad m_j = \left\lceil \frac{|\vec{m}|+1-j}{q} \right\rceil.
$$

We use the following compatible normalizations.

If $|\vec{m}| = |\vec{n}|+1$, write $|\vec{n}| = qr+l$, where $l\in\{0,\dots,q-1\}$, and normalize the type-II polynomials in (6) by

$$
B_{\vec{n},\vec{m}}^{(l+1)}[r] = 1. \tag{22}
$$

Thus $B_{\vec{n},\vec{m}}^{(l+1)}$ is monic of degree r.

If, on the other hand, $|\vec{n}| = |\vec{m}|+1$, write $|\vec{m}| = qr+l$, where $l\in\{0,\dots,q-1\}$, and use the type-I normalization

$$
\int_\Delta x^r A_{\vec{n},\vec{m}}(x) v_{l+1}(x)\,\mathrm{d}\,w(x) = 1. \tag{23}
$$

The step-line multi-indices in (21) can be ordered so that each modulus $|\vec{n}| = p\mathfrak{r}+\mathfrak{l}$ and $|\vec{m}| = qr+l$ increases by one along its corresponding sequence. We may therefore relabel the polynomials by the modulus of the corresponding multi-index. Thus, we can relabel the polynomials (6) as

$$B_n^{(j)} = B_{p\mathfrak{r}+\mathfrak{l}}^{(j)} \coloneqq B_{\vec{n},\vec{m}}^{(j)}, \qquad j \in \{1,\dots,q\}, \qquad n \in \mathbb{N}_0,$$

where $\vec{n} \in \mathbb{N}_0^p$ and $\vec{m} \in \mathbb{N}_0^q$ are the corresponding step-line multi-indices satisfying $|\vec{n}| = n$, $|\vec{m}| = n+1$. Similarly, we can relabel the polynomials (5) as

$$A_m^{(i)} = A_{qr+l}^{(i)} \coloneqq A_{\vec{n},\vec{m}}^{(i)}, \qquad i \in \{1,\dots,p\}, \qquad m \in \mathbb{N}_0,$$

where $\vec{n} \in \mathbb{N}_0^p$ and $\vec{m} \in \mathbb{N}_0^q$ are the corresponding step-line multi-indices satisfying $|\vec{n}| = m+1$, $|\vec{m}| = m$. In the same way, relabel the corresponding linear forms by

$$B_n(x) = \sum_{j=1}^{q} B_n^{(j)}(x) v_j(x), \qquad A_m(x) = \sum_{i=1}^{p} A_m^{(i)}(x) u_i(x), \qquad n,m \in \mathbb{N}_0.$$

Equation (7) gives the biorthogonality condition

$$\int_\Delta B_n(x) A_m(x)\, \mathrm{d}\, w(x) = \delta_{n,m}. \tag{24}$$

For the step-line multi-indices in (21), write

$$n+1 = qr_+ + l_+, \qquad r_+ \in \mathbb{N}_0, \qquad l_+ \in \{0,\dots,q-1\}.$$

The remainder l_+, rather than l, determines the two paths in the q-component type-II multi-index, because that multi-index has modulus $n+1$. Consider the following permutations:

$$\pi_s = \begin{bmatrix} 1 & 2 \cdots\cdots \mathfrak{l} & \mathfrak{l}+1 & \mathfrak{l}+2 \cdots\cdots p \\ \mathfrak{l} & \mathfrak{l}-1 \cdots\cdots 1 & p & p-1 \cdots\cdots \mathfrak{l}+1 \end{bmatrix},$$

$$\pi_{\mathfrak{s}} = \begin{bmatrix} 1 & 2 \cdots\cdots p-\mathfrak{l} & p-\mathfrak{l}+1 & p-\mathfrak{l}+2 \cdots\cdots p \\ \mathfrak{l}+1 & \mathfrak{l}+2 \cdots\cdots p & 1 & 2 \cdots\cdots \mathfrak{l} \end{bmatrix},$$

$$\pi_t = \begin{bmatrix} 1 & 2 \cdots\cdots q-l_+ & q-l_+ +1 & q-l_+ +2 \cdots\cdots q \\ l_+ +1 & l_+ +2 \cdots\cdots q & 1 & 2 \cdots\cdots l_+ \end{bmatrix},$$

$$\pi_{\mathfrak{t}} = \begin{bmatrix} 1 & 2 \cdots\cdots l_+ & l_+ +1 & l_+ +2 \cdots\cdots q \\ l_+ & l_+ -1 \cdots\cdots 1 & q & q-1 \cdots\cdots l_+ +1 \end{bmatrix}.$$

With these choices, $(\vec{n}+\vec{\mathfrak{s}}_j, \vec{m}+\vec{t}_j)$ is the type-II step-line pair associated with B_{n+j}, and $(\vec{n}-\vec{s}_i, \vec{m}-\vec{\mathfrak{t}}_i)$ is the pair associated with B_{n-i} whenever it is componentwise nonnegative. Empty blocks in the displayed permutations are omitted.

Using these quantities, we relabel the recurrence coefficients in (9) as

$$b_n^k = b_{p\mathfrak{r}+\mathfrak{l}}^k \coloneqq b_{\vec{n},\vec{m}}^k, \qquad k \in \{-p,\dots,-1,0,1,\dots,q\}, \qquad n \in \mathbb{N}_0,$$

where $\vec{n} \in \mathbb{N}_0^p$ and $\vec{m} \in \mathbb{N}_0^q$ are the corresponding step-line multi-indices satisfying $|\vec{n}| = n$, $|\vec{m}| = n+1$. The type-II recurrence (9) becomes

$$xB_n(x) = \sum_{j=1}^{q} b_n^j\, B_{n+j}(x) + b_n^0\, B_n(x) + \sum_{i=1}^{p} b_n^{-i}\, B_{n-i}(x), \qquad n \in \mathbb{N}_0. \tag{25}$$

The analogous type-I recurrence is

$$xA_m(x) = \sum_{j=1}^{q} a_m^{-j}\, A_{m-j}(x) + a_m^0\, A_m(x) + \sum_{i=1}^{p} a_m^i\, A_{m+i}(x), \qquad m \in \mathbb{N}_0.$$

Biorthogonality (24) gives

$$a_m^{-j} = b_{m-j}^{j}, \qquad a_m^0 = b_m^0, \qquad \text{and} \qquad a_m^i = b_{m+i}^{-i}$$

and therefore

$$xA_m(x) = \sum_{j=1}^{q} b_{m-j}^{j} A_{m-j}(x) + b_m^0 A_m(x) + \sum_{i=1}^{p} b_{m+i}^{-i} A_{m+i}(x), \qquad m \in \mathbb{N}_0. \tag{26}$$

Here $A_n = B_n = 0$ for every $n < 0$.

For $0 \leq n < p$, set $b_n^{-i} = 0$ when $i > n$. The proof of the preceding proposition applies after the negative-index forms are omitted. Indeed, the space determined by the same degree bounds and shifted moment conditions has dimension $q + 1 + n$, while $\{B_{n+1}, \ldots, B_{n+q}, B_n, B_{n-1}, \ldots, B_0\}$ consists of $q + 1 + n$ biorthogonally independent elements. Since xB_n belongs to this space, its expansion in that basis proves (25) at the boundary. The type-I boundary recurrence follows by duality.

Using (24), the recurrence coefficients, including the boundary coefficients just defined, are

$$b_n^k = \int_\Delta x\, A_{n+k}(x) B_n(x)\, \mathrm{d}\, w(x), \qquad k \in \{-p, \ldots, -1, 0, 1, \ldots, q\}, \qquad n \in \mathbb{N}_0. \tag{27}$$

With this notation, (25) and (26) take the matrix form

$$T \begin{bmatrix} B_0(x) \\ B_1(x) \\ B_2(x) \\ \vdots \end{bmatrix} = x \begin{bmatrix} B_0(x) \\ B_1(x) \\ B_2(x) \\ \vdots \end{bmatrix}, \qquad T^{\top} \begin{bmatrix} A_0(x) \\ A_1(x) \\ A_2(x) \\ \vdots \end{bmatrix} = x \begin{bmatrix} A_0(x) \\ A_1(x) \\ A_2(x) \\ \vdots \end{bmatrix}.$$

where the banded matrix is

$$T := \begin{bmatrix} b_0^0 & b_0^1 & \cdots & 1 & 0 & \cdots \\ b_1^{-1} & b_1^0 & b_1^1 & \ddots & \ddots & \ddots \\ \vdots & b_2^{-1} & b_2^0 & \ddots & \ddots & \ddots \\ b_p^{-p} & \ddots & \ddots & \ddots & \ddots & \ddots \\ 0 & \ddots & \ddots & \ddots & \ddots & \ddots \\ \vdots & \ddots & \ddots & \ddots & \ddots & \ddots \end{bmatrix}.$$

Notice that the normalization conditions (22), (23) imply that $b_n^q = 1, n \in \mathbb{N}_0$.

Equations (24) and (27) give the integral representation

$$(T^n)_{i,j} = \int_\Delta x^n B_{i-1}(x) A_{j-1}(x)\ \mathrm{d}\, w(x), \qquad n \in \mathbb{N}_0, \qquad i, j \in \mathbb{N}.$$

Its leading $m \times m$ principal truncation is

$$T_m := \begin{bmatrix} b_0^0 & b_0^1 & \cdots & 1 & 0 & \cdots & 0 \\ b_1^{-1} & b_1^0 & b_1^1 & & & & \vdots \\ \vdots & b_2^{-1} & b_2^0 & \ddots & & & 0 \\ b_p^{-p} & & & & \ddots & & 1 \\ 0 & \ddots & & & & \ddots & \vdots \\ \vdots & & \ddots & & & & b_{m-2}^1 \\ 0 & \cdots & 0 & b_{m-1}^{-p} & \cdots & b_{m-1}^{-1} & b_{m-1}^0 \end{bmatrix}, \quad m \in \mathbb{N}.$$

3.1. **The step line and the matrix of moments for a general matrix of measures**. We now pass to the moment matrix of a general matrix of measures. The Gauss–Borel factorization recovers mixed orthogonality on the step line and expresses the recurrence matrix in terms of triangular factors [15]. Bidiagonal factorizations induced by elementary Christoffel perturbations were studied in [17]; below we adapt that construction to the normalization used here.

Consider a general matrix of measures

$$\mathrm{d}\,\mu = \begin{bmatrix} \mathrm{d}\,\mu_{1,1} & \cdots & \mathrm{d}\,\mu_{1,p} \\ \vdots & & \vdots \\ \mathrm{d}\,\mu_{q,1} & \cdots & \mathrm{d}\,\mu_{q,p} \end{bmatrix},$$

where the measures $\mu_{i,j}$ are supported on the interval $\Delta \subseteq \mathbb{R}$. For $r \in \mathbb{N}$, define the matrix of monomials by

$$X_{[r]}(x) = \begin{bmatrix} I_r \\ xI_r \\ x^2 I_r \\ \vdots \end{bmatrix},$$

where I_r is the usual identity matrix of order r. The moment matrix is

$$\mathscr{M}(\mu) := \int_\Delta X_{[q]}(x)\,\mathrm{d}\mu(x)\,X_{[p]}^\top(x).$$

When the measure is clear from context, we simply write $\mathscr{M}$.

3.2. **Gauss–Borel factorization**. If all the leading principal submatrices $\mathscr{M}^{[k]}$ are nonsingular, then the Gauss–Borel factorization exists:

$$\mathscr{M} = \mathscr{L}^{-1}\mathscr{U}^{-1},$$

where $\mathscr{L}$ is a nonsingular lower unitriangular semi-infinite matrix and $\mathscr{U}$ is a nonsingular upper triangular matrix. When necessary, we will denote these triangular matrices as $\mathscr{L}(\mu)$ and $\mathscr{U}(\mu)$, indicating the measure μ from which they are constructed. All semi-infinite identities below are understood coefficientwise, or equivalently through their compatible finite leading truncations; bandedness ensures that every displayed matrix entry involves only finitely many terms.

3.3. **Mixed multiple orthogonal polynomials on the step-line**. Associated with the Gauss–Borel factorization, consider the following matrices of polynomials:

$$B(x) = \mathscr{L} X_{[q]}(x), \qquad A(x) = X_{[p]}^{\top}(x)\mathscr{U}.$$

We represent these matrices in terms of their polynomial entries as follows:

$$B = \begin{bmatrix} B_0^{(1)} & \cdots & B_0^{(q)} \\ B_1^{(1)} & \cdots & B_1^{(q)} \\ B_2^{(1)} & \cdots & B_2^{(q)} \\ \vdots & & \vdots \end{bmatrix}, \qquad A = \begin{bmatrix} A_0^{(1)} & A_1^{(1)} & A_2^{(1)} & \cdots \\ \vdots & \vdots & \vdots & \\ A_0^{(p)} & A_1^{(p)} & A_2^{(p)} & \cdots \end{bmatrix}.$$

We have the following relations:

$$\int_{\Delta} B(x)\, \mathrm{d}\mu(x)\, A(x) = I,$$

whose entries are given by the biorthogonality relations:

$$\int_{\Delta} \sum_{b=1}^{q} \sum_{a=1}^{p} B_n^{(b)}(x)\, \mathrm{d}\mu_{b,a}(x)\, A_m^{(a)}(x) = \delta_{n,m}.$$

The Gauss–Borel factorization also gives

$$\int_{\Delta} B(x)\, \mathrm{d}\mu(x) X_{[p]}^{\top}(x) = \mathscr{U}^{-1}, \qquad \int_{\Delta} X_{[q]}(x)\, \mathrm{d}\mu(x) A(x) = \mathscr{L}^{-1},$$

which, entrywise, give the following mixed multiple orthogonality relations on the step line:

$$\int_{\Delta} x^l \sum_{i=1}^{p} \mathrm{d}\mu_{j,i}(x) A_n^{(i)}(x) = 0, \qquad j \in \{1,\dots,q\}, \qquad l \in \left\{0,\dots,\left\lceil \frac{n-j+1}{q} \right\rceil - 1\right\},$$

$$\int_{\Delta} \sum_{j=1}^{q} B_n^{(j)}(x) \mathrm{d}\mu_{j,i}(x) x^l = 0, \qquad i \in \{1,\dots,p\}, \qquad l \in \left\{0,\dots,\left\lceil \frac{n-i+1}{p} \right\rceil - 1\right\}.$$

The component $A_n^{(i)}$ corresponds to $A_{\vec{n},\vec{m}}^{(i)}$, where $\vec{n} = (n_1,\dots,n_p)$ and $\vec{m} = (m_1,\dots,m_q)$ are defined by

$$n_i = \left\lceil \frac{|\vec{n}| + 1 - i}{p} \right\rceil = \left\lceil \frac{n+2-i}{p} \right\rceil \qquad \text{and} \qquad m_j = \left\lceil \frac{|\vec{m}| + 1 - j}{q} \right\rceil = \left\lceil \frac{n+1-j}{q} \right\rceil.$$

Lemma 2.1 of [15] gives $|\vec{n}| = n+1$ and $|\vec{m}| = n$. Similarly, $B_n^{(j)}$ corresponds to $B_{\vec{n},\vec{m}}^{(j)}$, with $\vec{n} = (n_1,\dots,n_p)$ and $\vec{m} = (m_1, m_2, \dots, m_q)$ given by

$$n_i = \left\lceil \frac{n+1-i}{p} \right\rceil \qquad \text{and} \qquad m_j = \left\lceil \frac{n+2-j}{q} \right\rceil.$$

In this case $|\vec{m}| = |\vec{n}| + 1$.

3.4. **Banded recurrence matrix**. For $r \in \mathbb{N}$, define the block shift matrix by

$$\Lambda_{[r]} \coloneqq \begin{bmatrix} 0_r & I_r & 0_r & \cdots & \\ & \ddots & \ddots & \ddots & \\ 0_r & 0_r & I_r & \ddots & \\ & \ddots & \ddots & \ddots & \ddots \\ \vdots & \ddots & \ddots & \ddots & \ddots \end{bmatrix},$$

For $r = 1$, write $\Lambda_{[1]} = \Lambda$. Then $\Lambda_{[r]} = \Lambda^r$, and

$$\Lambda_{[r]} X_{[r]}(x) = x X_{[r]}(x).$$

The moment matrix $\mathscr{M}$ satisfies the Hankel-type symmetry

$$\Lambda_{[q]} \mathscr{M} = \mathscr{M} \Lambda_{[p]}^{\top}.$$

This relation and the Gauss–Borel factorization give

$$T = \mathscr{L} \Lambda_{[q]} \mathscr{L}^{-1} = \mathscr{U}^{-1} \Lambda_{[p]}^{\top} \mathscr{U}, \tag{28}$$

Thus T is a (p, q)-banded matrix with p subdiagonals and q superdiagonals. Moreover,

$$T B(x) = x B(x), \qquad A(x) T = x A(x),$$

so B and A are, respectively, right and left eigenvectors of T. These identities encode the recurrence relations; hence T is the recurrence matrix.

3.5. **Elementary Christoffel perturbations**. Define the $r \times r$ polynomial matrix

$$\mathfrak{X}_{[r]}(x) = \begin{bmatrix} 0 & 1 & 0 & \cdots & 0 & 0 \\ 0 & 0 & 1 & \ddots & & 0 \\ \vdots & \vdots & \ddots & \ddots & \ddots & \vdots \\ 0 & 0 & & \ddots & 1 & 0 \\ 0 & 0 & \cdots & \cdots & 0 & 1 \\ x & 0 & \cdots & \cdots & 0 & 0 \end{bmatrix}.$$

It satisfies $\mathfrak{X}_{[r]}^r = x I_r$ and $X_{[r]} \mathfrak{X}_{[r]} = \Lambda X_{[r]}$.

For $k \in \mathbb{N}_0$, define the transformed matrices of measures by

$$\mathrm{d}\mu_L^{(k)} := \mathrm{d}\mu \left(\mathfrak{X}_{[p]}^k \right)^{\top}, \qquad \mathrm{d}\mu_R^{(k)} := \mathfrak{X}_{[q]}^k \mathrm{d}\mu.$$

For $k = 0$, we have $\mathrm{d}\mu_R^{(0)} = \mathrm{d}\mu_L^{(0)} = \mathrm{d}\mu$. These are the Christoffel perturbations of $\mathrm{d}\mu$. Their moment matrices satisfy

$$\mathscr{M}_L^{(k)} := \mathscr{M}\left(\mu_L^{(k)}\right) = \mathscr{M}(\mu) \left(\Lambda^k\right)^{\top}, \qquad \mathscr{M}_R^{(k)} := \mathscr{M}\left(\mu_R^{(k)}\right) = \Lambda^k \mathscr{M}(\mu).$$

Assume that the perturbed moment matrices admit Gauss–Borel factorizations for every left perturbation $k = 0, \ldots, p$ and every right perturbation $k = 0, \ldots, q$:

$$\mathscr{M}_L^{(k)} = \left(\mathscr{L}_L^{(k)}\right)^{-1} \left(\mathscr{U}_L^{(k)}\right)^{-1}, \qquad \mathscr{M}_R^{(k)} = \left(\mathscr{L}_R^{(k)}\right)^{-1} \left(\mathscr{U}_R^{(k)}\right)^{-1}.$$

In both cases, $\mathscr{L}_L^{(k)}$ and $\mathscr{L}_R^{(k)}$ are lower unitriangular matrices.

4. Bidiagonal factorizations of the banded recurrence matrix

We now consider the k-th left and right Christoffel perturbations.

The triangular factors of the perturbed moment matrices are connected by elementary bidiagonal matrices. More precisely,

Proposition 4.1. *The left and right transformations satisfy the following factorization identities.*

i) *For the k-th left Christoffel transformation, there exist lower bidiagonal matrices with unit diagonal $L_1, \ldots, L_k$ such that*

$$\mathscr{U}_L^{-1} \left(\Lambda^k\right)^{\top} = L_1 \cdots L_k \left(\mathscr{U}_L^{(k)}\right)^{-1}, \qquad \mathscr{L}_L^{(k)} = L_k^{-1} \cdots L_1^{-1} \mathscr{L}_L. \tag{29}$$

ii) *For the k-th right Christoffel transformation, there exist upper bidiagonal matrices with unit superdiagonal $U_1, \dots, U_k$ such that*

$$\Lambda^k \mathscr{L}_R^{-1} = \left(\mathscr{L}_R^{(k)}\right)^{-1} U_k \cdots U_1, \qquad \mathscr{U}_R^{(k)} = \mathscr{U}_R U_1^{-1} \cdots U_k^{-1}. \tag{30}$$

Proof. i) From the Gauss–Borel factorizations of $\mathscr{M}_L^{(1)}$ and $\mathscr{M}$, we obtain:

$$\left(\mathscr{L}_L^{(1)}\right)^{-1} \left(\mathscr{U}_L^{(1)}\right)^{-1} = \mathscr{L}_L^{-1} \mathscr{U}_L^{-1} \Lambda^\top.$$

The matrix $\mathscr{U}_L^{-1}\Lambda^\top$ is upper Hessenberg, with Gauss–Borel factorization

$$\mathscr{U}_L^{-1}\Lambda^\top = L_1 \left(\mathscr{U}_L^{(1)}\right)^{-1},$$

where L_1 is a lower bidiagonal matrix with unit diagonal. Consequently:

$$\mathscr{L}_L^{(1)} = L_1^{-1} \mathscr{L}_L.$$

In general, since

$$\mathscr{M}_L^{(k+1)} = \mathscr{M}_L^{(k)} \Lambda^\top,$$

we obtain:

$$\left(\mathscr{U}_L^{(k)}\right)^{-1} \Lambda^\top = L_{k+1} \left(\mathscr{U}_L^{(k+1)}\right)^{-1}, \quad \mathscr{L}_L^{(k+1)} = L_{k+1}^{-1} \mathscr{L}_L^{(k)},$$

where L_{k+1} is a lower bidiagonal matrix with unit main diagonal. This proves (29).

ii) Consider the right Christoffel perturbations. For the Gauss–Borel factorization of $\mathscr{M}_R^{(1)}$, we have:

$$\mathscr{M}_R^{(1)} = \Lambda \mathscr{L}_R^{-1} \mathscr{U}_R^{-1}.$$

The matrix $\Lambda\mathscr{L}^{-1}$ is a lower Hessenberg matrix whose Gauss–Borel factorization is:

$$\Lambda \mathscr{L}_R^{-1} = \left(\mathscr{L}_R^{(1)}\right)^{-1} U_1,$$

where U_1 is an upper bidiagonal matrix with unit superdiagonal. Consequently:

$$\mathscr{U}_R^{(1)} = \mathscr{U}_R U_1^{-1}.$$

In general, since

$$\mathscr{M}_R^{(k+1)} = \Lambda \mathscr{M}_R^{(k)},$$

we obtain:

$$\Lambda \left(\mathscr{L}_R^{(k)}\right)^{-1} = \left(\mathscr{L}_R^{(k+1)}\right)^{-1} U_{k+1}, \qquad \mathscr{U}_R^{(k+1)} = \mathscr{U}_R^{(k)} U_{k+1}^{-1}.$$

Consequently (30) follows. □

Iterating the elementary Christoffel connections produces the bidiagonal factorization of the recurrence matrix.

Theorem 4.2. *Suppose that the Gauss–Borel factorizations just specified exist for $k = 0, \dots, p$ on the left and $k = 0, \dots, q$ on the right. Then the banded recurrence matrix has the bidiagonal factorization*

$$T = L_1 \cdots L_p U_q \cdots U_1. \tag{31}$$

Proof. Equations (28) and (29) give

$$T = \mathscr{U}_L^{-1} (\Lambda^p)^\top \mathscr{U}_L = L_1 \cdots L_p \left(\mathscr{U}_L^{(p)}\right)^{-1} \mathscr{U}_L,$$

Since $\mathfrak{X}_{[p]}^p = xI_p$ and $\mathfrak{X}_{[q]}^q = xI_q$,

$$\mathrm{d}\,\mu_L^{(p)} = x\ \mathrm{d}\,\mu = \mathrm{d}\,\mu_R^{(q)}.$$

The uniqueness of the normalized Gauss–Borel factorization therefore gives $\mathscr{U}_L^{(p)} = \mathscr{U}_R^{(q)}$. Using (30) we obtain:

$$\left(\mathscr{U}_R^{(q)}\right)^{-1} \mathscr{U}_L = U_q \cdots U_1$$

which proves (31).

□

The proof yields the following corollary.

Corollary 4.3. *The bidiagonal matrices satisfy*

$$L_k = \left(\mathscr{U}_L^{(k-1)}\right)^{-1} \Lambda^\top \mathscr{U}_L^{(k)} = \mathscr{L}_L^{(k-1)} \left(\mathscr{L}_L^{(k)}\right)^{-1}, \qquad U_k = \mathscr{L}_R^{(k)} \Lambda \left(\mathscr{L}_R^{(k-1)}\right)^{-1} = \left(\mathscr{U}_R^{(k)}\right)^{-1} \mathscr{U}_R^{(k-1)},$$

and the corresponding entries are

$$(L_k)_{n+1,n} = \frac{(\mathscr{U}_L^{(k)})_{n,n}}{(\mathscr{U}_L^{(k-1)})_{n+1,n+1}} = (\mathscr{L}_L^{(k-1)})_{n+1,n} - (\mathscr{L}_L^{(k)})_{n+1,n}, \qquad (U_k)_{n,n} = \frac{(\mathscr{U}_R^{(k-1)})_{n,n}}{(\mathscr{U}_R^{(k)})_{n,n}}.$$

Let $r(n,p)$ denote the remainder when n is divided by p. After k left or right Christoffel perturbations, denote the corresponding mixed multiple orthogonal polynomials by $A_L^{(k)}$ and $A_R^{(k)}$, respectively.

Given a polynomial P we denote its leading coefficient by $\mathrm{LC}(P)$. Then, we can express the bidiagonal matrix entries as follows:

Corollary 4.4. *In terms of the leading coefficients of the Christoffel-perturbed mixed multiple orthogonal polynomials, we have:*

$$(L_k)_{n+1,n} = \frac{\mathrm{LC}\left((A_L^{(k)})_n^{(r(n,p)+1)}\right)}{\mathrm{LC}\left((A_L^{(k-1)})_{n+1}^{(r(n+1,p)+1)}\right)}, \qquad (U_k)_{n,n} = \frac{\mathrm{LC}\left((A_R^{(k-1)})_n^{(r(n,p)+1)}\right)}{\mathrm{LC}\left((A_R^{(k)})_n^{(r(n,p)+1)}\right)}.$$

Proof. Write $n = ps + r$, with $0 \le r < p$. From $A = X_{[p]}^\top \mathscr{U}$, the entry $(\mathscr{U})_{n,n}$ multiplies the monomial x^s in component $r+1$ of A_n. Upper triangularity implies that every other contribution to that component has degree at most $s-1$. Hence

$$(\mathscr{U})_{n,n} = \mathrm{LC}\big((A)_n^{(r(n,p)+1)}\big).$$

The same identity holds for each left and right perturbed factorization. Substitution into the two diagonal-entry ratios in the preceding corollary gives the displayed formulas. □

Let $T_L^{(k)}$, respectively $T_R^{(k)}$, denote the recurrence matrix after the k-th left, respectively right, Christoffel transformation. Both use the normalization fixed above. The preceding factorizations identify these transformations as Darboux rearrangements.

Proposition 4.5. *The perturbed recurrence matrices are the Darboux transformations of the recurrence matrix:*

$$T_L^{(k)} = L_{k+1} \cdots L_p U_q \cdots U_1 L_1 \cdots L_k, \qquad k \in \{1, \ldots, p\},$$
$$T_R^{(k)} = U_k \cdots U_1 L_1 \cdots L_p U_q \cdots U_{k+1}, \qquad k \in \{1, \ldots, q\}.$$

Proof. Equations (28), (29), and (30) give

$$\begin{aligned} T_L^{(k)} &= \left(\mathscr{U}_L^{(k)}\right)^{-1} (\Lambda^p)^\top \mathscr{U}_L^{(k)} \\ &= L_{k+1}\cdots L_p \left(\mathscr{U}_L^{(p)}\right)^{-1} (\Lambda^k)^\top \mathscr{U}_L^{(k)} \\ &= L_{k+1}\cdots L_p \left(\mathscr{U}_R^{(q)}\right)^{-1} (\Lambda^k)^\top \mathscr{U}_L^{(k)} \\ &= L_{k+1}\cdots L_p U_q \cdots U_1 \left(\mathscr{U}_R^{(0)}\right)^{-1} (\Lambda^k)^\top \mathscr{U}_L^{(k)} \\ &= L_{k+1}\cdots L_p U_q \cdots U_1 \left(\mathscr{U}_L^{(0)}\right)^{-1} (\Lambda^k)^\top \mathscr{U}_L^{(k)} \\ &= L_{k+1}\cdots L_p U_q \cdots U_1 L_1 \cdots L_k \end{aligned}$$

Again, from (28), (29) and (30)

$$\begin{aligned} T_R^{(k)} &= \left(\mathscr{U}_R^{(k)}\right)^{-1} (\Lambda^p)^\top \mathscr{U}_R^{(k)} \\ &= U_k \cdots U_1 \left(\mathscr{U}_R^{(0)}\right)^{-1} (\Lambda^p)^\top \mathscr{U}_R^{(k)} \\ &= U_k \cdots U_1 \left(\mathscr{U}_L^{(0)}\right)^{-1} (\Lambda^p)^\top \mathscr{U}_R^{(k)} \\ &= U_k \cdots U_1 L_1 \cdots L_p \left(\mathscr{U}_L^{(p)}\right)^{-1} \mathscr{U}_R^{(k)} \\ &= U_k \cdots U_1 L_1 \cdots L_p \left(\mathscr{U}_R^{(q)}\right)^{-1} \mathscr{U}_R^{(k)} \\ &= U_k \cdots U_1 L_1 \cdots L_p U_q \cdots U_{k+1} \end{aligned}$$

□

5. Mixed Piñeiro orthogonal polynomials

Let $\vec{\alpha} = \left[\alpha_1 \cdots\cdots \alpha_p,\right]$, $\vec{\beta} = \left[\beta_1 \cdots\cdots \beta_q\right]$, and introduce a mixed Piñeiro matrix of measures of the form

$$\mathrm{d}\,\mu_{\vec{\alpha},\vec{\beta}} = \begin{bmatrix} x^{\beta_1} \\ \vdots \\ x^{\beta_q} \end{bmatrix} \left[x^{\alpha_1} \cdots\cdots x^{\alpha_p}\right] \mathrm{d}\,x,$$

supported on $\Delta = [0,1]$. We assume $\alpha_i > -1$, $\beta_j > -1$, and

$$\alpha_i + \beta_j > -1, \qquad 1 \le i \le p, \qquad 1 \le j \le q,$$

so that every entry $x^{\alpha_i+\beta_j}\,\mathrm{d}\,x$ is integrable. We also assume $\alpha_i - \alpha_h \notin \mathbb{Z}$ for $i \neq h$ and $\beta_j - \beta_s \notin \mathbb{Z}$ for $j \neq s$. Under these hypotheses the two systems are AT systems and the corresponding matrix of measures is perfect; in particular, all admissible multi-indices are normal [20].

For $|\vec{m}| = |\vec{n}| - 1$, write the type-I linear form as $A_{\vec{n},\vec{m}}(x) = \sum_{i=1}^{p} A_{\vec{n},\vec{m}}^{(i)}(x) x^{\alpha_i}$ where $\deg A_{\vec{n},\vec{m}}^{(i)} \le n_i - 1$. It satisfies

$$\int_0^1 A_{\vec{n},\vec{m}}(x) x^{k+\beta_j}\,\mathrm{d}\,x = 0, \qquad 0 \le k \le m_j - 1, \qquad 1 \le j \le q.$$

Perfectness implies that these conditions determine the form up to a nonzero constant. Motivated by the contour method in [10], we obtain the following direct representation.

Proposition 5.1. *Up to a nonzero constant, the type-I form has the contour representation*

$$A_{\vec{n},\vec{m}}(x) = \int_{\Sigma} \frac{x^t \prod_{j=1}^{q} (t+\beta_j+1)_{m_j}}{\prod_{i=1}^{p} (\alpha_i - t)_{n_i}} \frac{dt}{2\pi i} \tag{32}$$

where $\Sigma \subset \mathbb{C}$ is a clockwise contour enclosing once every point

$$\alpha_i, \alpha_i+1, \ldots, \alpha_i+n_i-1, \qquad 1 \le i \le p,$$

and no other pole of the integrand. For $0 < x < 1$, we use $x^t = \exp(t \log x)$ with the real logarithm.

For later sign checks, recall that our clockwise convention gives

$$\frac{1}{2\pi \mathrm{i}} \int_{\Sigma} f(t)\, \mathrm{d}\, t = - \sum_{a \text{ inside } \Sigma} \operatorname{Res}_{t=a} f(t).$$

Proof. By the residue theorem, the contour integral is a finite linear combination of the functions x^{α_i+k}, $0 \le k < n_i$, and hence has the required type-I form and component degree bounds. We may therefore compute its Mellin transform term by term. Equivalently, for $s = \beta_j + h$, $1 \le h \le m_j$, we have

$$\int_0^1 A_{\vec{n},\vec{m}}(x) x^{s-1}\, \mathrm{d}\, x = \int_{\Sigma} \frac{\prod_{r=1}^{q} (t+\beta_r+1)_{m_r}}{\prod_{i=1}^{p} (\alpha_i - t)_{n_i}} \frac{1}{t+s} \frac{\mathrm{d}\, t}{2\pi \mathrm{i}}.$$

The numerator contains the factor $t + s = t + \beta_j + h$, which cancels the last denominator. Since $|\vec{m}| = |\vec{n}| - 1$, the resulting rational function is $O(t^{-2})$ at infinity, and hence its residue at infinity is zero. After cancellation its only finite poles are the points enclosed by Σ. The global residue theorem therefore gives

$$\sum_{a \text{ inside } \Sigma} \operatorname{Res}_{t=a} f(t) = - \operatorname{Res}_{t=\infty} f(t) = 0.$$

The clockwise residue convention then makes the displayed integral vanish. This proves all the required orthogonality conditions. The residue at $t = \alpha_i$ is nonzero whenever $n_i \ge 1$, under the standing parameter assumptions. Perfectness then identifies the resulting nonzero form, up to its common multiplicative constant, with $A_{\vec{n},\vec{m}}$. □

Taking residues in the preceding contour representation produces the hypergeometric form componentwise.

Corollary 5.2. *For every $i \in \{1, \ldots, p\}$ with $n_i \ge 1$, the integral representation gives the following explicit formula, up to a common nonzero constant:*

$$A^{(i)}_{\vec{n},\vec{m}}(x) = \frac{\prod_{j=1}^{q} (\alpha_i+\beta_j+1)_{m_j}}{(n_i-1)! \prod_{j=1, j\neq i}^{p} (\alpha_j - \alpha_i)_{n_j}} \times {}_{p+q}F_{p+q-1}\left[\begin{matrix} -n_i+1,\ \{\alpha_i+\beta_k+m_k+1\}_{k=1}^{q},\ \{\alpha_i-\alpha_k-n_k+1\}_{k=1,k\neq i}^{p} \\ \{\alpha_i+\beta_k+1\}_{k=1}^{q},\ \{\alpha_i-\alpha_k+1\}_{k=1,k\neq i}^{p} \end{matrix} ; x \right]. \tag{33}$$

The individual coefficients are

$$A^{(i)}_{\vec{n},\vec{m}}[k] = \frac{\prod_{j=1}^{q} (\alpha_i+k+\beta_j+1)_{m_j}}{\prod_{j=1, j\neq i}^{p} (\alpha_j - \alpha_i - k)_{n_j} k! (n_i-k-1)! (-1)^k}, \tag{34}$$

and the leading coefficient is

$$\operatorname{LC}\left(A^{(i)}_{\vec{n},\vec{m}}\right) = A^{(i)}_{\vec{n},\vec{m}}[n_i-1] = \frac{\prod_{j=1}^{q} (\alpha_i+n_i+\beta_j)_{m_j}}{\prod_{j=1, j\neq i}^{p} (\alpha_j - \alpha_i - n_i + 1)_{n_j} (n_i-1)! (-1)^{n_i-1}}.$$

Proof. Write $A^{(i)}_{\vec{n},\vec{m}}(x) = \sum_{k=0}^{n_i-1} A^{(i)}_{\vec{n},\vec{m}}[k]x^k$. The clockwise residues at $t = \alpha_i + k$ give (34); consequently,

$$A^{(i)}_{\vec{n},\vec{m}}(x) = \sum_{k=0}^{n_i-1} \frac{\prod_{j=1}^{q}(\alpha_i + k + \beta_j + 1)_{m_j}}{\prod_{j=1,j\neq i}^{p}(\alpha_j - \alpha_i - k)_{n_j} k!(n_i - k - 1)!(-1)^k} x^k \tag{35}$$

Using the identities

$$\begin{aligned}
\frac{1}{(n_i - k - 1)!(-1)^k} &= \frac{(-n_i + 1)_k}{(n_i - 1)!},\\
(\alpha_i + k + \beta_j + 1)_{m_j} &= \frac{(\alpha_i + \beta_j + 1)_{m_j+k}}{(\alpha_i + \beta_j + 1)_k} = \frac{(\alpha_i + \beta_j + 1)_{m_j}(\alpha_i + \beta_j + 1 + m_j)_k}{(\alpha_i + \beta_j + 1)_k},\\
\frac{1}{(\alpha_j - \alpha_i - k)_{n_j}} &= \frac{(-1)^{n_j}}{(-\alpha_j + \alpha_i + k - n_j + 1)_{n_j}} = \frac{(-1)^{n_j}(-\alpha_j + \alpha_i - n_j + 1)_k}{(-\alpha_j + \alpha_i - n_j + 1)_{n_j+k}}\\
&= \frac{(-1)^{n_j}(-\alpha_j + \alpha_i - n_j + 1)_k}{(-\alpha_j + \alpha_i - n_j + 1)_{n_j}(-\alpha_j + \alpha_i + 1)_k} = \frac{(-\alpha_j + \alpha_i - n_j + 1)_k}{(\alpha_j - \alpha_i)_{n_j}(-\alpha_j + \alpha_i + 1)_k}.
\end{aligned}$$

Equation (35) becomes

$$A^{(i)}_{\vec{n},\vec{m}}(x) = \frac{\prod_{j=1}^{q}(\alpha_i + \beta_j + 1)_{m_j}}{(n_i - 1)!\prod_{j\neq i}(\alpha_j - \alpha_i)_{n_j}} \sum_{k=0}^{\infty} \frac{(-n_i + 1)_k \prod_{j=1}^{q}(\alpha_i + \beta_j + 1 + m_j)_k \prod_{j\neq i}(\alpha_i - \alpha_j - n_j + 1)_k}{\prod_{j=1}^{q}(\alpha_i + \beta_j + 1)_k \prod_{j\neq i}(\alpha_i - \alpha_j + 1)_k} \frac{x^k}{k!},$$

which is (33). □

The same contour representation also gives all mixed moments of the type-I form.

Proposition 5.3. *For $s \in \{1, \dots, q\}$, these moments are*

$$\int_0^1 x^k x^{\beta_s} A_{\vec{n},\vec{m}}(x)\, \mathrm{d}x = \frac{\prod_{j=1}^{q}(\beta_j - \beta_s - k)_{m_j}}{\prod_{i=1}^{p}(\alpha_i + \beta_s + k + 1)_{n_i}}, \qquad k \in \mathbb{N}_0. \tag{36}$$

Proof. Using (32), we obtain

$$\int_0^1 x^k x^{\beta_s} A_{\vec{n},\vec{m}}(x) = \int_\Sigma \frac{\prod_{j=1}^{q}(t + \beta_j + 1)_{m_j}}{\prod_{i=1}^{p}(\alpha_i - t)_{n_i}} \frac{1}{k + t + \beta_s + 1} \frac{\mathrm{d}t}{2\pi\mathrm{i}}.$$

The rational integrand is $O(t^{-2})$ at infinity. If $k < m_s$, the factor $t + \beta_s + k + 1$ cancels and both sides vanish. If $k \geq m_s$, evaluation from the exterior of the clockwise contour leaves the single pole $t = -(k + \beta_s + 1)$; its residue is the quotient displayed above. □

For $|\vec{m}| = |\vec{n}| + 1$, write the type-II linear form as $B_{\vec{n},\vec{m}}(x) = \sum_{j=1}^{q} B^{(j)}_{\vec{n},\vec{m}}(x)x^{\beta_j}$. It satisfies

$$\int_0^1 B_{\vec{n},\vec{m}}(x) x^{k+\alpha_i}\, \mathrm{d}x = 0, \qquad 0 \leq k \leq n_i - 1, \qquad 1 \leq i \leq p.$$

Interchanging the two parameter families gives the dual contour representation.

Proposition 5.4. *Up to a nonzero constant, the type-II form has the contour representation*

$$B_{\vec{n},\vec{m}}(x) = \int_\Sigma \frac{x^t \prod_{i=1}^{p}(t + \alpha_i + 1)_{n_i}}{\prod_{j=1}^{q}(\beta_j - t)_{m_j}} \frac{dt}{2\pi i} \tag{37}$$

where Σ is a clockwise contour enclosing once every pole $\beta_j, \beta_j + 1, \dots, \beta_j + m_j - 1$.

Proof. This is the type-I argument above with $(\vec{\alpha},\vec{n})$ and $(\vec{\beta},\vec{m})$ interchanged. □

Evaluating the residues of the dual contour integral yields the type-II components explicitly.

Corollary 5.5. *For every $j \in \{1,\dots,q\}$ with $m_j \geq 1$, the integral representation gives the following explicit formula, up to a common nonzero constant:*

$$B^{(j)}_{\vec{n},\vec{m}}(x) = \frac{\prod_{i=1}^{p}(\alpha_i+\beta_j+1)_{n_i}}{(m_j-1)!\prod_{i=1,i\neq j}^{q}(\beta_i-\beta_j)_{m_i}} \times {}_{p+q}F_{p+q-1}\left[\begin{matrix}-m_j+1,\ \{\alpha_k+\beta_j+n_k+1\}_{k=1}^{p},\ \{\beta_j-\beta_k-m_k+1\}_{k=1,k\neq j}^{q}\\ \{\alpha_k+\beta_j+1\}_{k=1}^{p},\ \{\beta_j-\beta_k+1\}_{k=1,k\neq j}^{q}\end{matrix};x\right].$$

The individual coefficients are

$$B^{(j)}_{\vec{n},\vec{m}}[k] = \frac{\prod_{i=1}^{p}(\alpha_i+k+\beta_j+1)_{n_i}}{\prod_{i=1,i\neq j}^{q}(\beta_i-\beta_j-k)_{m_i}k!(m_j-k-1)!(-1)^k}, \tag{38}$$

$$\mathrm{LC}\left(B^{(j)}_{\vec{n},\vec{m}}\right) = B^{(j)}_{\vec{n},\vec{m}}[m_j-1] = \frac{\prod_{i=1}^{p}(\beta_j+m_j+\alpha_i)_{n_i}}{\prod_{i=1,i\neq j}^{q}(\beta_i-\beta_j-m_j+1)_{m_i}(m_j-1)!(-1)^{m_j-1}}. \tag{39}$$

Proof. Interchange $(\vec{\alpha},\vec{n})$ and $(\vec{\beta},\vec{m})$ in the residue calculation used for the type-I components. □

The dual contour representation likewise determines all mixed moments of the type-II form.

Proposition 5.6. *For $s \in \{1,\dots,p\}$, these moments are*

$$\int_0^1 x^k x^{\alpha_s} B_{\vec{n},\vec{m}}(x)\,\mathrm{d}x = \frac{\prod_{i=1}^{p}(\alpha_i-\alpha_s-k)_{n_i}}{\prod_{j=1}^{q}(\beta_j+\alpha_s+k+1)_{m_j}}, \qquad k\in\mathbb{N}_0. \tag{40}$$

Proof. This is the type-I moment calculation with the two parameter systems interchanged. □

5.1. **Step line.** For each $n\in\mathbb{N}_0$, let $A_n := A_{\vec{n},\vec{m}}$, where $|\vec{n}| = n+1$ and $|\vec{m}| = n$, with the contour representation (32). The components of $\vec{n} = (n_{1,n}, n_{2,n},\dots,n_{p,n})$ and $\vec{m} = (m_{1,n}, m_{2,n},\dots,m_{q,n})$ are

$$n_{i,n} = \left\lceil \frac{|\vec{n}|+1-i}{p}\right\rceil = \left\lceil \frac{n+2-i}{p}\right\rceil \qquad \text{and} \qquad m_{j,n} = \left\lceil \frac{|\vec{m}|+1-j}{q}\right\rceil = \left\lceil \frac{n+1-j}{q}\right\rceil. \tag{41}$$

Write $n = q\nu+\rho$, where $0\leq\rho<q$, and set $r(n,q) := \rho$. The quotient ν will not be used below. We first fix the normalization of the type-I form on the step line.

Proposition 5.7. *Impose the normalization*

$$\int_0^1 x^{m_{r(n,q)+1,n}} x^{\beta_{r(n,q)+1}} A_n(x)\,\mathrm{d}x = 1. \tag{42}$$

Then

$$A_n(x) = \frac{\prod_{i=1}^{p}(\alpha_i+\beta_{r(n,q)+1}+m_{r(n,q)+1,n}+1)_{n_{i,n}}}{\prod_{j=1}^{q}(\beta_j-\beta_{r(n,q)+1}-m_{r(n,q)+1,n})_{m_{j,n}}}\int_\Sigma \frac{x^t\prod_{j=1}^{q}(t+\beta_j+1)_{m_{j,n}}}{\prod_{i=1}^{p}(\alpha_i-t)_{n_{i,n}}}\frac{dt}{2\pi i},$$

where $n_{i,n}$ and $m_{j,n}$ are given by (41).

Proof. Set $s = r(n,q)+1$ and $k = m_{s,n}$ in (36). The displayed prefactor is exactly the reciprocal of that moment, so the normalized moment equals 1. □

Substitution of the step-line multi-indices into the type-I formula gives the normalized components.

Corollary 5.8. *For every $i \in \{1,\dots,p\}$ with $n_{i,n} \geq 1$, we obtain the explicit representation of the mixed polynomial $A_n^{(i)}(x)$, with the normalization condition* (42)

$$A_n^{(i)}(x) = \frac{\prod_{j=1}^{p}(\alpha_j+\beta_{r(n,q)+1}+m_{r(n,q)+1,n}+1)_{n_{j,n}}}{\prod_{j=1}^{q}(\beta_j-\beta_{r(n,q)+1}-m_{r(n,q)+1,n})_{m_{j,n}}} \frac{\prod_{j=1}^{q}(\alpha_i+\beta_j+1)_{m_{j,n}}}{(n_{i,n}-1)!\prod_{j=1,j\neq i}^{p}(\alpha_j-\alpha_i)_{n_{j,n}}} \times {}_{p+q}F_{p+q-1}\left[\begin{matrix}-n_{i,n}+1,\ \{\alpha_i+\beta_k+m_{k,n}+1\}_{k=1}^{q},\ \{\alpha_i-\alpha_k-n_{k,n}+1\}_{k=1,k\neq i}^{p}\\ \{\alpha_i+\beta_k+1\}_{k=1}^{q},\ \{\alpha_i-\alpha_k+1\}_{k=1,k\neq i}^{p}\end{matrix};x\right].$$

If $n_{i,n}=0$, then $A_n^{(i)} \equiv 0$. Here $n_{i,n}$ and $m_{j,n}$ are given by (41).

Expanding the normalized components gives their scalar coefficients.

Corollary 5.9. *For $n_{i,n} \geq 1$, $A_n^{(i)}(x) = \sum_{k=0}^{n_{i,n}-1} A_n^{(i)}[k]x^k$ where*

$$A_n^{(i)}[k] = \frac{\prod_{j=1}^{p}(\alpha_j+\beta_{r(n,q)+1}+m_{r(n,q)+1,n}+1)_{n_{j,n}}}{\prod_{j=1}^{q}(\beta_j-\beta_{r(n,q)+1}-m_{r(n,q)+1,n})_{m_{j,n}}} \frac{\prod_{j=1}^{q}(\alpha_i+k+\beta_j+1)_{m_{j,n}}}{\prod_{j=1,j\neq i}^{p}(\alpha_j-\alpha_i-k)_{n_{j,n}}k!(n_{i,n}-k-1)!(-1)^k}$$

and $n_{i,n}$ and $m_{j,n}$ given by (41).

The last coefficient in the preceding expansion gives the corresponding leading coefficient.

Corollary 5.10. *For $n_{i,n} \geq 1$,*

$$\operatorname{LC} A_n^{(i)} = (-1)^{n_{i,n}-1}\frac{\prod_{j=1}^{p}(\alpha_j+\beta_{r(n,q)+1}+m_{r(n,q)+1,n}+1)_{n_{j,n}}}{\prod_{j=1}^{q}(\beta_j-\beta_{r(n,q)+1}-m_{r(n,q)+1,n})_{m_{j,n}}} \frac{\prod_{j=1}^{q}(\alpha_i+n_{i,n}+\beta_j)_{m_{j,n}}}{\prod_{j=1,j\neq i}^{p}(\alpha_j-\alpha_i-n_{i,n}+1)_{n_{j,n}}(n_{i,n}-1)!}$$

where $n_{i,n}$ and $m_{j,n}$ are given by (41).

The same normalization also yields a closed expression for every mixed moment.

Corollary 5.11. *For $s \in \{1,\dots,q\}$, the moments of A_n are*

$$\int_0^1 x^k x^{\beta_s} A_n(x)\,\mathrm{d}x = \frac{\prod_{j=1}^{p}(\alpha_j+\beta_{r(n,q)+1}+m_{r(n,q)+1,n}+1)_{n_{j,n}}}{\prod_{j=1}^{q}(\beta_j-\beta_{r(n,q)+1}-m_{r(n,q)+1,n})_{m_{j,n}}} \frac{\prod_{j=1}^{q}(\beta_j-\beta_s-k)_{m_{j,n}}}{\prod_{i=1}^{p}(\alpha_i+\beta_s+k+1)_{n_{i,n}}},\ k\in\mathbb{N}_0,$$

where $n_{i,n}$ and $m_{j,n}$ are given by (41).

Analogously, for each $n \in \mathbb{N}_0$, set $B_n(x) := B_{\vec{n},\vec{m}}(x)$, where $|\vec{n}| = n$ and $|\vec{m}| = n+1$,

$$B_{\vec{n},\vec{m}}(x) = \sum_{j=1}^{q} B_{\vec{n},\vec{m}}^{(j)}(x)x^{\beta_j}.$$

Here $\vec{n} = (n_{1,n}, n_{2,n}, \dots, n_{p,n})$ and $\vec{m} = (m_{1,n}, m_{2,n}, \dots, m_{q,n})$ are defined by

$$n_{i,n} = \left\lceil \frac{|\vec{n}|+1-i}{p} \right\rceil = \left\lceil \frac{n+1-i}{p} \right\rceil \qquad \text{and} \qquad m_{j,n} = \left\lceil \frac{|\vec{m}|+1-j}{q} \right\rceil = \left\lceil \frac{n+2-j}{q} \right\rceil. \tag{43}$$

Again write $n = q\nu + \rho$, where $0 \leq \rho < q$, and set $r(n,q) := \rho$. We impose the normalization

$$\operatorname{LC} B_n^{(r(n,q)+1)} = 1. \tag{44}$$

This monic condition fixes the multiplicative constant in the type-II contour representation.

Proposition 5.12. *Imposing this normalization condition gives*

$$B_n(x) = \frac{\prod_{j=1,j\neq r(n,q)+1}^{q}(\beta_j - \beta_{r(n,q)+1} - m_{r(n,q)+1,n}+1)_{m_{j,n}}}{\prod_{i=1}^{p}(\beta_{r(n,q)+1}+m_{r(n,q)+1,n}+\alpha_i)_{n_{i,n}}} \times (m_{r(n,q)+1,n}-1)!(-1)^{m_{r(n,q)+1,n}-1}\int_{\Sigma}\frac{x^t\prod_{i=1}^{p}(t+\alpha_i+1)_{n_{i,n}}}{\prod_{j=1}^{q}(\beta_j-t)_{m_{j,n}}}\frac{dt}{2\pi i}$$

where $n_{i,n}$ and $m_{j,n}$ are given by (43).

Proof. To calculate B_n we can use the integral representation obtained in (37) up to a constant κ. In this case, taking into account (39), we have

$$\operatorname{LC} B_n^{(r(n,q)+1)} = \frac{\kappa\prod_{i=1}^{p}(\beta_{r(n,q)+1}+m_{r(n,q)+1,n}+\alpha_i)_{n_{i,n}}}{\prod_{j=1,j\neq(r(n,q)+1)}^{q}(\beta_j-\beta_{r(n,q)+1}-m_{r(n,q)+1,n}+1)_{m_{j,n}}(m_{r(n,q)+1,n}-1)!(-1)^{m_{r(n,q)+1,n}-1}}.$$

To determine this constant κ we impose the normalization $\operatorname{LC} B_n^{(r(n,q)+1)} = 1$, and we obtain the result. □

The preceding normalization constant gives every type-II component on the step line.

Corollary 5.13. *Put $j_0 = r(n,q)+1$ and*

$$C_n = (-1)^{m_{j_0,n}-1}(m_{j_0,n}-1)!\,\frac{\prod_{h\neq j_0}(\beta_h-\beta_{j_0}-m_{j_0,n}+1)_{m_{h,n}}}{\prod_{i=1}^{p}(\beta_{j_0}+m_{j_0,n}+\alpha_i)_{n_{i,n}}}.$$

With the normalization (44)*, the components are given, for $m_{j,n}\geq 1$, by*

$$B_n^{(j)}(x) = C_n\frac{\prod_{i=1}^{p}(\alpha_i+\beta_j+1)_{n_{i,n}}}{(m_{j,n}-1)!\prod_{h\neq j}(\beta_h-\beta_j)_{m_{h,n}}} \times {}_{p+q}F_{p+q-1}\left[\begin{matrix}-m_{j,n}+1,\ \{\alpha_i+\beta_j+n_{i,n}+1\}_{i=1}^{p},\ \{\beta_j-\beta_h-m_{h,n}+1\}_{h\neq j}\\ \{\alpha_i+\beta_j+1\}_{i=1}^{p},\ \{\beta_j-\beta_h+1\}_{h\neq j}\end{matrix};x\right].$$

If $m_{j,n}=0$, then $B_n^{(j)}\equiv 0$ by convention.

Expanding these components yields the following coefficient formula.

Corollary 5.14. *For $m_{j,n}\geq 1$, $B_n^{(j)}(x) = \sum_{k=0}^{m_{j,n}-1} B_n^{(j)}[k]x^k$ where*

$$B_n^{(j)}[k] = \frac{\prod_{j=1,j\neq r(n,q)+1}^{q}(\beta_j-\beta_{r(n,q)+1}-m_{r(n,q)+1,n}+1)_{m_{j,n}}(m_{r(n,q)+1,n}-1)!(-1)^{m_{r(n,q)+1,n}-1}}{\prod_{i=1}^{p}(\beta_{r(n,q)+1}+m_{r(n,q)+1,n}+\alpha_i)_{n_{i,n}}} \times\frac{\prod_{i=1}^{p}(\alpha_i+k+\beta_j+1)_{n_{i,n}}}{\prod_{i=1,i\neq j}^{q}(\beta_i-\beta_j-k)_{m_{i,n}}k!(m_{j,n}-k-1)!(-1)^k}.$$

The normalized type-II step-line form then has the following mixed moments.

Corollary 5.15. *For $s\in\{1,\dots,p\}$, these moments are*

$$\int_0^1 x^k x^{\alpha_s}B_n(x)\,\mathrm{d}x = \frac{\prod_{j=1,j\neq r(n,q)+1}^{q}(\beta_j-\beta_{r(n,q)+1}-m_{r(n,q)+1,n}+1)_{m_{j,n}}}{\prod_{i=1}^{p}(\beta_{r(n,q)+1}+m_{r(n,q)+1,n}+\alpha_i)_{n_{i,n}}} \times\frac{(m_{r(n,q)+1,n}-1)!(-1)^{m_{r(n,q)+1,n}-1}\prod_{i=1}^{p}(\alpha_i-\alpha_s-k)_{n_{i,n}}}{\prod_{j=1}^{q}(\beta_j+\alpha_s+k+1)_{m_{j,n}}},\qquad k\in\mathbb{N}_0.$$

where $n_{i,n}$ and $m_{j,n}$ are given by (43).

Next, consider the basic Christoffel transformations described in [17]. Define the affine cyclic shifts C_p and C_q by

$$C_p\vec{\alpha} = \begin{bmatrix}\alpha_2 & \cdots\cdots & \alpha_p & \alpha_1+1\end{bmatrix}, \quad C_q\vec{\beta} = \begin{bmatrix}\beta_2 & \cdots\cdots & \beta_q & \beta_1+1\end{bmatrix}.$$

For admissible parameter vectors, put

$$\vec{\alpha}^{(k)} := C_p^k\vec{\alpha}, \qquad\qquad \vec{\beta}^{(k)} := C_q^k\vec{\beta}.$$

Using the affine extensions

$$\alpha_{sp+i} = \alpha_i + s \quad \text{for} \quad 1 \le i \le p \quad \text{and} \quad \beta_{sq+j} = \beta_j + s \quad \text{for} \quad 1 \le j \le q, \tag{45}$$

this is equivalently

$$\vec{\alpha}^{(k)} = (\alpha_{1+k}, \alpha_{2+k}, \ldots, \alpha_{p+k}), \qquad\qquad \vec{\beta}^{(k)} = (\beta_{1+k}, \beta_{2+k}, \ldots, \beta_{q+k}),$$

and the transformed parameters remain in the admissible AT range. Indeed, with $u = (x^{\alpha_1}, \ldots, x^{\alpha_p})$ and $v = (x^{\beta_1}, \ldots, x^{\beta_q})$,

$$u\mathfrak{X}_{[p]}^{\top} = (x^{\alpha_2}, \ldots, x^{\alpha_p}, x^{\alpha_1+1}), \qquad\qquad \mathfrak{X}_{[q]}v^{\top} = (x^{\beta_2}, \ldots, x^{\beta_q}, x^{\beta_1+1})^{\top}.$$

Consequently, the Christoffel-transformed matrices of measures remain mixed Piñeiro matrices and satisfy

$$\left(\mathrm{d}\mu_{\vec{\alpha},\vec{\beta}}\right)_L^{(k)} = \mathrm{d}\mu_{\vec{\alpha}^{(k)},\vec{\beta}}, \qquad\qquad \left(\mathrm{d}\mu_{\vec{\alpha},\vec{\beta}}\right)_R^{(k)} = \mathrm{d}\mu_{\vec{\alpha},\vec{\beta}^{(k)}}.$$

For brevity, write $A_n^{(i)} = (A_{\vec{\alpha},\vec{\beta}})_n^{(i)}$. Then $\left(A^{(k)}_{L,\vec{\alpha},\vec{\beta}}\right)_n^{(i)} = \left(A_{\vec{\alpha}^{(k)},\vec{\beta}}\right)_n^{(i)}$ and $\left(A^{(k)}_{R,\vec{\alpha},\vec{\beta}}\right)_n^{(i)} = \left(A_{\vec{\alpha},\vec{\beta}^{(k)}}\right)_n^{(i)}$.

The cyclic left Christoffel shift can now be substituted directly into the normalized type-I formula.

Corollary 5.16. *For $n_{i,n} \ge 1$, the component $\left(A_{\vec{\alpha}^{(k)},\vec{\beta}}\right)_n^{(i)}$ has the following explicit representation with normalization* (42)*:*

$$\begin{aligned}\left(A_{\vec{\alpha}^{(k)},\vec{\beta}}\right)_n^{(i)}(x) = {} & \frac{\prod_{j=1}^{p}(\alpha_{j+k}+\beta_{r(n,q)+1}+m_{r(n,q)+1,n}+1)_{n_{j,n}}}{\prod_{j=1}^{q}(\beta_j-\beta_{r(n,q)+1}-m_{r(n,q)+1,n})_{m_{j,n}}}\,\frac{\prod_{j=1}^{q}(\alpha_{i+k}+\beta_j+1)_{m_{j,n}}}{(n_{i,n}-1)!\prod_{j=1,j\neq i}^{p}(\alpha_{j+k}-\alpha_{i+k})_{n_{j,n}}} \\ & \times {}_{p+q}F_{p+q-1}\left[\begin{matrix}-n_{i,n}+1,\ \{\alpha_{i+k}+\beta_s+m_{s,n}+1\}_{s=1}^{q},\ \{\alpha_{i+k}-\alpha_{s+k}-n_{s,n}+1\}_{s=1,s\neq i}^{p} \\ \{\alpha_{i+k}+\beta_s+1\}_{s=1}^{q},\ \{\alpha_{i+k}-\alpha_{s+k}+1\}_{s=1,s\neq i}^{p}\end{matrix};x\right]\end{aligned}$$

where $n_{i,n}$ and $m_{j,n}$ are given by (41).

Its leading coefficient follows by taking the highest-degree term.

Corollary 5.17. *For $n_{i,n} \ge 1$,*

$$\mathrm{LC}\left(A_{\vec{\alpha}^{(k)},\vec{\beta}}\right)_n^{(i)} = \frac{\prod_{j=1}^{p}(\alpha_{j+k}+\beta_{r(n,q)+1}+m_{r(n,q)+1,n}+1)_{n_{j,n}}}{\prod_{j=1}^{q}(\beta_j-\beta_{r(n,q)+1}-m_{r(n,q)+1,n})_{m_{j,n}}}\,\frac{(-1)^{n_{i,n}-1}\prod_{j=1}^{q}(\alpha_{i+k}+n_{i,n}+\beta_j)_{m_{j,n}}}{\prod_{j=1,j\neq i}^{p}(\alpha_{j+k}-\alpha_{i+k}-n_{i,n}+1)_{n_{j,n}}(n_{i,n}-1)!},$$

where $n_{i,n}$ and $m_{j,n}$ are given by (41).

The right Christoffel shift is obtained similarly by cycling the β-parameters.

Corollary 5.18. *For $n_{i,n} \geq 1$, the transformed component has the explicit representation $\left(A_{\vec{\alpha},\vec{\beta}^{(k)}}\right)_n^{(i)}$. If $s = r(n,q)+1$, its normalization is*

$$\int_0^1 x^{m_{s,n}} x^{\beta_{s+k}} \left(A_{\vec{\alpha},\vec{\beta}^{(k)}}\right)_n (x)\, \mathrm{d}x = 1.$$

$$\left(A_{\vec{\alpha},\vec{\beta}^{(k)}}\right)_n^{(i)}(x) = \frac{\prod_{j=1}^{p}(\alpha_j+\beta_{r(n,q)+1+k}+m_{r(n,q)+1,n}+1)_{n_{j,n}}}{\prod_{j=1}^{q}(\beta_{j+k}-\beta_{r(n,q)+1+k}-m_{r(n,q)+1,n})_{m_{j,n}}} \frac{\prod_{j=1}^{q}(\alpha_i+\beta_{j+k}+1)_{m_{j,n}}}{(n_{i,n}-1)!\prod_{j=1,j\neq i}^{p}(\alpha_j-\alpha_i)_{n_{j,n}}}$$
$$\times {}_{p+q}F_{p+q-1}\left[\begin{matrix}-n_{i,n}+1,\ \{\alpha_i+\beta_{s+k}+m_{s,n}+1\}_{s=1}^{q},\ \{\alpha_i-\alpha_s-n_{s,n}+1\}_{s=1,s\neq i}^{p}\\ \{\alpha_i+\beta_{s+k}+1\}_{s=1}^{q},\ \{\alpha_i-\alpha_s+1\}_{s=1,s\neq i}^{p}\end{matrix};x\right],$$

where $n_{i,n}$ and $m_{j,n}$ are given by (41).

Taking the highest-degree term gives the leading coefficient of the right Christoffel transform.

Corollary 5.19. *For $n_{i,n} \geq 1$,*

$$\mathrm{LC}\left(\left(A_{\vec{\alpha},\vec{\beta}^{(k)}}\right)_n^{(i)}\right) = \frac{\prod_{j=1}^{p}(\alpha_j+\beta_{r(n,q)+1+k}+m_{r(n,q)+1,n}+1)_{n_{j,n}}}{\prod_{j=1}^{q}(\beta_{j+k}-\beta_{r(n,q)+1+k}-m_{r(n,q)+1,n})_{m_{j,n}}} \frac{(-1)^{n_{i,n}-1}\prod_{j=1}^{q}(\alpha_i+n_{i,n}+\beta_{j+k})_{m_{j,n}}}{\prod_{j=1,j\neq i}^{p}(\alpha_j-\alpha_i-n_{i,n}+1)_{n_{j,n}}(n_{i,n}-1)!},$$

where $n_{i,n}$ and $m_{j,n}$ are given by (41).

Recalling Corollary 4.4, the coefficients in the bidiagonal factorization are given by

$$(L_k)_{n+1,n} = \frac{\mathrm{LC}\left(\left(A_{\vec{\alpha}^{(k)},\vec{\beta}}\right)_n^{(r(n,p)+1)}\right)}{\mathrm{LC}\left(\left(A_{\vec{\alpha}^{(k-1)},\vec{\beta}}\right)_{n+1}^{(r(n+1,p)+1)}\right)}, \qquad (U_k)_{n,n} = \frac{\mathrm{LC}\left(\left(A_{\vec{\alpha},\vec{\beta}^{(k-1)}}\right)_n^{(r(n,p)+1)}\right)}{\mathrm{LC}\left(\left(A_{\vec{\alpha},\vec{\beta}^{(k)}}\right)_n^{(r(n,p)+1)}\right)}.$$

These identities are exact consequences of Corollary 4.4: inserting the displayed leading coefficients gives the bidiagonal entries, and the product $L_1\cdots L_p U_q\cdots U_1$ is the recurrence matrix.

Substituting the contour-normalized polynomial coefficients (34) and (38), together with the moments (36) and (40), into (12) yields the following result.

In the next theorem we use the same contour normalization in (32) and (37): the unspecified common multiplicative constant in each of those two representations is set equal to 1. This convention is essential, because rescaling the type-II forms changes the off-diagonal recurrence coefficients by the corresponding ratios of scaling constants.

With this convention fixed, the recurrence coefficients can be stated explicitly.

Theorem 5.20. *The Piñeiro recurrence coefficients* (12) *that appear in the recurrence relations* (9) *are:*

$$b_{\vec{n},\vec{m}}^{j} = \prod_{i=1}^{p} \frac{(\alpha_i+\beta_{\pi_t(j)}+m_{\pi_t(j)}+1)_{(\vec{n}+\vec{\mathrm{s}}_{j+1})_i}}{(\alpha_i+\beta_{\pi_t(j)}+m_{\pi_t(j)}+1)_{(\vec{n}+\vec{\mathrm{s}}_j)_i}}$$
$$\times \sum_{k=j}^{q}\prod_{i=1}^{p} \frac{(\alpha_i+\beta_{\pi_t(k)}+m_{\pi_t(k)})_{n_i}}{(\alpha_i+\beta_{\pi_t(k)}+m_{\pi_t(k)}+1)_{(\vec{n}+\vec{\mathrm{s}}_{j+1})_i}} \frac{\prod_{i=1}^{q}\beta_{\pi_t(i)}-\beta_{\pi_t(k)}-m_{\pi_t(k)}}{\prod_{i=j,i\neq k}^{q}\beta_{\pi_t(i)}-\beta_{\pi_t(k)}-m_{\pi_t(k)}+m_{\pi_t(i)}},$$

$$b^{0}_{\vec{n},\vec{m}} = \prod_{j=1}^{q} \frac{\alpha_{\pi_{\mathfrak{s}}(1)} + \beta_j + n_{\pi_{\mathfrak{s}}(1)} + 1}{\alpha_{\pi_{\mathfrak{s}}(1)} + \beta_j + n_{\pi_{\mathfrak{s}}(1)} + m_j + 1} \prod_{i=1}^{p} \frac{\alpha_i - \alpha_{\pi_{\mathfrak{s}}(1)} - n_{\pi_{\mathfrak{s}}(1)} - 1}{\alpha_i - \alpha_{\pi_{\mathfrak{s}}(1)} - n_{\pi_{\mathfrak{s}}(1)} + n_i - 1}$$
$$+ \sum_{k=1}^{p} \frac{\left(\alpha_{\pi_{\mathfrak{s}}(1)} + \beta_{\pi_{\mathfrak{t}}(1)} + n_{\pi_{\mathfrak{s}}(1)} + m_{\pi_{\mathfrak{t}}(1)}\right)}{\left(\alpha_{\pi_{\mathfrak{s}}(k)} + \beta_{\pi_{\mathfrak{t}}(1)} + n_{\pi_{\mathfrak{s}}(k)} + m_{\pi_{\mathfrak{t}}(1)} - 1\right)\left(\alpha_{\pi_{\mathfrak{s}}(1)} - \alpha_{\pi_{\mathfrak{s}}(k)} - n_{\pi_{\mathfrak{s}}(k)} + n_{\pi_{\mathfrak{s}}(1)} + 1\right)}$$
$$\times \prod_{j=1}^{q} \frac{\alpha_{\pi_{\mathfrak{s}}(k)} + \beta_j + n_{\pi_{\mathfrak{s}}(k)}}{\alpha_{\pi_{\mathfrak{s}}(k)} + \beta_j + n_{\pi_{\mathfrak{s}}(k)} + m_j} \frac{\prod_{i=1}^{p} \alpha_i - \alpha_{\pi_{\mathfrak{s}}(k)} - n_{\pi_{\mathfrak{s}}(k)}}{\prod_{i=1, i\neq k}^{p} \alpha_{\pi_{\mathfrak{s}}(i)} - \alpha_{\pi_{\mathfrak{s}}(k)} - n_{\pi_{\mathfrak{s}}(k)} + n_{\pi_{\mathfrak{s}}(i)}},$$

$$b^{-i}_{\vec{n},\vec{m}} = \prod_{j=1}^{q} \frac{\left(\alpha_{\pi_s(i)} + \beta_j + n_{\pi_s(i)}\right)_{\left(\vec{m}-\vec{\mathfrak{t}}_i\right)_j}}{\left(\alpha_{\pi_s(i)} + \beta_j + n_{\pi_s(i)}\right)_{\left(\vec{m}-\vec{\mathfrak{t}}_{i+1}\right)_j}} \sum_{k=i}^{p} \prod_{j=1}^{q} \frac{\left(\alpha_{\pi_s(k)} + \beta_j + n_{\pi_s(k)}\right)_{\left(\vec{m}-\vec{\mathfrak{t}}_{i+1}\right)_j}}{\left(\alpha_{\pi_s(k)} + \beta_j + n_{\pi_s(k)} + 1\right)_{m_j}}$$
$$\times \frac{\prod_{j=1}^{p} \alpha_{\pi_s(j)} - \alpha_{\pi_s(k)} - n_{\pi_s(k)}}{\prod_{j=i, j\neq k}^{p} \alpha_{\pi_s(j)} - \alpha_{\pi_s(k)} - n_{\pi_s(k)} + n_{\pi_s(j)}},$$

for $j \in \{1, \dots, q\}$ and $i \in \{1, \dots, p\}$. The formula for $b^{-i}_{\vec{n},\vec{m}}$ is asserted when the two shifted multi-indices in that term are nonnegative; otherwise that backward term is absent.

Proof. The result is obtained by exact substitution of the polynomial coefficients and moments in (38), (36), (34), and (40) in the general coefficient formulas (18), (19) and (20). □

The cyclic notation in (45) gives a compact form of the preceding step-line recurrence coefficients. Fix $n \geq 0$ and write

$$n = p\mathfrak{r} + \mathfrak{l} = qr + l, \qquad 0 \leq \mathfrak{l} < p, \qquad 0 \leq l < q.$$

Let $\vec{n}$ be the p-component step-line multi-index of modulus n, and let $\vec{m}^{\,A}$ be the q-component step-line multi-index of modulus n. The type-II multi-index paired with $\vec{n}$ is $\vec{m}^{\,B} = \vec{m}^{\,A} + \vec{e}_{l+1}$, of modulus $n+1$. The relevant shifted components are

$$\left(\vec{n} + \vec{\mathfrak{s}}_j\right)_i = \left\lfloor \frac{|\vec{n}| + j + 1 - i}{p} \right\rfloor = \mathfrak{r} + \left\lceil \frac{\mathfrak{l} + j + 1 - i}{p} \right\rceil, \qquad j \in \{0, 1, \dots, q\},$$
$$\left(\vec{m}^{\,B} - \vec{\mathfrak{t}}_i\right)_j = \left\lfloor \frac{|\vec{m}^{\,A}| - i + 2 - j}{q} \right\rfloor = r + \left\lceil \frac{l - i + 2 - j}{q} \right\rceil, \qquad i \in \{0, 1, \dots, p\}.$$

To keep the two normalizations distinct, let $\widehat{B}_N$ denote the step-line type-II form obtained from (37) with multiplicative constant 1. Put

$$s_N := r(N, q) + 1, \qquad d_N := \left\lfloor \frac{N}{q} \right\rfloor, \qquad \widehat{\kappa}_N := \widehat{B}_N^{(s_N)}[d_N]. \tag{46}$$

By (38),

$$\widehat{\kappa}_N = \frac{\prod_{a=1}^{p} (\alpha_a + \beta_{s_N} + d_N + 1)_{n_{a,N}}}{\prod_{h\neq s_N} (\beta_h - \beta_{s_N} - d_N)_{m_{h,N}} d_N!\, (m_{s_N,N} - d_N - 1)!\, (-1)^{d_N}}.$$

Here $m_{s_N,N} = d_N + 1$, so the last factorial in the denominator is $(m_{s_N,N} - d_N - 1)! = 0! = 1$. The monic form used in (44) is therefore

$$B_N = \widehat{B}_N / \widehat{\kappa}_N. \tag{47}$$

Specializing the preceding recurrence formulas to the step line gives the monic coefficients.

Corollary 5.21. *With $n = p\mathfrak{r} + \mathfrak{l} = qr + l$ as above, set $b_n^{-i} = 0$ whenever $i > n$. For $j \in \{1, \dots, q\}$ and $1 \le i \le \min(p, n)$, the remaining Piñeiro recurrence coefficients on the step line are*

$$b_n^j = \frac{\widehat{\kappa}_{n+j}}{\widehat{\kappa}_n} \prod_{i=1}^{p} \frac{\left(\alpha_i + \beta_{l+1+j} + r + 1 + \mathfrak{r}\right)_{\left\lceil \frac{\mathfrak{l}+j+2-i}{p} \right\rceil}}{\left(\alpha_i + \beta_{l+1+j} + r + 1 + \mathfrak{r}\right)_{\left\lceil \frac{\mathfrak{l}+j+1-i}{p} \right\rceil}}$$
$$\times \sum_{k=j}^{q} \prod_{i=1}^{\mathfrak{l}} \left(\alpha_i + \beta_{k+l+1} + r\right) \prod_{i=1}^{p} \frac{\left(\alpha_{i+\mathfrak{l}} + \beta_{k+l+1} + r\right)_{\mathfrak{r}}}{\left(\alpha_i + \beta_{k+l+1} + r + 1\right)_{\mathfrak{r}+\left\lceil \frac{\mathfrak{l}+j+2-i}{p} \right\rceil}} \frac{\prod_{i=1}^{q} \left(\beta_i - \beta_{k+l+1} - r\right)}{\prod_{i=j, i\neq k}^{q} \left(\beta_{i+l+1} - \beta_{k+l+1}\right)},$$

$$b_n^0 = \prod_{j=1}^{q} \frac{\alpha_{\mathfrak{l}+1} + \beta_j + \mathfrak{r} + 1}{\alpha_{\mathfrak{l}+1} + \beta_{l+1+j} + r + \mathfrak{r} + 1} \prod_{i=1}^{p} \frac{\alpha_i - \alpha_{\mathfrak{l}+1} - \mathfrak{r} - 1}{\alpha_{\mathfrak{l}+i} - \alpha_{\mathfrak{l}+1} - 1}$$
$$+ \sum_{k=1}^{p} \frac{\alpha_{\mathfrak{l}+1} + \beta_{l+1} + \mathfrak{r} + r + 1}{\left(\alpha_{\mathfrak{l}+k} + \beta_{l+1} + \mathfrak{r} + r\right)\left(\alpha_{\mathfrak{l}+1} - \alpha_{\mathfrak{l}+k} + 1\right)} \prod_{j=1}^{q} \frac{\alpha_{\mathfrak{l}+k} + \beta_j + \mathfrak{r}}{\alpha_{\mathfrak{l}+k} + \beta_{l+1+j} + \mathfrak{r} + r} \frac{\prod_{i=1}^{p} \left(\alpha_i - \alpha_{\mathfrak{l}+k} - \mathfrak{r}\right)}{\prod_{i=1, i\neq k}^{p} \left(\alpha_{\mathfrak{l}+i} - \alpha_{\mathfrak{l}+k}\right)},$$

$$b_n^{-i} = \frac{\widehat{\kappa}_{n-i}}{\widehat{\kappa}_n} \prod_{j=1}^{q} \frac{\left(\alpha_{p+\mathfrak{l}-i+1} + \beta_j + \mathfrak{r} + r\right)_{\left\lceil \frac{l+2-i-j}{q} \right\rceil}}{\left(\alpha_{p+\mathfrak{l}-i+1} + \beta_j + \mathfrak{r} + r\right)_{\left\lceil \frac{l+1-i-j}{q} \right\rceil}}$$
$$\times \sum_{k=i}^{p} \prod_{j=1}^{l+1} \frac{1}{\alpha_{p+\mathfrak{l}-k+1} + \beta_j + \mathfrak{r} + 1} \frac{\prod_{j=1}^{q} \frac{\left(\alpha_{p+\mathfrak{l}-k+1} + \beta_j + \mathfrak{r}\right)_{r+\left\lceil \frac{l+1-i-j}{q} \right\rceil}}{\left(\alpha_{p+\mathfrak{l}-k+1} + \beta_{l+1+j} + \mathfrak{r} + 1\right)_r} \prod_{j=1}^{p} \left(\alpha_j - \alpha_{p+\mathfrak{l}-k+1} - \mathfrak{r}\right)}{\prod_{j=i, j\neq k}^{p} \left(\alpha_{p+\mathfrak{l}-j+1} - \alpha_{p+\mathfrak{l}-k+1}\right)}.$$

Here $(z)_a$ for a negative integer a has the meaning specified in (1).

Proof. We substitute the coefficients formulas obtained in Theorem 5.20, taking into account (46), and (45). This gives the three coefficients for the contour-normalized forms $\widehat{B}_N$, namely the displayed expressions before the pivot ratios are inserted. Equation (47) then multiplies every forward coefficient by $\widehat{\kappa}_{n+j}/\widehat{\kappa}_n$, leaves the central coefficient unchanged, and multiplies every backward coefficient by $\widehat{\kappa}_{n-i}/\widehat{\kappa}_n$.

□

6. Conclusion

We have related the step-line mixed multiple orthogonality problem to a banded recurrence matrix and, under the stated Gauss–Borel assumptions, to a product of lower and upper bidiagonal factors. For the mixed Piñeiro system, the contour representations yield terminating hypergeometric formulas and explicit recurrence and factorization coefficients. The distinction between the contour-normalized forms $\widehat{B}_n$ and the monic forms B_n is essential: their pivot ratios give the off-diagonal coefficients in the monic step-line recurrence. The bidiagonal factorization is then obtained exactly from the Christoffel leading-coefficient formulas; computational checks may verify the resulting expressions, but they are not used as a proof.

For the mixed Piñeiro system, it remains a topic for future research to exploit the bidiagonal factorization in order to derive sufficient conditions on the parameters defining the system that guarantee the nonnegativity of the associated recurrence matrix. Such a characterization is expected to have applications to the numerically stable computation of its eigenvalues and eigenvectors, and consequently to the construction of quadrature formulas associated with mixed multiple orthogonality.

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