PRÉ-PUBLICAÇÕES DMUC

CARTESIAN CLOSEDNESS OF LAX COMMA CATEGORIES

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Abstract. For a finitely complete category X, we prove that the lax comma category $\mathsf{Cat} /\!\!/ X$ is cartesian closed if and only if its free coproduct completion $\mathsf{Fam}(X)$ is cartesian closed. The proof uses a change-of-base theorem: postcomposition with a functor induces a left adjoint between lax comma categories precisely when that functor is a left multiadjoint. We describe the exponentials and derive examples from known results on family categories.

DMUC Preprint 26-59, 30 Oct 2026

Introduction

The study of lax comma categories in [9] was motivated by two-dimensional aspects of Janelidze-Galois theory [4] and their connection with lax orthogonal factorization systems. Subsequent work on lax comma categories of ordered sets [8] and categories [10] addressed exponentiability, extensivity, topologicity and descent. Related developments include the study of filtered preorders and ordered families [7, 11], as well as topological lax comma categories [6]. The latter also occur in the description of the arithmetic spaces used in the Stone–Gelfand duality of Abbadini, Marra and Spada [1].

The cartesian closedness result of [10, Theorem 2.4] constructs exponentials in $\mathsf{Cat} /\!\!/ X$ using exponentials and ends in the base. In particular, a complete cartesian closed X gives a cartesian closed $\mathsf{Cat} /\!\!/ X$, and the projection to Cat preserves these exponentials. However, the results on Σ-tractable monoidal structures in [21] show that $\mathsf{Cat} /\!\!/ \mathsf{CMon}$ is cartesian closed, although CMon is not. Thus the fibred construction does not exhaust the possibilities. The challenge is to characterize cartesian closedness of the total category $\mathsf{Cat} /\!\!/ X$ without requiring its exponentials to be preserved by the projection $\mathsf{Cat} /\!\!/ X \to \mathsf{Cat}$.
We answer this question in terms of families, that is to say, the free coproduct completion of X. For finitely complete X, we prove

$$\mathsf{Cat} /\!\!/ X \text{ is cartesian closed} \iff \mathsf{Fam}(X) \text{ is cartesian closed.}$$

The proof rests on a general change-of-base result: left multiadjoints are precisely the functors inducing left adjoints between lax comma categories. For the singleton embedding, this gives a full coreflection of $\mathsf{Cat} /\!\!/ X$ in $\mathsf{Cat} /\!\!/ \mathsf{Fam}(X)$. Together with the discrete-diagram embedding, it allows cartesian closedness to be transferred between families (objects of $\mathsf{Fam}(X)$) and diagrams (objects of $\mathsf{Cat} /\!\!/ X$).

Cartesian closedness of $\mathsf{Fam}(X)$ has already been studied in several settings, from the characterization of Adámek and Rosický [2] to the semantics of automatic differentiation [20] and free distributive and extensive completions [18, 22]. The final section applies these results to lax comma categories.

1. Families

Throughout, families and diagram shapes are small, while the categories X and Y may be large. Presheaf categories and the ordinary comma constructions recalled in Section 2 may be interpreted in a larger universe. We identify sets with discrete categories and denote the singleton set by $\mathbf{1}$. An object $x \in X$ also denotes the functor $\mathbf{1} \to X$ selecting it. When X has a terminal object, we denote it by 1.

For limits and cartesian closedness in the free coproduct completion $\mathsf{Fam}(X)$, see Hu and Tholen [15] and Adámek and Rosický [2], as well as [4, Sections 6.1 and 6.2]. Family categories

2020 *Mathematics Subject Classification.* Primary 18D15, 18A40; Secondary 18A30, 18D30.
Key words and phrases. lax comma category, cartesian closed category, free coproduct completion, multiadjoint.

[v1] Fri, 30 Oct 2026

occur in the semantics of automatic differentiation [20, 17], monoidal closure of Grothendieck constructions [21], and free distributive and extensive completions [18, 22]. Descent in categories of families is studied in [26, 10, 11].

An object of the free coproduct completion $\mathsf{Fam}(X)$ is a pair (A, F), where A is a set and $F : A \to X$ is a functor. A morphism $(u, \alpha) : (A, F) \to (B, G)$ consists of a function $u : A \to B$ and a natural transformation $\alpha : F \Rightarrow G \circ u$; thus its components are morphisms $\alpha_a : F(a) \to G(u(a))$. The singleton embedding $j_X : X \to \mathsf{Fam}(X)$, given by $j_X(x) = (\mathbf{1}, x)$, is fully faithful. We often write j for j_X.

Coproducts are given by disjoint unions of the indexing sets. If X has finite products, then $j(1) = (\mathbf{1}, 1)$ is terminal in $\mathsf{Fam}(X)$ and

$$(A, F) \times (B, G) = (A \times B, F \times G), \qquad (F \times G)(a, b) = F(a) \times G(b). \tag{1.1}$$

In particular, j preserves finite products. More generally, if the required products exist in X, then

$$\prod_{k \in K} (A_k, F_k) = \left(\prod_{k \in K} A_k, H \right), \qquad H(s) = \prod_{k \in K} F_k(s(k)).$$

If X has finite limits, so does $\mathsf{Fam}(X)$; see [2, Remarks 2.2(1) and 2.3].

The functor

$$\Phi_X : \mathsf{Fam}(X) \longrightarrow \mathsf{CAT}[X^{\mathrm{op}}, \mathsf{Set}], \qquad (A, F) \longmapsto \coprod_{a \in A} X(-, F(a)) \tag{1.2}$$

is fully faithful by Yoneda, with essential image the *familially representable* presheaves [5]. We recall Diers's multi-notions [12] in this language; see also [2, 15].

Definition 1.1. A *multilimit* of $M : A \to X$ is an object $(I, L) \in \mathsf{Fam}(X)$ representing its cone presheaf:

$$\mathsf{Fam}(X)(j(z), (I, L)) \cong \mathsf{CAT}[A, X](\Delta(z), M).$$

The category X is *multicomplete* if every diagram has a multilimit. For discrete diagrams we speak of *multiproducts*. If X has finite products, it is *cartesian multi-closed* if each functor $X(- \times x, y)$ is familially representable.

A functor $c : X \to Y$ is a *left multiadjoint* if, for every $y \in Y$, the presheaf $Y(c(-), y)$ is familially representable. Thus there is a family $\widehat{R}(y) \in \mathsf{Fam}(X)$ and an isomorphism

$$Y(c(x), y) \cong \mathsf{Fam}(X)(j(x), \widehat{R}(y)), \tag{1.3}$$

natural in x. Equivalently, $(c \downarrow y)$ has a set of connected components, each with a terminal object.

By full faithfulness of (1.2), the families $\widehat{R}(y)$ extend to a functor $\widehat{R} : Y \to \mathsf{Fam}(X)$, with (1.3) natural in both variables. Equivalently, $\mathsf{Fam}(c)$ has a right adjoint: its value at (B, G) is $\coprod_{b \in B} \widehat{R}(G(b))$. This is the family-completion formulation of multiadjointness; see [3, Section 2.6] and [2, Remark 2.7].

Lemma 1.2 ([2, Remark 2.2(1) and Lemma 2.6])**.** *The category $\mathsf{Fam}(X)$ is complete if and only if X is multicomplete. It has all products if and only if X has all multiproducts.*

2. Lax comma categories and change of base

We use the convention of [10]. An object of $\mathsf{Cat} \mathbin{/\!\!/} X$ is a pair (A, F), where A is a small category and $F : A \to X$. A morphism $(A, F) \to (B, G)$ is a pair (u, α) with

$$u : A \longrightarrow B, \qquad \alpha : F \Longrightarrow G \circ u.$$

Composition is $(v, \beta) \circ (u, \alpha) = (v \circ u, \gamma)$, where $\gamma_a = \beta_{u(a)} \circ \alpha_a$. When X has finite products, $(\mathbf{1}, 1)$ is terminal in $\mathsf{Cat} \mathbin{/\!\!/} X$ and

$$(A, F) \times (B, G) = (A \times B, F \times G), \qquad (F \times G)(a, b) = F(a) \times G(b). \tag{2.1}$$

For the recalled adjunction (2.2), Remark 2.1, and the comparison (2.4), we work in an ambient universe containing the comma objects and enlarge the slice and diagram categories accordingly.

For $c : X \to Y$, write $c_! : \mathsf{Cat} /\!\!/ X \to \mathsf{Cat} /\!\!/ Y$ for postcomposition and $I_X : \mathsf{Cat}/X \to \mathsf{Cat} /\!\!/ X$ for the inclusion of the strict slice. The comma construction of [9, Theorem 5.8], in its 2-cell-dual form, gives the adjunction

$$\mathsf{Cat} /\!\!/ Y \underset{c^{\Rightarrow}}{\overset{c_! \circ I_X}{\rightleftarrows}} \mathsf{Cat}/X \;, \qquad c^{\Rightarrow}(B, G) = (c \downarrow G \longrightarrow X). \tag{2.2}$$

Indeed, a strict map into the comma object consists of a functor $u : A \to B$ and a transformation $c \circ F \Rightarrow G \circ u$. In particular, $I_X \dashv K_X$, where $K_X(B, G) = (\mathrm{id}_X \downarrow G \to X)$.

Remark 2.1. The notation $\mathsf{Cat} /\!\!/ X$ and the above definition, with the same direction of the comparison transformation, already appear in Grothendieck [14, Section 11(b), p. 186]. This is also the category of small diagrams considered by Peschke and Tholen [25, Definition 2.1] and Perrone and Tholen [24, Definition 2.1]. Its projection to Cat is the split fibration associated with $A \mapsto \mathsf{CAT}[A, X]$, as recalled in [10, Section 1.2].

In the ambient universe, the adjunction $I_X \dashv K_X$ induces the free split fibration 2-monad on the strict slice:

$$T_X(F) = \big(\mathrm{id}_X \downarrow F \xrightarrow{\pi_X} X\big).$$

Its unit inserts identity arrows and its multiplication composes arrows in X. Its strict algebras are split fibrations over X; see Street [27, Section 2]. The corresponding comma 2-monad with varying base and its strict algebras are considered in [19]. A strict morphism $F \to T_X(G)$ over X is precisely a pair (u, α) with $\alpha : F \Rightarrow G \circ u$. Consequently, the underlying Kleisli category, restricted to objects with small domains, is $\mathsf{Cat} /\!\!/ X$. This is the 2-cell-dual form of [9, Theorem 5.11] and gives the connection with free fibrations recorded in [10, Section 4.3].

We ask when $c_!$ itself has a right adjoint. Its construction uses a diagram whose domain records the indices in a family-valued functor.

For $T : B \to \mathsf{Fam}(X)$, write $T(b) = (S(b), T_b)$, with $T_b : S(b) \to X$, and $T(v) = (S(v), \tau_v)$. Thus $S : B \to \mathsf{Set}$ is a functor and $\tau_v : T_b \Rightarrow T_{b'} \circ S(v)$ for $v : b \to b'$. Put

$$\mathsf{Gr}(B, T) = \left(\int_B S, \widetilde{T}\right), \qquad \widetilde{T}(b, s) = T_b(s). \tag{2.3}$$

An arrow $(b, s) \to (b', s')$ is an arrow $v : b \to b'$ with $S(v)(s) = s'$; its image is $(\tau_v)_s : T_b(s) \to T_{b'}(s')$. The category $\int_B S$ is the Grothendieck construction of S, regarded as a functor with discrete values, and its projection to B is a discrete opfibration. For the Grothendieck construction and the corresponding fibred and cofibred descriptions, see Grothendieck [14, Section 8], Gray [13] and Street [28, Section 1.10]; a recent account is given in [23, Section 2.1].

On a lax morphism $(v, \gamma) : (B, T) \to (B', T')$, the index maps of γ induce the functor between categories of elements, and its component maps give the comparison transformation. Thus Gr is functorial.

Theorem 2.2. *For a functor $c : X \to Y$, the following conditions are equivalent:*

1. *c is a left multiadjoint;*
2. *$\mathsf{Fam}(c)$ has a right adjoint;*
3. *$c_! : \mathsf{Cat} /\!\!/ X \to \mathsf{Cat} /\!\!/ Y$ has a right adjoint.*

If $\widehat{R}$ is as in (1.3)*, the right adjoint in* (3) *is*

$$R(B, G) = \mathsf{Gr}(B, \widehat{R} \circ G).$$

Proof. The equivalence of (1) and (2) was recalled in Section 1. Assuming (1), naturality of (1.3) gives

$$\begin{aligned}\mathsf{Cat}\mathbin{/\!\!/} Y((A, c\circ F),(B,G)) &\cong \coprod_{u:A\to B} \mathsf{CAT}[A,Y](c\circ F, G\circ u)\\ &\cong \coprod_{u:A\to B} \mathsf{CAT}[A,\mathsf{Fam}(X)](j\circ F, \widehat{R}\circ G\circ u)\\ &\cong \mathsf{Cat}\mathbin{/\!\!/} X((A,F),\mathsf{Gr}(B,\widehat{R}\circ G)).\end{aligned}$$

For the last bijection, write $\widehat{R}(G(b)) = (S(b), E_b)$. A transformation $j\circ F \Rightarrow \widehat{R}\circ G\circ u$ selects $s_a \in S(u(a))$ and maps $F(a) \to E_{u(a)}(s_a)$. Naturality says precisely that $a \mapsto (u(a), s_a)$ is a functor to $\int_B S$ and that the component maps form a natural transformation. The bijections are natural, proving (3).

Conversely, suppose $c_! \dashv R$ and write $R(\mathbf{1}, y) = (D, E)$. Singleton diagrams give

$$Y(c(x), y) \cong \mathsf{Cat}\mathbin{/\!\!/} X((\mathbf{1}, x),(D,E)) \cong \coprod_{d\in \mathsf{Ob}(D)} X(x, E(d)),$$

naturally in x. Since D is small, this proves (1). □

We can now compare this adjunction with (2.2). Since $I_X \dashv K_X$, the composite $c_!\circ I_X$ has right adjoint $K_X \circ R$. Uniqueness of right adjoints gives, whenever the comma objects are available,

$$c^{\Rightarrow} \cong K_X \circ R. \tag{2.4}$$

Thus the adjunction of [9] factors through the two lax comma categories precisely when c is a left multiadjoint.

The singleton embedding gives the following particular case.

Corollary 2.3. *The functor $j_X : X \to \mathsf{Fam}(X)$ is a left multiadjoint. It induces a fully faithful left adjoint, preserving finite products whenever these exist in X:*

$$\mathsf{Cat}\mathbin{/\!\!/}\mathsf{Fam}(X) \underset{\mathsf{Gr}}{\overset{J=(j_X)_!}{\leftarrow}} \mathsf{Cat}\mathbin{/\!\!/} X \quad (\perp) \tag{2.5}$$

Proof. For $(A,F) \in \mathsf{Fam}(X)$, the identity $\mathsf{Fam}(X)(j(x),(A,F)) = \coprod_{a\in A} X(x, F(a))$ shows that j is a left multiadjoint with $\widehat{R} = \mathrm{id}_{\mathsf{Fam}(X)}$. By Theorem 2.2, we have $J \dashv \mathsf{Gr}$. The unit is an isomorphism, since the index functor of $j\circ F$ is constantly a singleton. Product preservation follows from (1.1) and (2.1). □

Remark 2.4. These adjunctions extend to 2-adjunctions, in the usual Cat-enriched sense [16]. A 2-cell $(u,\alpha) \Rightarrow (v,\beta)$ is a transformation $\tau : u \Rightarrow v$ such that $G(\tau_a)\circ\alpha_a = \beta_a$ for every $a \in A$. The bijections above identify precisely these transformations. The ordinary adjunctions suffice for cartesian closedness.

3. Cartesian closedness

Families are diagrams with discrete domains: in this sense $\mathsf{Fam}(X) = \mathsf{Set}\mathbin{/\!\!/} X$. The adjunction $\mathsf{Disc} \dashv \mathsf{Ob}$ between sets and categories induces

$$\mathsf{Cat}\mathbin{/\!\!/} X \underset{U}{\overset{D}{\leftarrow}} \mathsf{Fam}(X)\ , \quad (\perp) \qquad U(A,F) = (\mathsf{Ob}(A), F\circ\iota_A), \tag{3.1}$$

where $\iota_A : \mathsf{Disc}(\mathsf{Ob}(A)) \to A$ is the canonical functor. Indeed, a morphism from a discrete diagram consists of an assignment of objects and a family of component maps. Thus D is fully

faithful and, when X has finite products, preserves them. Together with (2.5), this gives the two full coreflections needed below.

Lemma 3.1. *Let $L : \mathcal{A} \to \mathcal{B}$ be a fully faithful left adjoint preserving finite products, with right adjoint R. If $L(P)$ is exponentiable in $\mathcal{B}$, then P is exponentiable in $\mathcal{A}$, with $Q^P = R(L(Q)^{L(P)})$.*

Proof. For every $Z \in \mathcal{A}$,

$$\mathcal{A}(Z, R(L(Q)^{L(P)})) \cong \mathcal{B}(L(Z) \times L(P), L(Q)) \cong \mathcal{A}(Z \times P, Q).$$

□

Proposition 3.2. *If X has finite products and $\mathsf{Cat} /\!\!/ X$ is cartesian closed, then $\mathsf{Fam}(X)$ is cartesian closed.*

Proof. The functor D in (3.1) is a fully faithful left adjoint preserving finite products. The conclusion therefore follows from Lemma 3.1. □

For the converse, we use the products supplied by cartesian closedness of the family category.

Theorem 3.3. *Let X be a category with finite limits. Then $\mathsf{Cat} /\!\!/ X$ is cartesian closed if and only if $\mathsf{Fam}(X)$ is cartesian closed. Under these equivalent conditions, $\mathsf{Fam}(X)$ is complete.*

Proof. We first observe that if $\mathsf{Fam}(X)$ is cartesian closed, then it is complete. Indeed, by [2, Theorem 2.11], X has all multiproducts, so $\mathsf{Fam}(X)$ has all products by Lemma 1.2. Since X has finite limits, so does $\mathsf{Fam}(X)$. Products and equalizers therefore give all limits in $\mathsf{Fam}(X)$.

If $\mathsf{Cat} /\!\!/ X$ is cartesian closed, Proposition 3.2 gives cartesian closedness of $\mathsf{Fam}(X)$, and the preceding observation gives its completeness. Conversely, if $\mathsf{Fam}(X)$ is cartesian closed, it is also complete, so $\mathsf{Cat} /\!\!/ \mathsf{Fam}(X)$ is cartesian closed by [10, Theorem 2.4]. Corollary 2.3 and Lemma 3.1 then give cartesian closedness of $\mathsf{Cat} /\!\!/ X$. □

4. Semiadditive case

The known cartesian closedness of $\mathsf{Fam}(X)$ for complete semiadditive X was one of our main motivations for investigating the non-fibred case for $\mathsf{Cat} /\!\!/ X$. In fact, semiadditivity and products suffice [20, Proposition 74]; see also [21, Section 9]. The exponential formula in Corollary 4.1 recovers the lax comma construction of [21, Section 9].

The proof of Theorem 3.3 also describes the exponentials. Assume that X has finite products and that $\mathsf{Fam}(X)$ is complete and cartesian closed. For $(A, F), (B, G) \in \mathsf{Cat} /\!\!/ X$, define

$$T : \mathsf{Cat}[A, B] \longrightarrow \mathsf{Fam}(X), \qquad T(H) = \int_{a \in A} j(G(H(a)))^{j(F(a))}. \tag{4.1}$$

The end is that of the bifunctor $(a, a') \mapsto j(G(H(a')))^{j(F(a))}$. Postcomposition makes it functorial in H. The construction of [10] and Lemma 3.1 give

$$(B, G)^{(A,F)} = \mathsf{Gr}(\mathsf{Cat}[A, B], T). \tag{4.2}$$

Writing $T(H) = (S(H), T_H)$, the domain of (4.2) is a discrete opfibration over $\mathsf{Cat}[A, B]$ with fibre $S(H)$ at H. This projection need not be an isomorphism.

The description is particularly simple for semiadditive categories. Recall that these are categories with finite biproducts. In this case

$$X(z \times x, y) \cong X(z, y) \times X(x, y) \cong \coprod_{\beta : x \to y} X(z, y). \tag{4.3}$$

Thus X is cartesian multi-closed [2, Example 2.9(b)]. If X is complete, [2, Theorem 2.11] and Lemma 1.2 also give the completeness and cartesian closedness of $\mathsf{Fam}(X)$.

Corollary 4.1. *Let X be a complete semiadditive category. Then $\mathsf{Cat} /\!\!/ X$ is cartesian closed, with*

$$(B, G)^{(A,F)} = \Big((F \downarrow G_!),\ (H, \beta) \longmapsto \lim_A (G \circ H) \Big), \tag{4.4}$$

where $G_! : \mathsf{Cat}[A, B] \to \mathsf{CAT}[A, X]$ is postcomposition with G.

Proof. The cartesian closedness assertion is an immediate corollary of Theorem 3.3 and the known cartesian closedness of $\mathsf{Fam}(X)$ for semiadditive categories with products [20, Proposition 74]; see also [21, Section 9]. To identify the exponential, for each $H : A \to B$ the biproduct decomposition gives

$$\mathsf{CAT}[A, X](\Delta(z) \times F, G \circ H) \cong \coprod_{\beta : F \Rightarrow G \circ H} X\left(z, \lim_A (G \circ H)\right).$$

Thus $T(H)$ is the pair $(\mathsf{CAT}[A, X](F, G \circ H), \Delta(\lim_A(G \circ H)))$, where the second component is the constant functor. On a transformation $\eta : H \Rightarrow H'$, its index map sends β to the transformation with components $G(\eta_a) \circ \beta_a$, and its component map is induced between limits. Flattening therefore gives exactly the comma category and diagram in (4.4). Evaluation at $((H, \beta), a)$ is the biproduct map $[\pi_a, \beta_a] : (\lim_A(G \circ H)) \oplus F(a) \to G(H(a))$. □

For $X = \mathsf{CMon}$, the self-exponential of $(\mathbf{1}, \mathbb{N})$ is $(\mathsf{Disc}(\mathbb{N}), \Delta(\mathbb{N}))$, with evaluation over k given by $(m, n) \mapsto m + kn$. This recovers the non-fibred phenomenon in [21]. The same construction applies to complete additive categories, including Ab and Vect_k.

5. Cartesian closed families

We conclude by relating our characterization to known results on family categories, starting with the following theorem of Adámek and Rosický.

Theorem 5.1 ([2, Theorem 2.11])**.** *Let X be a category with finite products. Then $\mathsf{Fam}(X)$ is cartesian closed if and only if X has all multiproducts and is cartesian multi-closed.*

Explicitly, let $R(x, y) \in \mathsf{Fam}(X)$ represent $X(- \times x, y)$. For $P = (A, F)$ and $Q = (B, G)$, one has

$$Q^P = \prod_{a \in A} \coprod_{b \in B} R(F(a), G(b)). \tag{5.1}$$

Indeed, a map from $j(z)$ into (5.1) selects a function $u : A \to B$ and maps $z \times F(a) \to G(u(a))$. This is a map $j(z) \times P \to Q$, and the assertion extends to all families by coproducts.

Together with Lemma 1.2 and Theorem 3.3, this gives the following formulation in terms of X.

Corollary 5.2. *Let X be a category with finite limits. Then $\mathsf{Cat} /\!\!/ X$ is cartesian closed if and only if X is multicomplete and cartesian multi-closed.*

The examples of cartesian multi-closed categories in [2, Examples 2.9] include cartesian closed categories, semiadditive categories, and opposites of extensive categories. Whenever these are complete, Corollary 5.2 applies. In particular, it gives cartesian closedness of $\mathsf{Cat} /\!\!/ \mathsf{Set}^{\mathrm{op}}$, as well as the semiadditive examples above. Families of vector spaces and of opposite vector spaces also appear as cartesian closed semantic categories in *CHAD for expressive total languages* [20, Section 10.2]; the present result gives their corresponding lax comma examples.

Free completions provide another source. Let $\mathcal{L}(C)$ denote the free completion of C under limits. The cartesian closedness of $\mathsf{Fam}(\mathcal{L}(C))$ proved in [18, Theorem 4.4], together with Theorem 3.3, gives

$$\mathsf{Cat} /\!\!/ \mathcal{L}(C) \quad \text{cartesian closed.}$$

The same work treats exponentiable finite coproducts of connected objects in free infinitary lextensive categories [18, Theorem 4.7].

Finally, the free doubly-infinitary distributive category $\mathsf{Fam}(\mathsf{Fam}(C^{\mathrm{op}})^{\mathrm{op}})$ is cartesian closed [22, Theorem 3.11]. Whenever $\mathsf{Fam}(C^{\mathrm{op}})^{\mathrm{op}}$ has finite limits, Theorem 3.3 therefore gives cartesian closedness of $\mathsf{Cat} /\!\!/ \mathsf{Fam}(C^{\mathrm{op}})^{\mathrm{op}}$.

Acknowledgements. The authors acknowledge partial financial support by the Centre for Mathematics of the University of Coimbra (CMUC), through the Fundação para a Ciência e a Tecnologia (FCT), grants UID/00324/2025 and UID/PRR/00324/2025.

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