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Absolute descent and the Bénabou–Roubaud theorem

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Abstract

We continue the programme developed in *Pseudo-Kan extensions and descent theory* by studying descent for augmented pseudocosimplicial objects in their own right. Let $\mathcal{H}$ be a 2-category, and let $\mathrm{suc} = 1 + (-)$ be the successor monad on the augmented simplex category. Precomposition with suc induces the *décalage* pseudomonad $\mathbb{S}$ on the 2-category of augmented pseudocosimplicial objects with values in $\mathcal{H}$. We prove that every successor contraction makes the corresponding augmented pseudocosimplicial object effective for descent. This structure is preserved by postcomposition with every pseudofunctor. Consequently, the resulting descent object is an absolute bilimit.

A first application concerns split epimorphisms. We recover the Janelidze–Tholen characterization of absolute effective descent morphisms given in *Facets of Descent II*. Moreover, we prove that one direction holds for every pseudofunctor $A\colon C^{\mathrm{op}} \to \mathcal{H}$, where $\mathcal{H}$ is an arbitrary 2-category.

A further consequence of our main theorem is that every $\mathbb{S}$-pseudoalgebra is of absolute descent. In particular, this holds for every free $\mathbb{S}$-pseudoalgebra, and hence for every one-sided décalage. This makes the shift-effectiveness hypothesis in the Bénabou–Roubaud theorem of *Pseudo-Kan extensions and descent theory* automatic. Thus, under the adjointness and Beck–Chevalley hypotheses, the descent and Eilenberg–Moore factorizations are equivalent whenever they exist. We thus obtain a stronger version of the Bénabou–Roubaud theorem for arbitrary augmented pseudocosimplicial objects, which we call the formal Bénabou–Roubaud theorem. Applied, in particular, to indexed Čech objects associated with pseudofunctors $A\colon C^{\mathrm{op}} \to \mathcal{H}$, this theorem extends the classical Bénabou–Roubaud theorem to an arbitrary 2-category $\mathcal{H}$.

Keywords. pointed endopseudofunctor; pseudoalgebra; idempotent pseudomonad; absolute descent; successor contraction; décalage; pseudo-Kan extension; Bénabou–Roubaud theorem; Janelidze–Tholen theorem.

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1 Introduction

Descent theory studies the image of a functor, typically one induced by a change of base. In this interpretation, objects in the source play the role of global objects, while their images in the target play the role of local realizations. The descent problem is to recognize, from within the target, which local objects arise from global ones.

A bare description of the essential image rarely provides an intrinsic criterion for recognizing its objects. Instead, one equips objects in the target with coherent compatibility data and proves that they come from the source precisely when they admit such data. One must also identify the morphisms that preserve this structure, thereby reconstructing the source, up to equivalence, as a category of local objects equipped with descent data. Descent thus transforms an external problem about the image of a functor into an internal recognition principle.

This pattern occurs throughout mathematics. For modules, bundles, sheaves, and indexed categories, a global object determines local data together with canonical compatibility information. Effective descent provides the converse: every coherent system of such data arises, essentially uniquely, from a global object [3–6].

The programme developed in [12] places Grothendieck descent theory in a general two-dimensional setting, organized around two complementary principles:

1. *Bilimit viewpoint.* In an ambient 2-category, descent objects are conical bilimits of pseudocosimplicial objects; see [12, Theorem 4.11].
2. *Pseudomonadic viewpoint.* For pseudomonadic pseudofunctors, essential images can be recognized through the existence of suitable pseudoalgebra structures. Idempotent pseudomonads are particularly well suited to this purpose: their pseudoalgebra structures are essentially unique and may be characterized by pseudoretractions of the unit; see [12, Section 2].

Bilimit viewpoint. The first principle can be expressed in more general terms: descent is expressed by the appropriate notion of limit of a cosimplicial object. It therefore makes sense to develop descent for such objects in their own right, independently of the change-of-base constructions from which they may arise.

In dimension 1, the conical limit of a cosimplicial object in an ordinary category is the equalizer of its first two cofaces, whenever either exists. This is natural: compatibility in dimension 1 is expressed by equality, and the cosimplicial identities extend any morphism equalizing these two cofaces uniquely to a cone over the entire diagram. For example, the sheaf condition identifies sections over a covered open set with the equalizer of the two maps restricting families of local sections to pairwise overlaps: compatible local sections glue uniquely.

In dimension 2, in the pseudo setting, compatibility is expressed by isomorphisms satisfying coherence conditions. Here, the equalizing equation is replaced by a specified invertible 2-cell satisfying normalization and a cocycle condition. These are precisely the conditions defining Grothendieck descent [5, 6, 14]. Indeed, by [12, Theorem 4.11 and Remark 4.4], the usual descent object, defined by these data and conditions in the first three nonempty degrees, is equivalent to the conical bilimit of the entire pseudocosimplicial object.

More explicitly, let $\dot{\Delta}$ be the category of finite ordinals and monotone maps, including the empty ordinal $\mathsf{0}$, and let

$$j\colon \Delta \hookrightarrow \dot{\Delta}$$

be the full inclusion of the nonempty finite ordinals. We regard each ordinal

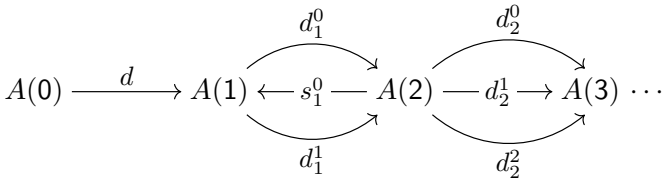
$$\mathsf{n} = \{0 \leq \ldots \leq n-1\}$$

as a category. An augmented pseudocosimplicial object in $\mathcal{H}$ is a pseudofunctor $A\colon \dot{\Delta} \longrightarrow \mathcal{H}$; see [11, Definition 2.1] for the definition of a pseudofunctor. We denote by

$$d_n^i\colon A(\mathsf{n}) \longrightarrow A(\mathsf{n}+1), \qquad s_n^i\colon A(\mathsf{n}+1) \longrightarrow A(\mathsf{n}) \tag{1.1}$$

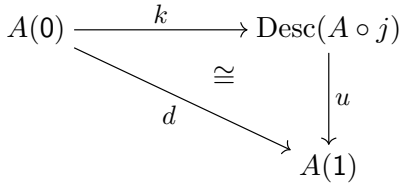
the coface induced by the monotone injection omitting i, for $0 \leq i \leq n$, and the codegeneracy induced by the monotone surjection identifying i and $i+1$, for $0 \leq i < n$.

Writing $d = d_0^0$, the relevant part of A is



where the positive part $A \circ j$ contains the descent data, while $A(\mathsf{0})$ is seen as the proposed reconstruction.

The augmented pseudocosimplicial structure equips d with a canonical descent datum for $A \circ j$. Hence, whenever $\mathrm{Desc}(A \circ j)$ exists in $\mathcal{H}$, its universal property induces the descent factorization



(1.2)

given by the canonical invertible 2-cell $u \circ k \cong d$. We write $k = k_A$ and $u = u_A$ when the diagram needs to be specified. Here u is the forgetful morphism, while

$$k\colon A(\mathsf{0}) \longrightarrow \mathrm{Desc}(A \circ j)$$

is the descent comparison. The augmented pseudocosimplicial object A is of *effective descent* precisely when k is an equivalence.

Pseudomonadic viewpoint. The second principle returns descent to an image problem, now in the 2-category

$$[\dot{\Delta}, \mathcal{H}]_{PS}$$

of augmented pseudocosimplicial objects, pseudonatural transformations, and modifications. The proposed reconstruction $A(\mathsf{0})$ can be replaced by the universal one, $\mathrm{Desc}(A \circ j)$, producing an augmented object of effective descent. In [12], this completion is formalized by an idempotent pseudomonad constructed from pointwise pseudo-Kan extensions.

More precisely, suppose that the relevant conical bilimits exist for all diagrams, so that restriction along j has a pointwise right pseudo-Kan-extension biadjoint

$$j^*\colon [\dot{\Delta}, \mathcal{H}]_{PS} \rightleftarrows [\Delta, \mathcal{H}]_{PS} : \mathrm{PsRan}_j\,.$$

The induced *descent pseudomonad* is

$$\mathbb{D} = \langle j \rangle = \mathrm{PsRan}_j \circ j^*.$$

Up to equivalence, $\mathbb{D}$ leaves the positive part of A unchanged and replaces $A(\mathsf{0})$ by $\mathrm{Desc}(A \circ j)$. Since j is fully faithful, $\mathbb{D}$ is idempotent [12, Theorem 3.30]. Its unit

$$\eta_A\colon A \longrightarrow \mathbb{D}(A)$$

is the descent comparison: its degree-zero component is k_A, while its positive-degree components are equivalences, and may be taken to be isomorphisms. Consequently,

$$\begin{aligned} A \text{ is of effective descent} &\iff \eta_A \colon A \longrightarrow \mathbb{D}(A) \text{ is an equivalence} \\ &\iff A \text{ admits a } \mathbb{D}\text{-pseudoalgebra structure.} \end{aligned}$$

The effective-descent objects are therefore precisely the objects in the essential image of the forgetful pseudofunctor

$$\mathrm{PsAlg}(\mathbb{D}) \longrightarrow [\dot{\Delta}, \mathcal{H}]_{PS},$$

or, equivalently, the objects fixed up to equivalence by descent completion. The pseudomonadic viewpoint thus provides an organizing principle for the bilimit viewpoint: it expresses the same reconstruction in terms of pseudoalgebra structure, while bringing effective descent back to the problem of characterizing the essential image of a forgetful pseudofunctor.

This viewpoint does not require descent objects for every $\mathcal{H}$-valued diagram. We may restrict to the full 2-subcategory of augmented pseudocosimplicial objects A for which $\mathrm{Desc}(A \circ j)$ exists. Descent completion preserves this subcategory and defines an idempotent pseudomonad on it, with the same characterization of effective descent by pseudoalgebra structure. We make this restriction precise in Proposition 2.2. For Theorem A below, even existence of the particular descent object will follow from the successor contraction.

Absolute descent

The present paper asks a stronger stability question. Effective descent asks whether reconstruction succeeds; absolute descent asks whether the same reconstruction survives every change of ambient 2-category. More precisely, an augmented pseudocosimplicial object $A \colon \dot{\Delta} \to \mathcal{H}$ is of *absolute effective descent* if $P \circ A$ is of effective descent for every pseudofunctor $P \colon \mathcal{H} \to \mathcal{K}$.

The ordinary case suggests what such a criterion should look like. A splitting expresses the universal property of an equalizer by equations preserved by every functor, and hence makes the equalizer absolute. Since ordinary descent reduces to an equalizer, the problem is to identify the coherent two-dimensional structure that plays the role of this splitting.

The required structure comes from the successor monad

$$\mathrm{suc} = 1 + (-) \colon \dot{\Delta} \longrightarrow \dot{\Delta},$$

which adjoins a new least element. Viewing ordinals as categories, this is the restriction to $\dot{\Delta}$ of the monad that freely adjoins an initial object. Its unit $\delta \colon \mathrm{id}_{\dot{\Delta}} \to \mathrm{suc}$ has components

$$\delta_n \colon \mathsf{n} \longrightarrow 1 + \mathsf{n}$$

given by inclusion into the second summand. Its multiplication has components

$$\mu_n \colon 1 + 1 + \mathsf{n} \longrightarrow 1 + \mathsf{n}$$

that identify the two adjoined least elements and act as the identity on the original n-summand.

For an augmented pseudocosimplicial object A, the components of

$$\mathrm{id}_A * \delta \colon A \longrightarrow A \circ \mathrm{suc}$$

are the first cofaces d^0_n. A *successor contraction* on A is a pseudonatural retraction of $\mathrm{id}_A * \delta$, namely a pseudonatural transformation

$$r \colon A \circ \mathrm{suc} \longrightarrow A$$

equipped with an invertible modification $r \circ (\mathrm{id}_A * \delta) \cong \mathrm{id}_A$. The main observation of this paper is the following.

Theorem A (Absolute descent). *Let $\mathcal{H}$ be any 2-category. If an augmented pseudocosimplicial object $A\colon \dot{\Delta} \to \mathcal{H}$ admits a successor contraction, then A is of absolute effective descent.*

Precomposition with the successor monad induces the *décalage pseudomonad*

$$\mathbb{S} = (-) \circ \mathrm{suc}$$

on the 2-category $[\dot{\Delta}, \mathcal{H}]_{PS}$ of augmented pseudocosimplicial objects. We call $A \circ \mathrm{suc}$ the *left décalage* of A, with *left* referring to the adjoined least element in the ordinal sum $1 + (-)$. The multiplication $\mathrm{id}_A * \mu$ equips it with the free $\mathbb{S}$-pseudoalgebra structure on A and, in particular, with a successor contraction.

Theorem A therefore gives the following consequence.

Theorem B (Absolute shifts). *For every augmented pseudocosimplicial object A, the left décalage*

$$A \circ \mathrm{suc}\colon \dot{\Delta} \longrightarrow \mathcal{H}$$

is of absolute effective descent.

We now place these results within a general mechanism for contractions of pointed endopseudofunctors. A successor contraction is precisely a contraction for the point of $\mathbb{S}$. The key is a canonical pointed comparison $\mathbb{D} \to \mathbb{S}$, which transfers this structure to the idempotent descent pseudomonad. Idempotence then gives effective descent, and preservation of successor contractions under postcomposition gives absoluteness.

Contractions and pseudoalgebras

The proof of Theorem A becomes transparent once the point of a pseudomonad is separated from its multiplication. A pointed endopseudofunctor $\mathbb{T} = (T, \iota)$ consists of an endopseudofunctor T on a 2-category $\mathcal{H}$ and a pseudonatural transformation $\iota\colon \mathrm{id}_{\mathcal{H}} \to T$. A $\mathbb{T}$-contraction on X is a pseudoretraction of the point:

$$T(X) \xrightarrow{x} X, \qquad x \circ \iota_X \cong \mathrm{id}_X.$$

There is no multiplication to respect and therefore no associativity axiom. We write $\mathrm{Contr}(\mathbb{T})$ for the 2-category of contractions of $\mathbb{T}$, with pseudomorphisms and compatible 2-cells. For a pseudomonad, this notation refers to its underlying pointed endopseudofunctor.

Whenever the descent pseudomonad is defined, we construct a canonical pointed comparison $\chi\colon \mathbb{D} \to \mathbb{S}$, where $\iota_A = \mathrm{id}_A * \delta$ is the décalage point. Restriction along χ induces a homomorphism

$$\chi^*\colon \mathrm{Contr}(\mathbb{S}) \longrightarrow \mathrm{Contr}(\mathbb{D}).$$

It sends a successor contraction $r\colon \mathbb{S}(A) \to A$ to the descent contraction

$$\mathbb{D}(A) \xrightarrow{\chi_A} \mathbb{S}(A) \xrightarrow{r} A.$$

The unit cell is the composite

$$r \circ \chi_A \circ \eta_A \cong r \circ \iota_A \cong \mathrm{id}_A.$$

Since $\mathbb{D}$ is idempotent, this contraction extends essentially uniquely to a $\mathbb{D}$-pseudoalgebra structure. The extension also respects pseudomorphisms and 2-cells, so it gives a biequivalence $\mathrm{Contr}(\mathbb{D}) \simeq \mathrm{PsAlg}(\mathbb{D})$.

It gives the hierarchy

$$\begin{array}{ccc} \mathrm{PsAlg}(\mathbb{S}) & \xrightarrow{\quad U_{\mathbb{S}} \quad} & \mathrm{Contr}(\mathbb{S}) \\ & & \Big\downarrow{\scriptstyle \chi^*} \\ \mathrm{PsAlg}(\mathbb{D}) & \xleftarrow[\simeq]{\text{idempotence}} & \mathrm{Contr}(\mathbb{D}) \end{array}$$

The first arrow forgets multiplication coherence. Restriction along χ then turns a successor contraction into a splitting of the descent unit, and idempotence promotes that splitting essentially uniquely to a descent pseudoalgebra. Theorem A begins with $\mathrm{Contr}(\mathbb{S})$, rather than with $\mathrm{PsAlg}(\mathbb{S})$, which is why it applies to contractions that are not associative.

The transferred pseudoalgebra makes the descent unit an equivalence, so the original augmented object is of effective descent. Successor contractions are preserved by postcomposition with every pseudofunctor; applying the same transfer to the image contraction therefore gives absoluteness. The existence of the descent objects needed for this argument, without an ambient completeness assumption, is established in Corollary 4.5.

The two completions in the Bénabou–Roubaud theorem

Absolute descent of successor shifts has a second consequence. It supplies, automatically and in the stronger absolute form, the shift-effectiveness hypothesis used in the effective-descent part of the formal Bénabou–Roubaud theorem of [12, Theorem 8.5]. It also makes visible the two pseudoalgebra structures governing that theorem.

The first coface

$$d = d^0_0 \colon A(\mathsf{0}) \longrightarrow A(\mathsf{1})$$

admits two natural reconstruction procedures. The pseudocosimplicial structure gives its descent factorization through $\mathrm{Desc}(A \circ j)$. If d^0_0 has a left adjoint, it also induces a monad

$$T_A = d^0_0 \circ (d^0_0)_!$$

on $A(\mathsf{1})$, and hence an Eilenberg–Moore factorization through $A(\mathsf{1})^{T_A}$. The Bénabou–Roubaud theorem asserts that, under the Beck–Chevalley condition, these two reconstructions agree.

Suppose, more generally, that all components of $\mathrm{id}_A * \delta$ have left adjoints and that the corresponding Beck–Chevalley mates are invertible. By doctrinal adjunction, these componentwise adjunctions assemble into an adjunction in

$$[\dot{\Delta}, \mathcal{H}]_{PS},$$

or equivalently into a two-variable pseudofunctor

$$C_A \colon \mathsf{Adj} \times \dot{\Delta} \longrightarrow \mathcal{H},$$

where Adj is the walking adjunction. This diagram carries two independent reconstruction procedures. Descent completion acts in the pseudocosimplicial variable, while Eilenberg–Moore completion acts in the adjunction variable. Both are idempotent pseudo-Kan completions, and Fubini for pointwise pseudo-Kan extensions identifies the two orders in which they may be performed, whenever the required objects exist.

The boundary of the resulting completion square is already effective. The shifted row $A \circ \mathrm{suc}$ is of absolute descent by Theorem B. For every $n \geq 1$, the right adjoint

$$d^0_n \colon A(\mathsf{n}) \longrightarrow A(\mathsf{n}+\mathsf{1})$$

is a pseudosection, with retraction s_n^0, and is therefore monadic. Thus the shifted row carries the required descent pseudoalgebra, while the positive-degree adjunctions carry the required Eilenberg–Moore pseudoalgebras. Fubini fills the only remaining corner in either order and produces an equivalence

$$\mathrm{Desc}(A \circ j) \simeq A(1)^{T_A}$$

compatible with the comparison morphisms from $A(0)$ and the forgetful morphisms to $A(1)$.

Three constructions occur here, but only two are interchanged. The successor pseudomonad supplies the absolute shifted row. The actual Fubini comparison is between the idempotent descent and Eilenberg–Moore completions.

Theorem C (Formal Bénabou–Roubaud). *Let $\mathcal{H}$ be any 2-category and let $A\colon \dot{\Delta} \to \mathcal{H}$ be augmented pseudocosimplicial. Suppose that the components of $\mathrm{id}_A * \delta$ have left adjoints and that the corresponding Beck–Chevalley mates are invertible. Whenever the descent object and the relevant Eilenberg–Moore object exist, the descent factorization of d is equivalent to its Eilenberg–Moore factorization. Hence A is of effective descent if and only if d is monadic.*

The existence of these two particular objects is the only ambient completeness requirement. Both always exist in **Cat**, but the theorem itself is valid in any 2-category in which they exist.

Indexed Čech descent

The preceding results concern pseudocosimplicial objects in an arbitrary 2-category. Classical descent theory is recovered by applying them to indexed Čech objects. Consider a pseudofunctor

$$\mathcal{L}\colon \mathcal{C}^{\mathrm{op}} \longrightarrow \mathcal{H},$$

where $\mathcal{C}$ has pullbacks, and a morphism $p\colon E \to B$ in $\mathcal{C}$. If the change-of-base 1-cells have left adjoints and $\mathcal{L}$ satisfies Beck–Chevalley for pullback squares, then Theorem C applied to the indexed Čech object $\mathcal{L} \circ D_p$ identifies the descent factorization of $\mathcal{L}(p)$ with its Eilenberg–Moore factorization, whenever the two objects exist. Consequently,

$$p \text{ is of effective } \mathcal{L}\text{-descent} \quad \Longleftrightarrow \quad \mathcal{L}(p) \text{ is monadic.}$$

This is the two-dimensional Bénabou–Roubaud theorem for arbitrary $\mathcal{H}$; taking $\mathcal{H} = \mathbf{Cat}$ yields its classical form [1, 6].

The final application uses the absolute-descent theorem directly and requires neither adjoints nor Beck–Chevalley. If p has a section, inserting it in the first coordinate contracts the entire augmented Čech nerve, rather than merely its kernel-pair fork. Theorem A then makes the resulting cone an absolute descent cone for every indexed pseudofunctor with values in any 2-category.

In the **Cat**-valued setting, every split epimorphism is therefore a morphism of absolute effective descent. Conversely, the single indexed category

$$\mathcal{L}_B(X) = \mathbf{2}^{\mathcal{C}(B,X)}$$

detects whether p has a section. This proves that the morphisms of absolute effective descent are precisely the split epimorphisms and recovers the Janelidze–Tholen characterization [7, Theorem 3.5].

Organization

Section 2 develops descent both as a bilimit and as an idempotent pseudo-Kan completion. Section 3 proves the transfer principle for contractions. Section 4 constructs the pointed comparison $\mathbb{D} \to \mathbb{S}$ and proves absolute descent for successor contractions and one-sided décalage. Section 5 presents the formal Bénabou–Roubaud theorem as an application of the comparison results of [12] and absolute descent of successor shifts. Section 6 applies the results to indexed Čech objects and proves the Janelidze–Tholen characterization. Section 7 gives the co-dual comparison with co-Eilenberg–Moore objects.

2 Descent as a bilimit and as an image

This section makes precise the two viewpoints announced in the introduction. We describe descent objects as bilimits and then package effective descent as pseudoalgebra structure for an idempotent pseudo-Kan completion.

2.1 Conventions

We work relative to a fixed hierarchy of universes and leave changes of universe implicit; see [10]. The symbol **Cat** denotes the 2-category of categories, functors, and natural transformations. Ordinary categories are regarded as locally discrete 2-categories. For 2-categories $\mathcal{J}$ and $\mathcal{H}$, we write $[\mathcal{J}, \mathcal{H}]_{PS}$ for the 2-category of pseudofunctors, pseudonatural transformations, and modifications, and $[\mathcal{J}, \mathcal{H}]_{PS}(F, G)$ for its hom-category. In particular, this notation applies when $\mathcal{J}$ is an ordinary category.

The compositor and unitor of a pseudofunctor F are oriented as

$$F(v) \circ F(u) \Longrightarrow F(v \circ u), \qquad \mathrm{id}_{F(i)} \Longrightarrow F(\mathrm{id}_i).$$

For a pseudonatural transformation $\tau \colon F \to G$, the pseudonaturality constraint associated with $u \colon i \to k$ is oriented as

$$G(u) \circ \tau_i \Longrightarrow \tau_k \circ F(u).$$

We write $\circ$ for composition of 1-cells and for vertical composition of 2-cells, read from right to left, and $*$ for horizontal composition. In particular, whiskering is written $\mathrm{id}_g * \alpha$ or $\alpha * \mathrm{id}_f$. Identities are denoted by id, with a subscript when needed. Pseudofunctor evaluation is written with parentheses, as in $F(f)$. Associators, unitors, and structural isomorphisms are suppressed when coherence determines them. See [8, 9] for background.

A 1-cell $f \colon X \to Y$ is called a *pseudosection* if it admits a retraction up to an invertible 2-cell: there are $r \colon Y \to X$ and $r \circ f \cong \mathrm{id}_X$. This terminology will be used both in a 2-category and in a pseudofunctor 2-category.

2.2 Pseudocosimplicial objects and descent data

Let $\dot{\Delta}$ be the category of finite ordinals and monotone maps, including the empty ordinal $\mathsf{0}$, and let Δ be its full subcategory on the nonempty ordinals. We write

$$j \colon \Delta \hookrightarrow \dot{\Delta}, \qquad \mathsf{n} = \{0, \dots, n-1\}, \tag{2.1}$$

where j is the inclusion. A pseudofunctor $G \colon \Delta \to \mathcal{H}$ is a *pseudocosimplicial object*; a pseudofunctor $A \colon \dot{\Delta} \to \mathcal{H}$ is an *augmented pseudocosimplicial object*. The coface $\delta^i_n \colon \mathsf{n} \to \mathsf{n}+1$ omits i, and the codegeneracy $\sigma^i_n \colon \mathsf{n}+1 \to \mathsf{n}$ identifies i and $i+1$. Their images are denoted by

$$d^i_n = A(\delta^i_n), \qquad s^i_n = A(\sigma^i_n).$$

The positive part begins as

$$A(1) \underset{d_1^1}{\overset{d_1^0}{\leftleftarrows}} A(2) \underset{d_2^2}{\overset{d_2^0}{\rightrightarrows}} A(3) \;\cdots\; . \tag{2.2}$$

Only the codegeneracy needed for the normalization condition is displayed.

For $T \in \mathcal{H}$, a descent datum for $G\colon \Delta \to \mathcal{H}$ with vertex T consists of a 1-cell $x\colon T \to G(1)$ and an invertible 2-cell

$$\theta\colon d_1^1 \circ x \Longrightarrow d_1^0 \circ x \tag{2.3}$$

satisfying normalization and the cocycle equation. In strict notation these read

$$\mathrm{id}_{s_1^0} * \theta = \mathrm{id}_x, \qquad (\mathrm{id}_{d_2^0} * \theta) \circ (\mathrm{id}_{d_2^2} * \theta) = \mathrm{id}_{d_2^1} * \theta; \tag{2.4}$$

for a pseudofunctor, the canonical compositor and unitor isomorphisms are inserted in (2.4). A morphism $(x, \theta) \to (y, \vartheta)$ is a 2-cell $\beta\colon x \Rightarrow y$ for which the following square commutes in $\mathcal{H}(T, G(2))$:

$$\begin{array}{ccc} d_1^1 \circ x & \xRightarrow{\;\theta\;} & d_1^0 \circ x \\ {\scriptstyle \mathrm{id}_{d_1^1} * \beta}\Big\Downarrow & & \Big\Downarrow{\scriptstyle \mathrm{id}_{d_1^0} * \beta} \\ d_1^1 \circ y & \xRightarrow[\;\vartheta\;]{} & d_1^0 \circ y. \end{array} \tag{2.5}$$

These data form a category, denoted $\mathscr{D}_G(T)$. Precomposition in $\mathcal{H}$ makes

$$\mathscr{D}_G\colon \mathcal{H}^{\mathrm{op}} \longrightarrow \mathbf{Cat} \tag{2.6}$$

a pseudofunctor.

Definition 2.1 (Descent object)**.** A *descent object* for $G\colon \Delta \to \mathcal{H}$ is an object $\mathrm{Desc}(G)$ representing (2.6); thus there are equivalences

$$\mathcal{H}(T, \mathrm{Desc}(G)) \simeq \mathscr{D}_G(T)$$

pseudonatural in T.

Equivalently, $\mathrm{Desc}(G)$ is the conical bilimit of G. Its description by descent data uses only the first three nonempty degrees: degree 1 contains the underlying object, degree 2 the descent isomorphism, the first codegeneracy its normalization, and degree 3 the cocycle equation [12, Theorems 4.10–4.11].

For an augmented object A, write $d = d_0^0$. Its canonical descent cell is

$$\kappa := a^{-1}_{\delta_1^0, \delta_0^0} \circ a_{\delta_1^1, \delta_0^0}\colon d_1^1 \circ d \Longrightarrow d_1^0 \circ d, \tag{2.7}$$

where $a_{v,u}\colon A(v) \circ A(u) \Rightarrow A(v \circ u)$ is the compositor of A. The pseudofunctor axioms give the normalization and cocycle identities for κ. Whenever $\mathrm{Desc}(A \circ j)$ exists, the canonical comparison induces a functor

$$\mathcal{H}(T, k_A)\colon \mathcal{H}(T, A(0)) \longrightarrow \mathcal{H}(T, \mathrm{Desc}(A \circ j)).$$

Under the equivalence defining $\mathrm{Desc}(A \circ j)$, this sends

$$z\colon T \to A(0) \quad \longmapsto \quad (d \circ z, \kappa * \mathrm{id}_z).$$

2.3 Pseudo-Kan completion

The bilimit description is equivalently a pseudo-Kan-extension description [12, Corollary 3.27 and Theorem 4.11]. Let $G\colon \Delta \to \mathcal{H}$. When the conical bilimit of G exists, its right pseudo-Kan extension $\mathrm{PsRan}_j(G)$ exists and is characterized by equivalences

$$[\dot{\Delta}, \mathcal{H}]_{PS}(X, \mathrm{PsRan}_j(G)) \simeq [\Delta, \mathcal{H}]_{PS}(X \circ j, G)$$

pseudonatural in $X\colon \dot{\Delta} \to \mathcal{H}$. Its values are

$$(\mathrm{PsRan}_j(G))(\mathsf{n}) \simeq \{\dot{\Delta}(\mathsf{n}, j(-)), G\}_{\mathrm{bi}}. \tag{2.8}$$

For $n \geq 1$, the weight in (2.8) is representable, so the corresponding value is equivalent to $G(\mathsf{n})$. Since $\mathsf{0}$ is initial in $\dot{\Delta}$, the weight at $\mathsf{0}$ is constant at the terminal category, and hence

$$(\mathrm{PsRan}_j(G))(\mathsf{0}) \simeq \mathrm{Desc}(G). \tag{2.9}$$

For $A\colon \dot{\Delta} \to \mathcal{H}$, the unit of the pseudo-Kan-extension biadjunction has degree-zero component

$$k_A\colon A(\mathsf{0}) \longrightarrow \mathrm{Desc}(A \circ j), \tag{2.10}$$

induced by the canonical cone. Its positive components are equivalences.

The domain of this completion can be specified without assuming all descent bilimits in $\mathcal{H}$. Let $[\Delta, \mathcal{H}]_{PS}^{\mathrm{desc}}$ be the full 2-subcategory of $[\Delta, \mathcal{H}]_{PS}$ on the diagrams G for which $\mathrm{Desc}(G)$ exists, and let $[\dot{\Delta}, \mathcal{H}]_{PS}^{\mathrm{desc}}$ be the full 2-subcategory of $[\dot{\Delta}, \mathcal{H}]_{PS}$ on those A with $A \circ j \in [\Delta, \mathcal{H}]_{PS}^{\mathrm{desc}}$.

Proposition 2.2 (The domain of descent completion)**.** *Restriction and right pseudo-Kan extension give a biadjunction*

$$j^*\colon [\dot{\Delta}, \mathcal{H}]_{PS}^{\mathrm{desc}} \rightleftarrows [\Delta, \mathcal{H}]_{PS}^{\mathrm{desc}} : \mathrm{PsRan}_j\,.$$

Its counit is a pseudonatural equivalence. Consequently,

$$\mathbb{D}_{\mathcal{H}} = \langle j \rangle_{\mathcal{H}} := \mathrm{PsRan}_j \circ j^* \tag{2.11}$$

is an idempotent pseudomonad on $[\dot{\Delta}, \mathcal{H}]_{PS}^{\mathrm{desc}}$ [12, Theorem 3.30].

We write $\eta\colon \mathrm{id} \to \mathbb{D}_{\mathcal{H}}$ for the unit and omit the subscript $\mathcal{H}$ when the ambient 2-category is fixed. The full subcategory $[\dot{\Delta}, \mathcal{H}]_{PS}^{\mathrm{desc}}$ is the domain of $\mathbb{D}_{\mathcal{H}}$ throughout.

Definition 2.3 (Effective descent)**.** An augmented pseudocosimplicial object A is *of effective j-descent* if $\mathrm{Desc}(A \circ j)$ exists and the comparison k_A in (2.10) is an equivalence.

Proposition 2.4 (Effective descent as a pseudoalgebra)**.** *For $A \in [\dot{\Delta}, \mathcal{H}]_{PS}^{\mathrm{desc}}$, the following conditions are equivalent:*

(i) *A is of effective j-descent;*

(ii) *$\eta_A\colon A \to \mathbb{D}_{\mathcal{H}}(A)$ is an equivalence;*

(iii) *A admits a $\mathbb{D}_{\mathcal{H}}$-pseudoalgebra structure.*

Such a pseudoalgebra structure is essentially unique.

Proof. The unit is already an equivalence in every positive degree, so the first two conditions are equivalent by (2.10). The remaining equivalence is the standard characterization of pseudoalgebras for an idempotent pseudomonad [12, Theorem 2.20]. □

Thus the forgetful pseudofunctor

$$\mathrm{PsAlg}(\mathbb{D}_{\mathcal{H}}) \longrightarrow [\dot{\Delta}, \mathcal{H}]^{\mathrm{desc}}_{PS}$$

identifies its essential image with the effective-descent objects in $[\dot{\Delta}, \mathcal{H}]^{\mathrm{desc}}_{PS}$. This formulation assumes only the descent objects occurring in that domain. To prove existence from a successor contraction, we shall apply the same pseudoalgebra argument representably in **Cat**, where descent completion is defined on every diagram.

2.4 Absolute descent

Definition 2.5 (Absolute descent)**.** An augmented pseudocosimplicial object $A\colon \dot{\Delta} \to \mathcal{H}$ is *of absolute effective descent*, or simply *of absolute descent*, if A is of effective j-descent and $F \circ A$ is of effective j-descent for every 2-category $\mathcal{K}$ and every pseudofunctor $F\colon \mathcal{H} \to \mathcal{K}$.

3 Contractions of pointed endopseudofunctors

The descent completion of the previous section is an idempotent pseudomonad. The left décalage construction of the next section is also a pseudomonad, but the absolute-descent theorem does not use its multiplication. To express exactly what the proof needs, we separate an endopseudofunctor with a point from a pseudomonad with a multiplication.

3.1 Pointed endopseudofunctors

Let $\mathcal{K}$ be a 2-category.

Definition 3.1 (Pointed endopseudofunctor)**.** A *pointed endopseudofunctor* on $\mathcal{K}$ is a pair

$$\mathbb{T} = (T, \iota), \qquad T\colon \mathcal{K} \to \mathcal{K}, \qquad \iota\colon \mathrm{id}_{\mathcal{K}} \to T,$$

where T is a pseudofunctor and ι is pseudonatural.

Definition 3.2 (Contraction)**.** A $\mathbb{T}$-*contraction* on $X \in \mathcal{K}$ consists of a 1-cell

$$x\colon T(X) \longrightarrow X$$

and an invertible 2-cell

$$\bar{x}\colon x \circ \iota_X \Longrightarrow \mathrm{id}_X. \tag{3.1}$$

Thus a contraction is a pseudoretraction of the point, with its invertible retraction cell specified.

A pseudomorphism of contractions $(f, \bar{f})\colon (X, x, \bar{x}) \to (Y, y, \bar{y})$ consists of a 1-cell $f\colon X \to Y$ and an invertible 2-cell

$$\bar{f}\colon f \circ x \Longrightarrow y \circ T(f)$$

compatible with the unit cells. Explicitly, if $\iota_f\colon T(f) \circ \iota_X \Rightarrow \iota_Y \circ f$ is the pseudonaturality cell of the point, the following diagram commutes:

$$\begin{array}{ccc} f \circ x \circ \iota_X & \xRightarrow{\bar{f} * \mathrm{id}_{\iota_X}} & y \circ T(f) \circ \iota_X \\ {\scriptstyle \mathrm{id}_f * \bar{x}}\Downarrow & & \Downarrow{\scriptstyle \mathrm{id}_y * \iota_f} \\ f & \xLeftarrow[\bar{y} * \mathrm{id}_f]{} & y \circ \iota_Y \circ f \end{array}$$

A 2-cell of contractions $\beta\colon (f,\bar{f}) \Rightarrow (g,\bar{g})$ is a 2-cell $\beta\colon f \Rightarrow g$ for which the following diagram commutes:

$$\begin{array}{ccc} f \circ x & \xRightarrow{\bar{f}} & y \circ T(f) \\ {\scriptstyle \beta * \mathrm{id}_x}\Big\Downarrow & & \Big\Downarrow{\scriptstyle \mathrm{id}_y * T(\beta)} \\ g \circ x & \xRightarrow[\bar{g}]{} & y \circ T(g) \end{array}$$

We denote the resulting 2-category by $\mathrm{Contr}(\mathbb{T})$. For a pseudomonad $\mathbb{T}$, this notation refers to contractions of its underlying pointed endopseudofunctor.

If $\mathbb{T} = (T, \iota, \mu)$ is a pseudomonad, a $\mathbb{T}$-pseudoalgebra also has an invertible multiplication cell

$$x \circ T(x) \Longrightarrow x \circ \mu_X$$

satisfying the unit and associativity axioms [2, 15]. Forgetting this cell gives a homomorphism

$$U_{\mathbb{T}}\colon \mathrm{PsAlg}(\mathbb{T}) \longrightarrow \mathrm{Contr}(\mathbb{T}). \tag{3.2}$$

Throughout, *pseudoalgebra* includes both unit and multiplication coherence; a contraction requires only the retraction cell (3.1).

3.2 Restriction along a pointed comparison

Definition 3.3 (Pointed comparison)**.** Let $\mathbb{T} = (T, \iota)$ and $\mathbb{S} = (S, \sigma)$ be pointed endopseudofunctors on $\mathcal{K}$. A *pointed comparison*

$$(\chi, \omega)\colon \mathbb{T} \longrightarrow \mathbb{S}$$

consists of a pseudonatural transformation $\chi\colon T \to S$ and an invertible modification

$$\omega\colon \chi \circ \iota \Longrightarrow \sigma. \tag{3.3}$$

At X, Condition (3.3) is the coherent triangle

$$\begin{array}{ccc} X & \xrightarrow{\iota_X} & T(X) \\ & {\scriptstyle \sigma_X} \searrow & \Big\downarrow{\scriptstyle \chi_X} \\ & & S(X) \end{array} \qquad \omega_X \colon \cong \tag{3.4}$$

Proposition 3.4 (Restriction of contractions)**.** *Every pointed comparison* $(\chi, \omega)\colon \mathbb{T} \to \mathbb{S}$ *induces a homomorphism over* $\mathcal{K}$

$$\chi^*\colon \mathrm{Contr}(\mathbb{S}) \longrightarrow \mathrm{Contr}(\mathbb{T}).$$

On objects,

$$(X, s, \bar{s}) \longmapsto \big(X, s \circ \chi_X, \bar{s} \circ (\mathrm{id}_s * \omega_X)\big). \tag{3.5}$$

3.3 Pseudoalgebras from contractions

Let $\mathbb{D} = (D, \eta, \mu)$ be an idempotent pseudomonad on $\mathcal{K}$, so that $\mu\colon D \circ D \to D$ is a pseudonatural equivalence. We recall the characterization of its pseudoalgebras [12, Theorems 2.18 and 2.20], expressed below in terms of contractions.

Proposition 3.5 (Pseudoalgebras for an idempotent pseudomonad)**.** *For $X \in \mathcal{K}$, the following conditions are equivalent:*

(i) *X admits a $\mathbb{D}$-pseudoalgebra structure;*

(ii) *$\eta_X\colon X \to D(X)$ is an equivalence;*

(iii) *η_X is a pseudosection;*

(iv) *X admits a $\mathbb{D}$-contraction.*

Forgetting multiplication gives a biequivalence over $\mathcal{K}$ whose inverse adds the uniquely determined multiplication cell and preserves the contraction data, pseudomorphisms and 2-cells:

$$U_{\mathbb{D}}\colon \mathrm{PsAlg}(\mathbb{D}) \longrightarrow \mathrm{Contr}(\mathbb{D}), \qquad E_{\mathbb{D}}\colon \mathrm{Contr}(\mathbb{D}) \longrightarrow \mathrm{PsAlg}(\mathbb{D}). \tag{3.6}$$

Theorem 3.6 (From contractions to pseudoalgebras)**.** *Let $\mathbb{D}$ be an idempotent pseudomonad and let $\mathbb{S} = (S, \iota)$ be a pointed endopseudofunctor on $\mathcal{K}$. A pointed comparison*

$$(\chi, \omega)\colon \mathbb{D} \longrightarrow \mathbb{S}$$

induces a homomorphism over $\mathcal{K}$

$$E_{\mathbb{D}} \circ \chi^*\colon \mathrm{Contr}(\mathbb{S}) \longrightarrow \mathrm{PsAlg}(\mathbb{D}).$$

For a contraction $(X, s, \bar{s})$, the resulting pseudoalgebra action is

$$D(X) \xrightarrow{\chi_X} S(X) \xrightarrow{s} X, \tag{3.7}$$

with unit cell

$$s \circ \chi_X \circ \eta_X \xRightarrow{\mathrm{id}_s * \omega_X} s \circ \iota_X \xRightarrow{\bar{s}} \mathrm{id}_X,$$

and its multiplication cell is uniquely determined by this contraction. On pseudomorphisms and 2-cells, the homomorphism acts by restriction along χ, leaving the underlying 1-cells and 2-cells unchanged.

Proof. Compose the homomorphism χ^* of Proposition 3.4 with $E_{\mathbb{D}}$ from Proposition 3.5. The latter supplies multiplication coherence and proves its compatibility with every restricted pseudomorphism and 2-cell. □

When $\mathbb{S}$ is a pseudomonad, this gives a diagram of homomorphisms between the corresponding 2-categories:

$$\begin{array}{ccc} \mathrm{PsAlg}(\mathbb{S}) & \xrightarrow{\quad U_{\mathbb{S}} \quad} & \mathrm{Contr}(\mathbb{S}) \\ & & \Big\downarrow{\scriptstyle \chi^*} \\ \mathrm{PsAlg}(\mathbb{D}) & \xleftarrow[\simeq]{\quad E_{\mathbb{D}} \quad} & \mathrm{Contr}(\mathbb{D}) \end{array} \tag{3.8}$$

Every arrow preserves the carrier. The bottom arrow supplies the uniquely determined multiplication coherence, on both objects and morphisms, and preserves the given contraction data.

4 The left décalage point and absolute descent

We now apply the transfer homomorphism of Section 3 to descent.

4.1 The left décalage pseudomonad

We define

$$\mathrm{suc} = 1 + (-)\colon \dot{\Delta} \longrightarrow \dot{\Delta} \tag{4.1}$$

by adjoining a new least element. Thus $\mathrm{suc}(\mathsf{n}) = \mathsf{n}+1$, and for $u\colon \mathsf{m} \to \mathsf{n}$,

$$\mathrm{suc}(u)(0) = 0, \qquad \mathrm{suc}(u)(i+1) = u(i) + 1.$$

There is a natural transformation

$$\delta\colon \mathrm{id}_{\dot{\Delta}} \longrightarrow \mathrm{suc}, \qquad \delta_n(i) = i+1. \tag{4.2}$$

Its component $\delta_n = \delta^0_n$ is the coface omitting the newly adjoined least element.

The successor is a monad. Its multiplication $\mu\colon \mathrm{suc} \circ \mathrm{suc} \to \mathrm{suc}$ has components

$$\mu_n = \sigma^0_{n+1}\colon \mathsf{n}+2 \longrightarrow \mathsf{n}+1, \tag{4.3}$$

which identify the two newly adjoined least elements. In coordinates, $\mu_n(0) = \mu_n(1) = 0$ and $\mu_n(i+2) = i+1$. Precomposition therefore defines the *left décalage pseudomonad*

$$\mathbb{S}_{\mathcal{H}} := (-) \circ \mathrm{suc} \tag{4.4}$$

on $[\dot{\Delta}, \mathcal{H}]_{PS}$. Its unit at A is

$$\iota_A = \mathrm{id}_A * \delta\colon A \longrightarrow A \circ \mathrm{suc}, \tag{4.5}$$

whose degree-n component is d^0_n. The adjective *left* records that suc adjoins the new element on the left. We retain the name *successor monad* for $(\mathrm{suc}, \delta, \mu)$ on $\dot{\Delta}$. Thus the successor monad acts on the indexing category, whereas the décalage pseudomonad acts on augmented pseudocosimplicial objects. In particular,

$$\mathbb{S}_{\mathcal{H}}(A)(\mathsf{n}) = A(\mathsf{n}+1), \qquad d^i_{\mathbb{S}(A),n} = d^{i+1}_{A,n+1}, \qquad s^i_{\mathbb{S}(A),n} = s^{i+1}_{A,n+1}.$$

The structural comparison cells in these identifications are those of A.

Definition 4.1 (Successor contraction)**.** A *successor contraction* on $A\colon \dot{\Delta} \to \mathcal{H}$ is an $\mathbb{S}_{\mathcal{H}}$-contraction. Explicitly, it consists of a pseudonatural transformation

$$r\colon A \circ \mathrm{suc} \longrightarrow A$$

and an invertible modification

$$\varepsilon\colon r \circ (\mathrm{id}_A * \delta) \Longrightarrow \mathrm{id}_A. \tag{4.6}$$

Writing $r_n\colon A(\mathsf{n}+1) \to A(\mathsf{n})$, the component of (4.6) is

$$\varepsilon_n\colon r_n \circ d^0_n \Longrightarrow \mathrm{id}_{A(\mathsf{n})}.$$

For every $u\colon \mathsf{m} \to \mathsf{n}$, pseudonaturality supplies an invertible 2-cell

$$A(u) \circ r_m \Longrightarrow r_n \circ A(\mathrm{suc}(u)). \tag{4.7}$$

These cells, together with the modification axiom for ε, are the higher coherence absent from a splitting of d^0_0 alone.

4.2 From décalage structure to descent structure

We work on the full 2-subcategory $[\dot{\Delta}, \mathcal{H}]^{\mathrm{desc}}_{PS}$ on which the descent pseudomonad $\mathbb{D}_{\mathcal{H}}$ is defined. The next proposition also establishes that the décalage pseudomonad restricts to this 2-subcategory; we denote its restriction by $\mathbb{S}^{\mathrm{desc}}_{\mathcal{H}}$.

Let $q\colon \dot{\Delta} \to \Delta$ be the corestriction of suc, so that

$$j \circ q = \mathrm{suc}\,. \tag{4.8}$$

Write $\epsilon^j\colon j^* \circ \mathrm{PsRan}_j \Rightarrow \mathrm{id}$ for the counit of the biadjunction, and write

$$\tau\colon (\epsilon^j * \mathrm{id}_{j^*}) \circ (\mathrm{id}_{j^*} * \eta) \Longrightarrow \mathrm{id}_{j^*}$$

for its invertible triangle modification.

Proposition 4.2 (The descent–décalage comparison)**.** *Every shifted diagram $A \circ \mathrm{suc}$ has descent object $A(1)$. Consequently, $\mathbb{S}_{\mathcal{H}}$ restricts to a pseudomonad $\mathbb{S}^{\mathrm{desc}}_{\mathcal{H}}$ on $[\dot{\Delta}, \mathcal{H}]^{\mathrm{desc}}_{PS}$. For A in this 2-category, the composite*

$$\begin{aligned} \chi_A\colon \mathbb{D}_{\mathcal{H}}(A) &\xrightarrow{\iota_{\mathbb{D}_{\mathcal{H}}(A)}} \mathbb{D}_{\mathcal{H}}(A) \circ \mathrm{suc} = (\mathbb{D}_{\mathcal{H}}(A) \circ j) \circ q \\ &\xrightarrow{\epsilon^j_{A\circ j} * \mathrm{id}_q} (A \circ j) \circ q = \mathbb{S}^{\mathrm{desc}}_{\mathcal{H}}(A) \end{aligned} \tag{4.9}$$

defines a pseudonatural transformation $\chi\colon \mathbb{D}_{\mathcal{H}} \Rightarrow \mathbb{S}^{\mathrm{desc}}_{\mathcal{H}}$. Its compatibility with the points is the invertible modification

$$\omega\colon \chi \circ \eta \Longrightarrow \iota, \tag{4.10}$$

whose component is the following pasting:

$$\begin{aligned} \omega_A\colon \chi_A \circ \eta_A &= (\epsilon^j_{A\circ j} * \mathrm{id}_q) \circ \iota_{\mathbb{D}_{\mathcal{H}}(A)} \circ \eta_A \\ &\xRightarrow{\cong} (\epsilon^j_{A\circ j} * \mathrm{id}_q) \circ (\eta_A * \mathrm{id}_{\mathrm{suc}}) \circ \iota_A \\ &\xRightarrow{(\tau_A * \mathrm{id}_q) * \mathrm{id}_{\iota_A}} \iota_A. \end{aligned} \tag{4.11}$$

Here the first cell is pseudonaturality of η_A at δ, and the second is the triangle modification of the biadjunction. Thus $(\chi, \omega)\colon (\mathbb{D}_{\mathcal{H}}, \eta) \to (\mathbb{S}^{\mathrm{desc}}_{\mathcal{H}}, \iota)$ is a pointed comparison.

Proof. In **Cat**, all descent objects exist, so the displayed formulas give the pointed comparison on the entire diagram 2-category. The multiplication of $\mathbb{S}_{\mathbf{Cat}}$ equips every shifted diagram $B \circ \mathrm{suc}$ with a successor contraction; Theorem 3.6 and Proposition 2.4 therefore make it effective for descent. Applying this to $B = \mathcal{H}(X, -) \circ A$, pseudonaturally in X, shows that $A(1)$ represents the descent data of $A \circ \mathrm{suc} \circ j$. Thus $\mathbb{S}_{\mathcal{H}}$ restricts to $[\dot{\Delta}, \mathcal{H}]^{\mathrm{desc}}_{PS}$, where the same formulas define χ and ω. □

Corollary 4.3 (Transfer to descent pseudoalgebras)**.** *The pointed comparison induces homomorphisms*

$$\mathrm{Contr}(\mathbb{S}^{\mathrm{desc}}_{\mathcal{H}}) \xrightarrow{\chi^*} \mathrm{Contr}(\mathbb{D}_{\mathcal{H}}) \xrightarrow{E_{\mathbb{D}_{\mathcal{H}}}} \mathrm{PsAlg}(\mathbb{D}_{\mathcal{H}}), \tag{4.12}$$

The transferred action and unit cell on a successor contraction (A, r, ε) are

$$a_A = r \circ \chi_A, \qquad \bar{a}_A = \varepsilon \circ (\mathrm{id}_r * \omega_A). \tag{4.13}$$

In particular, A is of effective j-descent.

4.3 Postcomposition and absolute descent

The algebraic structures just used are preserved under postcomposition. This holds for the décalage pseudomonad without any assumption about preservation of descent objects.

Proposition 4.4 (Postcomposition of contractions and pseudoalgebras)**.** *Every pseudofunctor $F\colon \mathcal{H} \to \mathcal{K}$ induces homomorphisms*

$$\begin{aligned} F_*\colon \mathrm{Contr}(\mathbb{S}_{\mathcal{H}}) &\longrightarrow \mathrm{Contr}(\mathbb{S}_{\mathcal{K}}), \\ F_*\colon \mathrm{PsAlg}(\mathbb{S}_{\mathcal{H}}) &\longrightarrow \mathrm{PsAlg}(\mathbb{S}_{\mathcal{K}}), \end{aligned} \tag{4.14}$$

compatible with the forgetful homomorphisms from pseudoalgebras to contractions.

Corollary 4.5 (Successor contractions give absolute descent)**.** *Let $\mathcal{H}$ be any 2-category. If $A\colon \dot{\Delta} \to \mathcal{H}$ admits a successor contraction, then $F \circ A$ is of effective descent for every pseudofunctor $F\colon \mathcal{H} \to \mathcal{K}$. Hence A is of absolute descent.*

Proof. Postcomposition preserves successor contractions by Proposition 4.4. Apply Corollary 4.3 in **Cat** to $\mathcal{K}(X, -) \circ F \circ A$ for every $X \in \mathcal{K}$; the resulting equivalences exhibit $F(A(\mathsf{0}))$ as the descent object of $F \circ A \circ j$. □

4.4 Free décalage pseudoalgebras and absolute shifts

The multiplication of the décalage pseudomonad supplies a canonical source of the structures to which the transfer applies.

Theorem 4.6 (Absolute left décalage)**.** *For every augmented pseudocosimplicial object $A\colon \dot{\Delta} \to \mathcal{H}$, the shift*

$$A^+ := A \circ \mathrm{suc} = \mathbb{S}_{\mathcal{H}}(A)$$

is the carrier of the free $\mathbb{S}_{\mathcal{H}}$-pseudoalgebra on A. Its action is

$$\mathrm{id}_A * \mu\colon A \circ \mathrm{suc} \circ \mathrm{suc} \longrightarrow A \circ \mathrm{suc}\,. \tag{4.15}$$

Its action and unit cell form a successor contraction. Consequently, A^+ is of absolute descent.

Corollary 4.7 (Absolute décalage)**.** *Let $G\colon \Delta \to \mathcal{H}$ be pseudocosimplicial, and let $q\colon \dot{\Delta} \to \Delta$ be the corestriction of* $\mathrm{suc} = 1 + (-)$*, so that $j \circ q = \mathrm{suc}$. Then $G \circ q\colon \dot{\Delta} \to \mathcal{H}$ carries a canonical $\mathbb{S}_{\mathcal{H}}$-pseudoalgebra structure and is of absolute descent. Its action has components*

$$G(\sigma^0_{n+1})\colon G(\mathsf{n}+2) \longrightarrow G(\mathsf{n}+1).$$

Proof. The multiplication μ corestricts to a natural transformation $\bar{\mu}\colon q \circ \mathrm{suc} \Rightarrow q$ in Δ. The identities

$$\bar{\mu} \circ (\mathrm{id}_q * \delta) = \mathrm{id}_q, \qquad \bar{\mu} \circ (\bar{\mu} * \mathrm{id}_{\mathrm{suc}}) = \bar{\mu} \circ (\mathrm{id}_q * \mu)$$

are the unit and associativity laws for the action of suc on q. Applying G gives the pseudoalgebra action $\mathrm{id}_G * \bar{\mu}$ and its coherence modifications. Forgetting multiplication and applying Corollary 4.5 proves the assertion. □

Corollary 4.8 (Restriction of the décalage pseudomonad)**.** *For arbitrary $\mathcal{H}$, the endopseudofunctor $\mathbb{S}_{\mathcal{H}}$ takes every object of $[\dot{\Delta}, \mathcal{H}]_{PS}$ into $[\dot{\Delta}, \mathcal{H}]^{\mathrm{desc}}_{PS}$. It therefore restricts to a pseudomonad $\mathbb{S}^{\mathrm{desc}}_{\mathcal{H}}$ on $[\dot{\Delta}, \mathcal{H}]^{\mathrm{desc}}_{PS}$. The comparison of Proposition 4.2 is a pointed comparison on this 2-category*

$$(\chi, \omega) \colon \mathbb{D}_{\mathcal{H}} \longrightarrow \mathbb{S}^{\mathrm{desc}}_{\mathcal{H}}.$$

The inclusion identifies the contractions and the pseudoalgebras of $\mathbb{S}^{\mathrm{desc}}_{\mathcal{H}}$ with those of $\mathbb{S}_{\mathcal{H}}$.

Thus the full transfer may be displayed as

$$\begin{array}{ccc} \mathrm{PsAlg}(\mathbb{S}_{\mathcal{H}}) & \xrightarrow{\quad U_{\mathbb{S}_{\mathcal{H}}} \quad} & \mathrm{Contr}(\mathbb{S}_{\mathcal{H}}) \\ & & \Big\downarrow{\scriptstyle \chi^*} \\ \mathrm{PsAlg}(\mathbb{D}_{\mathcal{H}}) & \xleftarrow[\simeq]{\quad E_{\mathbb{D}_{\mathcal{H}}} \quad} & \mathrm{Contr}(\mathbb{D}_{\mathcal{H}}) \end{array} \tag{4.16}$$

All arrows are homomorphisms of 2-categories. The lower arrow supplies the uniquely determined multiplication coherence; it also extends every pseudomorphism and 2-cell of contractions. Composing this diagram with the free-pseudoalgebra homomorphism

$$\mathsf{Fr}_{\mathbb{S}_{\mathcal{H}}} \colon [\dot{\Delta}, \mathcal{H}]_{PS} \longrightarrow \mathrm{PsAlg}(\mathbb{S}_{\mathcal{H}})$$

gives descent pseudoalgebras with underlying carriers $A \mapsto A^{+}$. The absolute-descent theorem gives the additional assertion that these carriers retain their descent universal property under every pseudofunctor.

5 Descent and Eilenberg–Moore completion

We first isolate the implication from effective descent to monadicity, which requires only a left adjoint to the augmentation [13]. For the converse and the comparison of factorizations, we apply the results of [12], using absolute décalage to supply the effective shifted row.

5.1 Effective descent implies monadicity

The implication from effective descent to monadicity belongs to a more general monadicity principle. The creation theorem of [13, Corollaries 2.6–2.7] shows that forgetful morphisms from descent objects create absolute Kan extensions and, in **Cat**, absolute limits and colimits. Creation of absolute coequalizers and Beck's monadicity theorem imply that a right adjoint which factors, up to natural isomorphism, as an equivalence followed by such a forgetful functor is monadic [13, Theorem 2.8].

Proposition 5.1 is the representable form of this principle. This implication requires no Beck–Chevalley condition; Beck–Chevalley enters the converse and the comparison of the descent and Eilenberg–Moore factorizations.

For a monad T on $Y \in \mathcal{H}$, its Eilenberg–Moore object Y^T represents $Z \mapsto \mathrm{Alg}_{\mathcal{H}(Z,T)}(\mathcal{H}(Z, Y))$. If $u \colon X \to Y$ has a left adjoint $u_!$, its counit induces the comparison $K^T \colon X \to Y^T$ for $T = u \circ u_!$; u is *monadic* when this comparison is an equivalence.

Proposition 5.1 (Effective descent implies monadicity)**.** *Let $A\colon \dot{\Delta} \to \mathcal{H}$ be of effective j-descent. If $d = d^0_0\colon A(0) \to A(1)$ has a left adjoint $d_!$, then d is monadic. More precisely, $A(0)$, together with d and the action $\mathrm{id}_d * \epsilon$, is an Eilenberg–Moore object for $d \circ d_!$, where $\epsilon\colon d_! \circ d \Rightarrow \mathrm{id}_{A(0)}$ is the counit.*

Proof. Apply [13, Theorem 2.8] to each $\mathcal{H}(Z, d)$. □

In particular, only the converse implication from monadicity to effective descent remains to be established under the additional hypotheses of the formal Bénabou–Roubaud theorem. Its comparison of the two factorizations is stronger: it identifies their intermediate objects even when their comparison morphisms are not equivalences.

Example 5.2 (Monadicity alone does not imply effective descent)**.** This is the constant-diagram construction of [13, Remark 4.5 and Lemma 4.6]. Let $U\colon \mathcal{B} \to \mathcal{C}$ be a monadic functor that is not an equivalence, such as the forgetful functor from pointed sets to sets. Augment the constant cosimplicial object at $\mathcal{C}$ by U: put $A(\mathsf{0}) = \mathcal{B}$, $A(\mathsf{n}) = \mathcal{C}$ for $n \geq 1$, send every map between nonempty ordinals to the identity, and send every map $\mathsf{0} \to \mathsf{n}$ with $n \geq 1$ to U. Normalization forces the descent isomorphism to be an identity, so $\mathrm{Desc}(A \circ j) \simeq \mathcal{C}$ and k_A identifies with U. Thus d^0_0 is monadic, but A is not of effective descent.

5.2 The Beck–Chevalley comparison

Let $\alpha\colon A \to B$ be a pseudonatural transformation of augmented pseudocosimplicial objects, and suppose that $\alpha_{n!} \dashv \alpha_n$ for every $n \geq 0$. For a morphism $v\colon \mathsf{m} \to \mathsf{n}$, orient its pseudonaturality cell as $B(v) \circ \alpha_m \Rightarrow \alpha_n \circ A(v)$. Its left mate is

$$\overline{\alpha}_v\colon \alpha_{n!} \circ B(v) \Longrightarrow A(v) \circ \alpha_{m!}. \tag{5.1}$$

Definition 5.3 (Beck–Chevalley)**.** The transformation α satisfies the *Beck–Chevalley condition* if all mates (5.1) are invertible.

Under this condition, the adjunctions assemble into $\alpha_! \dashv \alpha$ in $[\dot{\Delta}, \mathcal{H}]_{PS}$ [12, Lemma 7.3]. Applied representably, [12, Theorem 7.4] then gives, without an ambient completeness assumption: if B is of effective descent and α_n is monadic for every $n \geq 1$, then

$$A \text{ is of effective descent} \quad \Longleftrightarrow \quad \alpha_0 \text{ is monadic.} \tag{5.2}$$

The hypotheses concern the diagram

$$\begin{array}{ccccccc}
A(0) & \xrightarrow{d^0_0} & A(1) & \rightleftarrows & A(2) & \rightleftarrows & \cdots \\
\downarrow{\scriptstyle \alpha_0} & & \downarrow{\scriptstyle \alpha_1} & & \downarrow{\scriptstyle \alpha_2} & & \\
B(0) & \xrightarrow[d^0_0]{} & B(1) & \rightleftarrows & B(2) & \rightleftarrows & \cdots .
\end{array}$$

For the décalage point $\mathrm{id}_A * \delta\colon A \to A \circ \mathrm{suc}$, the corresponding mates are

$$(d^0_n)_! \circ A(\mathrm{suc}(v)) \Longrightarrow A(v) \circ (d^0_m)_!. \tag{5.3}$$

5.3 The formal Bénabou–Roubaud theorem

Let $A\colon \dot{\Delta} \to \mathcal{H}$ be augmented pseudocosimplicial, and put

$$d = d^0_0\colon A(0) \longrightarrow A(1).$$

If $\mathrm{Desc}(A \circ j)$ exists, the canonical cone gives the descent factorization

$$\begin{array}{ccc} A(0) & \xrightarrow{\;k_A\;} & \mathrm{Desc}(A \circ j) \\ & \searrow^{d} & \downarrow{\scriptstyle u_A} \\ & & A(1) \end{array} \tag{5.4}$$

Suppose $d_! \dashv d$, put

$$T_A = d \circ d_!, \tag{5.5}$$

and assume that the Eilenberg–Moore object $A(1)^{T_A}$ exists. The adjunction gives the factorization

$$\begin{array}{ccc} A(0) & \xrightarrow{\;K^{T_A}\;} & A(1)^{T_A} \\ & \searrow^{d} & \downarrow{\scriptstyle U^{T_A}} \\ & & A(1) \end{array} \tag{5.6}$$

The following theorem identifies the two factorizations under Beck–Chevalley. Its implication from effective descent to monadicity is already covered by Proposition 5.1; the additional hypotheses give the converse and the equivalence of the intermediate objects.

Theorem 5.4 (Formal Bénabou–Roubaud theorem)**.** *Let $\mathcal{H}$ be any 2-category and let $A\colon \dot{\Delta} \to \mathcal{H}$ be an augmented pseudocosimplicial object. Suppose that*

$$\mathrm{id}_A * \delta\colon A \longrightarrow A \circ \mathrm{suc}$$

has componentwise left adjoints and satisfies the Beck–Chevalley condition (5.3)*. Put*

$$d = d^0_0, \qquad d_! = (d^0_0)_!, \qquad T_A = d \circ d_!.$$

Whenever $\mathrm{Desc}(A \circ j)$ *and* $A(1)^{T_A}$ *exist, there is an equivalence*

$$\Phi_A\colon \mathrm{Desc}(A \circ j) \simeq A(1)^{T_A} \tag{5.7}$$

and invertible 2-cells

$$U^{T_A} \circ \Phi_A \cong u_A, \qquad \Phi_A \circ k_A \cong K^{T_A}. \tag{5.8}$$

Thus the ladder

$$\begin{array}{ccccc} A(0) & \xrightarrow{\;k_A\;} & \mathrm{Desc}(A \circ j) & \xrightarrow{\;u_A\;} & A(1) \\ \| & & \downarrow{\scriptstyle \Phi_A}\,\simeq & & \| \\ A(0) & \xrightarrow[K^{T_A}]{} & A(1)^{T_A} & \xrightarrow[U^{T_A}]{} & A(1) \end{array} \tag{5.9}$$

commutes up to the specified cells. In particular,

$$A \text{ is of effective } j\text{-descent} \quad\Longleftrightarrow\quad d^0_0 \text{ is monadic.}$$

Proof. Apply the comparison above to $\alpha = \mathrm{id}_A * \delta$. The shifted object $A \circ \mathrm{suc}$ is of effective descent by Theorem 4.6. For $n \geq 1$, the identity $s^0_n \circ d^0_n \cong \mathrm{id}_{A(\mathsf{n})}$ makes d^0_n a pseudosection, hence monadic by [12, Lemma 8.4]. The equivalence of factorizations is therefore [12, Theorem 8.5], whose shift-effectiveness hypothesis is supplied by Theorem 4.6. □

Applied to $\mathcal{H}(Z, -) \circ A$, this comparison identifies the descent-data and algebra pseudofunctors, pseudonaturally in Z. Thus either is representable precisely when the other is, giving the following consequence.

Corollary 5.5 (Local existence)**.** *Under the adjointness and Beck–Chevalley hypotheses of Theorem 5.4, either of* $\mathrm{Desc}(A \circ j)$ *and* $A(1)^{T_A}$ *exists if and only if the other does. In particular, existence of both need not be imposed independently.*

6 Indexed Čech descent

We apply the preceding results to the indexed Čech objects of Grothendieck descent. The formal Bénabou–Roubaud theorem compares their descent and Eilenberg–Moore factorizations. A section of the base morphism gives a décalage pseudoalgebra. Its underlying successor contraction then yields absolute descent, without any adjointness assumption, and the Janelidze–Tholen characterization.

6.1 The augmented Čech nerve

Let $\mathcal{C}$ be a category with pullbacks and let $p\colon E \to B$. Put

$$E^{[0]} = B, \qquad E^{[n]} = \underbrace{E \times_B \cdots \times_B E}_{n \text{ factors}} \quad (n \geq 1). \tag{6.1}$$

The augmented Čech nerve is the functor

$$D_p\colon \dot{\Delta}^{\mathrm{op}} \longrightarrow \mathcal{C}, \qquad D_p(\mathsf{n}) = E^{[n]}. \tag{6.2}$$

For $v\colon \mathsf{m} \to \mathsf{n}$, the morphism $D_p(v)\colon E^{[n]} \to E^{[m]}$ is characterized, for $m > 0$, by

$$\pi^{(m)}_i \circ D_p(v) = \pi^{(n)}_{v(i)} \qquad (0 \leq i < m).$$

For $m = 0 < n$, it is the common morphism $E^{[n]} \to B$.

We also regard D_p as a functor $\dot{\Delta} \to \mathcal{C}^{\mathrm{op}}$. Given a pseudofunctor

$$\mathcal{L}\colon \mathcal{C}^{\mathrm{op}} \longrightarrow \mathcal{H}, \tag{6.3}$$

put

$$\mathcal{L}_p := \mathcal{L} \circ D_p\colon \dot{\Delta} \longrightarrow \mathcal{H}. \tag{6.4}$$

When it exists, write

$$\mathrm{Desc}_{\mathcal{L}}(p) := \mathrm{Desc}(\mathcal{L}_p \circ j).$$

The comparison and forgetful 1-cells are

$$K^p_{\mathcal{L}}\colon \mathcal{L}(B) \longrightarrow \mathrm{Desc}_{\mathcal{L}}(p), \qquad U^p_{\mathcal{L}}\colon \mathrm{Desc}_{\mathcal{L}}(p) \longrightarrow \mathcal{L}(E). \tag{6.5}$$

They give the factorization

$$\begin{array}{ccc} \mathcal{L}(B) & \xrightarrow{K^p_{\mathcal{L}}} & \mathrm{Desc}_{\mathcal{L}}(p) \\ & \searrow^{\mathcal{L}(p)} & \downarrow{\scriptstyle U^p_{\mathcal{L}}} \\ & & \mathcal{L}(E) \end{array} \tag{6.6}$$

We say that p is of *effective $\mathcal{L}$-descent* if $\mathrm{Desc}_{\mathcal{L}}(p)$ exists and $K^p_{\mathcal{L}}$ is an equivalence. For $\mathcal{H} = \mathbf{Cat}$, this is the classical category of descent data and its usual factorization [7, 12]. Taking $\mathcal{L} = \mathcal{C}/(-)$ recovers the ordinary notion of an effective descent morphism.

Before imposing Beck–Chevalley, we record the implication that requires only a left adjoint to the single change-of-base morphism.

Corollary 6.1 (Effective indexed descent implies monadicity)**.** *If p is of effective $\mathcal{L}$-descent and $\mathcal{L}(p)$ has a left adjoint, then $\mathcal{L}(p)$ is monadic. No Beck–Chevalley condition is required.*

Proof. Apply Proposition 5.1 to $\mathcal{L}_p$, whose augmentation is $\mathcal{L}(p)$. □

For $\mathcal{H} = \mathbf{Cat}$, this is the implication established in [13, Theorem 4.7]. The preceding proof gives its representable form for an arbitrary 2-category.

6.2 The indexed Bénabou–Roubaud theorem

Assume that every change-of-base 1-cell $\mathcal{L}(f)$ has a left adjoint. For a pullback square

$$\begin{array}{ccc} X' & \xrightarrow{v'} & X \\ {\scriptstyle u'}\downarrow & \lrcorner & \downarrow{\scriptstyle u} \\ Y' & \xrightarrow[v]{} & Y \end{array} \tag{6.7}$$

pseudofunctoriality gives an invertible 2-cell

$$\mathcal{L}(v') \circ \mathcal{L}(u) \Longrightarrow \mathcal{L}(u') \circ \mathcal{L}(v).$$

With the adjunctions on the two vertical change-of-base 1-cells, its left mate is

$$\mathcal{L}(u')_! \circ \mathcal{L}(v') \Longrightarrow \mathcal{L}(v) \circ \mathcal{L}(u)_!. \tag{6.8}$$

We say that $\mathcal{L}$ satisfies *Beck–Chevalley* if this mate is invertible for every pullback square. Equivalently one may mate with respect to the horizontal sides: transposing the same pullback square gives that convention.

Fix p and put $A = \mathcal{L}_p$. For $n \geq 0$, let

$$f_n = D_p(\delta_n) \colon E^{[n+1]} \longrightarrow E^{[n]}.$$

For $v \colon \mathsf{m} \to \mathsf{n}$, naturality of δ gives the square

$$\begin{array}{ccc} E^{[n+1]} & \xrightarrow{D_p(\mathrm{suc}(v))} & E^{[m+1]} \\ {\scriptstyle f_n}\downarrow & \lrcorner & \downarrow{\scriptstyle f_m} \\ E^{[n]} & \xrightarrow[D_p(v)]{} & E^{[m]} \end{array} \tag{6.9}$$

This is a pullback. Indeed, under $E^{[n+1]} \cong E \times_B E^{[n]}$, it is obtained by pulling back $E \times_B E^{[m]} \to E^{[m]}$ along $D_p(v)$. After applying $\mathcal{L}$, its pseudofunctoriality cell is precisely the pseudonaturality cell of $\mathrm{id}_A * \delta$. Hence Beck–Chevalley for $\mathcal{L}$ implies the condition (5.3) for A.

Theorem 6.2 (Two-dimensional Bénabou–Roubaud)**.** *Let $\mathcal{C}$ be a category with pullbacks, let $\mathcal{H}$ be any 2-category, and let $\mathcal{L}\colon \mathcal{C}^{\mathrm{op}} \to \mathcal{H}$ be a pseudofunctor. Suppose that every $\mathcal{L}(f)$ has a left adjoint and that $\mathcal{L}$ satisfies Beck–Chevalley. For $p\colon E \to B$, put*

$$T_p = \mathcal{L}(p) \circ \mathcal{L}(p)_!.$$

The objects $\mathrm{Desc}_{\mathcal{L}}(p)$ and $\mathcal{L}(E)^{T_p}$ exist simultaneously. Whenever they exist, there is an equivalence

$$\Phi_p\colon \mathrm{Desc}_{\mathcal{L}}(p) \simeq \mathcal{L}(E)^{T_p} \tag{6.10}$$

with invertible 2-cells

$$U^{T_p} \circ \Phi_p \cong U^p_{\mathcal{L}}, \qquad \Phi_p \circ K^p_{\mathcal{L}} \cong K^{T_p}.$$

Consequently,

$$p \text{ is of effective } \mathcal{L}\text{-descent} \iff \mathcal{L}(p) \text{ is monadic.}$$

Proof. For $A = \mathcal{L}_p$, the component d^0_0 is $\mathcal{L}(p)$ and the induced monad is T_p. The pullback calculation (6.9) supplies Beck–Chevalley for $\mathrm{id}_A * \delta$. Apply Theorem 5.4 and Corollary 5.5. □

Corollary 6.3 (Bénabou–Roubaud)**.** *Let $\mathcal{C}$ be a category with pullbacks and let $\mathcal{L}\colon \mathcal{C}^{\mathrm{op}} \to \mathbf{Cat}$ be an indexed category whose change-of-base functors have left adjoints and satisfy Beck–Chevalley. Then, for every $p\colon E \to B$, the descent factorization of $\mathcal{L}(p)$ is equivalent to its Eilenberg–Moore factorization. In particular,*

$$p \text{ is of effective } \mathcal{L}\text{-descent} \iff \mathcal{L}(p) \text{ is monadic.}$$

Proof. The required descent categories and Eilenberg–Moore categories exist in **Cat**, so this is Theorem 6.2 with $\mathcal{H} = \mathbf{Cat}$. □

6.3 The Čech contraction of a split epimorphism

Suppose now that $p\colon E \to B$ has a section

$$s\colon B \longrightarrow E, \qquad p \circ s = \mathrm{id}_B. \tag{6.11}$$

Define

$$h_n\colon E^{[n]} \longrightarrow E^{[n+1]} \tag{6.12}$$

by $h_0 = s$ and, for $n \geq 1$, by

$$\pi^{(n+1)}_0 \circ h_n = s \circ p \circ \pi^{(n)}_0, \qquad \pi^{(n+1)}_{i+1} \circ h_n = \pi^{(n)}_i \quad (0 \leq i < n). \tag{6.13}$$

In element notation, this inserts the section in the first coordinate:

$$h_n(e_0, \ldots, e_{n-1}) = \big((s \circ p)(e_0), e_0, \ldots, e_{n-1}\big).$$

Lemma 6.4 (The Čech contraction)**.** *For every $v\colon \mathsf{m} \to \mathsf{n}$ in $\dot{\Delta}$,*

$$D_p(\mathrm{suc}(v)) \circ h_n = h_m \circ D_p(v), \tag{6.14}$$

$$D_p(\delta_n) \circ h_n = \mathrm{id}_{E^{[n]}}, \tag{6.15}$$

$$h_{n+1} \circ h_n = D_p(\mu_n) \circ h_n. \tag{6.16}$$

Consequently, regarding D_p as an augmented cosimplicial object in $\mathcal{C}^{\mathrm{op}}$, the opposites of the h_n form a strict $\mathbb{S}_{\mathcal{C}^{\mathrm{op}}}$-algebra, whose underlying contraction is the successor contraction.

Proof. The first two identities follow by comparing projections, using $p \circ s = \mathrm{id}_B$. For the third, both sides insert the section twice. □

In low degrees,

$$h_0 = s, \qquad h_1 = \langle s \circ p, \mathrm{id}_E \rangle\colon E \to E \times_B E.$$

The opposites of these equations give the ordinary split-equalizer identities. The higher maps h_n give the full successor algebra.

Postcomposition with $\mathcal{L}$ transports this algebra to a $\mathbb{S}_{\mathcal{H}}$-pseudoalgebra on $A = \mathcal{L}_p$. Its action is

$$r\colon A \circ \mathrm{suc} \longrightarrow A, \qquad r_n = \mathcal{L}(h_n). \tag{6.17}$$

The naturality squares (6.14), together with the compositors of $\mathcal{L}$, supply its pseudonaturality cells. Its unit and multiplication cells have components

$$\begin{aligned} r_n \circ d^0_n &= \mathcal{L}(h_n) \circ \mathcal{L}(D_p(\delta_n)) \cong \mathcal{L}(D_p(\delta_n) \circ h_n) \cong \mathrm{id}_{A(\mathsf{n})}, \\ r_n \circ r_{n+1} &= \mathcal{L}(h_n) \circ \mathcal{L}(h_{n+1}) \cong \mathcal{L}(h_{n+1} \circ h_n) \\ &= \mathcal{L}(D_p(\mu_n) \circ h_n) \cong \mathcal{L}(h_n) \circ \mathcal{L}(D_p(\mu_n)). \end{aligned} \tag{6.18}$$

The last expression is the degree-n component of $r \circ m_A$, where $m_A = \mathrm{id}_A * \mu$ is the décalage multiplication. Thus the second line gives the modification $r \circ \mathbb{S}(r) \Rightarrow r \circ m_A$. The strict algebra laws just proved and the pseudofunctor coherence axioms for $\mathcal{L}$ give the pseudoalgebra unit and associativity axioms. In particular, the multiplication coherence is part of the induced structure, rather than an additional choice.

Theorem 6.5 (Absolute indexed descent for split epimorphisms)**.** *Let $\mathcal{C}$ be a category with pullbacks and let $p\colon E \to B$ be a split epimorphism. For every 2-category $\mathcal{H}$ and every pseudofunctor $\mathcal{L}\colon \mathcal{C}^{\mathrm{op}} \to \mathcal{H}$, a section of p equips $\mathcal{L}_p$ with the $\mathbb{S}_{\mathcal{H}}$-pseudoalgebra* (6.17)–(6.18)*. Its underlying successor contraction induces a descent pseudoalgebra, and its augmented cone is an absolute descent cone with vertex $\mathcal{L}(B)$. In particular, p is of effective $\mathcal{L}$-descent.*

Proof. The preceding construction gives the décalage pseudoalgebra. Forgetting its multiplication cell and applying Corollary 4.5 proves existence and absoluteness of the descent cone. Thus $\mathcal{L}_p$ belongs to the domain of descent completion in Proposition 2.2. Its descent action is $r \circ \chi_{\mathcal{L}_p}\colon \mathbb{D}(\mathcal{L}_p) \to \mathcal{L}_p$, with unit induced by the pointed comparison and multiplication supplied by Proposition 3.5. This is the structure obtained by the transfer of contractions. □

6.4 The Janelidze–Tholen characterization

A morphism p is *of absolute effective descent* if it is of effective $\mathcal{L}$-descent for every indexed category $\mathcal{L}\colon \mathcal{C}^{\mathrm{op}} \to \mathbf{Cat}$ [7]. Effective $\mathcal{L}$-descent makes $\mathcal{L}(p)$ conservative, since the forgetful functor from descent data reflects isomorphisms.

For a set S, put $\mathbf{2}^S = [S_{\mathrm{disc}}, \mathbf{2}]$, where $\mathbf{2} = (0 \to 1)$.

Lemma 6.6. *For a function $v\colon S \to T$, restriction $v^*\colon \mathbf{2}^T \to \mathbf{2}^S$ is conservative if and only if v is surjective.*

Proof. Surjectivity detects invertibility at every coordinate. If a coordinate is missing, the transformation $0 \to 1$ there and identities elsewhere is noninvertible but restricts to an identity. □

Fix $B \in \mathcal{C}$ and define the indexed category

$$\mathcal{L}_B(X) = \mathbf{2}^{\mathcal{C}(B,X)}, \qquad \mathcal{L}_B(f) = \mathcal{C}(B, f)^*. \tag{6.19}$$

Proposition 6.7 (One indexed category detects splitting)**.** *If $p\colon E \to B$ is of effective $\mathcal{L}_B$-descent, then p is a split epimorphism.*

Proof. Conservativity of $\mathcal{L}_B(p)$ makes $\mathcal{C}(B, p)$ surjective by Lemma 6.6. A preimage of id_B is a section of p. □

Theorem 6.8 (Janelidze–Tholen)**.** *Let $\mathcal{C}$ be a category with pullbacks. For a morphism $p\colon E \to B$, the following conditions are equivalent:*

(i) *p is of absolute effective descent;*

(ii) *p is a split epimorphism.*

Proof. If p is of absolute effective descent, it is of effective $\mathcal{L}_B$-descent and therefore splits by Proposition 6.7. Conversely, if p splits, Theorem 6.5 with $\mathcal{H} = \mathbf{Cat}$ shows that it is of effective $\mathcal{L}$-descent for every indexed category $\mathcal{L}$. □

7 Co-duality and co-Eilenberg–Moore objects

Since the preceding results hold in an arbitrary 2-category, they also apply in $\mathcal{H}^{\mathrm{co}}$. Reversing 2-cells turns right adjoints into left adjoints and comonads into monads. Conical bilimits, and hence descent objects, are unchanged: inverting cone constraints identifies the cone categories with their opposites.

For an adjunction $d \dashv d_*$, with unit η and counit ϵ, write

$$Q_A = d \circ d_*, \qquad \epsilon\colon Q_A \Rightarrow \mathrm{id}_{A(1)}, \qquad \mathrm{id}_d * \eta * \mathrm{id}_{d_*}\colon Q_A \Rightarrow Q_A \circ Q_A, \tag{7.1}$$

where $d = d_0^0\colon A(0) \to A(1)$. The *co-Eilenberg–Moore object* $A(1)^{Q_A}$ represents Q_A-coalgebras; equivalently, it is the Eilenberg–Moore object of the corresponding monad in $\mathcal{H}^{\mathrm{co}}$. The coaction $\mathrm{id}_d * \eta$ induces the factorization

$$A(0) \xrightarrow{K^{Q_A}} A(1)^{Q_A} \xrightarrow{U^{Q_A}} A(1), \qquad U^{Q_A} \circ K^{Q_A} \cong d. \tag{7.2}$$

The 1-cell d is *comonadic* when K^{Q_A} is an equivalence.

Corollary 7.1 (Effective descent implies comonadicity)**.** *If A is of effective descent and d^0_0 has a right adjoint, then d^0_0 is comonadic. No Beck–Chevalley condition is required.*

Proof. Apply Proposition 5.1 in $\mathcal{H}^{\mathrm{co}}$. □

Suppose each d^0_n has a right adjoint r_n. The *right Beck–Chevalley condition* requires the right mates of the inverses of the pseudonaturality cells $\gamma_v \colon A(\mathrm{suc}(v)) \circ d^0_m \Rightarrow d^0_n \circ A(v)$ to be invertible:

$$\gamma_v^* \colon A(v) \circ r_m \Longrightarrow r_n \circ A(\mathrm{suc}(v)) \qquad (v \colon \mathsf{m} \to \mathsf{n}). \tag{7.3}$$

This is the left Beck–Chevalley condition in $\mathcal{H}^{\mathrm{co}}$.

Corollary 7.2 (Co-dual Bénabou–Roubaud)**.** *Let $A \colon \dot{\Delta} \to \mathcal{H}$ be an augmented pseudocosimplicial object. If each d^0_n has a right adjoint and the right Beck–Chevalley mates* (7.3) *are invertible, then* $\mathrm{Desc}(A \circ j)$ *and $A(1)^{Q_A}$ exist simultaneously. When they exist, there is an equivalence*

$$\Phi^{\mathrm{co}}_A \colon \mathrm{Desc}(A \circ j) \simeq A(1)^{Q_A}, \tag{7.4}$$

with $U^{Q_A} \circ \Phi^{\mathrm{co}}_A \cong u_A$ and $\Phi^{\mathrm{co}}_A \circ k_A \cong K^{Q_A}$. In particular, A is of effective descent if and only if d^0_0 is comonadic.

Proof. Apply Theorem 5.4 and Corollary 5.5 in $\mathcal{H}^{\mathrm{co}}$. □

For an indexed pseudofunctor $\mathcal{L} \colon \mathcal{C}^{\mathrm{op}} \to \mathcal{H}$ whose change-of-base 1-cells have right adjoints, with $\mathcal{C}$ admitting pullbacks, the corresponding mates for (6.7) are

$$\mathcal{L}(v) \circ \mathcal{L}(u)_* \Longrightarrow \mathcal{L}(u')_* \circ \mathcal{L}(v'). \tag{7.5}$$

They are the right mates of the inverses of the pseudofunctoriality cells used in (6.8).

Corollary 7.3 (Comonadic indexed descent)**.** *If the mates* (7.5) *are invertible for every pullback square, then the descent and co-Eilenberg–Moore factorizations of $\mathcal{L}(p)$ are equivalent whenever either exists. Thus p is of effective $\mathcal{L}$-descent if and only if $\mathcal{L}(p)$ is comonadic. For $\mathcal{H} = \mathbf{Cat}$, both objects always exist.*

Proof. Apply Theorem 6.2 to the indexed pseudofunctor with values in $\mathcal{H}^{\mathrm{co}}$. □

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