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THE INVERSE REDUCTION MAP OF A SYMPLECTIC COLUMN BY DECREASING THE RANK BY ONE

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Abstract. We have previously given a factorization of a symplectic column under the action of the parity involution which enabled to explicitly have written the inverse of the reduction map in the quantum Littlewood-Richardson bijection. Watanabe has written the reduction map as a composition of several maps, among them, combinatorial R-matrices on single columns and a reduction map on a shorter column with rank reduced by one. We now use this approach to explicitly write the inverse of the reduction map on a symplectic column by detaching the corresponding symplectic column of rank reduced by one and thus avoiding going through several map compositions.

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1. Introduction

Given $\lambda \in Par_{\leq 2n}$ a partition with at most $2n$ parts, the quantum Littlewood-Richardson (LR) map LR^{AII} [Wat25, Theorem 3.1.4] is an one-to-one assignment of a semi-standard tableau T of shape λ to a pair $(P^{II}(T), Q^{II}(T))$ consisting of a symplectic tableau in $SpT_{2n}(\mu)$, for some shape $\mu \in Par_{\leq n}$, a partition with at most n parts, where $\mu \subseteq \lambda$, and a recording tableau of skew shape λ/μ in $Rec_{2n}(\lambda/\mu)$,

$$SST_{2n}(\lambda) \xrightarrow{\sim} \bigsqcup_{\substack{\mu \in Par_{\leq n} \\ \mu \subseteq \lambda \\ Q \in Rec_{2n}(\lambda/\mu)}} SpT_{2n}(\mu) \times \{Q\}. \tag{1}$$

The set of recording tableaux $Rec_{2n}(\lambda/\mu)$ is in natural bijection [Wat25, Aze26b] with the set of LR-Sundaram tableaux $LRS_{2n}(\lambda/\mu)$ [Sun86],

$$Rec_{2n}(\lambda/\mu) \xrightarrow{\sim} LRS_{2n}(\lambda/\mu). \tag{2}$$

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For $T \in SST_{2n}(\lambda)$, the computation of the symplectic tableau $P^{II}(T) \in SpT_{2n}(\mu)$ requires in the first step the action of the reduction map, red, on its first column, say $\mathbf{a} \in SST_{2n}(\varpi_l)$ with l the length of λ. The action of the reduction map, red, to $\mathbf{a}$, returns a symplectic column $\mathrm{red}(\mathbf{a}) \in SpT_{2n}(\varpi_t)$ for some $t \in [0, n]$ such that $0 \leq t \leq min\{l, 2n-l\}$ and $l - t \in 2\mathbb{Z}$ [Wat25, Proposition 4.3.6.]. Next by column Schensted insertion [Ful97, Sta98], insert $\mathrm{red}(\mathbf{a})$ into T', $(T' \leftarrow \mathrm{red}(\mathbf{a}))$, to get the successor of T, $\mathrm{suc}(T)$,

$$T \mapsto (\mathrm{red}(\mathbf{a}), T') \mapsto \mathrm{suc}(T) := \mathrm{red}(\mathbf{a}) \otimes T' = (T' \leftarrow \mathrm{red}(\mathbf{a})). \tag{3}$$

If $\mathrm{suc}(T)$ is not a symplectic tableau it means that its first column is not symplectic and the previous procedure is iterated. The successor will have strictly less cells otherwise the successor stabilizes which means that its first column is symplectic. The termination of the procedure in a finite number of steps is guaranteed by the property $\mathrm{red}(\mathbf{a}) = \mathbf{a}$ if and only if $\mathbf{a}$ is a symplectic column.

The computation of the reduction map on a column $\mathbf{a} \in SST_{2n}(\varpi_l)$ is defined via the removal operation, rem, on the column $\mathbf{a}$, [Wat25, Definition 2.5.1] together with their properties [Wat25, Proposition 4.2.2, 4.2.7, 4.2.8] in the sense that $\mathrm{red}(\mathbf{a}) = \mathbf{a} \setminus \mathrm{rem}(\mathbf{a})$. The set $\mathrm{rem}(\mathbf{a})$ of removal entries of $\mathbf{a}$ consists of certain pairs of entries, an odd entry followed with an even entry (see Proposition 2), to be removed from $\mathbf{a}$ so that the resulting column is symplectic. The inverse of the reduction map consists in adding such pairs to a symplectic column so that they can be removed.

To prove the injectivity of the reduction map, Watanabe has written the reduction map as composition maps [Wat25, Corollary 4.4.3], among them R-combinatorial matrices on single columns [NY97] and a reduction map acting on columns of rank reduced by one (see Subsection 4.1): For a fixed $n \in \mathbb{N}$, let $l \in [0, 2n]$. The map

$$\mathrm{red} = \mathrm{red}_l : SST_{2n}(\varpi_l) \to \bigsqcup_{\substack{0 \leq t \leq min\{l, 2n-l\} \\ l-t \in 2\mathbb{Z}}} SpT_{2n}(\varpi_t), \quad \mathbf{a} \mapsto \mathrm{red}(\mathbf{a}) \tag{4}$$

is injective and is inductively defined as follows

(1) for $l = 0, 1$, either $t = 0$ or respectively $t = 1$, in which cases $\mathrm{red} = id$,
(2) for $l = 2$, if $\mathbf{a} = (1, 2)$, $\mathrm{red}(\mathbf{a}) = ()$ and $t = 0$, otherwise $\mathrm{red} = id$ and $t = 2$,
(3) for $l \geq 3$,

$$\mathrm{red} = \pi \circ \bigwedge \circ (K^{-1}, id) \circ R_{t-1,2n-1} \circ (\mathrm{red}_{l-1}, id) \circ R_{2n-1,l-1} \circ (K, id) \circ \bigvee, \tag{5}$$

where red_{l-1} is the reduction map acting on $SST_{2n}(\varpi_{l-1})$, $R_{t-1,2n-1}$ and $R_{2n-1,l-1}$ are R combinatorial matrices acting on single columns, and $\bigvee, \bigwedge$ are operators on columns in $SST_{2n}(\varpi_l)$ splitting it into two parts so that the last entry is detached respectively adjoining an entry to the bottom of a column in $SpT_{2n}(\varpi_{t-1})$, and K is based on the parity involution $\mathbf{s}$, Definition 6, and the $\vee$-operator $.^{\vee}$ (see [Wat25, Sections 4.3, 4.4]).

Recall that the parity involution $\mathbf{s}$, Definition 6, swaps an even number with the previous odd number and an odd number with the next even number. In [Aze26a, Theorem 1], we have decomposed a symplectic column by detaching its maximal nonempty intervals that are fixed by the parity involution $\mathbf{s}$, this consists of the part of the symplectic column that is fixed by that involution, and the remaining part consisting of the subword whose image under the parity involution is disjoint of the symplectic column. The decomposition is uniquely determined by the symplectic column and the symplectic conditions satisfied by those pieces encode the needed information to explicitly write the inverse of the reduction map in the quantum Littlewood-Richardson bijection. Therefore, we explicitly show the surjectivity and the inverse of the inverse of reduction map in [Aze26a, Theorem 7].

Here we consider the reduction map red (4) as a composition of the maps in (5) and prove in this framework that it is surjective in Proposition 6. The proof is lengthy and detailed. As a bonus we get in Theorem 3 an explicit presentation of the inverse reduction map on a symplectic column by detaching the corresponding symplectic column of rank reduced by one. This avoids the need to go through the composition of several maps, by detaching the corresponding symplectic column of rank reduced by one. In

this presentation, the computation of the inverse reduction map on a symplectic column essentially reduces to the computation of the inverse of the reduction map on a symplectic column of rank reduced by one.

1.1. **Organization.** The paper is organized into four sections. Section 2 is devoted to provide the decomposition of symplectic columns, under the action of the parity involution Definition 1, in Theorem Section 3 is devoted to providing a full procedure in Theorem to compute the inverse reduction map on a symplectic column. The full procedure runs across theorems with many illustrations. Last section discusses our methods and its combination with combinatorial R-matrices.

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2. Symplectic tableaux and symplectic columns

Throughout we fix the following notation. Given $a, b \in \mathbb{Z}$, $[a, b] := \{x \in \mathbb{Z} : a \leq x \leq b\}$. When $a \in \mathbb{Z}^+$, we just write $[a] := [1, a]$. Given $l \geq 0$, $\varpi_l := (\underbrace{1, \ldots, 1}_{l}) \in \mathbb{Z}^l$.

2.1. **Preliminaries.** A *partition* λ is a weakly decreasing sequence of nonnegative integers $\lambda_1 \geq \lambda_2 \geq \cdots$ such that $\lambda_k = 0$ for some $k \geq 1$. The maximal i such that $\lambda_i > 0$ is called the *number* of *parts* or *length* of λ, denoted $\ell(\lambda)$. For each $m \geq 0$, the set of partitions of length at most m is denoted by $Par_{\leq m}$. We assume the inclusion $Par_{\leq m} \subseteq Par_{\leq k}$ whenever $k \geq m$. Thus We often write the partition λ as a vector $\lambda = (\lambda_1, \lambda_2, \ldots, \lambda_k)$ for $k \geq \ell(\lambda)$. The empty partition is the empty sequence () and is regarded as the unique partition of length zero.

A partition λ is identified with its *Young diagram* $D(\lambda)$ which is a left and top justified collection of boxes (or cells) with λ_k many boxes in the kth row for all $k \in \mathbb{Z}_>0$. In particular, the empty Young diagram and the partition () are identified. The number of cells of $D(\lambda)$ is the sum of the parts of λ and is denoted by $|\lambda|$. The boxes or cells of the Young diagram of λ are identified by its coordinates (i, j) in the matrix style, that is, $1 \leq i \leq \ell(\lambda)$ and $1 \leq j \leq \lambda_i$.

A *tableau* T of shape λ is a map (or a filling of $D(\lambda)$)

$$T : D(\lambda) \to \mathbb{Z}_>0,\ (i, j) \mapsto T(i, j),$$

assigning a positive integer to each box of $D(\lambda)$. We say that the tableau T is *semi-standard* if in addition the assignment is such that it is weakly increasing as we go from left to right along a row and strictly increasing as we go from top to bottom along a column,

$$T(i, j) \leq T(i, j + 1),\ T(i, j) < T(i + 1, j),\ \text{for all } (i, j) \in D(\lambda),$$

where we set $T(a, b) := \infty$ if $(a, b) \notin D(\lambda)$. Usually $T(i, j)$ is just referred as the entry in the box (i, j). A positive integer $m \geq \ell(\lambda)$ will be fixed and $[m]$ will be used as a co-domain for the map T. We call $[m] := \{1, \ldots, m\}$ the alphabet of the semi-standard tableau T. In this case, we will denote the *set* of *semi-standard tableaux* of *shape* λ by $SST_m(\lambda)$. The *weight* or *content* of T is the nonnegative vector $\mathrm{wt}(T) = (T[1], \ldots, T[m])$, where $T[i] := \#\{(a, b) \in D(\lambda) : T(a, b) = i\}$ for $i \in [m]$, that is, $T[i]$ is the number of occurrences of i in the tableau T.

2.2. **Symplectic tableaux and symplectic columns.** We fix $n \in \mathbb{N}$. We first need the definition of the *parity* (*swapping*) *involution*.

Definition 1. For each $x \in \mathbb{Z}$, set [Wat25]

$$\mathbf{s}(x) = \begin{cases} x + 1, & \text{if } x \notin 2\mathbb{Z}, \\ x - 1, & \text{if } x \in 2\mathbb{Z}. \end{cases} \tag{6}$$

Indeed $\mathbf{s}^2(x) = x$, for each $x \in \mathbb{Z}$. We call it the parity (swapping) involution.

Definition 2. [Kin76] A semistandard tableau $G \in SST_{2n}(\lambda)$ is said to be symplectic if

$$G(k,1) \geq 2k-1, \text{ for all } k \in [\ell(\lambda)].$$

Let $SpT_{2n}(\lambda)$ denote the set of all symplectic tableaux of shape λ on the alphabet $[2n]$.

Proposition 1. Let $G \in SST_{2n}(\lambda)$.

(1) [Wat25, Proposition 2.4.2] If $SpT_{2n}(\lambda) \neq \emptyset$ then $\ell(\lambda) \leq n$.
(2) [Aze26a]If $G = (a_1, \ldots, a_t) \in SpT_{2n}(\varpi_t)$ then $t \leq n$ and $t \leq \lfloor \frac{a_t+1}{2} \rfloor$.
(3) [Wat25, Lemma 2.4.3.] If G is not symplectic, then there exists a unique $i \in [2, 2n]$ such that

$$G(i,1) < 2i-1 \text{ and } G(k,1) \geq 2k-1 \text{ for all } k \in [1, i-1]. \tag{7}$$

Moreover, we have

$$G(i-1,1) = 2i-3 = 2(i-1)-1 \text{ and } G(i,1) = 2i-2 = 2(i-1)\ . \tag{8}$$

(4) [Aze26a]If the first column G has no consecutive integers consisting of an odd number followed with an even number, that is, the first column of G does not contain any interval of the form $[x < \mathbf{s}(x)]$, then G is symplectic.
(5) [Aze26a]$SpT_{2n}(\varpi_1) = SST_{2n}(\varpi_1)$ and $SpT_{2n}(\varpi_2) = SST_{2n}(\varpi_2) \setminus \{(1,2)\}$.

Corollary 1. [Aze26b, Aze26a] The following holds:

(a) For $0 \leq t \leq n$, $SpT_{2n}(\varpi_t) \neq \emptyset$. Namely, $SpT_{2n}(()) = \{()\}$, and, for $1 \leq t \leq n$, $G = (1, 3, \ldots, 2t-1) \in SpT_{2n}(\varpi_t)$ or $H = (2, 4, \ldots, 2t) \in SpT_{2n}(\varpi_t)$.
(b) $SpT_{2n}(\lambda) \neq \emptyset$ if and only if $\ell(\lambda) \leq n$.
(c) If G' is obtained from $G \in SpT_{2n}(\varpi_t)$ by suppressing $0 \leq t_0 \leq t$ entries then $G' \in SpT_{2n}(\varpi_{t-t_0})$ and the suppressed part $G'' \in SpT_{2n}(\varpi_{t_0})$.
(d) Let $\lambda \in Par_{\leq} n$,

$$SpT_{2n}(\lambda) = \{S \in SST_{2n}(\lambda) | (S(1,1), \cdots, S(\ell(\lambda),1) \in SpT_{2n}(\varpi_{\ell(\lambda)})\}.$$

where $S(i,1)$ indicates the entry in row i and column 1 of S, for $1 \leq i \leq \ell(\lambda)$.

3. The reduction map

For the reader convenience this section recalls several properties of removal and reduction maps in [Wat25] and [Aze26b]. We fix $l \in [0, 2n]$ and $\mathbf{a} = (a_1, \ldots, a_l)$ a column in $SST_{2n}(\varpi_l)$. Recall often $\mathbf{a}$ is regarded as a set or its reading word top to bottom. The *removal subword* of $\mathbf{a}$ [Wat25] is defined to be the subword $\mathrm{rem}(\mathbf{a})$ of $\mathbf{a}$ obtained by the following recursive formula:

$$\mathrm{rem}(\mathbf{a}) := \begin{cases} \emptyset, & \text{if } l \leq 1, \\ \mathrm{rem}(a_l, \ldots, a_{l-2})(a_{l-1}, a_l), & \text{if } l \geq 2, a_l \in 2\mathbb{Z}, a_{l-1} = a_l - 1, \\ & \text{and } a_l < 2l - |\mathrm{rem}(a_1, \ldots, a_{l-2})| - 1, \\ \mathrm{rem}(a_1, \ldots, a_{l-1}) & \text{otherwise.} \end{cases} \tag{9}$$

Definition 3. [Wat25] For the column $\mathbf{a}$, the new column $\mathrm{red}(\mathbf{a})$, reduction of $\mathbf{a}$, is defined to be the one obtained from $\mathbf{a}$ by removing the entries in the set $\mathrm{rem}(\mathbf{a})$,

$$\mathrm{red}(\mathbf{a}) = \mathbf{a} \setminus \mathrm{rem}(\mathbf{a}) \in SST_{2n}(\varpi_{l-|\mathrm{rem}(\mathbf{a})|}).$$

In fact Proposition 3 below, [Wat25, Section 4.3, Proposition 4.3.6], shows that $\mathrm{red}(\mathbf{a}) \in SpT_{2n}(\varpi_{l-|\mathrm{rem}(\mathbf{a})|})$, where $l - |\mathrm{rem}(\mathbf{a})|$, satisfy further conditions. We first recall useful properties of removable entries in $\mathbf{a}$. In particular, we highlight the parity swapping involution $\mathbf{s}$ with a meaningful role in the removal of pairs and consequently in the reduction map and in our computation of the inverse reduction map red.

In the next proposition points (2) and (3) together are an alternative to the usage of Definition 3 to compute $\mathrm{rem}(\mathbf{a})$, and providing an explicit recognition of the removal pairs.

Proposition 2. [Wat25, Proposition 4.2.2, 4.2.3]

(1) If $i \in [1, l]$ and $a_j \notin \mathrm{rem}(\mathbf{a})$, for $j \in [i, l]$ then $\mathrm{rem}(\mathbf{a}) = \mathrm{rem}(a_1, \ldots, a_{i-1})$.

(2) for each $i \in [1,l]$, $a_i \in \mathrm{rem}(\mathbf{a})$ if and only if one of the following holds
   (a) a_i odd, $i < l$, $a_{i+1} = a_i + 1$ and $a_i < 2i - |\mathrm{rem}(a_1, \ldots, a_{i-1})|$
   (b) a_i even, $i > 1$, $a_i = a_{i-1} + 1$ and $a_i < 2i - |\mathrm{rem}(a_1, \ldots, a_{i-2})| - 1$
(3) $a_i \in \mathrm{rem}(\mathbf{a})$ if and only if $\mathbf{s}(a_i) \in \mathrm{rem}(\mathbf{a})$. Consequently, $|\mathrm{rem}(\mathbf{a})| \in 2\mathbb{Z}$.

Corollary 2. For $l \in [0, 2n]$,

(1) [Wat25, Lemma4.3.1] l even $\Rightarrow \mathrm{rem}(1,2,\ldots,l) = (1,2,\ldots,l) = \mathrm{red}(1,2,\ldots,l) = \emptyset$.
(2) l odd $\Rightarrow \mathrm{rem}(1,2,\ldots,l) = \mathrm{rem}(1,2,\ldots,l-1) = (1,2,\ldots,l-1) \Rightarrow \mathrm{red}(1,2,\ldots,l) = \{l\}$.
(3) $\mathrm{rem}(\mathbf{a}) = \mathbf{a}$ if and only if l is even and $\mathbf{a} = (1,2,\ldots,l)$.
(4) $\mathrm{red}(\mathbf{a}) = \emptyset$ if and only if l is even and $\mathbf{a} = (1,2,\ldots,l)$.

Example 1. $\mathrm{rem}(6,7,8,9,10,11,12) = (11,12)$, $\mathrm{red}(6,7,8,9,10,11,12) = (6,7,8,9,10)$
$\mathrm{rem}(1,2,3,4,5) = (1,2,3,4)$, $\mathrm{red}(1,2,3,4,5) = (5)$.

Proposition 3. [Wat25, NSW26] Let $l \in [0,2n]$ and $\mathbf{a} = (a_1,\ldots,a_l) \in SST_{2n}(\varpi_l)$. Then

(1) [Wat25, Proposition 4.3.6.] $\mathrm{red}(\mathbf{a})$ is symplectic.
(2) [Wat25, Proposition 4.3.7.] $\mathrm{red}(\mathbf{a}) = \mathbf{a}$ if and only if $\mathbf{a}$ is symplectic.
(3) [Wat25, Proposition 4.3.6., Corollary 4.4.3.] The reduction map red on $SST_{2n}(\varpi_l)$ is the injective assignment

$$\begin{aligned} \mathrm{red} = \mathrm{red}_l : SST_{2n}(\varpi_l) &\to \bigsqcup_{\substack{0 \le t \le min\{l, 2n-l\} \\ l-t \in 2\mathbb{Z}}} SpT_{2n}(\varpi_t) \\ \mathbf{a} &\mapsto \mathrm{red}(\mathbf{a}) = \mathbf{a} \setminus \mathrm{rem}(\mathbf{a}). \end{aligned} \tag{10}$$

4. The inverse reduction map of a symplectic column by decreasing the rank by one

In [Wat25, Sections 4.3, 4.4] the reduction map is given as composition of several maps, as below in Theorem 1, and among them combinatorial R-matrices [NY97] and a reduction map on lower length columns. In this section we study the inverse reduction map as a composition of maps and combinatorial R-matrices and explicitly exhibit the last component of the inverse reduction of a symplectic column in relation with the inverse reduction of that symplectic column without the last entry.

4.1. **Combinatorial R-matrices for single columns: Nakayashiki-Yamada pairing.** We recall the Nakayashiki-Yamada rule or pairing to describe the combinatorial R-matrix for single-columns in [NY97, Rule 3.10] as recursively presented in [Wat25, Section 4.1].

Definition 4. Let $k, l \in [0,m]$. The combinatorial R-matrix for single columns is the map

$$R = R_{k,l} : SST_m(\varpi_k) \times SST_m(\varpi_l) \to SST_m(\varpi_l) \times SST_m(\varpi_k)$$

defined as follows. Let $\mathbf{a} = (a_1, \ldots a_k) \in SST_m(\varpi_k)$ and $\mathbf{b} = (b_1, \ldots, b_l) \in SST_m(\varpi_l)$

(1) When $k \le l$.
   For each $r \in [1,k]$, define $i_r \in [1,l]$ inductively as follows. Set i_1 to be the minimum $i \in [1,l]$ such that $a_1 \le b_i$; when such i does not exist, we set $i_1 := 1$. Suppose that $r \ge 2$ and we have determined $i_1, \ldots, i_{r-1}$. Set i_r to be the minimum $i \in [1,l] \setminus \{i_1, \ldots, i_{r-1}\}$ such that $a_r \le b_i$; when such i does not exist, we set $i_r := min([1,l] \setminus \{i_1, \ldots, i_{r-1}\})$. Then, we set

$$R(\mathbf{a}, \mathbf{b}) := (\mathbf{a} \sqcup \mathbf{b}'', \mathbf{b}'),$$

   where

$$\mathbf{b}' := (b_{i_1}, \ldots, b_{i_k}),\ \mathbf{b}'' := \mathbf{b} \setminus \mathbf{b}'.$$

(2) When $k \ge l$.
   For each $r \in [1,l]$, define $i_r \in [1,k]$ inductively as follows. Set i_1 to be the maximum $i \in [1,k]$ such that $a_i \le b_1$; when such i does not exist, we set $i_1 := k$. Suppose that $r \ge 2$ and we have

determined $\{i_1, \ldots, i_{r-1}\}$. Set i_r to be the maximum $i \in [1,k] \setminus \{i_1, \ldots, i_{r-1}\}$ such that $a_i \leq b_r$; when such i does not exist, we set $i_r := max([1,k] \setminus \{i_1, \ldots, i_{r-1}\})$. Then, we set

$$R(\mathbf{a}, \mathbf{b}) := (\mathbf{a}', \mathbf{b} \sqcup \mathbf{a}''),$$

where

$$\mathbf{a}' := (a_{i_1}, \ldots, a_{i_l}),\ \mathbf{a}'' := \mathbf{a} \setminus \mathbf{a}'.$$

The map R_{kl} is a bijection and its inverse is R_{lk} [NY97, Proposition].
Given an integer $a \in [1,m]$, set

$$a^\vee := [1,m] \setminus \{a\} \in SST_m(\varpi_{m-1}). \tag{11}$$

Then

$$SST_m(\varpi_{m-1}) = \{a^\vee : a \in [m]\}$$

When $m = 2n$,

$$SST_{2n}(\varpi_{2n-1}) = \{a^\vee : a \in [2n]\} = \{\mathbf{s}(a)^\vee : a \in [2n]\}.$$

Example 2. Let $n = 4$.

(1) Let $\mathbf{a} = (5^\vee) = (1,2,3,4,6,7,8) \in SST_8(\varpi_7)$, $\mathbf{b} = (3,4,5) \in SST_8(\varpi_3)$

$\mathbf{a}$	$\mathbf{b}$		$\mathbf{a}'$	$\mathbf{b} \sqcup \mathbf{a}''$	
1				1	
2				**3**	
3	**3**		**2**	**4**	
4	**4**	$\xrightarrow{R_{7,3}}$	**3**	**5**	$\in SST_{2n}(\varpi_3) \times SST_{2n}(\varpi_7)$
	5		**4**		
6				6	
7				7	
8				8	

(2) Let $\mathbf{a} = (2) \in SST_8(\varpi_1)$, $\mathbf{b} = (2^\vee) = (1,3,4,5,6,7,8) \in SST_8(\varpi_7)$

$\mathbf{a}$	$\mathbf{b}$		$\mathbf{a} \sqcup \mathbf{b}''$	$\mathbf{b}'$	
	1		1		
2	**3**		**2**	**3**	
	4		4		
	5	$\xrightarrow{R_{1,7}}$	5		$\in SST_8(\varpi_7) \times SST_8(\varpi_1)$
	6		6		
	7		7		
	8		8		

(3) $\mathbf{a} = (3)$. $\mathbf{b} = (1,2)$, $R_{1\ 2}(\mathbf{a}, \mathbf{b}) = ((2,3),(1))$ and $R_{2\ 1}((2,3),(1)) = (\mathbf{a}, \mathbf{b})$.

4.2. **The inverse reduction map as a composition of maps.** We collect some convenient properties of the R-combinatorial matrix on single columns.

Proposition 4. (1) [Wat25, Lemma 4.1.5.] Let $\mathbf{a} = (a_1, \ldots, a_k) \in SST_m(\varpi_k)$ and $\mathbf{b} = (b_1, \ldots, b_l) \in SST_m(\varpi_l)$ with $k \leq l$. Suppose that for each $r \in [1,k]$, there exists $j_r \in [1,l]$ such that $a_r = b_{j_r}$, that is, $\mathbf{a} \subseteq \mathbf{b}$. Then, $R(\mathbf{a}, \mathbf{b}) = (\mathbf{b}, \mathbf{a})$ and $R(\mathbf{b}, \mathbf{a}) = (\mathbf{a}, \mathbf{b})$.

(2) Let $m \geq 1$.
   (a) [Wat25, Lemma 4.1.3.] $R(m, m^\vee) = (1^\vee, 1)$.
   (b) [Wat25, Lemma 4.1.4.] Let $t \in [1, m-1]$. Then, $R(t^\vee, (1, \ldots, t)) = ((1, \ldots, t-1, m), m^\vee)$.

(3) [Wat25, Lemma 4.1.6] Let $a = (a_1, \ldots, a_k) \in SST_m(\varpi_k)$, and $r, s \in [1,m]$ be such that $a_k + 1 < r \leq s$. Then, we have

$$R(s^\vee, a \sqcup [r,s]) = (a \sqcup [r-1, s-1], (r-1)^\vee) \tag{12}$$

and

$$R(a \sqcup [r-1, s-1], (r-1)^\vee) = (s^\vee, a \sqcup [r, s]). \tag{13}$$

We first recall Proposition 4.4.1 in [Wat25] that we phrase in the way convenient for us. Also recall that $SpT_{2n}(\varpi_1) = SST_{2n}(\varpi_1)$ and $SpT_{2n}(\varpi_2) = SST_{2n}(\varpi_2) \setminus \{(1,2)\}$.

Proposition 5. [Wat25, Proposition 4.4.1] Let $l \in [2, 2n]$ and $\mathbf{a} = (a_1, \ldots, a_l) \in SST_{2n}(\varpi_l)$. Let us write the following sequence of transformations,

$$\begin{aligned}
&\mathbf{a} = (a_1, \ldots, a_l) \longrightarrow ((a_l), (a_1, \ldots, a_{l-1})) \\
&\longrightarrow (\mathbf{s}(a_l)^\vee, (a_1, \ldots, a_{l-1})) \in SST_{2n}(\varpi_{2n-1}) \times SST_{2n}(\varpi_{l-1}) \\
&\overset{R_{2n-1,l-1}}{\longrightarrow} (\mathbf{c}, \mathbf{d}), \text{ for some } \mathbf{c} \in SST_{2n}(\varpi_{l-1}) \text{ and } \mathbf{d} \in SST_{2n}(\varpi_{2n-1}) \\
&\overset{\mathrm{red}_{l-1} \otimes id}{\longrightarrow} (\mathrm{red}_{l-1}(\mathbf{c}), \mathbf{d}) \in SpT_{2n}(\varpi_{t-1}) \times SST_{2n}(\varpi_{2n-1}) \quad \text{for some } t \geq 1 \\
&\overset{R_{t-1,2n-1}}{\longrightarrow} (\mathbf{s}(b_t)^\vee, \mathbf{b}' = (b_1, \ldots, b_{t-1})) \in SST_{2n}(\varpi_{2n-1}) \times SST_{2n}(\varpi_{t-1}), \\
&\text{for some } b_t > b_{t-1} \text{ and } \mathbf{b}' \in SST_{2n}(\varpi_{t-1}) \\
&\longrightarrow ((b_t), \mathbf{b}' = (b_1, \ldots, b_{t-1}) \\
&\longrightarrow \mathbf{b} = (b_1, \ldots, b_t).
\end{aligned} \tag{14}$$

Then

$$\mathbf{b} = \begin{cases} (1,2), & \text{if } a_l = l \in 2\mathbb{Z}, \\ \mathrm{red}(\mathbf{a}), & \text{otherwise.} \end{cases} \tag{15}$$

Next theorem is a consequence of this proposition for $l \in [2, 2n]$ with the following maps in [Wat25, Sections 4.3, 4.4] to complete the translation of the steps in (14):

$$\begin{aligned}
\bigvee : SST_{2n}(\varpi_l) &\longrightarrow \{((a), (a_1, \ldots, a_{l-1})) \in SST_{2n}(\varpi_1) \times SST_{2n}(\varpi_{l-1}) : a > a_{l-1}\} \\
(a_1, \ldots, a_{l-1}, a) &\mapsto ((a), (a_1, \ldots, a_{l-1})),
\end{aligned} \tag{16}$$

$$\begin{aligned}
K : SST_{2n}(\varpi_1) &\longrightarrow SST_{2n}(\varpi_{2n-1}) \\
(a) &\mapsto \mathbf{s}(a)^\vee,
\end{aligned} \tag{17}$$

$$\begin{aligned}
K^{-1} : SST_{2n}(\varpi_{2n-1}) &\longrightarrow SST_{2n}(\varpi_1) \\
\mathbf{s}(a)^\vee &\mapsto (a), \ a \in [2n],
\end{aligned} \tag{18}$$

and, for $t \geq 1$,

$$\begin{aligned}
\bigwedge : \{((a), a_1, \ldots, a_{t-1}) \in SST_{2n}(\varpi_1) \times SST_{2n}(\varpi_{t-1}) : a > a_{t-1}\} &\longrightarrow SST_{2n}(\varpi_t) \\
((a), (a_1, \ldots, a_{t-1})) &\mapsto (a_1, \ldots, a_{t-1}, a).
\end{aligned} \tag{19}$$

All these maps are bijections. Due to the first point of (15), we still need another map π such that $\pi(1,2) = ()$, and $\pi = id$ when restricted to $\bigsqcup_{k=0}^{2n} SST_{2n}(\varpi_k) \setminus \{(1,2)\}$:

$$\begin{aligned}
\pi : \bigsqcup_{k=0}^{2n} SST_{2n}(\varpi_k) &\longrightarrow \bigsqcup_{k=0}^{2n} SST_{2n}(\varpi_k) \\
\mathbf{a} &\mapsto \begin{cases} \mathbf{a}, & \text{if } \mathbf{a} \neq (1,2), \\ (), & \text{if } \mathbf{a} = (1,2). \end{cases}
\end{aligned} \tag{20}$$

Theorem 1. [Wat25, Corollary 4.4.3] For a fixed $n \in \mathbb{N}$, let $l \in [0, 2n]$. Then, the map

$$\mathrm{red} = \mathrm{red}_l : SST_{2n}(\varpi_l) \to \bigsqcup_{\substack{0 \le t \le min\{l, 2n-l\} \\ l-t \in 2\mathbb{Z}}} SpT_{2n}(\varpi_t), \quad \mathbf{a} \mapsto \mathrm{red}(\mathbf{a}) \tag{21}$$

is injective and is inductively defined as follows

(1) for $l = 0, 1$, either $t = 0$ or respectively $t = 1$, in which cases red $= id$,
(2) for $l = 2$, if $\mathbf{a} = (1, 2)$, red$(\mathbf{a}) = ()$ and $t = 0$, otherwise red $= id$ and $t = 2$,
(3) for $l \ge 3$,

$$\mathrm{red} = \pi \circ \bigwedge \circ (K^{-1}, id) \circ R_{t-1,2n-1} \circ (\mathrm{red}_{l-1}, id) \circ R_{2n-1,l-1} \circ (K, id) \circ \bigvee, \tag{22}$$

where $t - 1$ denotes the length of the obtained column after the action of the reduced map red_{l-1}. The output column has length $t = t - 1 + 1$ except when $\pi \neq id$ in which case $\mathbf{a} = (1, \dots, l)$, $l \in 2\mathbb{Z}$ and red$(\mathbf{a}) = ()$.

Example 3. Let $n = 4$. We compute red$(\mathbf{a})$ using previous theorem

(1) Let $l = 4 \in [0, 2n]$ and $\mathbf{a} = (3, 4, 5, 6) \in SST_{2n}(\varpi_l)$.

$$\begin{aligned}
&(3,4,5,6) \xrightarrow{\bigvee} ((6),(345)) \in SST_{2n}(\varpi_1) \times SST_{2n}(\varpi_{l-1}) \\
&\xrightarrow{K \otimes id} (\mathbf{s}(6)^\vee, (3,4,5)) \in SST_{2n}(\varpi_{2n-1}) \times SST_{2n}(\varpi_{l-1}) \\
&\xrightarrow{R_{2n-1,l-1}} ((2,3,4), 345 \sqcup \{1,6,7,8\}) \in ST_{2n}(\varpi_{l-1}) \times SST_{2n}(\varpi_{2n-1}) \\
&\xrightarrow{\mathrm{red}_{l-1} \otimes id} ((2), (3,4,5) \sqcup \{1,6,7,8\}) \in ST_{2n}(\varpi_{t-1}) \times SST_{2n}(\varpi_{2n-1}) \\
&\xrightarrow{R_{t-1,2n-1}} ((s(4)^\vee), (3)) \in SST_{2n}(\varpi_{2n-1}) \times SST_{2n}(\varpi_{t-1}) \\
&\xrightarrow{K^{-1} \otimes id} ((4),(3)) \in SST_{2n}(\varpi_1) \times SST_{2n}(\varpi_{t-1}) \\
&\xrightarrow{\bigwedge} (3,4) \in SST_{2n}(\varpi_{1+(t-1)}) \\
&\xrightarrow{\pi = id} (3,4) = \mathrm{red}(\mathbf{a}) \in SpT_{2n}(\varpi_2), \qquad t = 2 = 1 + (t-1).
\end{aligned} \tag{23}$$

$t = 2 \le min\{l, 2n - l\}$ and $l - t \in 2\mathbb{Z}$.

(2) Let $l = 6 \in [0, 2n]$ and $\mathbf{a} = (3, 4, 5, 6, 7, 8) \in SST_{2n}(\varpi_l)$

$$\begin{aligned}
&(3,4,5,6,7,8) \xrightarrow{\bigvee} ((8),(34567)) \in SST_{2n}(\varpi_1) \times SST_{2n}(\varpi_{l-1}) \\
&\xrightarrow{K \otimes id} (\mathbf{s}(8)^\vee, (3,4,5,6,7)) \in SST_{2n}(\varpi_{2n-1}) \times SST_{2n}(\varpi_{l-1}) \\
&\xrightarrow{R_{2n-1,l-1}} ((2,3,4,5,6),(3,4,5,6,7) \sqcup \{1,8\}) \in SST_{2n}(\varpi_{l-1}) \times SST_{2n}(\varpi_{2n-1}) \\
&\xrightarrow{\mathrm{red}_{l-1} \otimes id} ((2),(3,4,5,6,7) \sqcup \{1,8\}) \in SpT_{2n}(\varpi_{t-1}) \times SST_{2n}(\varpi_{2n-1}) \\
&\xrightarrow{R_{t-1,2n-1}} ((s(4)^\vee),(3)) \in SST_{2n}(\varpi_{2n-1}) \times SpT_{2n}(\varpi_{t-1}) \\
&\xrightarrow{K^{-1} \otimes id} ((4),(3)) \in SST_{2n}(\varpi_1) \times SST_{2n}(\varpi_{t-1}) \\
&\xrightarrow{\bigwedge} (3,4) \in SST_{2n}(\varpi_{1+t-1}) \\
&\xrightarrow{\pi = id} (3,4) \in SpT_{2n}(\varpi_t), \qquad t = 2 = 1 + (t-1).
\end{aligned} \tag{24}$$

$t = 2 \le min\{l, 2n - l\}$ and $l - t \in 2\mathbb{Z}$.

As we may check all these steps are reversible, and $\mathrm{red}^{-1}(3,4)$ from $SpT_{2n}(\varpi_2)$ to $SST_{2n}(\varpi_l)$.

This is conceptually very interesting because, in particular, it allows to reduce the computation to columns of lower length [Wat25, Proposition 4.4.1]. Combinatorial R-matrices are bijections and so we could use the inverse of these maps to compute the inverse reduction. However one still needs to compute the inverse reduction of shorter symplectic columns and thereby it is useful to have at hand the inverse.

We shall next use the previous inductive definition of the injective map red in Theorem 1 to prove that is also surjective. Indeed we already know from [Aze26a] the explicit inverse reduction map. The proof of the surjectivity of the reduction map in the frame work of those composition of maps will make explicit which shorter symplectic column should we consider in the inverse of the reduction map as a composition of those aforesaid maps.

Proposition 6. The reduction map red is surjective and therefore red^{-1} does exist.

Proof. Since $SpT_{2n}(\varpi_1) = SST_{2n}(\varpi_1)$ and $SpT_{2n}(\varpi_2) = SST_{2n}(\varpi_2) \setminus \{(1,2)\}$, clearly from previous theorem, the map red is surjective for $l = 0, 1, 2$. Let $l \geq 3$, and by induction assume that the map red is surjective for $l-1$.

Let $0 \leq t \leq min\{l, 2n-l\}$ such that $l - t \in 2\mathbb{Z}$. Given $\mathbf{b} \in SpT_{2n}(\varpi_t)$ let us prove that $\mathbf{b}$ is reached by the map red in $SST_{2n}(\varpi_l)$, that is, there exists $\mathbf{a} \in SST_{2n}(\varpi_l)$ such that $\mathrm{red}(\mathbf{a}) = \mathbf{b}$.

If $t = 0$ and $l \in 2\mathbb{Z}$, then $\mathbf{b} = ()$ and by Corollary 2, $\mathrm{red}(1, 2, \ldots, l) = ()$ with $l \in 2\mathbb{Z}$, and $\mathbf{a} = (1, 2, \ldots, l) \in SST_{2n}(\varpi_l)$.

If $t = 1$ and $2 \leq l - 1 \in 2\mathbb{Z}$, let $\mathbf{b} = (b) \in SpT_{2n}(\varpi_1)$ and define

$$\mathbf{a} = \begin{cases} [1, l_1] \sqcup (b) \sqcup [b+1, b+l_2], & \text{if } b \in 2\mathbb{Z} \\ [1, l_1]) \sqcup (b) \sqcup [b+1+1, b+1+l_2], & \text{if } b \notin 2\mathbb{Z} \end{cases}, \tag{25}$$

and

$$l_1 = \begin{cases} min\{b-2, l-1\}, & \text{if } b \in 2\mathbb{Z} \\ min\{b-1, l-1\}, & \text{if } b \notin 2\mathbb{Z}, \end{cases} \qquad \text{and } l_2 = l - 1 - l_1 \in 2\mathbb{Z}. \tag{26}$$

Henceforth, $\mathbf{a} \in SST_{2n}(\varpi_l)$ and $\mathrm{red}(\mathbf{a}) = \mathbf{b}$.

So far, one has $\mathrm{red}_l^{-1}() = [1, l]$ with $4 \leq l \in 2\mathbb{Z}$, and $\mathrm{red}_l^{-1}((b)) = \mathbf{a} \in SST_{2n}(\varpi_l)$, with $3 \leq l \notin 2\mathbb{Z}$, such that if

$$\hat{\mathbf{a}} := [1, l-1] = \mathrm{red}_{l-1}^{-1}() \tag{27}$$

then

$$\mathbf{a} = \begin{cases} (\hat{\mathbf{a}}, b), & \text{if } \hat{\mathbf{a}} \subseteq [1, b-1], \\ \hat{\mathbf{a}} \setminus \{b+1\} \sqcup [l, l+1], & \text{if } b \notin 2\mathbb{Z} \text{ and } b \in \hat{\mathbf{a}} = [1, l-1], \\ \hat{\mathbf{a}} \setminus \{b-1\} \sqcup [l, l+1], & \text{if } b \in 2\mathbb{Z} \text{ and } b \in \hat{\mathbf{a}} = [1, l-1], \end{cases} \tag{28}$$

and $\mathbf{a} \supseteq \mathbf{b}$.

Note that (28) coincides with (25).

Let $2 \leq b \in 2\mathbb{Z}$.

If $\hat{\mathbf{a}} \subseteq [1, b-1]$ holds in (28), $l - 1 < b - 1$ because $l - 1 \in 2\mathbb{Z}$ and $b - 1 \notin 2\mathbb{Z}$. It implies that $l - 1 \leq b - 2$ and $l_1 = l - 1$ and $l_2 = 0$ in (25). This means for $b \in 2\mathbb{Z}$, $\mathbf{a} = \hat{\mathbf{a}} \sqcup (b)$ in (25).

If $b \in \hat{\mathbf{a}} = [1, l-1]$ holds in (28), then $2 \leq b \leq l - 1$ and in (28) one has

$$\mathbf{a} = \hat{\mathbf{a}} \setminus \{b-1\} \sqcup [l, l+1] = [1, b-2] \sqcup [b, l-1] \sqcup [l, l+1] = [1, b-2] \sqcup (b) \sqcup [b+1, l+1].$$

.

In (25), one has $l_1 = b - 2$, $l_2 = l - 1 - b + 2 = l - b + 1$, and also

$$\mathbf{a} = [1, b-2] \sqcup (b) \sqcup [b+1, b+l-b+1] = [1, b-2] \sqcup (b) \sqcup [b+1, l+1].$$

Let $b \notin 2\mathbb{Z}$.

If $\hat{\mathbf{a}} \subseteq [1, b-1]$ holds in (28), then $l - 1 \leq b - 1$. It implies that $l_1 = l - 1$ and $l_2 = 0$ in (25). This means for $b \notin 2\mathbb{Z}$, $\mathbf{a} = \hat{\mathbf{a}} \sqcup (b)$ in (25).

If $b \in \hat{\mathbf{a}} = [1, l-1]$ holds in (28), then $b < l-1 \Leftrightarrow b+1 \leq l-1$ because $b \notin 2\mathbb{Z}$ and $l-1 \in 2\mathbb{Z}$. in (28) one has

$$\mathbf{a} = \hat{\mathbf{a}} \setminus \{b+1\} \sqcup [l, l+1] = [1, b] \sqcup [b+2, l-1] \sqcup [l, l+1] = [1, b-1] \sqcup (b) \sqcup [b+2, l+1].$$

.

In (25), one has $l_1 = b-1$, $l_2 = l-1-b+1 = l-b$, and also

$$\mathbf{a} = [1, b-1]) \sqcup (b) \sqcup [b+1+1, b+1+l-b] = [1, b-1] \sqcup (b) \sqcup [b+2, l+1].$$

Let $t \geq 2$. We now apply to $\mathbf{b} = (b_1, \ldots, b_t) \in SpT_{2n}(\varpi_t)$ the reverse procedure in (14).
One has

$$\begin{aligned}
&\mathbf{b} = (b_1, \ldots, b_t) \in SpT_{2n}(\varpi_t) \to ((b_t), (b_1, \ldots, b_{t-1})) \in SpT_{2n}(\varpi_1) \times SpT_{2n}(\varpi_{t-1}) \\
&\to (\mathbf{s}(b_t)^\vee, \mathbf{b}' = (b_1, \ldots, b_{t-1})) \in SST_{2n}(\varpi_{2n-1}) \times SpT_{2n}(\varpi_{t-1}) \\
&\overset{R_{2n-1,t-1}}{\longrightarrow} (\mathbf{c}', \mathbf{d}) \text{ for some } (\mathbf{c}', \mathbf{d}) \in SpT_{2n}(\varpi_{t-1}) \times SST_{2n}(\varpi_{2n-1}).
\end{aligned} \tag{29}$$

We now prove (29), that is, we exhibit $(\mathbf{c}', \mathbf{d}) \in SpT_{2n}(\varpi_{t-1}) \times SST_{2n}(\varpi_{2n-1})$, such that

$$R_{2n-1,t-1}(\mathbf{s}(b_t)^\vee, \mathbf{b}') = (\mathbf{c}', \mathbf{d}). \tag{30}$$

The proof depends on the parity of the last component b_t of $\mathbf{b}$: either $\mathbf{s}(b_t) = b_t+1$ if $b_t \notin 2\mathbb{Z}$, or $\mathbf{s}(b_t) = b_t-1$ if $b_t \in 2\mathbb{Z}$.

Case 1. $b_t \notin 2\mathbb{Z} \Leftrightarrow \mathbf{s}(b_t) = b_t + 1$.
If $b_t \notin 2\mathbb{Z}$ then $\mathbf{s}(b_t) = b_t + 1 \in 2\mathbb{Z}$ and $(\mathbf{s}(b_t)^\vee, \mathbf{b}') = ((b_t+1)^\vee, \mathbf{b}')$ with $\mathbf{b}' = (b_1, \ldots, b_{t-1}) \subseteq [1, b_{t-1}] \in SpT_{2n}(\varpi_{t-1})$. Hence $\mathbf{b}' \subseteq (b_t+1)^\vee$ and by Proposition 4, (1),

$$R_{2n-1,t-1}((b_t+1)^\vee, \mathbf{b}') = (\mathbf{b}', (b_t+1)^\vee) = (\mathbf{c}', \mathbf{d}) \in SpT_{2n}(\varpi_{t-1}) \times SST_{2n}(\varpi_{2n-1}). \tag{31}$$

Note $t-1 \leq min\{l-1, 2n-(l-1)\}$ and $l-t \in 2\mathbb{Z} \Leftrightarrow l-1-(t-1) \in 2\mathbb{Z}$. Then by induction on l, one has

$$\mathbf{b}' = \mathrm{red}_{l-1}(\hat{\mathbf{a}}), \text{ for some } \hat{\mathbf{a}} \in SST_{2n}(\varpi_{l-1}) \text{ with } \mathbf{b}' = (b_1, \ldots, b_{t-1}) \subseteq \hat{\mathbf{a}}. \tag{32}$$

That is, red_{l-1} is injective and surjective, and red_{l-1}^{-1} does exist,

$$\mathrm{red}_{l-1}^{-1}(\mathbf{b}') = \mathrm{red}_{l-1}^{-1}(b_1, \ldots, b_{t-1}) = \hat{\mathbf{a}} = (a_1, \ldots, a_{l-1}) \in SST_{2n}(\varpi_{l-1}) \text{ with } \mathbf{b}' = (b_1, \ldots, b_{t-1}) \subseteq \hat{\mathbf{a}}. \tag{33}$$

Since $t-1 \leq l-1$, by induction on $t \geq 1$, we then may write, for some column $\mathbf{a}'' \sqcup \{b_{t-1}\} := \hat{\mathbf{a}} \cap [1, b_{t-1}]$, with $b_0 := 0$,

$$\hat{\mathbf{a}} = \begin{cases} \mathbf{a}'' \sqcup \{b_{t-1}\} \sqcup [b_{t-1}+1, b_{t-1}+s], & \text{for some } 0 \leq s \in 2\mathbb{Z} \qquad \text{if } b_{t-1} \in 2\mathbb{Z} \\ \mathbf{a}'' \sqcup \{b_{t-1}\} \sqcup [b_{t-1}+1+1, b_{t-1}+1+s], & \text{for some } 0 \leq s \in 2\mathbb{Z} \quad \text{if } b_{t-1} \notin 2\mathbb{Z}, \end{cases} \tag{34}$$

such that $s + \ell(\hat{\mathbf{a}} \cap [1, b_{t-1}]) = l-1$.
Note if $s = 0$, it means the intervals in (34) are empty. If the intervals are not empty then $2 \leq s \in 2\mathbb{Z}$.
Hence, from (31), (33),

$$\begin{aligned}
&(\mathrm{red}_{l-1}^{-1} \otimes id) R_{2n-1,t-1}((b_t+1)^\vee, \mathbf{b}') = (\mathrm{red}_{l-1}^{-1} \otimes id)(\mathbf{b}', (b_t+1)^\vee) \\
&= (\hat{\mathbf{a}}, (b_t+1)^\vee), \\
&\text{with } \mathbf{a} \in SST_{2n}(\varpi_{l-1}) \text{ satisfying (34).}
\end{aligned} \tag{35}$$

From (34) and $\mathbf{s}(b_t) = b_t + 1 \in 2\mathbb{Z}$, we have to analyse two cases

- $\hat{\mathbf{a}} = (a_1, \ldots, a_{l-1}) \subseteq [1, b_t - 1]$.

From (31) and (35)

$$R_{l-1,2n-1}(\mathrm{red}_{l-1}^{-1}\otimes id)R_{2n-1,t-1}((b_t+1)^\vee,\mathbf{b}')=R_{l-1,2n-1}(\hat{\mathbf{a}},(b_t+1)^\vee)=((b_t+1)^\vee,\hat{\mathbf{a}})\Rightarrow$$

$$(K\otimes id)R_{l-1,2n-1}(\mathrm{red}_{l-1}^{-1}\otimes id)R_{2n-1,t-1}((b_t+1)^\vee,\mathbf{b}')=(K\otimes id)((b_t+1)^\vee,\hat{\mathbf{a}})$$
$$=(K\otimes id)(\mathbf{s}(b_t)^\vee,\hat{\mathbf{a}})=(b_t,\hat{\mathbf{a}})\longrightarrow(\hat{\mathbf{a}},b_t)\in SST_{2n}(\varpi_l).$$

Hence, for $b_t\notin 2\mathbb{Z}$ and $\hat{\mathbf{a}}\subseteq[1,b_t-1]$,

$$\mathrm{red}^{-1}(\mathbf{b})=\mathbf{a}=(\hat{\mathbf{a}},b_t)\supseteq\mathbf{b},\ \text{where }\hat{\mathbf{a}}=\mathrm{red}_{l-1}^{-1}(b_1,\ldots,b_{t-1}).\tag{36}$$

- $b_t\in\hat{\mathbf{a}}$.

Note $b_{t-1}+1\leq b_t$ with $b_{t-1}\in 2\mathbb{Z}$ and $b_{t-1}+1<b_t$ with $b_{t-1}\notin 2\mathbb{Z}$. Thus $b_{t-1}+1+1\leq b_t$ with $b_{t-1}\notin 2\mathbb{Z}$. On the other hand since $b_t\in\hat{\mathbf{a}}$, from (34), $b_t\in[b_{t-1}+1,b_{t-1}+s]$, with $2\leq s\in 2\mathbb{Z}$, if $b_{t-1}\in 2\mathbb{Z}$; and $b_t\in[b_{t-1}+1+1,b_{t-1}+1+s]$, with $2\leq s\in 2\mathbb{Z}$, if $b_{t-1}\notin 2\mathbb{Z}$. Additionally, these intervals start with an odd number and terminate with an even number, and are closed under the parity involution $\mathbf{s}$. Thus since b_t is odd, b_t,b_t+1 are together in the same interval.

Hence $[b_t,b_t+1]\subseteq\hat{\mathbf{a}}\backslash(\mathbf{a}''\sqcup\{b_{t-1}\})$ which implies $[b_t,b_t+1]\subseteq[b_{t-1}+1,b_{t-1}+s]$. Let $\mathbf{a}'''\sqcup\{b_t\}:=\hat{\mathbf{a}}\cap[1,b_t]$. We may then rewrite (34) as

$$\hat{\mathbf{a}}=\begin{cases}\mathbf{a}'''\sqcup\{b_t\}\sqcup[b_t+1,b_{t-1}+s],\ \text{ for some } 2\leq s\in 2\mathbb{Z} & \text{if } b_{t-1}\in 2\mathbb{Z}\\ \mathbf{a}'''\sqcup\{b_t\}\sqcup\ [b_t+1,b_{t-1}+1+s],\ \text{ for some } 2\leq s\in 2\mathbb{Z} & \text{if } b_{t-1}\notin 2\mathbb{Z},\end{cases}\tag{37}$$

such that $s+\ell(\hat{\mathbf{a}}\cap[1,b_{t-1}])=l-1$.

Remark 1. Note $\hat{\mathbf{a}}\cap[1,b_t]=\hat{\mathbf{a}}\cap[1,b_{t-1}]\sqcup\hat{\mathbf{a}}\cap[b_{t-1}+1,b_t]$, and considering (34) with $b_{t-1}\in 2\mathbb{Z}$,

$\hat{\mathbf{a}}=\hat{\mathbf{a}}\cap[1,b_{t-1}]\sqcup[b_{t-1}+1,b_{t-1}+s]=\hat{\mathbf{a}}\cap[1,b_{t-1}]\sqcup[b_{t-1}+1,b_t]\sqcup[b_t+1,b_{t-1}+s]=\hat{\mathbf{a}}\cap[1,b_t]\sqcup[b_t+1,b_{t-1}+s]$.

For $b_{t-1}\notin 2\mathbb{Z}$ is similar.

From (35), one has

$$R_{l-1,2n-1}(\mathrm{red}_{l-1}^{-1}\otimes id)R_{2n-1,t-1}((b_t+1)^\vee,\mathbf{b}')=R_{l-1,2n-1}(\hat{\mathbf{a}},(b_t+1)^\vee).\tag{38}$$

If $b_{t-1}\in 2\mathbb{Z}$, from Proposition 4, (3), (13), and (37)

$$\begin{aligned}&(38)\Leftrightarrow R_{l-1,2n-1}(\hat{\mathbf{a}},(b_t+1)^\vee)=R_{l-1,2n-1}(\mathbf{a}'''\sqcup\{b_t\}\sqcup[b_t+1,b_{t-1}+s],(b_t+1)^\vee)\\&=((b_{t-1}+s+1)^\vee,\mathbf{a}'''\sqcup\{b_t\}\sqcup[b_t+2,b_{t-1}+s+1])\\&=(\mathbf{s}(b_{t-1}+s+2)^\vee,\mathbf{a}'''\sqcup\{b_t\}\sqcup[b_t+2,b_{t-1}+s+1]),\\&\text{by } b_{t-1}+s+1\notin 2\mathbb{Z}\Rightarrow\mathbf{s}(b_{t-1}+s+2)=b_{t-1}+s+1,\end{aligned}\tag{39}$$

and, therefore, it follows

$$\begin{aligned}&(K\otimes id)R_{l-1,2n-1}(\mathrm{red}_{l-1}^{-1}\otimes id)R_{2n-1,t-1}((b_t+1)^\vee,\mathbf{b}')\\&=(K\otimes id)R_{l-1,2n-1}(\hat{\mathbf{a}},(b_t+1)^\vee)=(K\otimes id)R_{l-1,2n-1}(\mathbf{a}''\sqcup\{b_t\}\sqcup[b_t+1,b_{t-1}+s],(b_t+1)^\vee),\ \text{by (37)}\\&=(K\otimes id)(\mathbf{s}(b_{t-1}+s+2)^\vee,\mathbf{a}'''\sqcup\{b_t\}\sqcup[b_t+2,b_{t-1}+s+1])\\&=(b_{t-1}+s+2,\mathbf{a}'''\sqcup\{b_t\}\sqcup[b_t+2,b_{t-1}+s+1]),\ \text{by (39)}\\&\longrightarrow(\mathbf{a}'''\sqcup\{b_t\}\sqcup[b_t+2,b_{t-1}+s+1],b_{t-1}+s+2)=\mathbf{a}'''\sqcup\{b_t\}\sqcup[b_t+2,b_{t-1}+s+2]\in SST_{2n}(\varpi_l)\end{aligned}$$

Hence, for $b_{t-1}\in 2\mathbb{Z}$, $b_t\notin 2\mathbb{Z}$ and $b_t\in\hat{\mathbf{a}}$,

$$\mathrm{red}^{-1}(\mathbf{b})=\mathbf{a}=\mathbf{a}'''\sqcup\{b_t\}\sqcup[b_t+2,b_{t-1}+s+2]\supseteq\!\!\!=\mathbf{b}.$$

In other words,

$$\begin{aligned}&\mathrm{red}^{-1}(\mathbf{b})=\mathbf{a},\ \text{where}\\&\mathbf{a}=\hat{\mathbf{a}}\setminus\{b_t+1\}\sqcup[b_{t-1}+1+s,b_{t-1}+2+s]\in SST_{2n}(\varpi_l),\\&\ \text{with } 2\leq s\in 2\mathbb{Z} \text{ in (34) in the definition of}\\&\hat{\mathbf{a}}:=\mathrm{red}^{-1}(b_1,\dots,b_{t-1}).\end{aligned}\tag{40}$$

Remark 2. For $b_{t-1}\in 2\mathbb{Z}$, $[b_t+1,b_{t-1}+s]$ with $s\geq 2$, we have $b_t+1\leq b_{t-1}+s$ for an $s\geq 2$. Then $b_t+2\leq b_{t-1}+s+1<b_{t-1}+s+2$.

If $b_{t-1}\notin 2\mathbb{Z}$, the proof is similar. From Proposition 4, (3), (13), and (37)

$$\begin{aligned}&(38)\Leftrightarrow R_{l-1,2n-1}(\mathbf{a},(b_t+1)^\vee)=R_{l-1,2n-1}(\mathbf{a}'''\sqcup\{b_t\}\sqcup[b_t+1,b_{t-1}+s+1],(b_t+1)^\vee)\\&=((b_{t-1}+s+2)^\vee,\mathbf{a}'''\sqcup\{b_t\}\sqcup[b_t+2,b_{t-1}+s+2])\\&=(\mathbf{s}(b_{t-1}+s+3)^\vee,\mathbf{a}'''\sqcup\{b_t\}\sqcup[b_t+2,b_{t-1}+s+2]),\\&\text{by } b_{t-1}+s+2\notin 2\mathbb{Z}\Rightarrow\mathbf{s}(b_{t-1}+s+3)=b_{t-1}+s+2,\end{aligned}\tag{41}$$

and

$$\begin{aligned}&(K\otimes id)R_{l-1,2n-1}(\mathrm{red}_{l-1}^{-1}\otimes id)R_{2n-1,t-1}((b_t+1)^\vee,\mathbf{b}')\\&=(K\otimes id)R_{l-1,2n-1}(\hat{\mathbf{a}},(b_t+1)^\vee)=(K\otimes id)R_{l-1,2n-1}(\mathbf{a}'''\sqcup\{b_t\}\sqcup[b_t+1,b_{t-1}+s+1],(b_t+1)^\vee)\\&=(K\otimes id)(\mathbf{s}(b_{t-1}+s+3)^\vee,\mathbf{a}'''\sqcup\{b_t\}\sqcup[b_t+2,b_{t-1}+s+2])\text{ by (41)}\\&=(b_{t-1}+s+3,\mathbf{a}'''\sqcup\{b_t\}\sqcup[b_t+2,b_{t-1}+s+2])\\&\longrightarrow(\mathbf{a}'''\sqcup[b_t+2,b_{t-1}+s+2],b_{t-1}+s+3)=\mathbf{a}'''\sqcup\{b_t\}\sqcup[b_t+2,b_{t-1}+s+3]\in SST_{2n}(\varpi_l)\end{aligned}$$

Hence, for $b_{t-1}\notin 2\mathbb{Z}$, $b_t\notin 2\mathbb{Z}$ and $b_t\in\hat{\mathbf{a}}$,

$$\begin{aligned}&\mathrm{red}^{-1}(\mathbf{b})=\mathbf{a}=\mathbf{a}'''\sqcup\{b_t\}\sqcup[b_t+2,b_{t-1}+s+3]\supseteq\mathbf{b},\\&\ \text{and}\\&\mathbf{a}=\hat{\mathbf{a}}\setminus\{b_t+1\}\sqcup[b_{t-1}+s+2,b_{t-1}+s+3]\in SST_{2n}(\varpi_l),\\&\ \text{with } 2\leq s\in 2\mathbb{Z} \text{ in (37) in the definition of } \hat{\mathbf{a}}=\mathrm{red}^{-1}(b_1,\dots,b_{t-1}).\end{aligned}\tag{42}$$

Case 2. $b_t\in 2\mathbb{Z}\Leftrightarrow\mathbf{s}(b_t)=b_t-1$.

If $b_t\in 2\mathbb{Z}$, then $b_t\in 2\mathbb{Z}\Rightarrow\mathbf{s}(b_t)=b_t-1\notin 2\mathbb{Z}$ and $(\mathbf{s}(b_t)^\vee,\mathbf{b})=((b_t-1)^\vee,\mathbf{b}')$ with $\mathbf{b}'=(b_1,\dots,b_{t-1})\in SpT_{2n}(\varpi_{t-1})$.

We have two main cases: either $b_{t-1}<b_t-1$ or $b_{t-1}=b_t-1$.

(1) $b_{t-1}<b_t-1$.

If $b_{t-1}<b_t-1$, then $b_{t-1}\leq b_t-2$, if $b_{t-1}\in 2\mathbb{Z}$, and $b_{t-1}<b_t-2\Leftrightarrow b_{t-1}+2<b_t\Leftrightarrow b_{t-1}+2\leq b_t-1$ if $b_{t-1}\notin 2\mathbb{Z}$.

Hence, $\mathbf{b}'\subseteq(b_t-1)^\vee$ and by Proposition 4, (1),

$$R_{2n-1,t-1}((b_t-1)^\vee,\mathbf{b}')=(\mathbf{b}',(b_t-1)^\vee)=(\mathbf{c}',\mathbf{d})\in SpT_{2n}(\varpi_{t-1})\times SST_{2n}(\varpi_{2n-1}).\tag{43}$$

Then by induction on $l\geq 3$, one has

$$\mathbf{b}'=\mathrm{red}_{l-1}(\hat{\mathbf{a}}),\ \text{for some } \hat{\mathbf{a}}\in SST_{2n}(\varpi_{l-1}) \text{ such that } \mathbf{b}'=(b_1,\dots,b_{t-1})\subseteq\hat{\mathbf{a}}\tag{44}$$

and by induction on $t\geq 1$ we may write for some $\mathbf{a}''\sqcup\{b_{t-1}\}=\hat{\mathbf{a}}\cap[1,b_{t-1}]$, with $b_0:=0$,

$$\hat{\mathbf{a}}=\begin{cases}\mathbf{a}''\sqcup\{b_{t-1}\}\sqcup[b_{t-1}+1,b_{t-1}+s],\ \text{ for some } 0\leq s\in 2\mathbb{Z} & \text{if } b_{t-1}\in 2\mathbb{Z}\\ \mathbf{a}''\sqcup\{b_{t-1}\}\sqcup[b_{t-1}+1+1,b_{t-1}+1+s],\ \text{ for some } 0\leq s\in 2\mathbb{Z} & \text{if } b_{t-1}\notin 2\mathbb{Z}\end{cases},\tag{45}$$

such that $s+\ell(\hat{\mathbf{a}}\cap[1,b_{t-1}])=l-1$.

Again note if $s=0$ the intervals are empty in (45). If they are not empty, $s\geq 2$.

Hence

$$(\mathrm{red}_{l-1}^{-1}\otimes id)R_{2n-1,t-1}((b_t-1)^\vee,\mathbf{b}')=(\hat{\mathbf{a}},(b_t-1)^\vee)$$
$$\text{for some } \hat{\mathbf{a}}\in SST_{2n}(\varpi_{l-1}) \text{ satisfying (45).} \tag{46}$$

We have to analyse two cases

- $\hat{\mathbf{a}}=(a_1,\dots,a_{l-1})\subset[1,b_t-2]$.

$R_{l-1,2n-1}(\mathrm{red}_{l-1}^{-1}\otimes id)R_{2n-1,t-1}((b_t-1)^\vee,\mathbf{b}')=R_{l-1,2n-1}(\hat{\mathbf{a}},(b_t-1)^\vee)=((b_t-1)^\vee,\hat{\mathbf{a}})\Rightarrow$

$$(K\otimes id)R_{l-1,2n-1}(\mathrm{red}_{l-1}^{-1}\otimes id)R_{2n-1,t-1}((b_t-1)^\vee,\mathbf{b}')=(K\otimes id)((b_t-1)^\vee,\hat{\mathbf{a}})$$
$$=(K\otimes id)(\mathbf{s}(b_t)^\vee,\hat{\mathbf{a}})=(b_t,\hat{\mathbf{a}})\longrightarrow(\hat{\mathbf{a}},b_t)\in SST(\varpi_l).$$

Hence for $b_t\in2\mathbb{Z}$, $b_{t-1}<b_t-1$, and $\hat{\mathbf{a}}\subset[1,b_t-2]$.

$$\mathrm{red}^{-1}(\mathbf{b})=\mathbf{a}=(\hat{\mathbf{a}},b_t),\ \text{where } \hat{\mathbf{a}}=\mathrm{red}^{-1}(b_1,\dots,b_{t-1}). \tag{47}$$

- $b_t-1\in\hat{\mathbf{a}}\Leftrightarrow\mathbf{s}(b_t)\in\hat{\mathbf{a}}$.

By assumption, $b_t\in2\mathbb{Z}\Leftrightarrow b_t-1\notin2\mathbb{Z}$, and thus

$$b_{t-1}<b_t-1\Leftrightarrow b_{t-1}+1\leq b_t-1\Leftrightarrow b_{t-1}+1+1\leq b_t. \tag{48}$$

When $b_{t-1}\in2\mathbb{Z}$, since $b_t-1\in\hat{\mathbf{a}}$, it means in (45) that $s\geq2$. Therefore $[b_t-1,b_t]\subseteq[b_{t-1}+1,b_{t-1}+s]$ with $s\geq2$ in (45).

When $b_{t-1}\notin2\mathbb{Z}$ then $b_{t-1}+1+1\notin2\mathbb{Z}$ and in this case one has $b_{t-1}+1+1<b_t$ and $b_{t-1}+1+1\leq b_t-1$ with $b_t-1\in\hat{\mathbf{a}}$. Thus, it means in (45) that $s\geq2$. Therefore,

$$[b_t-1,b_t]\subseteq[b_{t-1}+1+1,b_{t-1}+1+s] \text{ with } s\geq2 \text{ in (45).}$$

Since $b_{t-1}\leq b_t-2$, from (48), let $\mathbf{a}'''\coloneqq\hat{\mathbf{a}}\cap[1,b_t-2]$, and we may then rewrite (45) as

$$\hat{\mathbf{a}}=\begin{cases}\mathbf{a}'''\sqcup[b_t-1,b_{t-1}+s], & \text{for some } 2\leq s\in2\mathbb{Z} \quad \text{if } b_{t-1}\in2\mathbb{Z}\\ \mathbf{a}'''\sqcup[b_t-1,b_{t-1}+1+s], & \text{for some } 2\leq s\in2\mathbb{Z} \quad \text{if } b_{t-1}\notin2\mathbb{Z},\end{cases} \tag{49}$$

such that $s+\ell(\hat{\mathbf{a}}\cap[1,b_{t-1}])=l-1$.

Remark 3. With $b_{t-1}\leq b_t-2$, $\hat{\mathbf{a}}\cap[1,b_t-2]=\hat{\mathbf{a}}\cap[1,b_{t-1}]\sqcup\hat{\mathbf{a}}\cap[b_{t-1}+1,b_t-2]$. Considering (45) with $b_{t-1}\in2\mathbb{Z}$, one has

$$\hat{\mathbf{a}}=\hat{\mathbf{a}}\cap[1,b_{t-1}]\sqcup[b_{t-1}+1,b_{t-1}+s]=\hat{\mathbf{a}}\cap[1,b_{t-1}]\sqcup[b_{t-1}+1,b_t-2]\sqcup[b_t-1,b_{t-1}+s]$$
$$=\hat{\mathbf{a}}\cap[1,b_t-2]\sqcup[b_t-1,b_{t-1}+s].$$

For $b_{t-1}\notin2\mathbb{Z}$ the proof is similar.

One has

$$R_{l-1,2n-1}(\mathrm{red}_{l-1}^{-1}\otimes id)R_{2n-1,t-1}((b_t-1)^\vee,\mathbf{b}')=$$
$$R_{l-1,2n-1}(\mathrm{red}_{l-1}^{-1}\otimes id)(\mathbf{b}',(b_t-1)^\vee)=R_{l-1,2n-1}(\hat{\mathbf{a}},(b_t-1)^\vee)$$
$$\text{for some } \hat{\mathbf{a}}\in SST_{2n}(\varpi_{l-1}) \text{ satisfying (49).} \tag{50}$$

If $b_{t-1}\in2\mathbb{Z}$, from Proposition 4, (3), (13) and (49)

$$(50)\Leftrightarrow R_{l-1,2n-1}(\hat{\mathbf{a}},(b_t-1)^\vee)=R_{l-1,2n-1}(\mathbf{a}'''\sqcup[b_t-1,b_{t-1}+s],(b_t-1)^\vee)$$
$$=((b_{t-1}+s+1)^\vee,\mathbf{a}'''\sqcup[b_t,b_{t-1}+s+1])$$
$$=(\mathbf{s}(b_{t-1}+s+2)^\vee,\mathbf{a}'''\sqcup[b_t,b_{t-1}+s+1]),$$
$$\text{by } b_{t-1}+s+1\notin2\mathbb{Z}\Rightarrow\mathbf{s}(b_{t-1}+s+2)=b_{t-1}+s+1, \tag{51}$$

and

$$
\begin{aligned}
&(K\otimes id)R_{l-1,2n-1}(\mathrm{red}_{l-1}^{-1}\otimes id)R_{2n-1,t-1}((b_t-1)^\vee,\mathbf{b}')\\
&=(K\otimes id)R_{l-1,2n-1}(\hat{\mathbf{a}},(b_t-1)^\vee)=(K\otimes id)R_{l-1,2n-1}(\mathbf{a}'''\sqcup[b_t-1,b_{t-1}+s],(b_t-1)^\vee)\\
&=(K\otimes id)(\mathbf{s}(b_{t-1}+s+2)^\vee,\mathbf{a}'''\sqcup[b_t,b_{t-1}+s+1])\\
&=(b_{t-1}+s+2,\mathbf{a}'''\sqcup[b_t,b_{t-1}+s+1]),\ \text{by (51)}\\
&\longrightarrow(\mathbf{a}'''\sqcup[b_t,b_{t-1}+s+1],b_{t-1}+s+2)\\
&=\mathbf{a}'''\sqcup[b_t,b_{t-1}+s+2]\in SST_{2n-1}(\varpi_l)
\end{aligned}
$$

Hence, for $b_t\in 2\mathbb{Z}$, $b_{t-1}\in 2\mathbb{Z}$ and $b_{t-1}<b_t-1$ and $\mathbf{s}(b_t)\in\hat{\mathbf{a}}$,

$$
\begin{aligned}
&\mathrm{red}^{-1}(\mathbf{b})=\mathbf{a}=\mathbf{a}'''\sqcup[b_t,b_{t-1}+s+2]\\
&=\mathbf{a}'''\sqcup\{b_t\}\sqcup[b_t+1,b_{t-1}+s+2]\supseteqq\mathbf{b},\\
&\mathbf{a}=\hat{\mathbf{a}}\setminus\{b_t-1\}\sqcup[b_{t-1}+s+1,b_{t-1}+s+2],\ \text{for some } 2\leq s\in 2\mathbb{Z} \text{ as in (45)}.\\
&\hat{\mathbf{a}}=\mathrm{red}^{-1}(b_1,\dots,b_{t-1}).
\end{aligned}
$$

If $b_{t-1}\notin 2\mathbb{Z}$, from Proposition 4, (3), (13) and (49)

$$
\begin{aligned}
&(50)\Leftrightarrow R_{l-1,2n-1}(\hat{\mathbf{a}},(b_t-1)^\vee)=R_{l-1,2n-1}(\mathbf{a}'''\sqcup[b_t-1,b_{t-1}+s+1],(b_t-1)^\vee)\\
&=((b_{t-1}+s+2)^\vee,\mathbf{a}'''\sqcup[b_t,b_{t-1}+s+2])\\
&=(\mathbf{s}(b_{t-1}+s+3)^\vee,\mathbf{a}'''\sqcup[b_t,b_{t-1}+s+2]),\\
&\text{by } b_{t-1}+s+2\notin 2\mathbb{Z}\Rightarrow\mathbf{s}(b_{t-1}+s+3)=b_{t-1}+s+2, \qquad (52)
\end{aligned}
$$

and

$$
\begin{aligned}
&(K\otimes id)R_{l-1,2n-1}(\mathrm{red}_{l-1}^{-1}\otimes id)R_{2n-1,t-1}((b_t-1)^\vee,\mathbf{b}')\\
&=(K\otimes id)R_{l-1,2n-1}(\mathbf{a},(b_t-1)^\vee)=(K\otimes id)R_{l-1,2n-1}(\mathbf{a}'''\sqcup[b_t-1,b_{t-1}+s+1],(b_t-1)^\vee)\\
&=(K\otimes id)(\mathbf{s}(b_{t-1}+s+3)^\vee,\mathbf{a}'''\sqcup[b_t,b_{t-1}+s+2])\\
&=(b_{t-1}+s+3,\mathbf{a}'''\sqcup[b_t,b_{t-1}+s+2]),\ \text{by (52)}\\
&\longrightarrow(\mathbf{a}'''\sqcup[b_t,b_{t-1}+s+2],b_{t-1}+s+3)=\mathbf{a}'''\sqcup[b_t,b_{t-1}+s+3]\in SST_{2n-1}(\varpi_l)
\end{aligned}
$$

Hence, for $b_{t-1}\notin 2\mathbb{Z}$, $b_t\in 2\mathbb{Z}$, and $b_{t-1}\leq b_t-2$, and $\mathbf{s}(b_t)\in\hat{\mathbf{a}}$,

$$
\begin{aligned}
&\mathrm{red}^{-1}(\mathbf{b})=\mathbf{a}=\mathbf{a}'''\sqcup\{b_t\}\sqcup[b_t+1,b_{t-1}+s+3]\supseteqq\mathbf{b},\\
&\mathbf{a}=\hat{\mathbf{a}}\setminus\{b_t-1\}\sqcup[b_{t-1}+s+2,b_{t-1}+s+3],\ \text{for some } 2\leq s\in 2\mathbb{Z} \text{ as in (45)},\\
&\hat{\mathbf{a}}=\mathrm{red}^{-1}(b_1,\dots,b_{t-1}). \qquad (53)
\end{aligned}
$$

(2) $b_{t-1}=b_t-1\notin 2\mathbb{Z}$

Recall $\mathbf{b}=(b_1,\dots,b_t)\in SpT_{2n}(\varpi_t)$ and $\mathbf{b}'=(b_1,\dots,b_{t-1}=b_t-1)\in SpT_{2n}(\varpi_{t-1})$. Unlike the previous cases we have

$$\mathbf{b}'\subsetneq\mathbf{s}(b_t)^\vee=(b_t-1)^\vee.$$

Set $r:=max\{i\in[1,t-1]:b_{t-i}=b_t-i\}$ and $\mathbf{b}'':=(b_1,\dots,b_{t-r-1})$. Note $r\geq 1$ and

$$b_{t-r-1}<b_t-r-1<b_t-r=b_{t-r}. \qquad (54)$$

Then, we have $\mathbf{b}=\mathbf{b}''\sqcup[b_t-r,b_t]$, $\mathbf{b}'=\mathbf{b}''\sqcup[b_t-r,b_t-1]\in SpT_{2n}(\varpi_{t-1})$, $\mathbf{s}(b_t)=b_t-1$, and from Proposition 4, (3), (12),

$$(30) \Leftrightarrow R_{2n-1,t-1}((b_t-1)^\vee, \mathbf{b}') = R_{2n-1,t-1}((b_t-1)^\vee, \mathbf{b}'' \sqcup [b_t-r, b_t-1])$$
$$= (\mathbf{b}'' \sqcup [b_t-r-1, b_t-2], (b_t-r-1)^\vee) \in SpT_{2n}(\varpi_{t-1}) \times SST_{2n}(\varpi_{2n-1}).$$

We have to show that

$$\bar{\mathbf{b}} := \mathbf{b}'' \sqcup [b_t-r-1, b_t-2] = \mathbf{b}' \setminus \{b_t-1\} \sqcup \{b_t-r-1\} \in SpT_{2n}(\varpi_{t-1}). \tag{55}$$

- For $r \notin 2\mathbb{Z}$ and $b_t \in 2\mathbb{Z}$, one has $b_t - r \notin 2\mathbb{Z}$, and $b_t - r - 1 \in 2\mathbb{Z}$.

We then write (55) as

$$(55) \Leftrightarrow \mathbf{b}'' \sqcup \{b_t-r-1\} \sqcup [b_t-r, b_t-2].$$

If $b_t - r - 1 \in 2\mathbb{Z}$ is removable in (55), by Proposition 2, $\mathbf{s}(b_t-r-1) = b_t-r-2 \notin 2\mathbb{Z}$ is removable. Since, from (54), $b_{t-r-1} \leq b_t-r-2$: either $b_{t-r-1} = b_t-r-2$ which implies b_{t-r-1} removable in $\mathbf{b}'' := (b_1, \ldots, b_{t-r-1}) \subseteq \mathbf{b}' \in SpT_{2n}(\varpi_{t-1})$, a contradiction; or $b_{t-r-1} < b_t-r-2$ and in this case $b_t-r-2 \notin \mathbf{b}''$ and thus not removable from $\mathbf{b}'' \sqcup [b_t-r-1, b_t-2]$.

Let $r \geq 3$. If $b_t - r$ is removable in (55), by Proposition 2,

$$b_t - r < 2(t-r-1+2) \Leftrightarrow b_t < 2t-r+2,\ r \geq 3 \Rightarrow b_t < 2t-1.$$

A contradiction with $\mathbf{b} = (b_1, \ldots, b_t) \in SpT_{2n}(\varpi_t)$. Therefore , $b_t - r, b_t - (r-1)$ are not removable in (55). By reverse induction, assume for $5 \leq j \leq r$ that $b_t - j, b_t - (j-1)$ are not removable whenever $j \notin 2\mathbb{Z}$. Let us prove the claim for $j-2$ with $j \notin 2\mathbb{Z}$. By Proposition 2, suppose that we had

$$b_t - (j-2) < 2(t-r-1+1+r-(j-2)+1) = 2(t-(j-2)+1)$$
$$\Leftrightarrow b_t - (j-2) < 2t - 2(j-2) + 2 \Leftrightarrow b_t < 2t-(j-2)+2$$
$$\Leftrightarrow b_t < 2t-j+4$$
$$\Leftrightarrow b_t < 2t-1, \text{ by } j \geq 5,$$

which is a contradiction with $\mathbf{b} = (b_1, \ldots, b_t)$ symplectic. Hence, in this case, (55) holds.

- For $2 \leq r \in 2\mathbb{Z}$ with $b_t \in 2\mathbb{Z}$, one has $b_t - r \in 2\mathbb{Z}$ and $b_t - r - 1 \notin 2\mathbb{Z}$.

If $b_t - r - 1 \notin 2\mathbb{Z}$ is removable in (55), by Proposition 2, $\mathbf{s}(b_t-r-1) = b_t - r \in 2\mathbb{Z}$ is removable in (55). Therefore, by Proposition 2

$$b_t - r < 2(t-r-1+1+1) - 1 \Leftrightarrow b_t - r < 2t - 2r + 2 - 1$$
$$\Leftrightarrow b_t < 2t-r+1 \leq 2t-1, \text{ by } r \geq 2$$
$$\Leftrightarrow b_t < 2t-1$$

which is a contradiction with $\mathbf{b} = (b_1, \ldots, b_t)$ symplectic. Hence $b_t - r - 1, b_t - r$ are not removable in (55).

By reverse induction, assume for $4 \leq j \leq r$ that $b_t - j - 1, b_t - j$ are not removable whenever $j \in 2\mathbb{Z}$. Let us prove the claim for $j-2$ with $j \in 2\mathbb{Z}$. By Proposition 2, suppose that we had

$$b_t - (j-2) < 2(t-r-1+r+1-(j-2)+1) - 1 = 2(t-(j-2)+1) - 1$$
$$\Leftrightarrow b_t - (j-2) < 2t - 2(j-2) + 1 \Leftrightarrow b_t < 2t-(j-2)+1$$
$$\Leftrightarrow b_t < 2t-j+3$$
$$\Leftrightarrow b_t < 2t-1, \text{ by } j \geq 4,$$

which is a contradiction with $\mathbf{b} = (b_1, \ldots, b_t)$ symplectic. Hence, also in this case, (55) holds.

Since $\bar{\mathbf{b}} = \mathbf{b}' \setminus \{b_t - 1\} \sqcup \{b_t - r - 1\} = (b_1, \dots, b_{t-r-1}) \sqcup [b_t - r - 1, b_t - 2] \in SpT_{2n}(\varpi_{t-1})$, by induction on l,

$$\begin{aligned} &\mathrm{red}_{l-1}^{-1}(\bar{\mathbf{b}}) = \mathrm{red}_{l-1}^{-1}((b_1, \dots, b_{t-r-1}) \sqcup [b_t - r - 1, b_t - 2]) \\ &= \bar{\mathbf{a}} \in SST_{2n}(\varpi_{l-1}), \quad \text{and } \bar{\mathbf{b}} = (b_1, \dots, b_{t-r-1}) \sqcup [b_t - r - 1, b_t - 2] \subseteq \bar{\mathbf{a}}. \end{aligned} \tag{56}$$

By induction on $t \geq 0$, recalling (54), define the column

$$\mathbf{x} := \bar{\mathbf{a}} \cap [1, b_t - r - 2] \supseteq \mathbf{b}'' = (b_1, \dots, b_{t-r-1}),$$

and one has

$$\bar{\mathbf{a}} = \mathbf{x} \sqcup [b_t - r - 1, b_t - 2 + s], \text{ for some } 0 \leq s \in 2\mathbb{Z} \, . \tag{57}$$

such that $\ell(\bar{\mathbf{a}} \cap [1, b_t - r - 2]) + r + s = l - 1$.

Therefore

$$\begin{aligned} &(K \otimes id) R_{l-1,2n-1}(\mathrm{red}_{l-1}^{-1} \otimes id)(\bar{\mathbf{b}}, (b_t - r - 1)^\vee) \\ &= (K \otimes id) R_{l-1,2n-1}(\mathbf{x} \sqcup [b_t - r - 1, b_t - 2 + s], (b_t - r - 1)^\vee), \text{ by (56), (57)} \\ &(K \otimes id)((b_t - 2 + s + 1)^\vee, \mathbf{x} \sqcup [b_t - r, b_t - 2 + s + 1]), \text{ by (13)} \\ &= (K \otimes id)(\mathbf{s}(b_t - 2 + s + 2)^\vee, \mathbf{x} \sqcup [b_t - r, b_t - 2 + s + 1]), \\ &\text{by } b_t - 2 + s + 2 \in 2\mathbb{Z} \Rightarrow \mathbf{s}(b_t - 2 + s + 2) = b_t - 2 + s + 1, \\ &= (b_t - 2 + s + 2, \mathbf{x} \sqcup [b_t - r, b_t - 2 + s + 1]) \\ &\longrightarrow (\mathbf{x} \sqcup [b_t - r, b_t - 2 + s + 1], b_t - 2 + s + 2) = \mathbf{x} \sqcup [b_t - r, b_t - 2 + s + 2] \in SST_{2n}(\varpi_l). \end{aligned}$$

Hence, for $b_t \in 2\mathbb{Z}$ and $\mathbf{b} = (b_1, \dots, b_{t-r-1}) \sqcup [b_t - r, b_t]$ for some $r \geq 1$, with $b_{t-r-1} < b_t - r - 1$,

$$\begin{aligned} &\mathrm{red}^{-1}(\mathbf{b}) = \mathbf{a} = \bar{\mathbf{a}} \cap [1, b_t - r - 2] \sqcup [b_t - r, b_t - 2 + s + 2] \supseteq \mathbf{b} = (b_1, \dots, b_{t-r-1}) \sqcup [b_t - r, b_t], \\ &\text{and} \\ &\mathbf{a} = \bar{\mathbf{a}} \setminus \{b_t - r - 1\} \sqcup [b_t - 2 + s + 1, b_t - 2 + s + 2], \text{ with } 0 \leq s \in 2\mathbb{Z} \text{ in (57)} \\ &\text{where } \bar{\mathbf{a}} = \mathrm{red}^{-1}((b_1, \dots, b_{t-r-1}) \sqcup [b_t - r - 1, b_t - 2]). \end{aligned} \tag{58}$$

□

As a consequence of the previous theorem, the injectivity of the reduction map, and Proposition 4.4.1 in [Wat25], the inverse reduction map red^{-1} for $t \geq 2$ also decomposes, in reverse order, into the inverses of the maps in (22) (on suitable domains). For those inverses on the suitable domains, we now explicitly write the inverses needed in next theorem.

Let $l \geq t \geq 2$,

$$\begin{aligned} \bigwedge : \{((a), a_1, \dots, a_{t-1}) \in SST_{2n}(\varpi_1) \times SST_{2n}(\varpi_{t-1}) : a > a_{t-1}\} &\longrightarrow SST_{2n}(\varpi_t) \\ ((a), (a_1, \dots, a_{t-1})) &\mapsto (a_1, \dots, a_{t-1}, a), \end{aligned} \tag{59}$$

$$\begin{aligned} \bigwedge^{-1} : SST_{2n}(\varpi_t) &\longrightarrow \{((a), a_1, \dots, a_{t-1}) \in SST_{2n}(\varpi_1) \times SST_{2n}(\varpi_{t-1}) : a > a_{t-1}\} \\ (a_1, \dots, a_{t-1}, a) &\mapsto ((a), (a_1, \dots, a_{t-1})), \end{aligned} \tag{60}$$

$$\begin{aligned} K : SST_{2n}(\varpi_1) &\longrightarrow SST_{2n}(\varpi_{2n-1}) \\ (a) &\mapsto \mathbf{s}(a)^\vee, \end{aligned} \tag{61}$$

$$K^{-1} : SST_{2n}(\varpi_{2n-1}) \longrightarrow SST_{2n}(\varpi_1)$$
$$\mathbf{s}(a)^\vee \mapsto (a),\ a \in [2n], \tag{62}$$

$$\bigvee : SST_{2n}(\varpi_l) \longrightarrow \{((a),(a_1,\dots,a_{l-1})) \in SST_{2n}(\varpi_1) \times SST_{2n}(\varpi_{l-1}) : a > a_{l-1}\}$$
$$(a_1,\dots,a_{l-1},a) \mapsto ((a),(a_1,\dots,a_{l-1})), \tag{63}$$

$$\bigvee^{-1} : \{((a),(a_1,\dots,a_{l-1})) \in SST_{2n}(\varpi_1) \times SST_{2n}(\varpi_{l-1}) : a > a_{l-1}\} \longrightarrow SST_{2n}(\varpi_l)$$
$$((a),(a_1,\dots,a_{l-1})) \mapsto (a_1,\dots,a_{l-1},a), \tag{64}$$

Theorem 2. For a fixed $n \in \mathbb{N}$, let $l \in [0,2n]$ and $t \in [0,n]$ such that $0 \le t \le min\{l, 2n-l\}$ and $l - t \in 2\mathbb{Z}$. Then

$$\mathrm{red}^{-1} = \mathrm{red}_t^{-1} : SpT_{2n}(\varpi_t) \to SST_{2n}(\varpi_l) \tag{65}$$

is inductively defined as follows

(1) for $t = 0$, $l \in 2\mathbb{Z}$ in which case $\mathrm{red}^{-1}(()) = (1,\dots,l)$,
(2) for $t = 1$, $l \notin 2\mathbb{Z}$ in which case for $\mathbf{b} = (b) \in SpT_{2n}(\varpi_1)$, one has

$$\mathrm{red}^{-1}(\mathbf{b}) = (1,\dots,l_1)(b)T_1 \in SST_{2n}(\varpi_l),$$

where

$$(b)T_1 = \begin{cases} (b)(b+1,\dots,b+l_2), & \text{if } b \in 2\mathbb{Z} \\ (b)(b+1+1, b+1+2, \dots, b+1+l_2), & \text{if } b \notin 2\mathbb{Z} \end{cases}, \tag{66}$$

and

$$l_1 = \begin{cases} min\{b-2, l-1\}, & \text{if } b \in 2\mathbb{Z} \\ min\{b-1, l-1\}, & \text{if } b \notin 2\mathbb{Z}, \end{cases} \qquad \text{and } l_2 = l - 1 - l_1. \tag{67}$$

(3) for $t \ge 2$, one has $l \ge t \ge 2$, and

$$\mathrm{red}^{-1} = \bigvee^{-1} \circ (K^{-1}, id) \circ R_{l-1,2n-1} \circ (\mathrm{red}_{t-1}^{-1}, id) \circ R_{2n-1,t-1} \circ (K, id) \circ \bigwedge^{-1}. \tag{68}$$

Proof. The composition (68) is well defined by Proposition 4.4.1 in [Wat25] and because $(1,2) \notin SpT_{2n}(\varpi_2)$. Also in this case $\pi = id$ and $\bigvee^{-1}, \bigwedge^{-1}$, restrict to suitable domains as in (60) and (63).

Assertion (1) follows from Corollary 2. Assertion (2) follows from Proposition 2. The computations show $\mathrm{rem}((1,\dots,l_1)(b)T_1) = (1,\dots,l_1)T_1$ and $\mathrm{red}((1,\dots,l_1)(b)T_1) = (b)$.

We prove assertion 3. Consider the maps above and Proposition 4.4.1 in [Wat25]. Let $l \ge t \ge 2$ and $\mathbf{b} = (b_1,\dots,b_t) \in SpT_{2n}(\varpi_t)$. Then $\mathrm{red}^{-1}(\mathbf{b})$ is obtained as follows:

$$\begin{aligned}
&\mathbf{b}=(b_1,\dots,b_t)\in SpT_{2n}(\varpi_t)\xrightarrow{\bigwedge^{-1}}((b_t),(b_1,\dots,b_{t-1}))\in SpT_{2n}(\varpi_1)\times SpT_{2n}(\varpi_{t-1})\\
&\xrightarrow{K\otimes id}(\mathbf{s}(b_t)^\vee,\mathbf{b}')\in SST_{2n-1}(\varpi_{2n-1})\times SpT_{2n}(\varpi_{t-1}),\ \mathbf{b}'=(b_1,\dots,b_{t-1})\\
&\xrightarrow{R_{2n-1,t-1}}(\mathbf{c}',\mathbf{d}),\ \text{for some } (\mathbf{c}',\mathbf{d})\in SpT_{2n}(\varpi_{t-1})\times SST_{2n}(\varpi_{2n-1})\\
&\xrightarrow{\mathrm{red}_{l-1}^{-1}\otimes id}(\mathbf{c},\mathbf{d})\in SST_{2n}(\varpi_{l-1})\times SST_{2n}(\varpi_{2n-1})\\
&\xrightarrow{R_{l-1,2n-1}}(\mathbf{s}(a_l)^\vee,\mathbf{a}'=(a_1,\dots,a_{l-1}))\in SST_{2n}(\varpi_{2n-1})\times SST_{2n}(\varpi_{l-1})\ \text{for some } a_l>a_{l-1}\\
&\xrightarrow{K^{-1}\otimes id}((a_l),\mathbf{a}'=(a_1,\dots,a_{l-1}))\in SST_{2n}(\varpi_1)\times SST_{2n}(\varpi_{l-1}),\\
&\xrightarrow{\bigvee^{-1}}\mathbf{a}=(a_1,\dots,a_l)\in SST_{2n}(\varpi_l).
\end{aligned}\tag{69}$$

□

4.3. **Main Theorem.** As a bonus of the lengthy proof of Proposition 6 we get the Main Theorem below which collects the information in (28), (36), (40), (42), (47), (53),and (58).

Theorem 3. (Main Theorem) For a fixed $n\in\mathbb{N}$, let $l\in[0,2n]$. The inverse reduction map

$$\mathrm{red}^{-1}=\mathrm{red}_l^{-1}:\bigsqcup_{\substack{0\le t\le min\{l,2n-l\}\\ l-t\in 2\mathbb{Z}}}SpT_{2n}(\varpi_t)\to SST_{2n}(\varpi_l),\quad \mathbf{b}=(b_1,\dots,b_{t-1},b_t)\mapsto \mathrm{red}^{-1}(\mathbf{b})=\mathbf{a}\tag{70}$$

is such that

(1) for $l=0,1$, either $t=0$ or respectively $t=1$, in which cases $\mathrm{red}^{-1}=id$, where $SpT_{2n}(\varpi_0)=SST_{2n}(\varpi_0)=\{()\}$ and $SpT_{2n}(\varpi_1)=SST_{2n}(\varpi_1)$.
(2) for $l=2$ either $t=0$ in which case $\mathbf{b}=()$ and $\mathrm{red}^{-1}()=\mathbf{a}=(1,2)\in SST_{2n}(\varpi_2)$; or respectively $t=2$ in which case $\mathbf{b}=(b_1,b_2)\in SpT_{2n}(\varpi_2)=SST_{2n}(\varpi_2)\setminus\{(1,2)\}$, and $\mathrm{red}^{-1}(\mathbf{b})=\mathbf{b}$.
(3) for $l\ge 3$, if $t=0$ and $l\in 2\mathbb{Z}$, then $\mathbf{b}=()$ and $\mathbf{a}=(1,2,\dots,l)\in SST_{2n}(\varpi_l)$; and if $t=1$ and $3\le l\notin 2\mathbb{Z}$, then $\mathbf{b}=(b)\in SpT_{2n}(\varpi_1)$, and $\mathrm{red}_l^{-1}((b))=\mathbf{a}\in SST_{2n}(\varpi_l)$ is such that if

$$\hat{\mathbf{a}}:=[1,l-1]=\mathrm{red}_{l-1}^{-1}()\tag{71}$$

then

$$\mathbf{a}=\begin{cases}(\hat{\mathbf{a}},b), & \text{if } \hat{\mathbf{a}}\subseteq[1,b-1],\\ \hat{\mathbf{a}}\setminus\{b+1\}\sqcup[l,l+1], & \text{if } b\notin 2\mathbb{Z} \text{ and } b\in\hat{\mathbf{a}}=[1,l-1],\\ \hat{\mathbf{a}}\setminus\{b-1\}\sqcup[l,l+1], & \text{if } b\in 2\mathbb{Z} \text{ and } b\in\hat{\mathbf{a}}=[1,l-1],\end{cases}\tag{72}$$

(4) Let $l\ge 3$ and $t\ge 2$.
- for $\mathbf{b}\in SpT_{2n}(\varpi_t)$ without a nonempty interval fixed by the parity involution $\mathbf{s}$ as a suffix, define

$$\hat{\mathbf{a}}:=\mathrm{red}_{l-1}^{-1}(b_1,\dots,b_{t-1})\in SST_{2n}(\varpi_{l-1}).$$

Then

$$\hat{\mathbf{a}}=\begin{cases}\hat{\mathbf{a}}\cap[1,b_{t-1}-1]\sqcup\{b_{t-1}\}\sqcup[b_{t-1}+1,b_{t-1}+s], & \text{for some } 0\le s\in 2\mathbb{Z} \quad \text{if } b_{t-1}\in 2\mathbb{Z}\\ \hat{\mathbf{a}}\cap[1,b_{t-1}-1]\sqcup\{b_{t-1}\}\sqcup[b_{t-1}+1+1,b_{t-1}+1+s], & \text{for some } 0\le s\in 2\mathbb{Z} \quad \text{if } b_{t-1}\notin 2\mathbb{Z},\end{cases}\tag{73}$$

such that $s+\ell(\hat{\mathbf{a}}\cap[1,b_{t-1}])=l-1$ and

$$\mathbf{a}=\begin{cases}(\hat{\mathbf{a}},b_t), & \text{if } \hat{\mathbf{a}}\subseteq[1,b_t-1] \text{ and } b_t\notin 2\mathbb{Z},\\ \hat{\mathbf{a}}\setminus\{\mathbf{s}(b_t)\}\sqcup[b_{t-1}+1+s,b_{t-1}+2+s], & \text{if } b_t\in\hat{\mathbf{a}}\ ,\ b_{t-1}\in 2\mathbb{Z},\ b_t\notin 2\mathbb{Z},\\ \hat{\mathbf{a}}\setminus\{\mathbf{s}(b_t)\}\sqcup[b_{t-1}+2+s,b_{t-1}+3+s], & \text{if } b_t\in\hat{\mathbf{a}}\ ,\ b_{t-1}\notin 2\mathbb{Z},\ b_t\notin 2\mathbb{Z},\\ (\hat{\mathbf{a}},b_t), & \text{if } \hat{\mathbf{a}}\subseteq[1,b_t-2] \text{ and } b_{t-1}<b_t-1,\ b_t\in 2\mathbb{Z},\\ \hat{\mathbf{a}}\setminus\{\mathbf{s}(b_t)\}\sqcup[b_{t-1}+s+1,b_{t-1}+s+2], & \text{if } \mathbf{s}(b_t)\in\hat{\mathbf{a}} \text{ and } b_{t-1}<b_t-1,\ b_{t-1}\in 2\mathbb{Z},\ b_t\in 2\mathbb{Z},\\ \hat{\mathbf{a}}\setminus\{\mathbf{s}(b_t)\}\sqcup[b_{t-1}+s+2,b_{t-1}+s+3], & \text{if } \mathbf{s}(b_t)\in\hat{\mathbf{a}} \text{ and } b_{t-1}<b_t-1,\ b_{t-1}\notin 2\mathbb{Z},\ b_t\in 2\mathbb{Z}.\end{cases}\tag{74}$$

where $0\leq s\in 2\mathbb{Z}$ and $s+\ell(\hat{\mathbf{a}}\cap[1,b_{t-1}])=l-1$.

- otherwise, $\mathbf{b}=(b_1,\dots,b_{t-r-1})\sqcup[b_t-r,b_t]\in SpT_{2n}(\varpi_t)$, for some $1\leq r<t$, with $b_t\in 2\mathbb{Z}$, $b_{t-r-1}<b_t-r-1<b_{t-r}=b_t-r$, and define

$$\mathbf{b}\setminus\{b_t,\mathbf{s}(b_t)\}\sqcup\{b_t-r-1\}=(b_1,\dots,b_{t-r-1})\sqcup[b_t-r-1,b_t-2]\in SpT_{2n}(\varpi_{t-1}),\ \text{and}$$
$$\overline{\mathbf{a}}:=\mathrm{red}_{l-1}^{-1}((b_1,\dots,b_{t-r-1})\sqcup[b_t-r-1,b_t-2])\in SST_{2n}(\varpi_{l-1}).$$

Then

$$\bar{\mathbf{a}}=\bar{\mathbf{a}}\cap[1,b_t-r-2]\sqcup[b_t-r-1,b_t-2+s],\ \text{for some } 0\leq s\in 2\mathbb{Z}\ ,\tag{75}$$

such that $\ell(\overline{\mathbf{a}}\cap[1,b_t-r-2])+r+s=l-1$, and

$$\mathbf{a}=\overline{\mathbf{a}}\setminus\{b_t-r-1\}\sqcup[b_t-2+s+1,b_t-2+s+2],\ \text{for some } 0\leq s\in 2\mathbb{Z},\tag{76}$$

such that $\ell(\overline{\mathbf{a}}\cap[1,b_t-r-2])+r+s=l-1$.

Remark 4. To compute $\hat{\mathbf{a}}$ or $\overline{\mathbf{a}}$ we may get assisted by the explicit theorems in [Aze26a].

4.4. **Illustration of the Main Theorem 3.** Our method in [Aze26a, Theorem 7] gives a general procedure on how to expand a given general symplectic column to a column of an admissible given length as in the conditions above. A compromise is a combination of our procedures with the composition above. See Example 4, (3) in [Aze26a], and [Wat25, Example 4.4.2].

In Example 4, (3) in [Aze26a], for $n=9$, $l=10$, $t_1=4$, and $t=6<min\{l=10,18-10\}$, $l-t=4\in 2\mathbb{Z}$, we have considered $\mathbf{b}=1,7,(11,12,13,14)\in SpT_{2n}(\varpi_6)$, and

$$\mathrm{red}_t^{-1}(\mathbf{b})=\mathbf{1}(3,4)\mathbf{7}(\mathbf{11},\mathbf{12},\mathbf{13},\mathbf{14})(15,16)\in SST_{18}(\varpi_{10}),$$

is easily computed using our explicit formulas in [Aze26a, Theorem 3] while we may compare it with the inverse reduction of

$$1,7,10,(11,12)\in SpT_{2n}(\varpi_9).$$

We now compute $\mathrm{red}^{-1}(\mathbf{b})$ using red^{-1} as a composition of several maps (68) and will see the usefulness of explicit formulas when large symplectic columns are considered. Then we avoid the composition of several maps and just use Theorem 3 where the computation remains essentially on the computation of the inverse reduction map on a symplectic column of rank reduced by one.

Example 4. With the previous setup, Example 4, (3) in [Aze26a], $n=9$, $t=6$, and $l=10$,

- using Theorem 2,

$$\begin{aligned}&\mathbf{b}=(1,7,11,12,13,14)\in SpT_{2n}(\varpi_t)\overset{\pi^{-1}=id}{\longrightarrow}\mathbf{a}\in SpT_{2n}(\varpi_t)\\ &\overset{\bigwedge^{-1}}{\longrightarrow}((14),(1,7,11,12,13))\in SpT_{2n}(\varpi_1)\times SpT_{2n}(\varpi_{t-1})\\ &\overset{K\otimes id}{\longrightarrow}(\mathbf{s}(14)^\vee;(1,7,11,12,13))\in SST_{2n}(\varpi_{2n-1})\times SpT_{2n}(\varpi_{t-1})\\ &\overset{R_{2n-1,t-1}}{\longrightarrow}((1,7,10,11,12);10^\vee)\in SpT_{2n}(\varpi_{t-1})\times SST_{2n}(\varpi_{2n-1}),\ \text{by } [1,9]\sqcup[11,18]=10^\vee,\\ &\overset{\mathrm{red}_{t-1}^{-1}\otimes id}{\longrightarrow}((1,3,4,7)\sqcup[10,14];10^\vee)\in SST_{2n}(\varpi_{l-1})\times SST_{2n}(\varpi_{2n-1}),\end{aligned}\tag{77}$$

by [Aze26a, Theorem 3], $\mathrm{red}_{t-1}^{-1}(1,7,10,11,12) = 1,(3,4)7,10,11,12,(13,14) = (1,3,4,7) \sqcup [10,14]$, (78)

$\overset{R_{l-1,2n-1}}{\longrightarrow} (\mathbf{s}^{\vee}(16); (1,(3,4),7)) \sqcup [11,15]) \in SST_{2n}(\varpi_{2n-1}) \times SST_{2n}(\varpi_{l-1})$, by Proposition 4, (3),

$\overset{K^{-1}\otimes id}{\longrightarrow} (16; 1,(3,4),7,(11,12,13,14),15) \in SST_{2n}(\varpi_1) \times SST_{2n}(\varpi_{l-1})$,

by $(1,(3,4),7,(11,12,13,14),15)) = (1,(3,4),7)) \sqcup [11,15]$,

$\overset{\bigvee^{-1}}{\longrightarrow} (1,(3,4),7,(11,12,13,14),15,16) = \mathrm{red}^{-1}(\mathbf{b}) \in SST_{2n}(\varpi_l)$, (79)

- We now check with Theorem 3, (75). One has $l = 10$, $r = 3$, $t = 6$, $t - r - 1 = 6 - 3 - 1 = 2$ and $\mathbf{b} = (1,7) \sqcup [11,14]$ with $b_6 = 14 \in 2\mathbb{Z}$, $b_2 = 7 < b_6 - 2 - 1 = 11 < b_6 - 3 = 11 = b_3$. One has $b_t - r - 1 = 10$ and $b_t - r - 2 = 9$.

  Define

  $$\mathbf{b} \setminus \{b_t, \mathbf{s}(b_t)\} \sqcup \{b_t - r - 1\} = (1,7) \sqcup [10,12] = (1,7,10) \sqcup [11,12] \in SpT_{2n}(\varpi_{t-1}), \text{ and}$$

  $$\begin{aligned}\overline{\mathbf{a}} &:= \mathrm{red}_{l-1}^{-1}((b_1,\dots,b_{t-r-1}) \sqcup [b_t - r - 1, b_t - 2]) = \mathrm{red}_{l-1}^{-1}((1,7,10) \sqcup [11,12]) \\ &= (1,3,4,7) \sqcup [10,14] \in SST_{2n}(\varpi_{l-1}) \text{ by [Aze26a, Theorem 3].}\end{aligned}$$

  Then

  $$\begin{aligned}\mathrm{red}_l^{-1}(\mathbf{b}) = \mathbf{a} &= \overline{\mathbf{a}} \setminus \{b_t - r - 1\} \sqcup [b_t - 2 + s + 1, b_t - 2 + s + 2] \\ &= (1,3,4,7) \sqcup [11,14] \sqcup [15,16] = (79)\end{aligned} \quad (80)$$

  such that $0 \leq s \in 2\mathbb{Z}$ and

  $$\ell(\overline{\mathbf{a}} \cap [1, b_t - r - 2]) + r + s = l - 1 \Leftrightarrow \ell(\overline{\mathbf{a}} \cap [1,9]) + 3 + s = 9 \Leftrightarrow 4 + 3 + s = 9$$
  $$\Leftrightarrow s = 2.$$

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