PRÉ-PUBLICAÇÕES DMUC

GRAPHICAL FREENESS AND COHERENCE FOR LINEAR CATEGORIES

NORIHIRO YAMADA

DMUC Preprint 26-63, 09 Nov 2026

Abstract. A *linear category* axiomatises a canonical interplay between linear and nonlinear functions recurring across logic and mathematics. Whilst *graphical calculi* such as Joyal and Street's *string diagrams* provide complete representations of a broad range of monoidal structures, to our knowledge, no graphical calculus for linear categories has emerged. A challenge here is to exactly represent the complex interaction of the comonad with the symmetric monoidal and the cartesian structures of a linear category by a *purely* geometric or graph-theoretic means without quotients or other extra structures. We fill this long-standing gap by building, from any small category $\mathcal{B}$, a calculus that consists solely of labelled finite graphs and forms a *free* linear category over $\mathcal{B}$. It comes with *reduction* that strongly normalises the graphs in a confluent way by graph-theoretic operations–copying, splitting and yanking–not by orienting categorical axioms. The calculus turns categorical constructs and equations into finitary geometric calculations, making the equality in the free linear category *decidable* if so is that in $\mathcal{B}$. In this way, the calculus is not only a language; it *reduces algebra to geometry*: two morphisms in the free linear category are equal if and only if their representatives are the same labelled graph. It also yields a necessary-and-sufficient *coherence* criterion for free linear categories, and as a corollary Kelly and Mac Lane's criterion for free closed categories.

1. Introduction

We provide a graphical presentation of a *free* linear category, reducing the algebraic structure to the geometry of finite graphs, and use it to prove a *coherence* theorem for linear categories.

A *linear category* [Ben94, Bie95, MMDPR05] is a symmetric monoidal closed category with finite products together with a comonad that *mediates* the worlds of linear and nonlinear maps, *i.e.* its *linear-nonlinear adjunction* [Ben94] is monoidal. This structure captures the recurring situation in mathematics, ranging from analysis and geometry to logic and computation (§1.1), that linear maps are distinguished, but nonlinear ones are also important and linked canonically with linear ones. For this ubiquity, linear categories act as a basic framework in various fields.

Graphical calculi [Sel11] aim to reduce categorical structures to geometric or graph-theoretic means. This programme traces back to Eilenberg, Kelly and Mac Lane [EK66, KM71] as well as Joyal and Street [JS91]. A variety of graphical calculi have been proposed in the literature, and they completely represent a broad range of monoidal structures [Sel11], including weakly distributive categories [BCST96] and star-autonomous categories [Hug12], but to our knowledge not linear categories. The *completeness* of a representation means that morphisms are equal if and only if their representatives coincide. Since the birth of linear categories in the early 1990s, a graphical calculus for a free linear category, especially without quotients or any extra means, *e.g.* hierarchies of *boxed* graphs equipped with rewritings [CS99, Mel06], has been missing. Even a recent calculus for a free linear category (without finite products) [Has20] is not inherently geometric or graph-theoretic but categorical and axiomatic as it takes numerous quotients and rewritings given by orienting categorical axioms. We rather seek a calculus that turns categorical axioms into more primitive, geometric or graph-theoretic principles as in [EK66, KM71, JS91]. The main challenge here comes from the intricate interplay of the comonad with the symmetric monoidal and the cartesian structures of a linear category [Bie95, pp. 82–86].

[v1] Mon, 09 Nov 2026

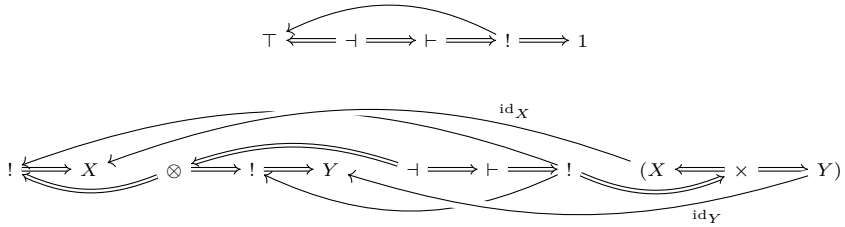


FIGURE 1. A (simplified) graphical representation of the Seely isomorphisms

We solve this long-standing problem by building, from any small category $\mathcal{B}$, a calculus that consists solely of labelled finite graphs (whose labels do not contain nontrivial equalities except those coming from $\mathcal{B}$) and showing that this construction extends to a left adjoint to the forgetful functor that maps small *strict* (*i.e.* with structures chosen) linear categories to their underlying small categories, *i.e.* the calculus forms a *free* strict linear category over $\mathcal{B}$, without quotients or any extra means. The labels on the graphs are to faithfully represent $\mathcal{B}$, pairing and contraction in the free strict linear category. The calculus comes with *reduction* $\rightsquigarrow$ that strongly normalises the graphs in a confluent manner by copying, splitting and yanking. All the graphs, including non-normalised ones, are characterised not inductively but geometrically.

To get a glimpse of our graphical calculus, we recall that every linear category $\mathcal{L}$ contains the *Seely isomorphisms* [See89, Bie95]

$$\top \xrightarrow{\sim} !1, \qquad\qquad !X \otimes !Y \xrightarrow{\sim} !(X \times Y)$$

natural in $X, Y \in \mathcal{L}$, where $\otimes$ and $\times$ are the tensor and the cartesian products, respectively, $\top$ and 1 are their respective units,[1] and ! is the comonad. Assume that $\mathcal{L}$ is the free strict one over $\mathcal{B}$. Our calculus represents the isomorphisms by the (slightly abbreviated) labelled finite graphs in Figure 1, where $\dashv$ and $\vdash$ are labels on vertices that separate the domain and the codomain of the isomorphisms, the thick edges $\Rightarrow$ are concerned with the objects, and the thin ones $\to$ with the isomorphisms, labelled with a morphism χ in $\mathcal{B}$ if it represents χ. The object part consists of the *parse trees* of formal free objects, and the morphism part extends *Eilenberg–Kelly–Mac Lane graphs* [EK66, KM71] with constructs for the finite products and the comonad. We often omit thick edges when they are evident. Note that the directions of thin edges are *opposite* to those of morphisms; they fit certain graph traversals that are central in this work (§2.2).

Our labelled graph is not quotiented but *saturated* under *Trimble rewiring* [BCST96, Tri94] (*i.e.* we take the union of all the labelled graphs related by Trimble rewiring) to yield the free linear category without quotients, and crucially the reduction works on the saturated graphs *themselves*. For instance, as in every monoidal category, $\mathcal{L}$ has the isomorphism

$$\top \otimes X \xrightarrow{\sim} X \tag{1}$$

natural in $X \in \mathcal{L}$, and the calculus represents it, its inverse and their composite (or *gluing*) by the upper left, the upper right and the lower labelled finite graphs in Figure 2, respectively, where the new vertex labelled with $\triangleright$ links the component graphs, the additional thin edge between the linked parts is dotted for clarity, and the dashed thin edge is added for the saturation. Our graphical composition is made of such a gluing of labelled graphs with the saturation, followed by the reduction. The reduction normalises the last diagram into the one representing the identity on $\top \otimes X$ in $\mathcal{L}$ by *yanking* the dotted edge and deleting the vertices becoming redundant as in

[1] Girard [Gir87] swaps the notations for these units.

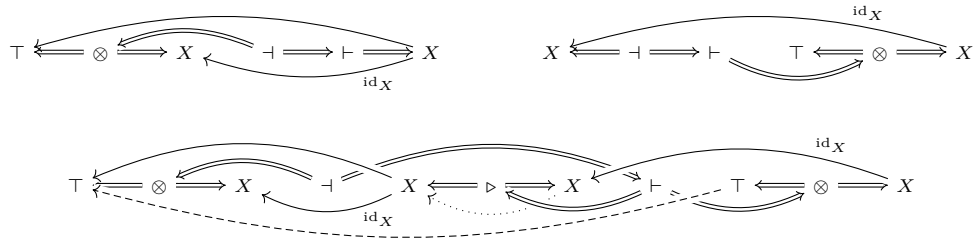


FIGURE 2. Saturation of the left-unit composite under Trimble rewiring

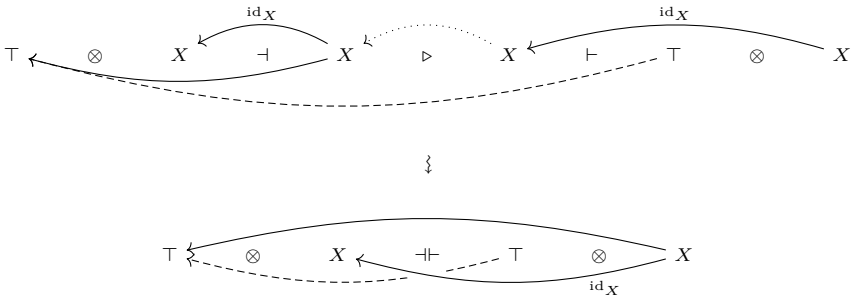


FIGURE 3. Normalisation of a saturated composite by yanking

Figure 3, where we omit thick edges and a space between $\dashv$ and $\vdash$. By yanking, two edges are combined into a single one, labelled with the composition $\zeta \circ \chi$ in $\mathcal{B}$ if and only if the component edges are labelled with morphisms χ and ζ in $\mathcal{B}$, respectively. Crucially, the normalised diagram would not be the identity without the dashed edge, which illustrates the role of the saturation. The reduction also normalises the other composite of the isomorphism (1) and its inverse by *splitting* elements under the construct $\triangleright$ (with respect to thick edges) and yanking as in Figure 4, where we informally insert parenthesis to indicate where omitted thick edges should occur.

Compared to Eilenberg–Kelly–Mac Lane graphs, a main feature of our graphical calculus is the construct ! for the comonad, which allows a vertex to have a degree more than one (with respect to thin edges) as long as the relevant thin edges have different labels, called *colours*. The reduction normalises a graph containing such a vertex by *copying* and yanking as in Figure 5, where we omit a space between the vertices labelled with ! and X, respectively, represent colours informally by blue and green, and in the last step consolidate multiple steps. Copying before yanking enables the reduction to always yank a pair of *single* edges. In the body of this article, we define a precise notion of colours in such a way that we do not have to take the quotient of labelled graphs modulo renamings of colours, so that our graphical calculus is quotient-free.

Why does the freeness matter? It is because, by the freeness, our calculus reasons about all and only equations valid in *all* linear categories, turning categorical constructs and axioms into finitary geometric calculations. In addition to this *completeness*, the freeness also subsumes the *full completeness* of the calculus, *i.e.* each of our labelled graphs corresponds to a morphism in the formal free linear category, similar to *definability* in model theory. Our full completeness is unusually strong: it covers *all* the labelled graphs, not only normalised ones. By the finitary nature of the calculus, the equality in the free linear category is *decidable* if so is the one in $\mathcal{B}$.

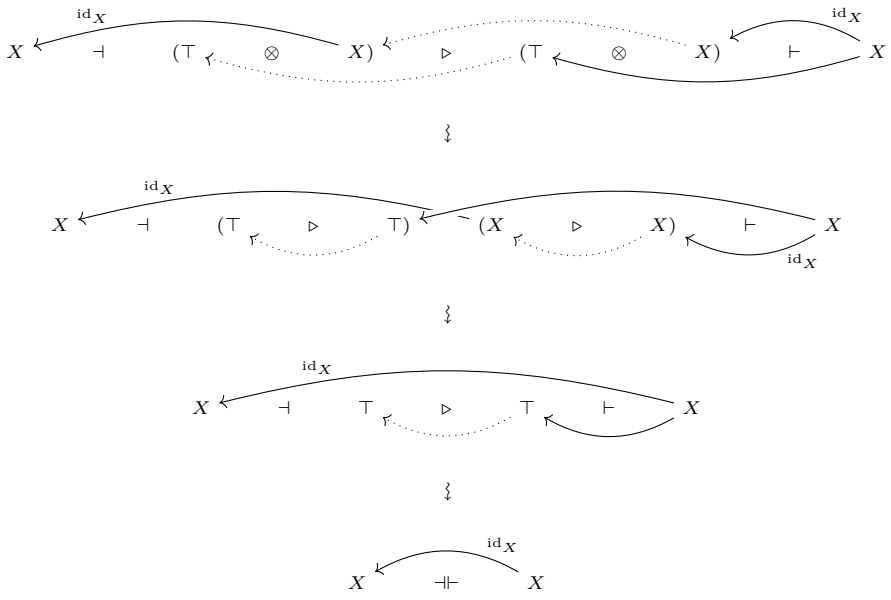


FIGURE 4. Normalisation of another composite by splitting and yanking

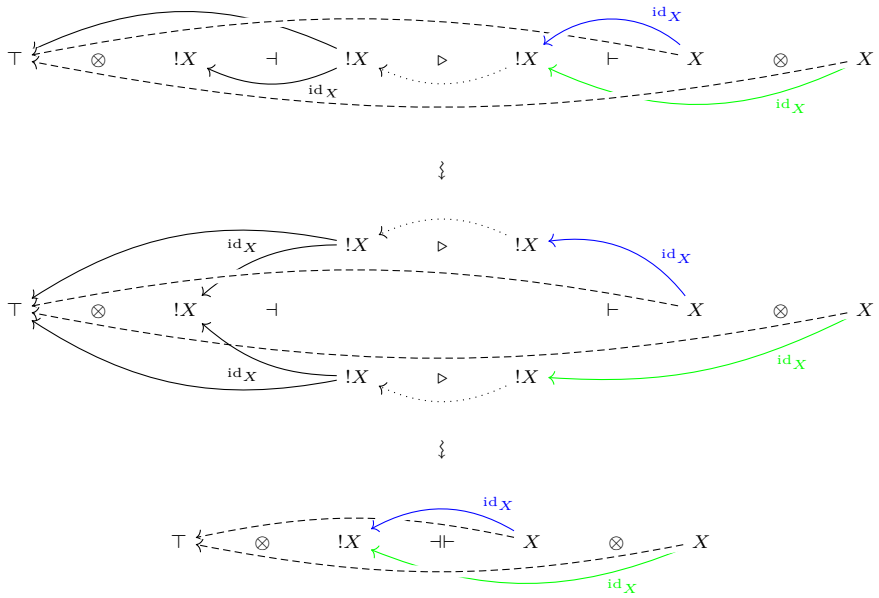
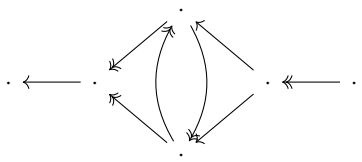


FIGURE 5. Normalisation by copying and yanking

The calculus also leads to a graphical *coherence* criterion, which characterises the equality of canonical or *allowable* natural transformations for the free linear category, where the allowable ones are built inductively from identities and coherence natural isomorphisms by constructs of the free linear category. By restriction, this covers the same coherence for symmetric monoidal

closed categories too. Roughly, our criterion is the absence of a certain subgraph of the form

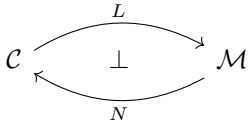
(2) 

in the union of the graphical representatives of two allowable natural transformations, where a double-head arrow $\twoheadrightarrow$ abbreviates a finite sequence of edges. That is, the two allowable natural transformations are equal if and only if the union does not contain such a subgraph.

Unlike the sufficient coherence criterion for (symmetric monoidal) closed categories [KM71], our criterion is *necessary and sufficient* for the equality of allowable natural transformations. It further extends to free linear categories and recovers the sufficient criterion as an immediate corollary. Our criterion (2) is simple and provides a *geometric* understanding of the coherence, which stands in contrast to the traditional, proof-theoretic method [Sol97].

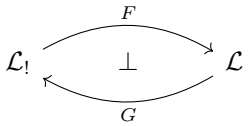
Why does the coherence matter? Initiated by Mac Lane [Mac63], coherence in general has become a basic principle in category theory and led to profound results, *e.g.* [KM71, BCST96, Sol97]. Here, coherence means that choices involved in building morphisms, parenthesising tensor product, inserting/deleting unit, using symmetry, and so forth, do not create ambiguity. It thus justifies the usual practice of suppressing associators, unitors and so forth. Hence, the coherence on linear categories allows them to act as a robust framework for the interplay between linear and nonlinear maps. Since the interplay abounds, it has significance beyond category theory.

1.1. **Background and motivation.** A *linear-nonlinear adjunction* [Ben94]



between a symmetric monoidal (or *linear*) category $\mathcal{M}$ and a cartesian (or *nonlinear*) category $\mathcal{C}$ abounds in logic and mathematics. Its examples include the adjunction between the cartesian category of sets and the symmetric monoidal category of vector spaces [ML13, VZ14].

Of our particular interest is a symmetric monoidal closed category $\mathcal{L}$ with finite products together with a comonad ! on $\mathcal{L}$, called *exponential*, such that the linear-nonlinear adjunction

(3) 

between $\mathcal{L}$ and the associated (cartesian closed [BBDPH93]) coKleisli category $\mathcal{L}_!$ is monoidal. Such categories, known as *linear categories* (originally called *new Seely categories* [Bie95]), constitute categorical semantics of intuitionistic linear logic [Bie95, MMDPR05]. A linear category is said to be *pointed* if it has a distinguished object, called *bottom*.[2] A (pointed) linear category is said to be *strict* if it comes with chosen finite products and closure (or internal hom).

Recall that every linear category $\mathcal{L} = (\mathcal{L}, \top, \otimes, !)$ induces the Seely isomorphisms

$$!1 \cong \top, \qquad\qquad !(X \times Y) \cong !X \otimes !Y$$

[2]There is no axiom for the bottom $\perp$ of a pointed linear category to satisfy. One can think of it as the distinguished element in a pointed set or a pointed topological space.

natural in $X, Y \in \mathcal{L}$, and the exponential ! gives rise to a symmetric monoidal comonad. In the sense of the isomorphisms, the exponential *mediates* the symmetric monoidal and the cartesian structures of $\mathcal{L}$. The isomorphisms illustrate a delicate interaction of the exponential with the symmetric monoidal and the cartesian structures. The subtlety of the interaction is illustrated furthermore by the commutativity of a number of diagrams in [Bie95, pp. 82–86].

Linear categories and their variants are found not only in logic but also in many other areas of mathematics, *e.g.* analysis and differential geometry [BCS06, BCLS20, FGH26], algebraic geometry [Mel22], topology and functional analysis [BPS94, KKR96, Gir99], foundations of mathematics and quantum computation [SS23, Ril22], computability and computational complexity [Gir98, Laf04, DJ03, Hof03, BG20]. Concretely, examples of a linear category include the category of modules over a commutative ring, that of sheaves of modules over a commutative ringed space, that of unbounded chain complexes of modules and that of representations of a discrete group (§4). For this ubiquity of linear categories, their canonical way to connect the worlds of linear and nonlinear maps, where one pays special attention to linear maps without losing the generality of nonlinear maps (*e.g.* as in functional analysis), is of broad interest.

On the other hand, *graphical* tools are effective for equational reasoning, and they were informally used by mathematicians and physicists [P$^+$71, Pen71]. This custom changed when Joyal and Street [JS91] established a mathematical ground for the graphical means; since then a variety of graphical yet rigorous calculi, called *string diagrams*, have been designed for a family of monoidal structures [Sel11] and widely used in mathematics and beyond [BS10]. In standard string diagrams, objects are represented by labelled wires, unit by the empty wire, morphisms by labelled boxes, identity morphisms by the trivial box, composition by sequential concatenation, and tensor product by parallel juxtaposition. For instance, the equality



between string diagrams models the equation

$$\mathrm{id}_\top \otimes (f \circ \mathrm{id}_A) \otimes (\mathrm{id}_D \circ g) = (\mathrm{id}_B \circ f) \otimes (g \circ \mathrm{id}_C) \otimes \mathrm{id}_\top$$

valid in any strict monoidal category. Note that these traditional string diagrams work only for *strict* monoidal structures due to its representation of unit and tensor product.

Another origin of graphical methods is *linear logic* [Gir87]. Standard tools in proof theory [Gen35, TS00] or type theory [Chu40, GTL89] are not ideal for linear logic; they have superfluous syntactic details. For instance, a λ-calculus for intuitionistic linear logic needs seventy intricate formal rewritings, called *commuting conversions*, to equate inessentially different λ-terms [Bie94, Figures 3.7–3.11]. Yet, this method only takes a quotient of λ-terms modulo inessential details; it does not solve the problem of establishing an ideal representation of linear logic (or linear categories) or capturing equality between proofs (or morphisms) in a mathematically clarifying fashion.[3] To solve this *bureaucracy of syntax* [Gir89], Girard [Gir87] invented *proof nets*, which can be seen as an extension of string diagrams to more than one tensor product [Sel11, §4.9].

However, these graphical methods have limitations. For instance, recall that standard string diagrams represent *strict* monoidal structures; they cover weak monoidal structures only *via*

[3]Girard *et al.* [GTL89, §10.1] describe the same problem on natural deduction systems as follows: "The true deductions are nothing more than equivalence classes of deductions modulo commutation rules."

coherence.[4] They also need to quotient graphs for monoidal *closed* categories (*i.e.* the categorical axioms are not captured by a purely geometric means) as in Alvarez-Picallo *et al.* [APGSZ22].

Another obstacle is unit $\top$: much of the study of coherence on symmetric monoidal closed categories has been devoted to unit [KM71, Sol91, Sol97, MS07].[5] The past graphical methods [Tri94, BCST96, KO99, LS06, Hug12] that admit unit take quotients of graphs, losing a *canonical* representation of morphisms. Due to the Seely isomorphism $\top \cong !1$, equality of morphisms related to exponential is at least as complex as equality concerned with unit. Besides, the exponential of a linear category interacts with the symmetric monoidal and the cartesian structures in a subtle way, making it substantially harder to realise a graphical calculus for linear categories than symmetric monoidal closed categories, especially without quotients or any extrinsic means such as hierarchies of *boxed* graphs equipped with rewritings [CS99, Mel06].

Similarly, proof nets have been mostly on *fragments* of linear logic (or linear categories), and Hughes [Hug06, p. 1070] regards them as *overly syntactic* because they have labels for not only logical constants or connectives but also logical rules in a sequent calculus.

As a result, to our knowledge, a graphical calculus for linear categories has been missing since the birth of linear categories, string diagrams or proof nets, while a variety of graphical calculi have been established in their extensive literature. We aim to solve this long-standing problem and extend the beautiful interplay between algebra and geometry to linear categories.

1.2. **Main results.** To this end, we build, from a given small category $\mathcal{B}$, a calculus, called the *$\ell_{\mathcal{B}}$-calculus*, that consists solely of labelled finite graphs, called *$\ell_{\mathcal{B}}$-nets*, together with reduction, called *$\ell_{\mathcal{B}}$-reduction*, that copies, splits and yanks $\ell_{\mathcal{B}}$-nets. An $\ell_{\mathcal{B}}$-net is said to be *normalised* if it is fully reduced. We state our main results informally below and precisely in later sections.

First, $\ell_{\mathcal{B}}$-reduction enjoys *confluence* and *strong normalisation*, therefore:

Theorem 1.2.1. *Every $\ell_{\mathcal{B}}$-net between a pair of objects is $\ell_{\mathcal{B}}$-reduced to a unique normalised one between the same pair of objects within a finite number of steps.*

We may extend the $\ell_{\mathcal{B}}$-calculus with *bottom* $\bot$ as in [TRO92, §2.5].[6] The extension is called the *$\ell^{\bullet}_{\mathcal{B}}$-calculus*, and the theorem holds for the extension of $\ell_{\mathcal{B}}$-reduction, called *$\ell^{\bullet}_{\mathcal{B}}$-reduction*.

Let the superscript $(_)^{\circ}$ range over the one $(_)^{\bullet}$ and no superscript $(_)$. For instance, the *$\ell^{\circ}_{\mathcal{B}}$-calculus* ranges over the $\ell^{\bullet}_{\mathcal{B}}$-calculus and the $\ell_{\mathcal{B}}$-calculus. We drop the subscript $(_)_{\mathcal{B}}$ when $\mathcal{B}$ is the empty category. Let s(p)LCat be the category of small strict (pointed) linear categories and functors strictly preserving the structure of (pointed) linear categories, and $U : \mathrm{s(p)LCat} \to \mathrm{Cat}$ the forgetful functor that maps a small strict (pointed) linear category to its underlying small category. Without the strictness, the next *freeness* would hold only in a 2-categorical sense.

Theorem 1.2.2. *The $\ell^{\circ}_{\mathcal{B}}$-calculus forms a free strict (pointed) linear category $\tilde{\mathrm{Cl}}(\ell^{\circ}_{\mathcal{B}})$ over $\mathcal{B}$ without quotients or any other extra means, whose objects are the parse trees of formal free objects, and morphisms are normalised $\ell_{\mathcal{B}}$-nets. Moreover, the construction $\mathcal{B} \mapsto \tilde{\mathrm{Cl}}(\ell^{\circ}_{\mathcal{B}})$ extends to a left adjoint to the forgetful functor U,* i.e. *the isomorphism*

$$\mathrm{s(p)LCat}(\tilde{\mathrm{Cl}}(\ell^{\circ}_{\mathcal{B}}), \mathcal{L}) \cong \mathrm{Cat}(\mathcal{B}, U\mathcal{L})$$

holds naturally in $\mathcal{B} \in \mathrm{Cat}$ and $\mathcal{L} \in \mathrm{s(p)LCat}$. In particular, $\tilde{\mathrm{Cl}}(\ell^{\circ})$ is initial in s(p)LCat.

Since a free strict (pointed) linear category over $\mathcal{B}$ is *unique* up to isomorphisms, the two theorems, combined with the finitary nature of $\ell^{\circ}_{\mathcal{B}}$-nets, imply:

[4] One may argue that this limitation is irrelevant because every (weak) monoidal category is *equivalent* to a strict one by MacLane's coherence theorem [ML13, §7]. However, the traditional string diagrams cannot be used for proving the coherence itself. Our aim is to establish a graphical method that overcomes this limitation.

[5] For instance, Mèhats and Soloviev [MS07, p. 129] say that unit makes symmetric monoidal closed categories (presented as deductive systems) surprisingly complex with respect to equivalence of derivations.

[6] Troelstra instead writes 0 for bottom and calls it *zero* (which we do not follow).

Corollary 1.2.3. *Given morphisms $f, f' : A \rightrightarrows B$ in any free strict (pointed) linear category over $\mathcal{B}$, it is decidable whether f and f' are equal provided that this decidability holds for $\mathcal{B}$.*

The freeness of Theorem 1.2.2 includes the *full completeness*: each hom-set in $\mathrm{Cl}(\ell^\circ_{\mathcal{B}})$ is isomorphic to the corresponding hom-set in the formal free strict (pointed) linear category over $\mathcal{B}$. Notably, this full completeness extends from normalised $\ell^\circ_{\mathcal{B}}$-nets to *arbitrary* ones:

Theorem 1.2.4. *Every $\ell^\circ_{\mathcal{B}}$-net can be freely generated from morphisms in $\mathcal{B}$ by the standard inductive constructions used in the definition of the formal free strict (pointed) linear category.*

We finally illustrate an application of our graphical approach to category theory by:

Corollary 1.2.5. *Two allowable natural transformations between functors into any free linear (respectively, symmetric monoidal closed) category coincide if and only if they satisfy the graphical criterion (2), from which the sufficient coherence criterion* [KM71, Theorem 2.4] *follows.*

1.3. **Our contributions and related work.** Our first contribution is to provide a graphical calculus for a free (pointed) linear category, whose reduction is strongly normalising and confluent. Its technical significance is the *freeness/initiality*, which precisely captures equality between morphisms in *any* (pointed) linear category not axiomatically but *purely* graph-theoretically, *viz.* by equality between labelled finite graphs, without quotients or any other means. In this sense, we not only present a graphical language but also *reduce algebra to geometry*, furnishing a geometric understanding of (pointed) linear categories and a graphical tool for them. A main innovation behind this result is the novel concept of *valid* paths/sequences (§2.2).

The effectivity of this tool is illustrated by our second contribution: a graphical characterisation of *coherence* on free linear categories. Our characterisation and its proof are simple, and the proof provides a geometric explanation for the coherence. In contrast, the previous method [Sol97] has to handle a number of formal details such as permutations of logical rules in derivations. Our coherence result advances the study of linear categories, and it is important for them to serve as a robust framework for the interplay between linear and nonlinear maps.

Mac Lane's celebrated coherence theorem for monoidal categories [Mac63] initiated the programme to reduce an infinite set of categorical equations to finitary combinatorial or graphical criteria. Kelly and Mac Lane [KM71] along with Eilenberg and Kelly [EK66] extended the programme to symmetric monoidal closed categories. They observed that, unlike Mac Lane's result, coherence on symmetric monoidal closed categories does not hold unconditionally, and instead established a sufficient criterion for the coherence through *Eilenberg–Kelly–Mac Lane graphs* [EK66, KM71]. This classic result was the beginning of the extensive literature on the coherence and related issues. Our graphical calculus can be seen as an extension of Eilenberg–Kelly–Mac Lane graphs, and our coherence theorem as part of Mac Lane's programme.

Voreadou [Vor77] tried to strengthen the sufficient criterion into a necessary-and-sufficient one, but one of her lemmata was false [Sol97]. Yet, based on her idea, Soloviev [Sol97] established a necessary-and-sufficient criterion in terms of Gentzen's *sequent calculus* [Gen35]. Such a proof-theoretic method has been used extensively for coherence problems in the literature, *e.g.* [BCST96, Sol91]. We instead employ a *graphical* method and provide a geometric proof.

Our graphical realisation of unit and weakening is based on the game-semantic presentation of a free symmetric monoidal closed category [MO03]. Specifically, we require, for the freeness theorem, that every $\ell^\circ_{\mathcal{B}}$-net is *saturated* under an equivalence relation based on [MO03, Definition 42], which is a game-semantic counterpart of *Trimble rewiring* [BCST96, Tri94]. A point is that $\ell^\circ_{\mathcal{B}}$-reduction *respects* this saturation and thus well-defined on $\ell^\circ_{\mathcal{B}}$-nets *themselves*.

Hughes [Hug12] extended Eilenberg–Kelly–Mac Lane graphs to a graphical presentation of a free star-autonomous category over a small category $\mathcal{B}$ [Bar06]. Through this presentation, he proved, as a main result, the decidability of equality of morphisms in the free star-autonomous

category when the decidability holds for $\mathcal{B}$. While three different graphical presentations of a free star-autonomous category were already given before, his method enjoys a particularly simple composition of morphisms, *viz.* the path composition, as in Eilenberg–Kelly–Mac Lane graphs and our calculus. Sharing almost the same lineage, the symmetric monoidal closed part of our graphical calculus and his graphical presentation are similar, except that only the latter quotients graphs under Trimble rewiring. A main difference between the two is that our calculus captures the complex interaction of exponential with symmetric monoidal and cartesian structures, and for this task we utilise some novel concepts, including the valid paths/sequences.

Essential nets [Lam08] constitute the first graphical calculus for the unit-free fragment of intuitionistic linear logic. Even for that fragment, essential nets and $\ell_{\mathcal{B}}$-nets are different. For instance, essential nets use not only labels for logical constants or connectives but also those for logical rules, *e.g.* weakening and contraction (so they are *overly syntactic* (§1.1)), while $\ell_{\mathcal{B}}$-nets do not employ the latter labels. We are not aware of a categorical study of essential nets.

Reader and Di Giorgio [RDG26] presented a graphical calculus for a free (in a 2-categorical sense) strict symmetric monoidal closed category. Their morphisms are string diagrams quotiented by formal equations. Another graphical calculus for monoidal categories by Wilson *et al.* [WGZ24] is given formally *via* generators and relations on string diagrams. We note that even these modern graphical calculi rely on formal quotients.

Last but not least, Fiore *et al.* [FGH26][7] recently introduced a *bicategorical* account of linear categories, which, unlike ordinary 1-categorical models, admits non-normalised morphisms (as 1-cells) and reduction (as 2-cells). We conjecture that our graphical calculus gives rise to this bicategorical structure, in particular a *free* one in a suitable sense, and leave it as future work.

1.4. **Article structure.** We first provide an informal explanation of the $\ell^{\circ}_{\mathcal{B}}$-calculus in §2. We next present its precise definition and show its basic properties in §3. We then prove that the $\ell^{\circ}_{\mathcal{B}}$-calculus forms a free strict (pointed) linear category over $\mathcal{B}$ in §4, and finally illustrate an application of this graphical tool by the graphical coherence theorem in §5.

2. Overview of main ingredients

In this section, we overview main ingredients of the present work. Their precise definitions begin in the next section, while the present section focuses on their main ideas.

Throughout this and the next three sections, we fix an arbitrary small category $\mathcal{B}$. We first explain components of the $\ell^{\circ}_{\mathcal{B}}$-calculus in §2.1, including *pre-$\ell^{\circ}_{\mathcal{B}}$-nets* that act as 'raw' morphisms. We then sketch how to carve out, in a non-inductive, quotient-free fashion, the pre-$\ell^{\circ}_{\mathcal{B}}$-nets that serve as morphisms in a free (pointed) linear category $\tilde{\mathrm{Cl}}(\ell^{\circ}_{\mathcal{B}})$ over $\mathcal{B}$, called *$\ell^{\circ}_{\mathcal{B}}$-nets*, in §2.2. In this task, *valid* paths/sequences in a pre-$\ell^{\circ}_{\mathcal{B}}$-net play central roles by determining if a pre-$\ell^{\circ}_{\mathcal{B}}$-net represents a morphism, *i.e.* if it is *correct*, and if two such pre-$\ell^{\circ}_{\mathcal{B}}$-nets represent the same morphism, *i.e.* if they have the same *prime form* up to an extension $\approx$ of Trimble rewiring. The extension is nontrivial due to exponential !; see Examples 3.2.38–3.2.39 in a later section. To be accurate, an $\ell^{\circ}_{\mathcal{B}}$-net is a pre-$\ell^{\circ}_{\mathcal{B}}$-net that is the saturation of correct pre-$\ell^{\circ}_{\mathcal{B}}$-nets in prime form with respect to $\approx$. In this way, the novel concepts of valid paths/sequences solve subtle issues and enable $\ell^{\circ}_{\mathcal{B}}$-nets to constitute the free (pointed) linear category. Table 1 lists the ingredients.

Notation. We employ the following notations:

- Let $\mathbb{N}$ be the set of all natural numbers, and $\mathbb{N}_+ := \mathbb{N} \setminus \{0\}$;
- Arbitrarily fix a countably infinite set $\mathcal{V}$ of *variables* (which are any elements), together with a distinguished variable $v_0 \in \mathcal{V}$;
- Let $\wp_{\mathrm{f}}(X)$ be the set of all finite subsets of a set X;

[7] This work has been accepted for publication in *Journal of the European Mathematical Society*.

Concept	Pre-$\ell^\circ_{\mathcal{B}}$-nets	Valid paths/sequences	Correctness	Prime form
Intuition	Raw graphs	Legal traces/traversals	Criterion for free generation	Canonical form
Role	Raw morphisms	Essence of pre-$\ell^\circ_{\mathcal{B}}$-nets	Non-inductive filter	Representative

TABLE 1. Main ingredients of the present work

- Let $\mathrm{Col} := \wp_{\mathrm{f}}(\mathcal{V} \times \mathbb{N}_+ \times \mathcal{V})$, and its elements are called *colours*;
- Let $\overline{0} := \emptyset$ and $\overline{n} := \{1, 2, \dots, n\}$ $(n \in \mathbb{N}_+)$;
- Let $\mathrm{Obj}(\mathcal{B})$ and $\mathrm{Mor}(\mathcal{B})$ respectively be the sets of all objects and morphisms in $\mathcal{B}$, where, by suitably choosing symbols $\top$, 1 and $\bot$, assume that $\mathrm{Obj}(\mathcal{B})$ is disjoint from $\{\top, 1, \bot\}$.

2.1. **Ingredients.** Objects or $\ell^\circ_{\mathcal{B}}$*-trees* in the free (pointed) linear category $\tilde{\mathrm{Cl}}(\ell^\circ_{\mathcal{B}})$ are parse trees freely generated from objects in $\mathcal{B}$,[8] and morphisms or $\ell^\circ_{\mathcal{B}}$*-nets* are finite graphs labelled with morphisms in $\mathcal{B}$, variables, colours and integers. Composition is made of gluing, saturation and $\ell^\circ_{\mathcal{B}}$-reduction; the first two have been illustrated in Figure 2, and the last in Figure 3–5.

More concretely, an $\ell^\circ_{\mathcal{B}}$-tree is the parse tree of an object in the formal free (pointed) linear category over $\mathcal{B}$. To be precise, an $\ell^\bullet_{\mathcal{B}}$*-tree* is the parse tree, written $\mathfrak{G}T$, of an expression T, called an $\ell^\bullet_{\mathcal{B}}$*-type*, defined by the grammar (in *Backus–Naur form* [Bac59])

$$T := X_{[v]} \mid \top_{[v]} \mid 1_{[v]} \mid \bot_{[v]} \mid T \otimes T' \mid T \times T' \mid T \multimap T' \mid !_{[v]}T \quad (X \in \mathrm{Obj}(\mathcal{B}), v \in \mathcal{V}),$$

where the construct $\spadesuit_{[v]}$ $(\spadesuit \in \{X, \top, 1, \bot, !\})$ is interpreted as a vertex v, called an *end*, labelled with $\spadesuit$, and the one $\clubsuit \in \{\otimes, \times, \multimap\}$ as an anonymous vertex, not an end, labelled with $\clubsuit$. An $\ell_{\mathcal{B}}$*-type* is an $\ell^\bullet_{\mathcal{B}}$-type free from the symbol $\bot$, and an $\ell_{\mathcal{B}}$*-tree* is the parse tree of an $\ell_{\mathcal{B}}$-type. In general, a vertex (or end) labelled with a symbol $\diamondsuit$ is called a $\diamondsuit$*-vertex* (or $\diamondsuit$*-end*).

To define pre-$\ell^\circ_{\mathcal{B}}$-nets, we first introduce what enables us to glue them together, or $\ell^\circ_{\mathcal{B}}$*-ports*. An $\ell^\bullet_{\mathcal{B}}$*-port* is the parse tree $\mathfrak{G}M$ of an expression M, called an $\ell^\bullet_{\mathcal{B}}$*-cut*, defined by

$$M := \top_{[v]} \mid T \triangleright \hat{T} \mid M \otimes M' \mid M \times M' \mid !_{[v]}M \quad (v \in \mathcal{V}),$$

where $\triangleright$ is an arbitrarily fixed symbol to label vertices that glue pre-$\ell^\circ_{\mathcal{B}}$-nets together, and the $\ell^\bullet_{\mathcal{B}}$-types T and $\hat{T}$ in the construct $T \triangleright \hat{T}$ are required to coincide up to renamings of ends. An $\ell_{\mathcal{B}}$*-cut* is an $\ell^\bullet_{\mathcal{B}}$*-cut* free from $\bot$, and an $\ell_{\mathcal{B}}$*-port* is the parse tree of an $\ell_{\mathcal{B}}$-cut.

We then combine $\ell^\circ_{\mathcal{B}}$-trees and $\ell^\circ_{\mathcal{B}}$-ports into an ambient structure for a pre-$\ell^\circ_{\mathcal{B}}$-net, called an $\ell^\circ_{\mathcal{B}}$*-diagram*, in which an edge is represented by a thick arrow $\Rightarrow$. As a compact notation for an $\ell^\bullet_{\mathcal{B}}$-diagram, we define an $\ell^\bullet_{\mathcal{B}}$*-sequent* to be an expression of the form

$$T_1, T_2, \dots, T_n \dashv M_1, M_2, \dots, M_m \vdash T, \tag{4}$$

where $\dashv$ and $\vdash$ are arbitrarily fixed symbols, and it is called an $\ell_{\mathcal{B}}$*-sequent* if it is free from $\bot$. It denotes the rooted tree in Figure 6, obtained from the $\ell^\circ_{\mathcal{B}}$-trees $\mathfrak{G}T_1, \mathfrak{G}T_2, \dots, \mathfrak{G}T_n, \mathfrak{G}T$ and the $\ell^\circ_{\mathcal{B}}$-ports $\mathfrak{G}M_1, \mathfrak{G}M_2, \dots, \mathfrak{G}M_m$ by the addition of the two anonymous vertices labelled with $\dashv$ and $\vdash$, respectively, and the thick edges, each of which points to the root of a subtree. Then, an $\ell^\bullet_{\mathcal{B}}$*-diagram* is such a rooted tree represented by an $\ell^\bullet_{\mathcal{B}}$-sequent, and it is called an $\ell_{\mathcal{B}}$*-diagram* if it is represented by an $\ell_{\mathcal{B}}$-sequent. For example, the $\ell_{\mathcal{B}}$-diagrams in Figure 7 are the precise versions of those in Figure 2, represented respectively by the $\ell_{\mathcal{B}}$-sequents

$$\top_{[v]} \otimes X_{[x]} \dashv\vdash X_{[y]}, \qquad X_{[z]} \dashv\vdash \top_{[u]} \otimes X_{[w]}, \qquad \top_{[v]} \otimes X_{[x]} \dashv X_{[y]} \triangleright X_{[z]} \vdash \top_{[u]} \otimes X_{[w]}.$$

[8] Strictly speaking, the objects are slightly different parse trees (given in Definition 4.1.12) for adopting the *strict* equality between (labelled) graphs, not graph isomorphisms, but the difference does not matter for now.

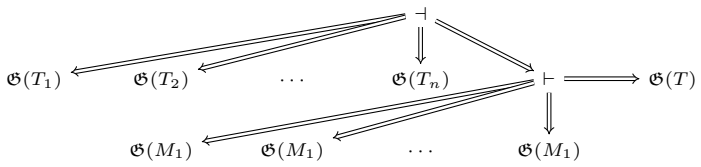


FIGURE 6. An $\ell^{\circ}_{\mathcal{B}}$-diagram

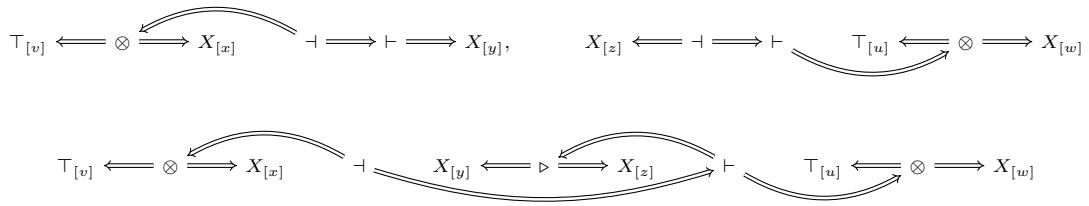


FIGURE 7. Gluing of $\ell^{\circ}_{\mathcal{B}}$-diagrams

In §3, $\ell^{\circ}_{\mathcal{B}}$-trees, $\ell^{\circ}_{\mathcal{B}}$-ports and $\ell^{\circ}_{\mathcal{B}}$-diagrams are defined non-inductively without recourse to $\ell^{\circ}_{\mathcal{B}}$-types, $\ell^{\circ}_{\mathcal{B}}$-cuts or $\ell^{\circ}_{\mathcal{B}}$-sequents; the latter three act as compact formal notations for the former three. For adopting the *strict* equality between these (labelled) graphs, not graph isomorphisms, we arbitrarily assign *canonical* vertices to each $\ell^{\circ}_{\mathcal{B}}$-tree, $\ell^{\circ}_{\mathcal{B}}$-port or $\ell^{\circ}_{\mathcal{B}}$-diagram. When we glue $\ell^{\circ}_{\mathcal{B}}$-trees, $\ell^{\circ}_{\mathcal{B}}$-ports or $\ell^{\circ}_{\mathcal{B}}$-diagrams together, we suitably rename vertices so that they are pairwise distinct, and after the gluing we rename the vertices into the canonical ones.

We now define a *pre-$\ell^{\bullet}_{\mathcal{B}}$-net* (respectively, a *pre-$\ell_{\mathcal{B}}$-net*) to be the union, written $\mathfrak{r} :: \mathfrak{S}$, of an $\ell^{\bullet}_{\mathcal{B}}$-diagram (respectively, an $\ell_{\mathcal{B}}$-diagram) $\mathfrak{S}$ and a labeled finite acyclic directed graph $\mathfrak{r}$, called a *$\mathcal{B}$-realiser*, whose vertices are exactly ends of $\mathfrak{S}$, and edges or *$\mathcal{B}$-arcs* are labelled as either

$$u \xrightarrow[\langle i\rangle]{\langle\kappa|\epsilon\rangle} v, \qquad u \xrightarrow[\langle i\rangle]{\chi\langle\kappa|\epsilon\rangle} v \qquad (u, v \in \mathcal{V}, \kappa \in \wp_{\mathrm{f}}(\mathcal{V}), \epsilon \in \wp_{\mathrm{f}}(\mathrm{Col} \times \mathcal{V}), i \in \mathbb{N}_+, \chi \in \mathrm{Mor}(\mathcal{B})),$$

where edges of $\mathfrak{S}$ are often omitted when we draw $\mathfrak{r} :: \mathfrak{S}$ since they are evident. Let $u \xrightarrow[\langle i\rangle]{(\chi)\langle\kappa|\epsilon\rangle} v$ range over the two forms of a $\mathcal{B}$-arc. The three labels $\langle\kappa|$, $|\epsilon\rangle$ and (i) are respectively omitted when $\kappa = \emptyset$, $\epsilon = \emptyset$ and $i = 1$. We write $\langle x_1, x_2, \ldots, x_n|$ for $\langle\kappa|$ if $\kappa = \{x_i\}_{i=1}^n$, and similarly for ϵ.

The label χ, said to be *arrowy*, is to embed $\mathcal{B}$ into $\tilde{\mathrm{Cl}}(\ell^{\circ}_{\mathcal{B}})$. For example, given a morphism $\chi : X \to Y$ in $\mathcal{B}$, the last step in the calculation of the composition

$$\top \otimes X \xrightarrow{\sim} X \xrightarrow{\chi} Y$$

in $\tilde{\mathrm{Cl}}(\ell^{\circ}_{\mathcal{B}})$ is as in Figure 8, where the direction of χ is *reversed* (to fit valid paths/sequences introduced below), and the $\mathcal{B}$-arc between vertices of the subtree of $\mathfrak{S}$ rooted at $\triangleright$ is dotted. Recall that a $\mathcal{B}$-arc yielded by yanking is labelled with the composition $\zeta \circ \chi$ in $\mathcal{B}$ if and only if the component undotted $\mathcal{B}$-arcs are labelled with morphisms χ and ζ in $\mathcal{B}$, respectively.

The label $\langle\kappa|$, said to be *cartesian*, is to distinguish between morphisms containing the pairing $C \to A \times B$ of morphisms $C \to A$ and $C \to B$ in $\tilde{\mathrm{Cl}}(\ell^{\circ}_{\mathcal{B}})$. For this, roughly, κ records the sides (*i.e.* left or right) of certain $\times$-vertices to which the $\mathcal{B}$-arc labelled with $\langle\kappa|$ belongs. For instance, the pre-$\ell^{\circ}_{\mathcal{B}}$-nets in Figure 9 (which are also $\ell^{\circ}_{\mathcal{B}}$-nets) would coincide without cartesian labels, where instead of thick edges we informally add parentheses to the $\ell^{\circ}_{\mathcal{B}}$-diagrams for readability.

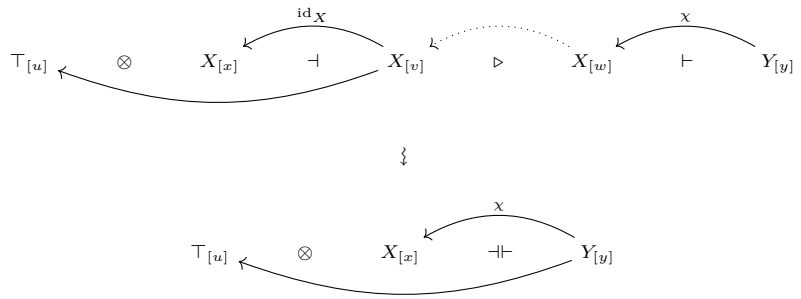


FIGURE 8. $\ell^{\circ}_{\mathcal{B}}$-reduction for graphical composition

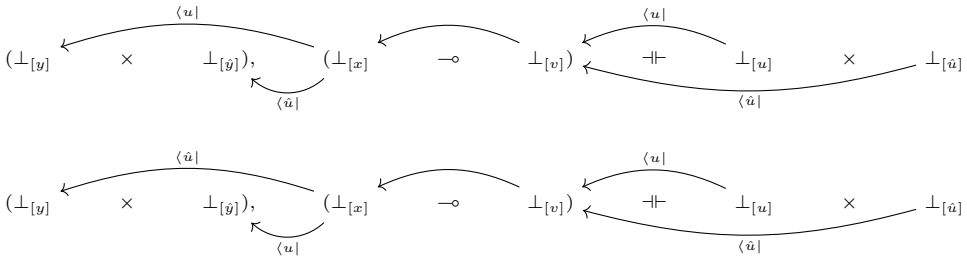


FIGURE 9. Pre-$\ell^{\circ}_{\mathcal{B}}$-nets distinguished by cartesian labels

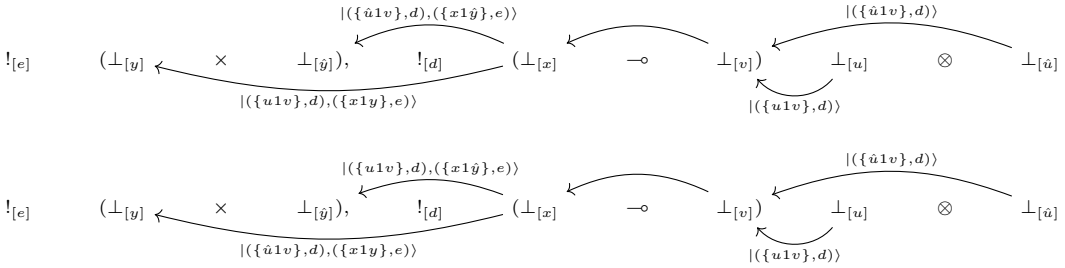


FIGURE 10. Pre-$\ell^{\circ}_{\mathcal{B}}$-nets distinguished by exponential labels

The label $|\epsilon\rangle$, said to be *exponential*, is to distinguish between morphisms including the contraction $!D \to C$ of a morphism $!D \otimes !D \to C$ in $\tilde{\mathrm{C}}\mathrm{l}(\ell^{\circ}_{\mathcal{B}})$, where a pair $(c, e) \in \epsilon$ signifies a colour c on an !-vertex e (*e.g.* e corresponds to the outermost ! in $!D$ in the present case) assigned to the $\mathcal{B}$-arc labelled with $|\epsilon\rangle$. The contraction $!D \to C$ yields a *partition* of $\mathcal{B}$-arcs on $!D$, due to the identification of the two copies of $!D$ in $!D \otimes !D \to C$, and colours encode such a partition: if a $\mathcal{B}$-arc a is labelled with $|\epsilon\rangle$ such that $(c, e) \in \epsilon$ and $uiv \in c$, then every $\mathcal{B}$-arc of the form

$$u \xrightarrow[\langle i\rangle]{(\chi')\langle\kappa'|\epsilon'\rangle} v$$

is classified into the same group with a in the partition with respect to e. For example, the pre-$\ell^{\circ}_{\mathcal{B}}$-nets in Figure 10 (which are also $\ell^{\circ}_{\mathcal{B}}$-nets) would coincide without exponential labels.

The last label (i), said to be *identifier*, is to distinguish between different copies of the same $\mathcal{B}$-arc (*n.b.* the same without the identifier label). For instance, the $\mathcal{B}$-arcs

$$u \xrightarrow[\langle i \rangle]{|(\{uiv\},e)\rangle} v,$$

where i ranges over the finite set $\overline{n}$ for some fixed $n \in \mathbb{N}_+$, of the pre-$\ell^\circ_{\mathcal{B}}$-net in Example 3.2.3 (which is also an $\ell^\circ_{\mathcal{B}}$-net) would be all identified without identifier labels.

The distinctions between $\mathcal{B}$-arcs made by labels, as illustrated above, are necessary for our graphical calculus to form a free (pointed) linear category over $\mathcal{B}$.

2.2. **Correctness and prime form.** There remain two technical problems:

(1) At this point, the free generation of pre-$\ell^\circ_{\mathcal{B}}$-nets over $\mathcal{B}$ (*i.e.* which ones represent morphisms in a free (pointed) linear category over $\mathcal{B}$) is defined only inductively;
(2) The equality between pre-$\ell^\circ_{\mathcal{B}}$-nets (even modulo $\ell^\circ_{\mathcal{B}}$-reduction) does not match the equality between morphisms in the free (pointed) linear category, as illustrated below.

We solve these problems by focusing on a class of pre-$\ell^\circ_{\mathcal{B}}$-nets, or $\ell^\circ_{\mathcal{B}}$-*nets*, defined geometrically and non-inductively in terms of *valid* paths/sequences in pre-$\ell^\circ_{\mathcal{B}}$-nets. For brevity, we sketch a simplified account of valid paths/sequences below, leaving their full details to later sections.

First, a *valid* path in a pre-$\ell^\circ_{\mathcal{B}}$-net $\mathfrak{r} :: \mathfrak{S}$ is a finite path

$$v_0 \xrightarrow{(\chi_1)\langle\kappa_1|\epsilon_1\rangle} v_1 \xrightarrow{(\chi_2)\langle\kappa_2|\epsilon_2\rangle} v_2 \cdots v_{n-1} \xrightarrow{(\chi_n)\langle\kappa_n|\epsilon_n\rangle} v_n$$

in $\mathfrak{r}$ satisfying certain axioms, to some of which labels of $\mathcal{B}$-arcs are relevant. The finite sequence

$$v_0 v_1 \dots v_n,$$

labelled (or paired) with the composition $\chi_1 \circ \chi_2 \circ \cdots \circ \chi_n$ in $\mathcal{B}$ if and only if all the arrowy labels χ_i exist, is said to be *valid*. Instead of giving the general definition of valid paths or sequences, we illustrate them with an example: maximal valid sequences in the first pre-$\ell^\bullet_{\mathcal{B}}$-net in Figure 9 are $uvxy$ and $\hat{u}vx\hat{y}$, both unlabelled, whose underlying valid paths are

$$u \xrightarrow{\langle u|} v \to x \xrightarrow{\langle u|} y, \qquad\qquad \hat{u} \xrightarrow{\langle \hat{u}|} v \to x \xrightarrow{\langle \hat{u}|} \hat{y},$$

respectively; similarly, those of the second are $uvx\hat{y}$ and $\hat{u}vxy$, both unlabelled.

Let us then define *correctness* of a pre-$\ell^\circ_{\mathcal{B}}$-net *via* valid sequences, which characterises if it is freely generated (up to *prime form* introduced below). To this end, we employ two operations on pre-$\ell^\circ_{\mathcal{B}}$-nets: *slicing* and *shredding*. We will see that the characterisation fails without slicing or shredding. We divide a pre-$\ell^\circ_{\mathcal{B}}$-net into its *slices*, which corresponds to the *reverse* of pairing. This idea is due to Girard [Gir87, §IV.6]. We also *reverse* contraction by copying subgraphs rooted at !-ends, similarly to $\ell^\circ_{\mathcal{B}}$-reduction in Figure 5, turning a pre-$\ell^\circ_{\mathcal{B}}$-net into its *shreddings*.

Leaving the general definitions of slicing and shredding to later sections, we illustrate them with examples here. For instance, the first pre-$\ell^\circ_{\mathcal{B}}$-net in Figure 9 has the two slices



while the second has those





FIGURE 11. (In)correctness of pre-$\ell^{\circ}_{\mathcal{B}}$-nets

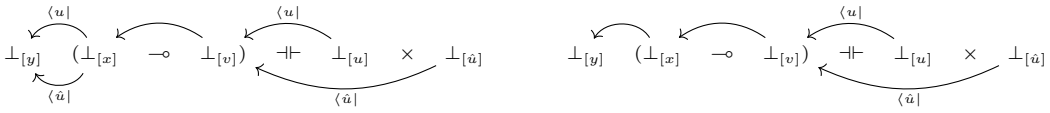
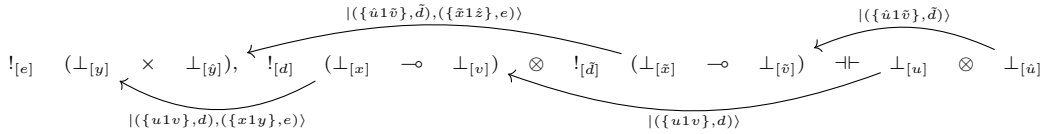
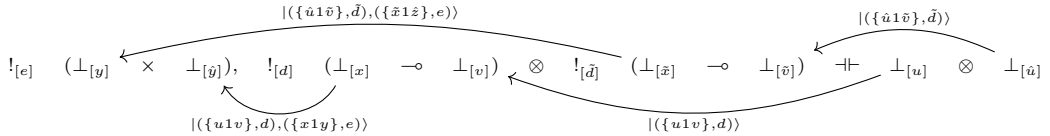


FIGURE 12. Standardisation of cartesian labels

As another example, the first pre-$\ell^{\circ}_{\mathcal{B}}$-net in Figure 10 has the shredding



where $!_{[\tilde{d}]}(\bot_{[\tilde{x}]} \multimap \bot_{[\tilde{v}]})$ is an arbitrary copy of $!_{[d]}(\bot_{[x]} \multimap \bot_{[v]})$, while the second has the one



A pre-$\ell^{\circ}_{\mathcal{B}}$-net is said to be *sliced* if all its slices coincide with itself, and *shredded* if all its shreddings with itself. Slicing and shredding are idempotent: slices are sliced, and shreddings are shredded.

Then, a pre-$\ell^{\circ}_{\mathcal{B}}$-net is said to be *correct* if any shredding of any slice of it satisfies certain axioms defined in terms of their valid sequences, including the axiom that requires, with certain exceptions, every end of the $\ell^{\circ}_{\mathcal{B}}$-diagram to occur in some valid sequence in the pre-$\ell^{\circ}_{\mathcal{B}}$-net. For instance, consider the pre-$\ell^{\circ}_{\mathcal{B}}$-nets in Figure 11, which are both sliced and expanded. The left one is correct and has the maximal valid sequence $\hat{x}x\hat{y}y$ that contains all the ends. In contrast, the right one is incorrect as it has only the maximal valid sequence $\hat{x}x$, which does not contain the end $\hat{y}$ or y. We will see that the left one is freely generated, while the right one is not.

Next, to fix the equality between correct pre-$\ell^{\circ}_{\mathcal{B}}$-nets, we focus on those in *prime form*. The idea is to see valid sequences in a pre-$\ell^{\circ}_{\mathcal{B}}$-net as its *essence*, identify correct pre-$\ell^{\circ}_{\mathcal{B}}$-nets if they have the same set of valid sequences, and among them focus on a *prime* representative, which is necessarily unique. Roughly, a correct pre-$\ell^{\circ}_{\mathcal{B}}$-net is said to be in *prime form* if

(1) Its vertices and identifier labels are canonically chosen ones;
(2) Its cartesian labels contain exactly the variables *relevant* to its valid sequences;
(3) Its exponential labels partition *relevant* $\mathcal{B}$-arcs;
(4) No *smaller* (with respect to the sum of the number $\mathcal{B}$-arcs and the cardinalities of sets in cartesian and exponential labels) correct pre-$\ell^{\circ}_{\mathcal{B}}$-net has the same set of valid sequences.

The first axiom is merely to identify correct pre-$\ell^{\circ}_{\mathcal{B}}$-nets in prime form *on the nose*, not up to (labelled) graph isomorphisms, since our aim is to establish a *quotient-free* graphical calculus.

The second axiom is to standardise cartesian labels. For instance, consider the pre-$\ell^{\circ}_{\mathcal{B}}$-nets in Figure 12, which are both correct. They must be equal since otherwise they would not form a (pointed) linear category; however, they are not. Meanwhile, the two share the same set of valid



FIGURE 13. Non-redundancy in prime form

sequences, whose maximal elements are $uvxy$ and $\hat{u}vxy$. As the ends u and $\hat{u}$ are *relevant* to the respective valid sequences in a sense made precise later, only the left one satisfies the axiom.

The last axiom is illustrated by the pre-$\ell^{\circ}_{\mathcal{B}}$-nets in Figure 13, which are both correct, where the 1-ends make an exception that no ends have to be in a valid sequence. Both of the pre-$\ell^{\circ}_{\mathcal{B}}$-nets have only the empty valid sequence. Since an 1-end represents a *terminal* object, the pre-$\ell^{\circ}_{\mathcal{B}}$-nets must be equal, but they are not. Then, only the right one satisfies the second axiom (where the remaining $\mathcal{B}$-arc $z \to x$ cannot be removed for a reason clarified later).

Finally, an *$\ell^{\circ}_{\mathcal{B}}$-net* is the saturation of a correct pre-$\ell^{\circ}_{\mathcal{B}}$-net in prime form with respect to an extension $\approx$ of Trimble rewiring. The necessity of this *$\approx$-saturation*, along with composition, has been illustrated in Figures 2–3. As we will see, $\ell^{\circ}_{\mathcal{B}}$-nets *fully* and *faithfully* represent morphisms in the formal free (pointed) linear category over $\mathcal{B}$, solving the two problems posed above.

3. A graphical calculus for free linear categories

In this section, we present the precise definition of our graphical calculus built from $\mathcal{B}$. We define its objects and raw morphisms in §3.1, morphisms in §3.2, and reduction in §3.3.

Notation. We henceforth employ the following notations:

- Given a finite set S, let $|S| \in \mathbb{N}$ denote the cardinality of S;
- Given a (binary) relation R, let R^* denote its reflexive, transitive closure, *i.e.*

$$R^n := \{\, (x_0, x_n) \mid \forall i \in \overline{n}.\, (x_{i-1}, x_i) \in R \,\} \quad (n \in \mathbb{N}), \qquad R^* := \bigcup_{n \in \mathbb{N}} R^n;$$

- Bold letters, $\boldsymbol{s}, \boldsymbol{t}, \boldsymbol{u}$, *etc.*, represent finite sequences, except that the Greek letter ε the empty sequence, and ordinary letters, a, b, x, *etc.*, elements of finite sequences;
- We also write $x_1 x_2 \dots x_{|\boldsymbol{s}|}$ for a finite sequence $\boldsymbol{s} = (x_1, x_2, \dots, x_{|\boldsymbol{s}|}) = (x_i)_{i=1}^{|\boldsymbol{s}|}$, where $|\boldsymbol{s}|$ is the length of $\boldsymbol{s}$,[9] and define $\boldsymbol{s}(i) := x_i$ and $\underline{\boldsymbol{s}} := \{\, \boldsymbol{s}(i) \mid i \in \overline{|\boldsymbol{s}|} \,\}$;
- For a set S of finite sequences, let $S^{\mathrm{Odd}} \subseteq S$ (respectively, $S^{\mathrm{Even}} \subseteq S$) be the set of all odd-length (respectively, even-length) elements of S, respectively;
- The concatenation of finite sequences $\boldsymbol{s}$ and $\boldsymbol{t}$ is represented by their juxtaposition $\boldsymbol{st}$, also written $\boldsymbol{s}.\boldsymbol{t}$, and a concatenation of the form $\boldsymbol{u}(c)\boldsymbol{v}$ is abbreviated as $\boldsymbol{ucv}$.

3.1. Graphical objects and raw morphisms.

We begin with objects and raw morphisms of our graphical calculus. Our graphs are *directed*; some of them have *labels*. Vertices with a label ♠ are called *♠-vertices*. For a graph $\mathfrak{G}$, let $V_{\mathfrak{G}}$ and $E_{\mathfrak{G}}$ respectively denote the sets of all vertices and edges of $\mathfrak{G}$. For an edge a, let $\mathrm{src}(a)$ and $\mathrm{tgt}(a)$ respectively denote the source and the target of a. For a rooted tree $\mathfrak{T}$ and its vertex v, let $\mathfrak{T}^v$ be the subtree of $\mathfrak{T}$ rooted at v.

We first define the graphical objects:

Definition 3.1.1 ($\ell^{\circ}_{\mathcal{B}}$-trees)**.** An *$\ell^{\bullet}_{\mathcal{B}}$-tree* is a finite rooted tree, in which vertices are pairwise distinct variables, such that

(1) Each of its leaves is labelled with an element of the set $\mathrm{Obj}(\mathcal{B}) \cup \{\top, 1, \bot\}$;
(2) Each of its non-leaf vertices satisfies either of the following:

[9] This notation $|_|$ is the same as the one for the cardinality of a finite set defined above, but in practice this overloading of notation does not bring confusion.

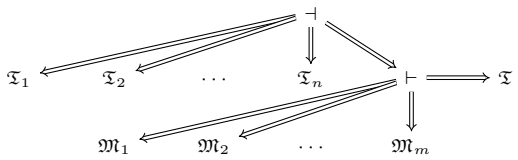


FIGURE 14. A graphical representation of an $\ell^{\circ}_{\mathcal{B}}$-diagram

(a) It is labelled with the symbol $\otimes$, $\times$ or $\multimap$ and has two children, distinguished as the *left* and the *right* ones, respectively;
(b) It is labelled with the symbol ! and has one child,

and an *$\ell_{\mathcal{B}}$-tree* is an $\ell^{\bullet}_{\mathcal{B}}$-tree in which no vertex has the label $\bot$.

As already illustrated in previous sections, we also need a graphical construct that *glues* two graphical (raw) morphisms together:

Definition 3.1.2 ($\ell^{\circ}_{\mathcal{B}}$-ports)**.** An *$\ell^{\bullet}_{\mathcal{B}}$-port* is a finite rooted tree, whose vertices are pairwise distinct variables, such that each of its vertices satisfies one of the following:

(1) It is labelled with the symbol $\top$ and has no children;
(2) It is labelled with the symbol $\otimes$ or $\times$ and has two children, distinguished as the *left* and the *right* ones, respectively;
(3) It is labelled with the symbol ! and has one child;
(4) It is labelled with the symbol $\triangleright$ and has two children, distinguished as the *left* and the *right* ones, respectively, such that the two subtrees rooted at the respective children are isomorphic $\ell^{\bullet}_{\mathcal{B}}$-trees, where the graph isomorphism preserves labels on vertices, and a pair of isomorphic vertices are called *mates* to each other,

and an *$\ell_{\mathcal{B}}$-port* is an $\ell^{\bullet}_{\mathcal{B}}$-port in which no vertex has the label $\bot$.

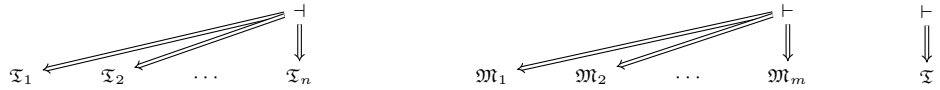
We then combine these two classes of vertex-labelled finite rooted trees into one that serves as an ambient structure for the graphical raw morphisms:

Definition 3.1.3 ($\ell^{\circ}_{\mathcal{B}}$-diagrams)**.** An *$\ell^{\bullet}_{\mathcal{B}}$-diagram* is a finite rooted tree, written

$$\mathfrak{T}_1, \mathfrak{T}_2, \ldots, \mathfrak{T}_n \dashv \mathfrak{M}_1, \mathfrak{M}_2, \ldots, \mathfrak{M}_m \vdash \mathfrak{T}, \tag{5}$$

obtained from $\ell^{\bullet}_{\mathcal{B}}$-trees $\mathfrak{T}_1, \mathfrak{T}_2, \ldots, \mathfrak{T}_n, \mathfrak{T}$ and $\ell^{\bullet}_{\mathcal{B}}$-ports $\mathfrak{M}_1, \mathfrak{M}_2, \ldots, \mathfrak{M}_m$ ($n, m \in \mathbb{N}$) by the addition of two vertices labelled with arbitrarily fixed symbols $\dashv$ and $\vdash$, respectively, an edge from the $\dashv$-vertex to the root of each $\mathfrak{T}_i$, an edge from the $\dashv$-vertex to the $\vdash$-vertex, an edge from the $\vdash$-vertex to the root of each $\mathfrak{M}_j$ and an edge from the $\vdash$-vertex to the root of $\mathfrak{T}$, together with the orders $\mathfrak{T}_1 < \mathfrak{T}_2 < \cdots < \mathfrak{T}_n$ and $\mathfrak{M}_1 < \mathfrak{M}_2 < \cdots < \mathfrak{M}_m$, and an *$\ell_{\mathcal{B}}$-diagram* is an $\ell^{\bullet}_{\mathcal{B}}$-diagram in which no vertex has the label $\bot$.

The $\ell^{\circ}_{\mathcal{B}}$-diagram (5) is drawn as in Figure 14, where an edge is represented by the thick arrow $\Rightarrow$ to be distinguished from other classes of edges given later, and the left and the right children of a vertex, if any, is placed on the left- and the right-hand sides, respectively. The subtrees



are respectively called the *domain*, the *centre* and the *codomain* of the $\ell^{\circ}_{\mathcal{B}}$-diagram (5), and their immediate rooted subtrees–$\mathfrak{T}_i$, $\mathfrak{M}_j$ and $\mathfrak{T}$ ($i \in \overline{n}$ and $j \in \overline{m}$)–their respective *factors*.

Let $\mathfrak{T}$ be a rooted subtree of an $\ell^{\circ}_{\mathcal{B}}$-diagram $\mathfrak{S}$. A vertex of $\mathfrak{T}$ is called an *end* if it is a leaf or labelled with !. Let $\mathrm{End}_{\mathfrak{T}}$ be the set of all ends of $\mathfrak{T}$. An end with a label $\diamondsuit$ is called a $\diamondsuit$-*end*. A vertex of $\mathfrak{T}$ is said to be *turning* if it is labelled with $\multimap$, $\triangleright$, $\dashv$ or $\vdash$, and *joker* if with $\top$ or !.

We shall define *valid* paths and sequences in labelled finite graphs, for which the following auxiliary concepts are used. An end $v \in \mathrm{End}_{\mathfrak{T}}$ is said to be *negative* in $\mathfrak{T}$ if $\#_{\mathfrak{T}}(v) < 0$, where

$$\#_{\mathfrak{T}}(v) := (-1)^{n+1} \tag{6}$$

is called the *parity* of v in $\mathfrak{T}$, and n the number of turning vertices in the path from the root of $\mathfrak{T}$ to v in $\mathfrak{T}$ whose left children occur in the path, *positive* in $\mathfrak{T}$ if $\#_{\mathfrak{T}}(v) > 0$, and *initial* in $\mathfrak{T}$ if $n = 0$. Note that every initial vertex is negative. In addition, $\mathfrak{T}$ is said to be *negative*, *positive* and *initial* in $\mathfrak{S}$, respectively, if so is the root of $\mathfrak{T}$ in $\mathfrak{S}$. Let

$$\mathrm{End}^{-}_{\mathfrak{T}}, \mathrm{End}^{+}_{\mathfrak{T}}, \mathrm{End}^{\mathrm{Ini}}_{\mathfrak{T}}, \mathrm{End}^{\mathrm{Jkr}}_{\mathfrak{T}}, \mathrm{End}^{!}_{\mathfrak{T}} \subseteq \mathrm{End}_{\mathfrak{T}}$$

denote the set of all negative, positive, initial, joker and !-ends of $\mathfrak{T}$, respectively. Another end $u \in V_{\mathfrak{T}}$ is said to *justify* v in $\mathfrak{T}$ if v and u are initial in the subtrees of $\mathfrak{T}$ rooted respectively at the left and the right children of some turning vertex of $\mathfrak{T}$.

Example 3.1.4. In the last $\ell^{\circ}_{\mathcal{B}}$-diagram in Figure 7, u and w are initial, v, x and z are positive, and y, u and w are negative. The positive ends are justified by u and w, and y by z.

For convenience, we next introduce a formal notation for $\ell^{\circ}_{\mathcal{B}}$-trees, called $\ell^{\circ}_{\mathcal{B}}$-*types*, which are divided into *negative* and *positive* ones in a way that they match the division of $\ell^{\circ}_{\mathcal{B}}$-trees:

Definition 3.1.5 ($\ell^{\circ}_{\mathcal{B}}$-types)**.** A *negative* and a *positive* $\ell^{\bullet}_{\mathcal{B}}$-*types* are the expressions N and P, respectively, defined by the mutual recursion

$$N := X_{[v]} \mid \top_{[v]} \mid 1_{[v]} \mid \bot_{[v]} \mid N \otimes N' \mid N_v{\times}_{v'}N' \mid P \multimap N \mid !_{[v]}N,$$
$$P := X_{[v]} \mid \top_{[v]} \mid 1_{[v]} \mid \bot_{[v]} \mid P \otimes P' \mid P \times P' \mid N \multimap P \mid !_{[v]}P \quad (X \in \mathrm{Obj}(\mathcal{B}), v, v' \in \mathcal{V}),$$

where $v = u$ if N in $N_v{\times}_{v'}N'$ is $X_{[u]}$, $\top_{[u]}$, $1_{[u]}$, $\bot_{[u]}$ or $!_{[u]}N$, $v' = u'$ if N' in $N_v{\times}_{v'}N'$ is $X_{[u']}$, $\top_{[u']}$, $1_{[u']}$, $\bot_{[u']}$ or $!_{[u']}N$, variables in N or P are pairwise distinct, and bracketed variables are called *ends*. An $\ell^{\bullet}_{\mathcal{B}}$-*type* is a negative or positive $\ell^{\bullet}_{\mathcal{B}}$-type, and said to be *atomic* if it is of the form $X_{[v]}$, $\top_{[v]}$, $1_{[v]}$ or $\bot_{[v]}$. An $\ell_{\mathcal{B}}$-*type* is an $\ell^{\bullet}_{\mathcal{B}}$-type in which $\bot$ does not occcur. An $\ell^{\bullet}_{\mathcal{B}}$- and an $\ell_{\mathcal{B}}$-*subtypes* of an $\ell^{\circ}_{\mathcal{B}}$-type T are respectively an $\ell^{\bullet}_{\mathcal{B}}$- and an $\ell_{\mathcal{B}}$-types that occur in T.

We divide $\ell^{\circ}_{\mathcal{B}}$-types into negative and positive ones for restricting the attachment of variables to the symbol $\times$; we can attach variables to all occurrences of $\times$, but it yields redundancy. We omit this attachment of variables to $\times$ in §2.1 since at that point it was not necessary.

We interpret each $\ell^{\circ}_{\mathcal{B}}$-type T by the $\ell^{\circ}_{\mathcal{B}}$-tree, written $\mathfrak{G}T$, that is the *parse tree* of T, where the left and the right children of the root of its subtree of the form $\mathfrak{G}(U \spadesuit U')$ ($\spadesuit \in \{\otimes, \times, {}_v{\times}_{v'}, \multimap\}$) are the roots of $\mathfrak{G}U$ and $\mathfrak{G}U'$, respectively. A vertex of this $\ell^{\circ}_{\mathcal{B}}$-tree $\mathfrak{G}T$ is

(1) The specified variable v labelled with $\clubsuit$ if it interprets $\clubsuit_{[v]}$ in T;
(2) An arbitrarily chosen *fresh* (*i.e.* not chosen yet) variable if it interprets $\otimes$, $\times$, ${}_v{\times}_{v'}$ or $\multimap$ in T, whose left and right children are respectively v and v' if it interprets ${}_v{\times}_{v'}$.[10]

Let End_T be the set of all ends in an $\ell^{\circ}_{\mathcal{B}}$-type T. By induction on T, we see that the equation

$$\mathrm{End}_T = \mathrm{End}_{\mathfrak{G}T}$$

holds, which justifies the overloading of the notation $\mathrm{End}_$. We also define

$$\mathrm{End}^{-}_T := \mathrm{End}^{-}_{\mathfrak{G}T}, \qquad \mathrm{End}^{+}_T := \mathrm{End}^{+}_{\mathfrak{G}T}, \qquad \mathrm{End}^{\mathrm{Ini}}_T := \mathrm{End}^{\mathrm{Ini}}_{\mathfrak{G}T},$$

$$\mathrm{End}^{\mathrm{Jkr}}_T := \mathrm{End}^{\mathrm{Jkr}}_{\mathfrak{G}T}, \qquad \mathrm{End}^{!}_T := \mathrm{End}^{!}_{\mathfrak{G}T},$$

[10]This last case is possible thanks to the axiom on the operator ${}_v{\times}_{v'}$ in $\ell^{\circ}_{\mathcal{B}}$-types (Definition 3.1.5).

overloading *negative*, *positive*, *initial* and *joker* ends too. We see by induction on T that, if T is negative (respectively, positive), then an end $v \in \mathrm{End}_T$ is negative in $\mathfrak{G}T$ if and only if the $\ell^\circ_{\mathcal{B}}$-subtype U of T such that $\mathfrak{G}U = (\mathfrak{G}T)^v$ is negative (respectively, positive). Moreover,

(1) $\mathrm{End}^{\mathrm{Ini}}_T = \mathrm{End}_T$ if T is atomic;
(2) $\mathrm{End}^{\mathrm{Ini}}_{T \times T'} = \mathrm{End}^{\mathrm{Ini}}_T \cup \mathrm{End}^{\mathrm{Ini}}_{T'}$ and $\mathrm{End}^{\mathrm{Ini}}_{T \times_{v,v'} T'} = \mathrm{End}^{\mathrm{Ini}}_T \cup \mathrm{End}^{\mathrm{Ini}}_{T'}$;
(3) $\mathrm{End}^{\mathrm{Ini}}_{T \multimap T'} = \mathrm{End}^{\mathrm{Ini}}_{T'}$;
(4) $\mathrm{End}^{\mathrm{Ini}}_{![e]T} = \{e\} \cup \mathrm{End}^{\mathrm{Ini}}_T$.

The operation $\mathrm{End}^{\heartsuit}_{_}$ ($\heartsuit \in \{-, +, \mathrm{Ini}, \mathrm{Jkr}, !\}$) is lifted to a finite sequence Γ of $\ell^\circ_{\mathcal{B}}$-types by

$$\mathrm{End}^{\heartsuit}_{\varepsilon} := \{v_0\}, \qquad \mathrm{End}^{\heartsuit}_{\Gamma.T} := \mathrm{End}^{\heartsuit}_{\Gamma} \cup \mathrm{End}^{\heartsuit}_{T}.$$

Notation. In $\ell^\circ_{\mathcal{B}}$-types, $\multimap$ is right associative; the other binary operators are left associative. Each unary operator precedes binary ones, and $\multimap$ is preceded by the other. For an $\ell^\circ_{\mathcal{B}}$-type T, let $\underline{T}$ be the expression yielded from T by the deletion of variables and bracketed variables (*n.b.* $\underline{T}$ is an object in the formal free (pointed) linear category over $\mathcal{B}$).

We then extend this formal notation for $\ell^\circ_{\mathcal{B}}$-trees to $\ell^\circ_{\mathcal{B}}$-ports and $\ell^\circ_{\mathcal{B}}$-diagrams:

Definition 3.1.6 ($\ell^\circ_{\mathcal{B}}$-cuts)**.** An $\ell^\bullet_{\mathcal{B}}$*-cut* is an expression M defined by

$$M := \top_{[v]} \mid N \triangleright P \mid M \otimes M' \mid M \times M' \mid !_{[v]}M \quad (\underline{N} = \underline{P}, v \in \mathcal{V}),$$

where variables are pairwise distinct, N and P are respectively a negative and a positive $\ell^\bullet_{\mathcal{B}}$-types, and bracketed variables are called *ends*. An $\ell_{\mathcal{B}}$*-cut* is an $\ell^\bullet_{\mathcal{B}}$-cut that is free from $\bot$.

Definition 3.1.7 ($\ell^\circ_{\mathcal{B}}$-sequents)**.** An $\ell^\bullet_{\mathcal{B}}$*-sequent* is an expression of the form

$$\Gamma \dashv \Sigma \vdash N,$$

where the *domain* $\Gamma = (P_i)_{i=1}^n$ is a finite sequence of positive $\ell^\bullet_{\mathcal{B}}$-types, the *centre* $\Sigma = (M_j)_{j=1}^m$ is that of $\ell^\bullet_{\mathcal{B}}$-ports, the *codomain* N is a negative $\ell^\bullet_{\mathcal{B}}$-type, variables are pairwise distinct, and bracketed variables are called *ends*. An $\ell_{\mathcal{B}}$*-sequent* is an $\ell^\bullet_{\mathcal{B}}$-sequent that is free from $\bot$.

Convention. An $\ell^\bullet_{\mathcal{B}}$- and an $\ell_{\mathcal{B}}$-types occurring in an $\ell^\circ_{\mathcal{B}}$-cut/sequent K are called an $\ell^\bullet_{\mathcal{B}}$- and an $\ell_{\mathcal{B}}$*-subtypes* of K, respectively. We abbreviate an $\ell^\circ_{\mathcal{B}}$-sequent $\Gamma \dashv \Sigma \vdash \bot_{v_0}$ as $\Gamma \dashv \Sigma \vdash$.

Just like the interpretation $\mathfrak{G}$ of $\ell^\circ_{\mathcal{B}}$-types, we interpret each $\ell^\circ_{\mathcal{B}}$-cut M by its parse tree $\mathfrak{G}M$, except that its vertices may be labelled with $\triangleright$, and each $\ell^\circ_{\mathcal{B}}$-sequent $S = (P_i)_{i=1}^n \dashv (M_j)_{j=1}^m \vdash N$ by the rooted tree (14), written $\mathfrak{G}S$, where $\mathfrak{T}_i := \mathfrak{G}P_i$, $\mathfrak{M}_j := \mathfrak{G}M_j$ and $\mathfrak{T} := \mathfrak{G}N$. Vertices of these rooted trees $\mathfrak{G}M$ and $\mathfrak{G}S$ are labelled just like those of $\mathfrak{G}T$, except that $\triangleright$, $\dashv$ and $\vdash$ in M and S are also interpreted by arbitrarily chosen fresh variables labelled with those symbols.

Again, overloading the notation $\mathrm{End}_{_}$, we define End_S to be the set of all ends in S, and then the equation

$$\mathrm{End}_S = \mathrm{End}_{\mathfrak{G}S}$$

holds. Moreover, we define

$$\mathrm{End}^-_S := \mathrm{End}^-_{\mathfrak{G}S}, \qquad \mathrm{End}^+_S := \mathrm{End}^+_{\mathfrak{G}S}, \qquad \mathrm{End}^{\mathrm{Ini}}_S := \mathrm{End}^{\mathrm{Ini}}_{\mathfrak{G}S},$$

$$\mathrm{End}^{\mathrm{Jkr}}_S := \mathrm{End}^{\mathrm{Jkr}}_{\mathfrak{G}S}, \qquad \mathrm{End}^{!}_S := \mathrm{End}^{!}_{\mathfrak{G}S},$$

again overloading *negative*, *positive*, *initial*, *joker* and !-ends too. Clearly, $v \in \mathrm{End}_S$ is negative (respectively, positive) in $\mathfrak{G}S$ if and only if so is the $\ell^\circ_{\mathcal{B}}$-subtype U of S such that $\mathfrak{G}U = (\mathfrak{G}S)^v$.

We next define raw morphisms in our graphical calculus, called *pre-*$\ell^\circ_{\mathcal{B}}$*-nets*.

Definition 3.1.8 ($\mathcal{B}$-arcs)**.** A *$\mathcal{B}$-arc* on an $\ell^{\circ}_{\mathcal{B}}$-diagram $\mathfrak{S}$ is a labelled edge in either of the forms

$$u \xrightarrow[\langle i\rangle]{\langle\kappa|\epsilon\rangle} v, \qquad u \xrightarrow[\langle i\rangle]{\chi\langle\kappa|\epsilon\rangle} v \quad (u, v \in \mathrm{End}_{\mathfrak{S}}, \kappa \in \wp_{\mathrm{f}}(V_{\mathfrak{S}}), \epsilon \in \wp_{\mathrm{f}}(\mathrm{Col} \times \mathrm{End}^{+!}_{\mathfrak{S}}), i \in \mathbb{N}_+, \chi \in \mathrm{Mor}(\mathcal{B})),$$

where $\mathrm{End}^{+!}_{\mathfrak{S}} := \mathrm{End}^{+}_{\mathfrak{S}} \cap \mathrm{End}^{!}_{\mathfrak{S}}$, $\mathfrak{S}X_{[v]}$ and $\mathfrak{S}Y_{[u]}$ occur in $\mathfrak{S}$, with v and u positive and negative, respectively, if and only if $\chi : X \to Y$ lies in $\mathcal{B}$, and the labels χ, $\langle\kappa|$, $|\epsilon\rangle$ and $\langle i\rangle$ are respectively said to be *arrowy*, *cartesian*, *exponential* and *identifier*, such that

(1) For each $e \in \mathrm{End}^{+!}_{\mathfrak{S}}$ such that $e \neq v$ and $v \in \mathrm{End}_{\mathfrak{S}^e}$, some colour c satisfies $(c, e) \in \epsilon$;
(2) Every $(c, e) \in \epsilon$ satisfies $uiv \in c$ and $y \in \mathrm{End}^{\mathrm{Ini}}_{\mathfrak{S}^e} \wedge y \neq e$ for all $xjy \in c$.

See Footnote 11 on why y is initial in $\mathfrak{S}^e$, and Example 3.2.2 for the axiom $y \neq e$. We keep using the notations on $\mathcal{B}$-arcs introduced in §2.1. A *$\mathcal{B}$-web* on $\mathfrak{S}$ is a finite set of $\mathcal{B}$-arcs on $\mathfrak{S}$.

Notation. Given a $\mathcal{B}$-arc $a_0 = \left(u \xrightarrow[\langle i\rangle]{(\chi)\langle\kappa|\epsilon\rangle} v\right)$, let

$$\mathrm{arr}(a_0) := \chi, \qquad \mathrm{car}(a_0) := \kappa, \qquad \mathrm{exp}(a_0) := \epsilon, \qquad \mathrm{ide}(a_0) := i.$$

For a $\ell^{\circ}_{\mathcal{B}}$-diagram $\mathfrak{S}$, let $<^{!+}_{\mathfrak{S}}$ be the relation between positive !-ends of $\mathfrak{S}$ such that $e <^{!+}_{\mathfrak{S}} f$ holds if $e \neq f$, $e \in \mathfrak{S}^f$, and any positive !-end $e' \in \mathfrak{S}^f$ with $e \neq e'$ and $e \in \mathfrak{S}^{e'}$ is f. We also define the colour $\mathrm{Col}(\mathscr{A})$ of a $\mathcal{B}$-web $\mathscr{A}$ by

$$\mathrm{Col}(a_0) := uiv, \qquad \mathrm{Col}(\mathscr{A}) := \left\{\, \mathrm{Col}(a) \mid a \in \mathscr{A} \,\right\}.$$

Definition 3.1.9 ($\mathcal{B}$-realisers)**.** A *pseudo-$\mathcal{B}$-realiser* on an $\ell^{\circ}_{\mathcal{B}}$-diagram $\mathfrak{S}$ is a graph $\mathfrak{r}$ such that $V_{\mathfrak{r}} = \mathrm{End}_{\mathfrak{S}}$, its edges are $\mathcal{B}$-arcs a on $\mathfrak{S}$ from negative to positive ends whose exponential label $|\epsilon\rangle$ satisfies, for all $(c, e) \in \epsilon$, $c \subseteq \mathrm{Col}(\{\, a \in \mathfrak{r} \mid \mathrm{tgt}(a) \in \mathrm{End}_{\mathfrak{S}^e}, \mathrm{tgt}(a) \neq e, \mathrm{src}(a) \notin \mathrm{End}_{\mathfrak{S}^e} \,\})$. The *completion* of $\mathfrak{r}$ in $\mathfrak{S}$ is the graph $\bar{\mathfrak{r}}_{\mathfrak{S}}$ yielded from $\mathfrak{r}$ by the addition of $\mathcal{B}$-arcs $p \to o$ on $\mathfrak{S}$ such that o is negative, justified by p, and the mate of p, if p has, all in $\mathfrak{S}$.

A pseudo-$\mathcal{B}$-realiser $\mathfrak{r}$ on $\mathfrak{S}$ is called a *$\mathcal{B}$-realiser* on $\mathfrak{S}$ if

(1) There is a finite rooted forest $\mathfrak{F}$ such that $V_{\mathfrak{F}}$ is a partition of the set of all nonempty colours occurring in the exponential labels of $\mathcal{B}$-arcs in $\mathfrak{r}$ with $\bigcup V_{\mathfrak{F}} \subseteq \mathrm{Col}(\mathfrak{r})$, in which
   (a) A vertex is a union of some colours assigned to the same positive !-end, and called the *junction* of the vertex;
   (b) An edge $S \to S'$ exists only if $S \supseteq S'$ holds, and the junctions e and e' of S and S', respectively, satisfy $e <^{!+}_{\mathfrak{S}} e'$;
(2) The completion $\bar{\mathfrak{r}}_{\mathfrak{S}}$ is acyclic.

Remark. The examples of $\mathcal{B}$-realisers given in previous sections are, strictly speaking, completions. From now on, (pseudo-)$\mathcal{B}$-realisers refer to those defined in Definition 3.1.9.

Convention. As in previous sections, we represent a $\mathcal{B}$-arc between mates by a dotted arrow.

Definition 3.1.10 (pre-$\ell^{\circ}_{\mathcal{B}}$-nets)**.** A *pre-$\ell^{\bullet}_{\mathcal{B}}$-net* on an $\ell^{\bullet}_{\mathcal{B}}$-diagram $\mathfrak{S}$ is a graph, written $\mathfrak{r} :: \mathfrak{S}$, that is the union of $\mathfrak{S}$ and a $\mathcal{B}$-realiser $\mathfrak{r}$ on $\mathfrak{S}$. It is called a *pre-$\ell_{\mathcal{B}}$-net* if $\mathfrak{S}$ is an $\ell_{\mathcal{B}}$-diagram, and said to be *normalised* if the centre of $\mathfrak{S}$ has no factors.

Example 3.1.11. The examples of pre-$\ell^{\circ}_{\mathcal{B}}$-nets given in previous sections conform to this definition (except that $\mathcal{B}$-realisers are replaced with their completions).

Remark. In the rest of this article, we draw some involved examples of pre-$\ell^{\circ}_{\mathcal{B}}$-nets. Hence, for simplicity, we usually draw their $\mathcal{B}$-realisers, not completions, and (abbreviated) $\ell^{\circ}_{\mathcal{B}}$-diagrams.

Convention. We write $\mathfrak{r}, \mathfrak{r}' :: \mathfrak{S}$ for two pre-$\ell^{\circ}_{\mathcal{B}}$-nets $\mathfrak{r} :: \mathfrak{S}$ and $\mathfrak{r}' :: \mathfrak{S}$, and $\mathfrak{r} R \mathfrak{r}' :: \mathfrak{S}$ if a relation R between pre-$\ell^{\circ}_{\mathcal{B}}$-nets holds for the pair of $\mathfrak{r} :: \mathfrak{S}$ and $\mathfrak{r}' :: \mathfrak{S}$.

We then introduce a compact notation for pre-$\ell^\circ_{\mathcal{B}}$-nets, or *pre-$\ell^\circ_{\mathcal{B}}$-terms*. First, an *$\ell^\bullet_{\mathcal{B}}$-context* on an $\ell^\bullet_{\mathcal{B}}$-sequent $S = (P_i)_{i=1}^n \dashv (M_j)_{j=1}^m \vdash N$ is a finite sequence $(\mathscr{P}_i : P_i)_{i=1}^n$ of pairs, written $\mathscr{P}_i : P_i$, of a $\mathcal{B}$-web $\mathscr{P}_i$ and the positive $\ell^\bullet_{\mathcal{B}}$-type P_i such that each $\mathcal{B}$-arc in $\mathscr{P}_i$ is from a negative to a positive ends of S with the target occurring in P_i, where the $\ell^\bullet_{\mathcal{B}}$-context is written $\gamma : \Gamma$ if $\gamma = (\mathscr{P}_i)_{i=1}^n$ and $\Gamma = (P_i)_{i=1}^n$. Similarly, an *$\ell^\bullet_{\mathcal{B}}$-interaction* on S is a finite sequence $(\mathscr{M}_j : M_j)_{j=1}^m$ of pairs, written $\mathscr{M}_j : M_j$, of a $\mathscr{B}$-web $\mathscr{M}_j$ and the $\ell^\bullet_{\mathcal{B}}$-cut M_j such that each $\mathcal{B}$-arc in $\mathscr{M}_j$ is from a negative to a positive ends of S with the target occurring in M_j, where the $\ell^\bullet_{\mathcal{B}}$-interaction is written $\sigma : \Sigma$ if $\sigma = (\mathscr{M}_j)_{j=1}^m$ and $\Sigma = (M_j)_{j=1}^m$. An *$\ell_{\mathcal{B}}$-context* and an *$\ell_{\mathcal{B}}$-interaction* are an $\ell^\bullet_{\mathcal{B}}$-context and an $\ell^\bullet_{\mathcal{B}}$-interaction, respectively, that have no occurrences of $\bot$. We then define:

Definition 3.1.12 (pseudo-$\ell^\circ_{\mathcal{B}}$-terms)**.** A *pseudo-$\ell^\bullet_{\mathcal{B}}$-term* on an $\ell^\bullet_{\mathcal{B}}$-sequent $S = \Gamma \dashv \Sigma \vdash N$ is an expression of the form

$$\gamma : \Gamma \dashv \sigma : \Sigma \vdash \mathscr{N} : N, \tag{7}$$

where $\gamma : \Gamma$ is an $\ell^\bullet_{\mathcal{B}}$-context on S, $\sigma : \Sigma$ is an $\ell^\bullet_{\mathcal{B}}$-interaction on S, $\mathscr{N}$ is a $\mathscr{B}$-web such that each $\mathcal{B}$-arc in $\mathscr{N}$ is from a negative to a positive ends of S with the target occurring in N. It is said to be *normalised* if $\sigma : \Sigma$ is empty, and called a *pseudo-$\ell_{\mathcal{B}}$-term* if S is an $\ell_{\mathcal{B}}$-sequent.

The pseudo-$\ell^\circ_{\mathcal{B}}$-term (7) is written $r :: S$ if r is the $\mathcal{B}$-web of all $\mathcal{B}$-arcs in γ, σ and $\mathscr{N}$ (*i.e.* $r = \bigcup \underline{\gamma.\sigma.N}$), and we interpret it as the graph, written $\mathfrak{G}(r :: S)$ or $\mathfrak{G}r :: \mathfrak{G}S$, that is the union of $\mathfrak{G}S$ and the pseudo-$\mathcal{B}$-realiser $\mathfrak{G}r$ on $\mathfrak{G}S$ that consists exactly of $\mathcal{B}$-arcs in r.

Definition 3.1.13 (pre-$\ell^\circ_{\mathcal{B}}$-terms)**.** A *pre-$\ell^\bullet_{\mathcal{B}}$-term* is a pseudo-$\ell^\bullet_{\mathcal{B}}$-term $r :: S$ such that $\mathfrak{G}(r :: S)$ is a pre-$\ell^\bullet_{\mathcal{B}}$-net, and a *pre-$\ell_{\mathcal{B}}$-term* is a pre-$\ell^\bullet_{\mathcal{B}}$-term whose $\ell^\bullet_{\mathcal{B}}$-sequent is an $\ell_{\mathcal{B}}$-sequent.

Without loss of generality, we can represent our graphical concepts–$\ell^\circ_{\mathcal{B}}$-trees/ports/diagrams and pre-$\ell^\circ_{\mathcal{B}}$-nets–by the formal notations–$\ell^\circ_{\mathcal{B}}$-types/cuts/sequents and pre-$\ell^\circ_{\mathcal{B}}$-terms–*via* the interpretations $\mathfrak{G}$, where the vertices of the graphs that are not ends are left anonymous in the notations. For simplicity, we informally represent the $\ell^\circ_{\mathcal{B}}$-diagram of a pre-$\ell^\circ_{\mathcal{B}}$-net in a diagram by the corresponding $\ell^\circ_{\mathcal{B}}$-sequent; this convention is already used in previous sections.

Notation. The pseudo-$\ell^\circ_{\mathcal{B}}$-term (7) is also written

$$\begin{array}{cccc} \gamma : \Gamma & \gamma : \Gamma \dashv & \gamma : \Gamma & \gamma : \Gamma \dashv \sigma : \Sigma \\ \dashv \sigma : \Sigma \vdash & \sigma : \Sigma & \dashv \sigma : \Sigma \vdash \mathscr{N} : N, & \vdash \mathscr{N} : N, \\ \mathscr{N} : N, & \vdash \mathscr{N} : N, & & \end{array}$$

and so on. It is also written $\gamma : \Gamma \dashv \sigma : \Sigma \vdash$ if $\mathscr{N} = \emptyset$ and $N = \bot_{[v_0]}$. Given two $\ell^\circ_{\mathcal{B}}$-contexts $\delta : \Delta = (\mathscr{R}_i : D_i)_{i=1}^k$ and $\delta' : \Delta' = (\mathscr{R}'_j : D'_j)_{j=1}^{k'}$, we concatenate them into the one

$$\begin{aligned} (\delta : \Delta), (\delta' : \Delta') &:= \delta\delta' : \Delta\Delta' \\ &= \mathscr{R}_1 : D_1, \mathscr{R}_2 : D_2, \ldots, \mathscr{R}_k : D_k, \mathscr{R}'_1 : D'_1, \mathscr{R}'_2 : D'_2, \ldots, \mathscr{R}'_{k'} : D'_{k'}. \end{aligned}$$

3.2. **Graphical morphisms.** We next define a class of pre-$\ell^\circ_{\mathcal{B}}$-nets, called *$\ell^\circ_{\mathcal{B}}$-nets*, that act as our graphical morphisms. As an auxiliary step towards this end, we first *freely generate* another class of pre-$\ell^\circ_{\mathcal{B}}$-nets from morphisms in $\mathcal{B}$. For ease of notation, we describe the latter class *via* the corresponding free generation of pre-$\ell^\circ_{\mathcal{B}}$-terms, though it is only a matter of notation.

We will see *via* Definition 4.2.1 and Theorem 4.1.16 that our free generation is equivalent to the one in terms of the constructs of (pointed) linear categories. Our free generation is based on the sequent calculus for intuitionistic linear logic [Gir87, Bie94], which is technically convenient. We will characterise the free generation non-inductively and geometrically by Theorem 3.2.44.

Notation. For a $\mathcal{B}$-arc $a = (u \xrightarrow[\langle i\rangle]{(\chi)\langle\kappa|\epsilon\rangle} v)$, finite sets $\kappa_0 \in \wp_{\mathrm{f}}(\mathcal{V})$ and $\epsilon_0 \in \wp_{\mathrm{f}}(\mathrm{Col} \times \mathcal{V})$, and $\ell^{\circ}_{\mathcal{B}}$-types T and $\hat{T}$ such that $\underline{T} = \underline{\hat{T}}$, we obtain the $\mathcal{B}$-arcs

$$a + \langle\kappa_0| := (u \xrightarrow[\langle i\rangle]{(\chi)\langle\kappa\cup\kappa_0|\epsilon\rangle} v), \qquad a - \langle\kappa_0| := (u \xrightarrow[\langle i\rangle]{(\chi)\langle\kappa\setminus\kappa_0|\epsilon\rangle} v),$$

$$a + |\epsilon_0\rangle := (u \xrightarrow[\langle i\rangle]{(\chi)\langle\kappa|\epsilon\cup\epsilon_0\rangle} v), \qquad a - |\epsilon_0\rangle := (u \xrightarrow[\langle i\rangle]{(\chi)\langle\kappa|\epsilon\setminus\epsilon_0\rangle} v),$$

and the one $a\{T/\hat{T}\}$ from a by the replacement of ends of $\hat{T}$ occurring in a with the corresponding ones of T (under $\underline{T} = \underline{\hat{T}}$). We lift the operators $(_) + \langle\kappa_0|$, $(_) - \langle\kappa_0|$, $(_) + |\epsilon_0\rangle$, $(_) - |\epsilon_0\rangle$ and $(_)\{T/\hat{T}\}$ on $\mathcal{B}$-arcs to $\mathcal{B}$-webs and further to finite sequences of $\mathcal{B}$-webs in the pointwise fashion.

Remark. We will eventually focus on pre-$\ell^{\circ}_{\mathcal{B}}$-nets/terms in *prime form* (Definition 3.2.28) whose $\ell^{\circ}_{\mathcal{B}}$-diagrams/sequents and identifier labels are assigned uniquely fixed ends and positive integers, respectively. Thus, although the following free generation of pre-$\ell^{\circ}_{\mathcal{B}}$-nets/terms requires suitable renamings of ends and identifier labels in pre-$\ell^{\circ}_{\mathcal{B}}$-nets/terms, they do not matter in the end.

Definition 3.2.1 (free generation)**.** A pre-$\ell^{\circ}_{\mathcal{B}}$-term is said to be *freely generated* (from $\mathcal{B}$) if it is derivable inductively by *typing rules* in Figure 15 (as in proof theory [TS00]), where

(1) In each typing rule, we assume that the ends of the $\ell^{\circ}_{\mathcal{B}}$-sequents are pairwise distinct by suitably renaming them, and fresh ends in the conclusion, if any, are arbitrary;
(2) In the typing rule !D (respectively, !M), $r \updownarrow P$ (respectively, $r \updownarrow M$) is the set of $\mathcal{B}$-arcs in r whose source or target occurs in P (respectively, M), and $r \uparrow P$ (respectively, $r \uparrow M$) is the set of $\mathcal{B}$-arcs in r whose target occurs in P (respectively, M);[11]
(3) In the typing rule !C, we assume that the union $\mathscr{P} \cup \hat{\mathscr{P}}\{!_{[w]}P/!_{[\hat{w}]}\hat{P}\}$ is a disjoint union by suitably renaming the identifier labels (i) of $\mathcal{B}$-arcs and the corresponding occurrences of the number i in colours of the exponential labels in $\mathscr{P}$ and $\hat{\mathscr{P}}$,

and a pre-$\ell^{\circ}_{\mathcal{B}}$-net is said to be *freely generated* if so is the corresponding pre-$\ell^{\circ}_{\mathcal{B}}$-term (under $\mathfrak{G}$).

Remark. By the typing rules $\top$M, $\otimes$M and $\times$M, we can always modify the centres of a pair of $\ell^{\circ}_{\mathcal{B}}$-sequents to apply the typing rule $\times$R. We will see that the first three typing rules do not modify pre-$\ell^{\circ}_{\mathcal{B}}$-nets/sequents up to $\ell^{\circ}_{\mathcal{B}}$-reduction, so the one $\times$R does not lose generality.

We define a *derivation (tree)* of a freely generated pre-$\ell^{\circ}_{\mathcal{B}}$-net $\mathfrak{G}(r :: S)$ to be that of the pre-$\ell^{\circ}_{\mathcal{B}}$-term $r :: S$. Unlike proof theory, derivations of pre-$\ell^{\circ}_{\mathcal{B}}$-terms/nets are *not* central or important in the present work; they only play the auxiliary role to define freely generated pre-$\ell^{\circ}_{\mathcal{B}}$-terms/nets. We often omit the typing rules XL and XM in derivations.

Example 3.2.2. Consider the freely generated pre-$\ell^{\circ}_{\mathcal{B}}$-term derived as

$$\dfrac{\dfrac{\dfrac{\dfrac{}{\{f \to e, x \xrightarrow{|(\{x1y\},e)\rangle} y\} : !_{[e]}X_{[y]} \vdash\!\dashv \emptyset : !_{[f]}X_{[x]}} \quad \dfrac{}{\{g \to h, u \xrightarrow{|(\{u1v\},h)\rangle} v\} : !_{[h]}X_{[v]} \vdash\!\dashv \emptyset : !_{[g]}X_{[u]}}\,(\otimes\mathrm{R})}{\{f \to e, x \xrightarrow{|(\{x1y\},e)\rangle} y\} : !_{[e]}X_{[y]}, \{g \to h, u \xrightarrow{|(\{u1v\},h)\rangle} v\} : !_{[h]}X_{[v]} \vdash\!\dashv \emptyset : !_{[f]}X_{[x]} \otimes !_{[g]}X_{[u]}}\,(!\mathrm{C})}{\{f \to e, x \xrightarrow{|(\{x1y\},e)\rangle} y, g \to e, u \xrightarrow{|(\{u1y\},e)\rangle} y\} : !_{[e]}X_{[y]} \vdash\!\dashv \emptyset : !_{[f]}X_{[x]} \otimes !_{[g]}X_{[u]}}\,(!\mathrm{R})}{\{k \to e, f \to e, x \xrightarrow{|(\{x1y\},e)\rangle} y, g \to e, u \xrightarrow{|(\{u1y\},e)\rangle} y\} : !_{[e]}X_{[y]} \vdash\!\dashv \emptyset : !_{[k]}(!_{[f]}X_{[x]} \otimes !_{[g]}X_{[u]})}$$

where the double lines abbreviate evident derivations. It explains the axiom $y \neq e$ (2) below.

[11] In this typing rule, we add colours from $r \uparrow P$ (respectively, $r \uparrow M$), not $r \updownarrow P$ (respectively, $r \updownarrow M$), partly because the choice makes the exponential labels of freely generated pre-$\ell^{\circ}_{\mathcal{B}}$-nets/terms simpler, and because otherwise the component $\rightsquigarrow_5$ of $\ell^{\circ}_{\mathcal{B}}$-reduction in Figure 18 would be more involved.

$$\text{(XL)}\ \frac{\gamma : \Gamma, \mathscr{P} : P, \mathscr{R} : R, \delta : \Delta \dashv \sigma : \Sigma \vdash \mathscr{N} : N}{\gamma : \Gamma, \mathscr{R} : R, \mathscr{P} : P, \delta : \Delta \dashv \sigma : \Sigma \vdash \mathscr{N} : N} \qquad \text{(XM)}\ \frac{\gamma : \Gamma \dashv \sigma : \Sigma, \mathscr{M} : M, \mathscr{L} : L, \pi : \Pi \vdash \mathscr{N} : N}{\gamma : \Gamma \dashv \sigma : \Sigma, \mathscr{L} : L, \mathscr{M} : M, \pi : \Pi \vdash \mathscr{N} : N}$$

$$\text{(Cut)}\ \frac{\gamma : \Gamma \dashv \sigma : \Sigma \vdash \mathscr{N} : N \qquad \delta : \Delta, \mathscr{P} : P \dashv \pi : \Pi \vdash \mathscr{Q} : Q}{\gamma : \Gamma, \delta : \Delta \dashv \sigma : \Sigma, \mathscr{N} \cup \mathscr{P} : N \triangleright P, \pi : \Pi \vdash \mathscr{Q} : Q}\ (\underline{N} = \underline{P})$$

$$\text{(BC)}\ \frac{}{\{u \xrightarrow{\chi} v\} : X_{[v]} \dashv\vdash \emptyset : Y_{[u]}}\ (\chi \in \mathcal{B}(X, Y)) \qquad \text{(!W)}\ \frac{\gamma : \Gamma \dashv \sigma : \Sigma \vdash \mathscr{N} : N}{\gamma : \Gamma, \{v \to w\} : !_{[w]}P \dashv \sigma : \Sigma \vdash \mathscr{N} : N}\ (v \in \mathrm{End}_N^{\mathrm{Ini}})$$

$$\text{(!D)}\ \frac{r :: (\Gamma, P \dashv \Sigma \vdash N)}{(r \setminus (r \updownarrow P)) \cup (\{\, a + |(\mathrm{Col}(r \uparrow P), w)\rangle \mid a \in (r \updownarrow P)\,\}) :: (\Gamma, !_{[w]}P \dashv \Sigma \vdash N)}$$

$$\text{(!DM)}\ \frac{r : (\Gamma \dashv \Sigma, M \vdash N)}{(r \setminus (r \updownarrow M)) \cup (\{\, a + |(\mathrm{Col}(r \uparrow M), w)\rangle \mid a \in (r \updownarrow M)\,\}) :: \Gamma \dashv \Sigma, !_{[w]}M \vdash N}$$

$$\text{(!C)}\ \frac{\gamma : \Gamma, \mathscr{P} : !_{[w]}P, \hat{\mathscr{P}} : !_{[\dot{w}]}\hat{P} \dashv \sigma : \Sigma \vdash \mathscr{N} : N}{(\gamma\{!_{[w]}P/!_{[\dot{w}]}\hat{P}\} : \Gamma, \mathscr{P} \cup \hat{\mathscr{P}}\{!_{[w]}P/!_{[\dot{w}]}\hat{P}\} : !_{[w]}P \dashv \sigma\{!_{[w]}P/!_{[\dot{w}]}\hat{P}\} : \Sigma \vdash \mathscr{N}\{!_{[w]}P/!_{[\dot{w}]}\hat{P}\} : N)}\ (\underline{P} = \underline{\hat{P}})$$

$$\text{(!R)}\ \frac{(\mathscr{P}_i : !_{[w_i]}P_i)_{i=1}^n \dashv (\mathscr{M}_j : !_{[v_j]}M_j)_{j=1}^m \vdash \mathscr{N} : N}{(\mathscr{P}_i \cup \{u \to w_i\} : !_{[w_i]}P_i)_{i=1}^n \dashv (\mathscr{M}_j \cup \{u \to v_j\} : !_{[v_j]}M_j)_{j=1}^m \vdash \mathscr{N} : !_{[u]}N} \qquad \text{(1R)}\ \frac{}{(\emptyset : P_i)_{i=1}^n \dashv\vdash \emptyset : 1_{[u]}}$$

$$\text{(}\top\text{L)}\ \frac{\gamma : \Gamma \dashv \sigma : \Sigma \vdash \mathscr{N} : N}{\gamma : \Gamma, \{u \to v\} : \top_{[v]} \dashv \sigma : \Sigma \vdash \mathscr{N} : N}\ (u \in \mathrm{End}_N^{\mathrm{Ini}})$$

$$\text{(}\top\text{M)}\ \frac{\gamma : \Gamma \dashv \sigma : \Sigma \vdash \mathscr{N} : N}{\gamma : \Gamma \dashv \sigma : \Sigma, \{u \to v\} : \top_{[v]} \vdash \mathscr{N} : N}\ (u \in \mathrm{End}_N^{\mathrm{Ini}}) \qquad \text{(}\top\text{R)}\ \frac{}{\dashv\vdash \emptyset : \top_{[u]}}$$

$$\text{(}\bot\text{L)}\ \frac{}{\{v_0 \to v\} : \bot_{[v]} \dashv\vdash} \qquad \text{(}\bot\text{R)}\ \frac{\gamma : \Gamma \dashv \sigma : \Sigma \vdash}{\gamma\{v/v_0\} : \Gamma \dashv \sigma\{v/v_0\} : \Sigma \vdash \emptyset : \bot_{[v]}}$$

$$\text{(}\otimes\text{L)}\ \frac{\gamma : \Gamma, \mathscr{P} : P, \mathscr{R} : R \dashv \sigma : \Sigma \vdash \mathscr{N} : N}{\gamma : \Gamma, \mathscr{P} \cup \mathscr{R} : P \otimes R \dashv \sigma : \Sigma \vdash \mathscr{N} : N} \qquad \text{(}\otimes\text{M)}\ \frac{\gamma : \Gamma \dashv \sigma : \Sigma, \mathscr{L} : L, \mathscr{M} : M \vdash \mathscr{N} : N}{\gamma : \Gamma \dashv \sigma : \Sigma, \mathscr{L} \cup \mathscr{M} : L \otimes M \vdash \mathscr{N} : N}$$

$$\text{(}\otimes\text{R)}\ \frac{\gamma : \Gamma \dashv \sigma : \Sigma \vdash \mathscr{N} : N \qquad \delta : \Delta \dashv \pi : \Pi \vdash \mathscr{Q} : Q}{\gamma : \Gamma, \delta : \Delta \dashv \sigma : \Sigma, \pi : \Pi \vdash \mathscr{N} \cup \mathscr{Q} : N \otimes Q}$$

$$\text{(}\times\text{L)}\ \frac{\gamma : \Gamma, \mathscr{P} : P \dashv \sigma : \Sigma \vdash \mathscr{N} : N}{\gamma : \Gamma, \mathscr{P} : P \times R \dashv \sigma : \Sigma \vdash \mathscr{N} : N} \qquad \text{(}\times\text{L)}\ \frac{\gamma : \Gamma, \mathscr{R} : R \dashv \sigma : \Sigma \vdash \mathscr{N} : N}{\gamma : \Gamma, \mathscr{R} : P \times R \dashv \sigma : \Sigma \vdash \mathscr{N} : N}$$

$$\text{(}\times\text{M)}\ \frac{\gamma : \Gamma \dashv \sigma : \Sigma, \mathscr{L} : L \vdash \mathscr{N} : N}{\gamma : \Gamma \dashv \sigma : \Sigma, \mathscr{L} : L \times M \vdash \mathscr{N} : N} \qquad \text{(}\times\text{M)}\ \frac{\gamma : \Gamma \dashv \sigma : \Sigma, \mathscr{M} : M \vdash \mathscr{N} : N}{\gamma : \Gamma \dashv \sigma : \Sigma, \mathscr{M} : L \times M \vdash \mathscr{N} : N}$$

$$\text{(}\times\text{R)}\ \frac{(\mathscr{P}_i : P_i)_{i=1}^n \dashv \sigma : \Sigma \vdash \mathscr{N} : N \qquad (\hat{\mathscr{P}}_i : P_i)_{i=1}^n \dashv \hat{\sigma} : \Sigma \vdash \mathscr{Q} : Q}{((\mathscr{P}_i + \langle x|) \cup (\hat{\mathscr{P}}_i + \langle y|) : P_i)_{i=1}^n \dashv (\sigma + \langle x|) \cup (\hat{\sigma} + \langle y|) : \Sigma \vdash (\mathscr{N} + \langle x|) \cup (\mathscr{Q} + \langle y|) : N_x \times_y Q}$$

$$\text{(}\multimap\text{L)}\ \frac{\gamma : \Gamma \dashv \sigma : \Sigma \vdash \mathscr{N} : N \qquad \delta : \Delta, \mathscr{P} : P \dashv \pi : \Pi \vdash \mathscr{Q} : Q}{\gamma : \Gamma, \delta : \Delta, \mathscr{N} \cup \mathscr{P} : N \multimap P \dashv \sigma : \Sigma, \pi : \Pi \vdash \mathscr{Q} : Q} \qquad \text{(}\multimap\text{R)}\ \frac{\gamma : \Gamma, \mathscr{P} : P \dashv \sigma : \Sigma \vdash \mathscr{N} : N}{\gamma : \Gamma \dashv \sigma : \Sigma \vdash \mathscr{P} \cup \mathscr{N} : P \multimap N}$$

FIGURE 15. Typing rules for free generation

The exponential labels of the $\mathcal{B}$-arcs $x \xrightarrow{|(\{x1y\},e)\rangle} y$ and $u \xrightarrow{|(\{u1y\},e)\rangle} y$ distinguish between them as they come from different hypotheses (or components) of the typing rule !C.

In contrast, the $\mathcal{B}$-arc $k \to e$ has the trivial exponential label, which makes sense as it does not belong to any of the hypotheses of !C. However, the $\mathcal{B}$-arc $f \to e$ or $g \to e$ has no nontrivial exponential label either, though they come from different hypotheses of !C. Yet, it is not a problem since the first two $\mathcal{B}$-arcs have enough information to tell (*via* the parts $!_{[f]}X_{[x]}$ and $!_{[g]}X_{[u]}$ and valid sequences) the hypothesis of !C from which each of $f \to e$ and $g \to e$ comes.

$$\dfrac{\dfrac{(\top\mathrm{R})\ \dfrac{}{\dashv\!\vdash \emptyset : \top_{[v]}}}{(!\mathrm{W})\ \dfrac{}{\{v \to w\} : !_{[w]}1_{[z]} \dashv\!\vdash \emptyset : \top_{[v]}}} \qquad \dfrac{(1\mathrm{R})\ \dfrac{}{\dashv\!\vdash \emptyset : 1_{[v]}}}{(!\mathrm{R})\ \dfrac{}{\dashv\!\vdash \emptyset : !_{[e]}1_{[v]}}}}{(\top\mathrm{L})\ \dfrac{}{\{e \to z\} : \top_{[z]} \dashv\!\vdash \emptyset : !_{[e]}1_{[v]}}}$$

$$\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{(\mathrm{BC})\ \dfrac{}{\{x \xrightarrow{\mathrm{id}_X} \hat{x}\} : X_{[\hat{x}]} \dashv\!\vdash \emptyset : X_{[x]}}}{(\times\mathrm{L})\ \dfrac{}{\{x \xrightarrow{\mathrm{id}_X} \hat{x}\} : X_{[\hat{x}]} \times Y_{[\hat{y}]} \dashv\!\vdash \emptyset : X_{[x]}}}}{(!\mathrm{D})\ \dfrac{}{\{x \xrightarrow{\mathrm{id}_X \mid (\{x1\hat{x}\},\hat{z})\rangle} \hat{x}\} : !_{[\hat{z}]}(X_{[\hat{x}]} \times Y_{[\hat{y}]}) \dashv\!\vdash \emptyset : X_{[x]}}}}{(!\mathrm{R})\ \dfrac{}{\{x \xrightarrow{\mathrm{id}_X \mid (\{x1\hat{x}\},\hat{z})\rangle} \hat{x}, v \to \hat{z}\} : !_{[\hat{z}]}(X_{[\hat{x}]} \times Y_{[\hat{y}]}) \dashv\!\vdash \emptyset : !_{[v]}X_{[x]}}} \qquad \dfrac{\dfrac{\dfrac{\dfrac{(\mathrm{BC})\ \dfrac{}{\{y \xrightarrow{\mathrm{id}_Y} \check{y}\} : Y_{[\hat{y}]} \dashv\!\vdash \emptyset : Y_{[y]}}}{(\times\mathrm{L})\ \dfrac{}{\{y \xrightarrow{\mathrm{id}_Y} \check{y}\} : X_{[\check{x}]} \times Y_{[\check{y}]} \dashv\!\vdash \emptyset : Y_{[y]}}}}{(!\mathrm{D})\ \dfrac{}{\{y \xrightarrow{\mathrm{id}_Y \mid (\{y1\check{y}\},\check{z})\rangle} \check{y}\} : !_{[\check{z}]}(X_{[\check{x}]} \times Y_{[\check{y}]}) \dashv\!\vdash \emptyset : Y_{[y]}}}}{(!\mathrm{R})\ \dfrac{}{\{y \xrightarrow{\mathrm{id}_Y \mid (\{y1\check{y}\},\check{z})\rangle} \check{y}, w \to \check{z}\} : !_{[\check{z}]}(X_{[\check{x}]} \times Y_{[\check{y}]}) \dashv\!\vdash \emptyset : !_{[w]}Y_{[y]}}}}{(\otimes\mathrm{R})\ \dfrac{}{\{x \xrightarrow{\mathrm{id}_X \mid (\{x1\hat{x}\},\hat{z})\rangle} \hat{x}, v \to \hat{z}\} : !_{[\hat{z}]}(X_{[\hat{x}]} \times Y_{[\hat{y}]}), \{y \xrightarrow{\mathrm{id}_Y \mid (\{y1\check{y}\},\check{z})\rangle} \check{y}, w \to \check{z}\} : !_{[\check{z}]}(X_{[\check{x}]} \times Y_{[\check{y}]}) \dashv\!\vdash \emptyset : !_{[v]}X_{[x]} \otimes !_{[w]}Y_{[y]}}}}{(!\mathrm{C})\ \dfrac{}{\{x \xrightarrow{\mathrm{id}_X \mid (\{x1\hat{x}\},\hat{z})\rangle} \hat{x}, v \to \hat{z}, y \xrightarrow{\mathrm{id}_Y \mid (\{y1\hat{y}\},\hat{z})\rangle} \hat{y}, w \to \hat{z}\} : !_{[\hat{z}]}(X_{[\hat{x}]} \times Y_{[\hat{y}]}) \dashv\!\vdash \emptyset : !_{[v]}X_{[x]} \otimes !_{[w]}Y_{[y]}}}$$

$$\dfrac{\dfrac{\dfrac{\dfrac{(\mathrm{BC})\ \dfrac{}{\{x \xrightarrow{\mathrm{id}_X} \hat{x}\} : X_{[\hat{x}]} \dashv\!\vdash \emptyset : X_{[x]}}}{(!\mathrm{D})\ \dfrac{}{\{x \xrightarrow{\mathrm{id}_X \mid (\{x1\hat{x}\},\hat{z})\rangle} \hat{x}\} : !_{[\hat{z}]}X_{[\hat{x}]} \dashv\!\vdash \emptyset : X_{[x]}}}}{(!\mathrm{W})\ \dfrac{}{\{x \xrightarrow{\mathrm{id}_X \mid (\{x1\hat{x}\},\hat{z})\rangle} \hat{x}\} : !_{[\hat{z}]}X_{[\hat{x}]}, \{x \to \hat{w}\} : !_{[\hat{w}]}Y_{[\hat{y}]} \dashv\!\vdash \emptyset : X_{[x]}}}}{(\otimes\mathrm{L})\ \dfrac{}{\{x \xrightarrow{\mathrm{id}_X \mid (\{x1\hat{x}\},\hat{z})\rangle} \hat{x}, x \to \hat{w}\} : !_{[\hat{z}]}X_{[\hat{x}]} \otimes !_{[\hat{w}]}Y_{[\hat{y}]} \dashv\!\vdash \emptyset : X_{[x]}}} \qquad \dfrac{\dfrac{\dfrac{\dfrac{(\mathrm{BC})\ \dfrac{}{\{y \xrightarrow{\mathrm{id}_Y} \hat{y}\} : Y_{[\hat{y}]} \dashv\!\vdash \emptyset : Y_{[y]}}}{(!\mathrm{D})\ \dfrac{}{\{y \xrightarrow{\mathrm{id}_Y \mid (\{y1\hat{y}\},\hat{w})\rangle} \hat{y}\} : !_{[\hat{w}]}Y_{[\hat{y}]} \dashv\!\vdash \emptyset : Y_{[y]}}}}{(!\mathrm{W})\ \dfrac{}{\{y \xrightarrow{\mathrm{id}_Y \mid (\{y1\hat{y}\},\hat{w})\rangle} \hat{y}\} : !_{[\hat{w}]}Y_{[\hat{y}]}, \{y \to \hat{z}\} : !_{[\hat{z}]}X_{[\hat{x}]} \dashv\!\vdash \emptyset : Y_{[y]}}}}{(\otimes\mathrm{L})\ \dfrac{}{\{y \xrightarrow{\mathrm{id}_Y \mid (\{y1\hat{y}\},\hat{w})\rangle} \hat{y}, y \to \hat{z}\} : !_{[\hat{z}]}X_{[\hat{x}]} \otimes !_{[\hat{w}]}Y_{[\hat{y}]} \dashv\!\vdash \emptyset : Y_{[y]}}}}{(\times\mathrm{R})\ \dfrac{}{\{x \xrightarrow{\mathrm{id}_X \langle x \mid (\{x1\hat{x}\},\hat{z})\rangle} \hat{x}, x \xrightarrow{\langle x|} \hat{w}, y \xrightarrow{\mathrm{id}_Y \langle y \mid (\{y1\hat{y}\},\hat{w})\rangle} \hat{y}, y \xrightarrow{\langle y|} \hat{z}\} : !_{[\hat{z}]}X_{[\hat{x}]} \otimes !_{[\hat{w}]}Y_{[\hat{y}]} \dashv\!\vdash \emptyset : X_{[x]} \times_y Y_{[y]}}}$$

FIGURE 16. Freely generated pre-$\ell^{\circ}_{\mathcal{B}}$-terms for the Seely isomorphisms

Example 3.2.3. The $\ell^{\circ}_{\mathcal{B}}$-diagram $\mathfrak{G}(!_{[e]}\top_{[v]} \dashv\!\vdash \top_{[u]})$ has the freely generated pre-$\ell^{\circ}_{\mathcal{B}}$-net

$$\mathfrak{G}\Big(\big\{\, u \xrightarrow[\langle i\rangle]{|(\{uiv\},e)\rangle} v \mid i \in \overline{n} \,\big\} : !_{[e]}\top_{[v]} \dashv\!\vdash \emptyset : \top_{[u]}\Big)$$

for each $n \in \mathbb{N}_+$, whose derivation is left as an exercise. These pre-$\ell^{\circ}_{\mathcal{B}}$-nets would be all identified without the identifier labels (i) (and the corresponding integers i in the colours). We shall see that the distinction between the pre-$\ell^{\circ}_{\mathcal{B}}$-nets is vital for our graphical calculus to form the free (pointed) linear category. This example thus explains why identifier labels are necessary. This complication comes from the combination of the unit and the exponential of a (pointed) linear category, which does not exist in monoidal, star-autonomous or weakly distributive categories.

Example 3.2.4. The pre-$\ell^{\circ}_{\mathcal{B}}$-terms for the Seely isomorphisms

$$!1 \cong \top, \qquad\qquad !(X \times Y) \cong !X \otimes !Y \quad (X, Y \in \mathrm{Obj}(\mathcal{B}))$$

and their inverses are freely generated in Figure 16.

Instead of the upper right pre-$\ell^{\circ}_{\mathcal{B}}$-term $\{e \to z\} : \top_{[z]} \dashv\!\vdash \emptyset : !_{[e]}1_{[v]}$, we may freely generate the one $\{v \to z\} : \top_{[z]} \dashv\!\vdash \emptyset : !_{[e]}1_{[v]}$ too. Nevertheless, the choice between the two does not matter: they both correspond to the same $\ell^{\circ}_{\mathcal{B}}$-net $\{e \to z\} : \top_{[z]} \dashv\!\vdash \emptyset : !_{[e]}1_{[v]}$, as we shall see later.

By Definition 4.1.8, the objects $X, Y \in \mathrm{Obj}(\mathcal{B})$ in the pre-$\ell^{\circ}_{\mathcal{B}}$-terms can be generalised to any $\ell^{\circ}_{\mathcal{B}}$-types, but Theorem 4.1.15 reduces the generalised case to the present case.

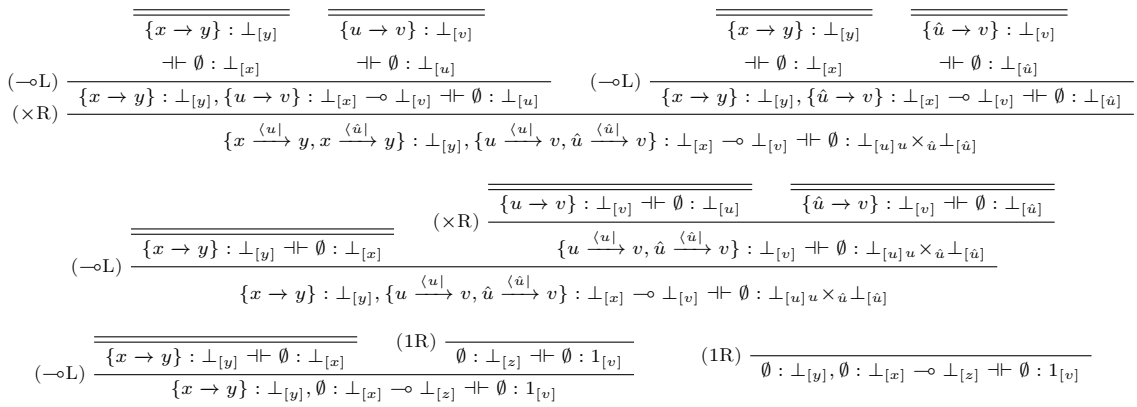


FIGURE 17. Superficially different freely generated pre-$\ell^{\circ}_{\mathcal{B}}$-terms

As seen in Example 3.2.4, freely generated pre-$\ell^{\circ}_{\mathcal{B}}$-nets/terms do *not* faithfully represent morphisms in a free (pointed) linear category over $\mathcal{B}$. Another instance of this sort has been given in Figures 12–13; the corresponding pre-$\ell^{\circ}_{\mathcal{B}}$-terms are freely generated in Figure 17.

As sketched in §2.2, we solve this problem by focusing on $\ell^{\circ}_{\mathcal{B}}$-nets through *valid* sequences in pre-$\ell^{\circ}_{\mathcal{B}}$-nets. For an end v of an $\ell^{\circ}_{\mathcal{B}}$-diagram $\mathfrak{S}$, let $\times^{-}_{\mathfrak{S}}(v)$ be the set of the children of all negative $\times$-vertices in the finite path in $\mathfrak{S}$ from the root of $\mathfrak{S}$ to v; *i.e.* $\times^{-}_{\mathfrak{S}}(v)$ records the binary choices on the negative occurrences of cartesian product $\times$ in $\mathfrak{S}$ made on the side of v. For instance, if $\mathfrak{S} = \mathfrak{G}(((\top_{[w]w} \times_v \bot_{[v]} \multimap \bot_{[u]}) \times \top_{[\hat{u}]} \multimap \bot_{[o]})_o \times_q \bot_{[q]})$, then $\times^{-}_{\mathfrak{S}}(v) = \{o, v\}$. We first need:

Definition 3.2.5 (valid paths)**.** For a pre-$\ell^{\circ}_{\mathcal{B}}$-net $\mathfrak{r} :: \mathfrak{S}$, a finite path in the completion $\bar{\mathfrak{r}}_{\mathfrak{S}}$, *i.e.*

- $v_1 \xrightarrow[\langle i_1\rangle]{(\chi_1)\langle\kappa_1|\epsilon_1\rangle} v_2 \longrightarrow v_3 \xrightarrow[\langle i_2\rangle]{(\chi_2)\langle\kappa_2|\epsilon_2\rangle} v_4 \longrightarrow \cdots \longrightarrow v_{2n-1}$ or
- $v_1 \xrightarrow[\langle i_1\rangle]{(\chi_1)\langle\kappa_1|\epsilon_1\rangle} v_2 \longrightarrow v_3 \xrightarrow[\langle i_2\rangle]{(\chi_2)\langle\kappa_2|\epsilon_2\rangle} v_4 \longrightarrow \cdots \longrightarrow v_{2n-1} \xrightarrow[\langle i_n\rangle]{(\chi_n)\langle\kappa_n|\epsilon_n\rangle} v_{2n}$,

is said to be *valid*, where the $\mathcal{B}$-arcs *consumed* in the path, if, for all $i, i' \in \overline{2n-1}$ and $j, j' \in \overline{n}$,

(1) v_1 is initial in $\mathfrak{S}$, and no 1-ends occur in the path (as some of v_k for $k \in \overline{2n}$);
(2) v_{2j-1} has a label $Y \in \mathrm{Obj}(\mathcal{B})$ if and only if v_{2j} has $X \in \mathrm{Obj}(\mathcal{B})$ with some $\chi_j : X \to Y$ in $\mathcal{B}$ being the arrowy label of the $\mathcal{B}$-arc from v_{2j-1} to v_{2j}, and v_{2j} is joker if so is v_{2j-1};
(3) For any $\ell^{\circ}_{\mathcal{B}}$-tree $\mathfrak{T}$ that is a subtree of $\mathfrak{S}$ and $i < i'$, if v_i and $v_{i'}$ are ends of $\mathfrak{T}$ such that no $v_{i''}$ with $i < i'' < i'$ is of $\mathfrak{T}$, then v_i and $v_{i'}$ have different parities (6) in $\mathfrak{T}$;
(4) $\kappa_j \subseteq \bigcup_{k=1}^{2j-1} \times^{-}_{\mathfrak{S}}(v_k)$;
(5) For any positive !-end e of $\mathfrak{S}$, if v_{2j-1} or v_{2j} is of $\mathfrak{S}^e$ yet not e, then $(c, e) \in \epsilon_j$ for some colour c, where $v_{2j-1} i_j v_{2j} \in c$ if $v_{2j-1} \notin \mathfrak{S}^e$ and $v_{2j} \in \mathfrak{S}^e$, and $(c', e) \in \epsilon_{j'}$ implies $c = c'$.

By the first axiom, v_1 is initial, and v_i and v_{i+1} have labels different from 1. Note that v_i and v_{i+1} must have different parities. This alternation holds *locally* as well by the third axiom; see Example 3.2.9. Similarly to [MO03, Axiom (j1), §8], the first part of the second axiom respects X-ends in the both directions, and the second part joker ends in the forward direction. The last two axioms come respectively from the typing rules $\times$R and !C; see Examples 3.2.10–3.2.11.

Convention. A valid path is a certain finite path in the *completion* of the $\mathcal{B}$-realiser of a pre-$\ell^{\circ}_{\mathcal{B}}$-net, but by convention we also say that it is in the pre-$\ell^{\circ}_{\mathcal{B}}$-net itself.

We now define valid sequences based on valid paths, for which *moves*–certain finite sequences of ends of length one or two–are introduced. The main idea here, which is justified ultimately by

our freeness result (Theorem 4.2.6), is as follows. First, an $\ell^{\circ}_{\mathcal{B}}$-tree of the form $\mathfrak{G}(!_{[e]}T)$, where $\mathfrak{T} := \mathfrak{G}T$, is to be seen informally as the infinite tree

$$\mathfrak{G}(T \otimes T \otimes \cdots). \tag{8}$$

As it suffices for a morphism in a free (pointed) linear category to handle each copy of $\mathfrak{T}$ in (8) *in the same way*, we may focus on the first copy (or the *base case*) and the next copy among those not used yet (or the *inductive step*), similarly to mathematical induction. Let a move $v \in \mathrm{End}_{\mathfrak{T}}$ of length one belong to the base case, and a move ev of length two to the inductive step. Yet, as multiple !-ends may occur in an $\ell^{\circ}_{\mathcal{B}}$-tree in a nested way, apparently we also need $e_1e_2 \dots e_nv$ for $n > 1$, where each e_i ($i \in \overline{n}$) is an !-end. However, it suffices for our aim to focus on *causality* or $\mathcal{B}$-arcs between !-ends, with respect to which the !-ends e_i are pairwise *orthogonal*. As a result, we may decompose the finite sequence $e_1e_2 \dots, e_nv$ into those e_iv and thus focus on the case of $n \in \overline{2}$; see Example 3.2.12. In contrast, e_i and v in e_iv are *inseparable*, as the third axiom in Definition 3.2.6 is on moves, not their elements separately.

Definition 3.2.6 (valid sequences)**.** A *valid sequence* in a pre-$\ell^{\circ}_{\mathcal{B}}$-net $\mathfrak{r} :: \mathfrak{S}$ is a finite sequence $\boldsymbol{s} = (\boldsymbol{v}_i)_{i=1}^{n}$ of nonempty sequences $\boldsymbol{v}_i$, called *moves*, of ends of $\mathfrak{S}$ of length at most two, whose first and last elements are written $!\boldsymbol{v}_i$ and $\sharp\boldsymbol{v}_i$, called its *outer* and *core*, respectively,[12] such that

(1) There is a valid path in $\mathfrak{r} :: \mathfrak{S}$ whose vertices are the cores $\sharp\boldsymbol{v}_i$ for $i = 1, 2, \dots, n$;
(2) If $\boldsymbol{v}_i = e_ix_i$ ($i \in \overline{n}$), then e_i is an !-end with $x_i \in \mathrm{End}_{\mathfrak{S}^{e_i}}$ and $e_i \neq x_i$, where $\boldsymbol{v}_j = e_jx_j$ with $x_i \in \mathrm{End}_{\mathfrak{S}^{e_j}}$ and $e_j \neq x_i$ for some $j < i$ implies $|\boldsymbol{v}_i| = 2$;
(3) If $1 < 2i \leqslant n$, then $|\boldsymbol{v}_{2i-1}| \geqslant |\boldsymbol{v}_{2i}|$, and $|\boldsymbol{v}_{2i-1}| = 2$ implies
  (a) Unless $!\boldsymbol{v}_{2i-1}$ is initial, $!\boldsymbol{v}_{2i-1} = !\boldsymbol{v}_{2j}$ or $!\boldsymbol{v}_{2i-1}$ is justified by $\sharp\boldsymbol{v}_{2j}$ for some $j < i$;
  (b) For every $j \leqslant i$ such that $|\boldsymbol{v}_{2j-1}| = 2$, we have $\sharp\boldsymbol{v}_{2i} \in \mathrm{End}^{\mathrm{nIni}}_{\mathfrak{S}^{!v_{2k}}}$ and $\sharp\boldsymbol{v}_{2i} \neq !\boldsymbol{v}_{2k}$ for some $j \leqslant k \leqslant i$ such that $\mathfrak{r}$ has a $\mathcal{B}$-arc from $!\boldsymbol{v}_{2j-1}$ to $!\boldsymbol{v}_{2k}$ whose exponential label $|\kappa\rangle$ satisfies $\kappa \subseteq \bigcup_{l=1}^{2j-1} \times^{-}_{\mathfrak{S}}(\sharp\boldsymbol{v}_l)$;
  (c) Whenever $1 < 2i < n$, if $\sharp\boldsymbol{v}_{2i}$ (respectively, $!\boldsymbol{v}_{2i}$) has a mate in $\mathfrak{S}$ (Definition 3.1.2), then $\sharp\boldsymbol{v}_{2i+1}$ (respectively, $!\boldsymbol{v}_{2i+1}$) is the mate with $|\boldsymbol{v}_{2i}| = |\boldsymbol{v}_{2i+1}|$,

labelled with the finite sequence of the arrowy labels of the $\mathcal{B}$-arcts in the valid path (1), where the $\mathcal{B}$-arcs consumed in the valid path and the one in (3b) are said to be *consumed* in $\boldsymbol{s}$. Two valid sequences are *equal* if they are the same finite sequence with the same label.

Convention. We often suppress the label part of a valid sequence, especially when it is empty, unless it is essential. If a $\mathcal{B}$-arc a is consumed in a valid sequence $\boldsymbol{s}$ for the first or the third axiom in Definition 3.2.6, where the cartesian label $\langle\kappa|$ of a satisfies the required condition $\kappa \subseteq \bigcup_{k=1}^{2j-1} \times^{-}_{\mathfrak{S}}(v_k)$, then the set $\bigcup_{k=1}^{2j-1} \times^{-}_{\mathfrak{S}}(v_k)$ is called the *cartesian condition* for a in $\boldsymbol{s}$.

The finite sequence $\sharp\boldsymbol{s} := \sharp\boldsymbol{v}_1.\sharp\boldsymbol{v}_2 \dots \sharp\boldsymbol{v}_n$ is called the *core* of a valid sequence $\boldsymbol{s}$. A move $\boldsymbol{v}$ is said to be *initial* (respectively, *negative*, *positive* and *joker*) in $\mathfrak{S}$ if so is its core $\sharp\boldsymbol{v}$.

Notation. Since a valid sequence is a finite sequence of finite sequences (*viz.* moves), we write $\boldsymbol{v}_1.\boldsymbol{v}_2 \dots \boldsymbol{v}_n$ for it, where a dot indicates a boarder between moves. Let $\mathrm{VS}(\mathfrak{r} :: \mathfrak{S})$ denote the set of all valid sequences in a pre-$\ell^{\circ}_{\mathcal{B}}$-net $\mathfrak{r} :: \mathfrak{S}$.

So far, for simplicity, we illustrated valid sequences containing only moves of length one. The first axiom in Definition 3.2.6 says that the core of a valid sequence comes from a valid path, and the second imposes the form of moves of length two. The equation $|\boldsymbol{v}_{2i-1}| = |\boldsymbol{v}_{2i}|$ of the last axiom means that, when a new copy of a negative $\ell^{\circ}_{\mathcal{B}}$-tree is used, then a new copy of a positive $\ell^{\circ}_{\mathcal{B}}$-tree must be used. The condition (3a) says that the outer of a negative move of length two

[12] We thus have $!\boldsymbol{v}_i = \sharp\boldsymbol{v}_i$ when $|\boldsymbol{v}_i| = 1$.

must be initial or justified, and the one (3b) that the outer of a positive move of length two must be calculated by $\mathcal{B}$-arcs in $\mathfrak{r}$. We illustrate these axioms by examples below.

Example 3.2.7. Check that examples of valid paths/sequences given in previous sections conform to Definitions 3.2.5–3.2.6.

Example 3.2.8. Consider the four pre-$\ell^{\bullet}_{\mathcal{B}}$-nets



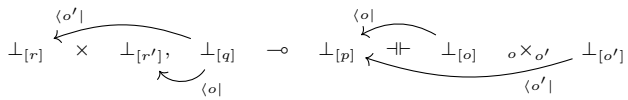
By the first two axioms in Definition 3.2.5, the finite sequence $x.y$ is valid in all but the second pre-$\ell^{\bullet}_{\mathcal{B}}$-net from the left. Note that all but the second one are freely generated.

Example 3.2.9. Consider the two pre-$\ell^{\bullet}_{\mathcal{B}}$-nets



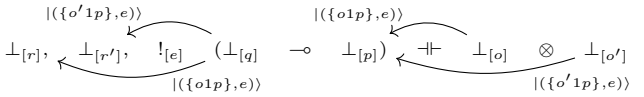
The finite path $i \to j \to k \to l$ is invalid in the left one by the third axiom in Definition 3.2.5 with respect to the $\ell^{\bullet}_{\mathcal{B}}$-tree $\mathfrak{G}(\bot_{[j]} \otimes \bot_{[l]})$, but it is valid in the right one (and so is the sequence $i.j.k.l$). As we will see, this difference captures that only the right one is freely generated.

Example 3.2.10. Consider the freely generated pre-$\ell^{\bullet}_{\mathcal{B}}$-net



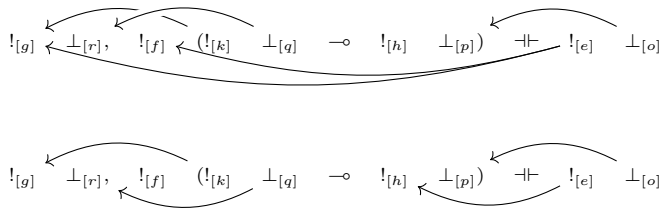
whose derivation is left to the reader. The finite sequences $o.p.q.r'$ and $o'.p.q.r$ are valid in this pre-$\ell^{\bullet}_{\mathcal{B}}$-net, but not the one $o.p.q.r$ or $o'.p.q.r'$ by the fourth axiom in Definition 3.2.5.

Example 3.2.11. Consider the freely generated pre-$\ell^{\bullet}_{\mathcal{B}}$-net



whose derivation is left to the reader. Its maximal valid sequences include $o.p.q.r$ and $o'.p.q.r'$, but not $o.p.q.r'$ or $o'.p.q.r$ by the last axiom in Definition 3.2.5.

Example 3.2.12. Consider the two freely generated pre-$\ell^{\bullet}_{\mathcal{B}}$-nets



whose derivations are left to the reader. The upper one has the maximal valid sequences

$e.f$, $e.g$, $o.p.k.g$, $o.p.q.r$, $o.p.kq.gr$, $eo.fp.fk.g$, $eo.fp.fq.gr$, $eo.fp.kq.gr$.

The move gr in $eo.fp.q.gr$ illustrates the point in the axiom (3b) that the outer of a move is not necessarily triggered by the immediately preceding move (*i.e.* the axiom has $k \leqslant i$, not $k = i$). Those q and kq in $eo.fp.k.g$, $eo.fp.q.gr$ or $eo.fp.kq.gr$ are meant to be abbreviations of fq and fkq, respectively (as discussed before Definition 3.2.6). These abbreviations are harmless by the focus on *initial* ends in the second axiom in Definition 3.2.6; *e.g.* k, not f, triggers g in $eo.fp.k.g$.

In contrast, the lower pre-$\ell^{\bullet}_{\mathcal{B}}$-net has the maximal valid sequences

$$e.h.k.g, \qquad e.h.q.r, \qquad e.h.kq.gr, \qquad o.p.k.g, \qquad o.p.q.r, \qquad o.p.kq.gr,$$
$$eo.hp.k.g, \qquad eo.hp.q.r, \qquad eo.hp.kq.gr.$$

We now proceed to a geometric characterisation of freely generated pre-$\ell^{\bullet}_{\mathcal{B}}$-nets, up to prime form, in terms of valid sequences. The first step towards this goal is:

Definition 3.2.13 (colourings)**.** A *colouring* on a pre-$\ell^{\circ}_{\mathcal{B}}$-net $\mathfrak{r} :: \mathfrak{S}$ is a function ι that maps each positive !-end e of $\mathfrak{S}$ to a colour ιe such that the pair $(\iota e, e)$ occurs in the exponential label of some $\mathcal{B}$-arc in $\mathfrak{r}$, and a valid sequence $\boldsymbol{s}$ in $\mathfrak{r} :: \mathfrak{S}$ is said to *at* ι if every pair (c, e) in the exponential label of a $\mathcal{B}$-arc used in $\boldsymbol{s}$ satisfies $c = \iota e$.

Definition 3.2.14 (coverage)**.** A pre-$\ell^{\circ}_{\mathcal{B}}$-net $\mathfrak{r} :: \mathfrak{S}$ is said to *cover* a positive end $v \in \mathrm{End}^{+}_{\mathfrak{S}}$ *with respect to a colouring* ι *on* $\mathfrak{r} :: \mathfrak{S}$ if at least one of the following holds:

(1) v occurs (as an element of one of the moves) in a valid sequence in $\mathfrak{r} :: \mathfrak{S}$ at ι;
(2) $\mathfrak{S}$ has a negative 1-end o that justifies a positive end p such that $v \in \mathrm{End}_{\mathfrak{S}^p}$;
(3) $\mathfrak{S}$ has a subtree of the form $\mathfrak{G}(U \times U')$, where both U and U' are either $\ell^{\circ}_{\mathcal{B}}$-types or $\ell^{\circ}_{\mathcal{B}}$-cuts, rooted at a positive $\times$-vertex such that one of $\mathfrak{G}U$ and $\mathfrak{G}U'$ contains v, and the other has an end occurring in a valid sequence in $\mathfrak{r} :: \mathfrak{S}$ at ι;
(4) $\mathfrak{S}$ has a positive !-end $e \neq v$ occurring in a valid sequence in $\mathfrak{r} :: \mathfrak{S}$ at ι with $v \in \mathrm{End}_{\mathfrak{S}^e}$;
(5) v is an !-end, and $\mathfrak{S}^v$ has a positive end $p \neq v$ in a valid sequence in $\mathfrak{r} :: \mathfrak{S}$ at ι,

and $\mathfrak{r} :: \mathfrak{S}$ is said to *cover* v if it covers v for every colouring on it.

The idea is that a freely generated pre-$\ell^{\circ}_{\mathcal{B}}$-net $\mathfrak{r} :: \mathfrak{S}$ has enough valid sequences to visit all positive ends of $\mathfrak{S}$ except those related to the typing rules 1R, $\times$L, !W and !D, corresponding to the last four cases in Definition 3.2.14.

Example 3.2.15. Consider the pre-$\ell^{\bullet}_{\mathcal{B}}$-term freely generated by

$$(1\mathrm{R})\ \frac{}{\emptyset : (\bot_{[v]} \multimap \bot_{[u]}) \multimap \bot_{[p]} \dashv\vdash \emptyset : 1_{[o]}}$$

The positive ends v and p are both covered by the empty valid sequence ε in the corresponding freely generated pre-$\ell^{\bullet}_{\mathcal{B}}$-net due to the second axiom in Definition 3.2.14.

Example 3.2.16. Consider the pre-$\ell^{\bullet}_{\mathcal{B}}$-term freely generated by

$$(!\mathrm{W})\ \frac{(!\mathrm{D})\ \dfrac{(\times\mathrm{L})\ \dfrac{\overline{\overline{\{q \to r\} : \bot_{[r]} \dashv\vdash \emptyset : \bot_{[q]}}}}{\{q \to r\} : \bot_{[r]} \times \bot_{[r']} \dashv\vdash \emptyset : \bot_{[q]}}}{\{q \xrightarrow{|(\{q1r\},e)\rangle} r\} : !_{[e]}(\bot_{[r]} \times \bot_{[r']}) \dashv\vdash \emptyset : \bot_{[q]}}}{\{q \xrightarrow{|(\{q1r\},e)\rangle} r\} : !_{[e]}(\bot_{[r]} \times \bot_{[r']}), \{q \to e'\} : !_{[e']}\bot_{[r'']} \dashv\vdash \emptyset : \bot_{[q]}}$$

The positive ends e, r and r' are covered by the valid sequence $q.r$ in the corresponding pre-$\ell^{\bullet}_{\mathcal{B}}$-net due to the last, the first and the third axiom in Definition 3.2.14, respectively, while those e' and r'' by the valid sequence $q.e'$ due to the first and the fourth axiom, respectively.

We shall then *graphically* characterise freely generated pre-$\ell^{\circ}_{\mathcal{B}}$-nets up to *prime form* by *correctness* and *injectivity*. For the injectivity of pre-$\ell^{\circ}_{\mathcal{B}}$-nets to work well, we need two auxiliary idempotent operators on pre-$\ell^{\circ}_{\mathcal{B}}$-nets–*slicing* and *shredding*–as illustrated in Example 3.2.27. The two operators *reverse* the typing rules $\times$R and !C for pairing and contraction, respectively.

Definition 3.2.17 (slices)**.** A *slice assignment* on an $\ell^{\circ}_{\mathcal{B}}$-diagram $\mathfrak{S}$ is an assignment ϑ to each negative $\times$-vertex v of $\mathfrak{S}$ one ϑv of its children, where let $\mathrm{Chi}(\times^{-}_{\mathfrak{S}})$ denote the set of all these

children. A *slice* of a pre-$\ell^{\circ}_{\mathcal{B}}$-net $\mathfrak{r} :: \mathfrak{S}$ at ϑ is the pre-$\ell^{\circ}_{\mathcal{B}}$-net, written $\mathfrak{r}/_{\vartheta} :: \mathfrak{S}$, defined by

$$E_{\mathfrak{r}/_{\vartheta}} := \Big\{ (o \xrightarrow[\langle i \rangle]{(\chi)\langle \kappa | \epsilon \rangle} p) \in E_{\mathfrak{r}} \mid \kappa \cap (\mathrm{Chi}(\times^{-}_{\mathfrak{S}}) \setminus \mathrm{Ran}(\vartheta)) = \emptyset, o, p \in \mathrm{End}_{\mathfrak{S}/_{\vartheta}} \Big\}. \tag{9}$$

A pre-$\ell^{\circ}_{\mathcal{B}}$-net is said to be *sliced* if any of its slices is the pre-$\ell^{\circ}_{\mathcal{B}}$-net itself. Slicing is clearly an *idempotent* operation in the sense that every slice of a pre-$\ell^{\circ}_{\mathcal{B}}$-net is sliced.

Example 3.2.18. The example of slicing in §2.2 conforms to Definition 3.2.17, where the two pre-$\ell^{\bullet}_{\mathcal{B}}$-nets in Figure 12 share the same two slices. The condition $\kappa \cap (\mathrm{Chi}(\times^{-}_{\mathfrak{S}}) \setminus \mathrm{Ran}(\vartheta)) = \emptyset$ in the formula (9) cannot be replaced with the simpler one $\kappa \cap \mathrm{Ran}(\vartheta) \neq \emptyset$ since the latter does not work for the right pre-$\ell^{\bullet}_{\mathcal{B}}$-net in Figure 12; this point is crucial for the *injectivity* of pre-$\ell^{\bullet}_{\mathcal{B}}$-nets (Definition 3.2.25) to work, as illustrated in Example 3.2.27.

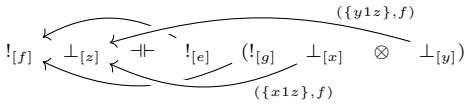
Definition 3.2.19 (shredding)**.** A *shredding assignment* on a pre-$\ell^{\circ}_{\mathcal{B}}$-net $\mathfrak{r} :: \mathfrak{S}$ is a function φ that maps a positive !-end e of $\mathfrak{S}$ to a colour $\varphi(e)$ such that the pair $(\varphi(e), e)$ is in the exponential label of a $\mathcal{B}$-arc in $\mathfrak{r}$, and the *shredding* of $\mathfrak{r} :: \mathfrak{S}$ at φ is the pre-$\ell^{\circ}_{\mathcal{B}}$-net $\mathfrak{r}/^{\varphi} :: \mathfrak{S}$ is defined by

$$E_{\mathfrak{r}/^{\varphi}} := \bigcup_{e \in \mathrm{End}^{+!}_{\mathfrak{S}}} \{\, a \in \mathfrak{r} \mid a \text{ is not consumed in an element of } \mathrm{VS}(\mathfrak{r} :: \mathfrak{S})_{(c,e)} \text{ for any } c \neq \varphi(e) \,\},$$

where $\mathrm{VS}(\mathfrak{r} :: \mathfrak{S})_{(c,e)}$ is the set of all valid sequences in $\mathfrak{r} :: \mathfrak{S}$ in which every consumed $\mathcal{B}$-arc with its exponential label containing a pair of the form (c', e) satisfies $c = c'$.

Example 3.2.20. The example of shredding in §2.2 conforms to Definition 3.2.19,

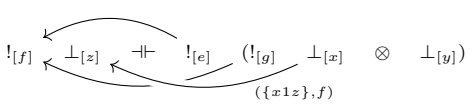
Example 3.2.21. The freely generated pre-$\ell^{\circ}_{\mathcal{B}}$-net

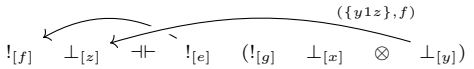


has the two shredding assignments

$$f \mapsto \{x1z\}, \qquad\qquad f \mapsto \{y1z\},$$

and the corresponding shreddings are

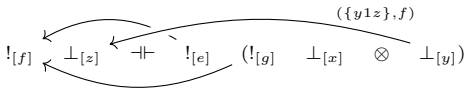




If we defined the edge set $E_{\mathfrak{r}/^{\varphi}}$ in Definition 3.2.19 instead by

$$E_{\mathfrak{r}/^{\varphi}} := \bigcup_{e \in \mathrm{End}^{+!}_{\mathfrak{S}}} \{\, a \in \mathfrak{r} \mid a \text{ is consumed in an element of } \mathrm{VS}(\mathfrak{r} :: \mathfrak{S})_{(\varphi(e),e)} \,\},$$

then the second shredding would be (incorrectly)



The last point is important for the injectivity of pre-$\ell^{\bullet}_{\mathcal{B}}$-nets to work well; see Example 3.2.27.

A pre-$\ell^{\circ}_{\mathcal{B}}$-net is said to be *shredded* if it coincides with all its shredding. Shredding is an idempotent operator in the sense that every shredding of a pre-$\ell^{\circ}_{\mathcal{B}}$-net is shredded.

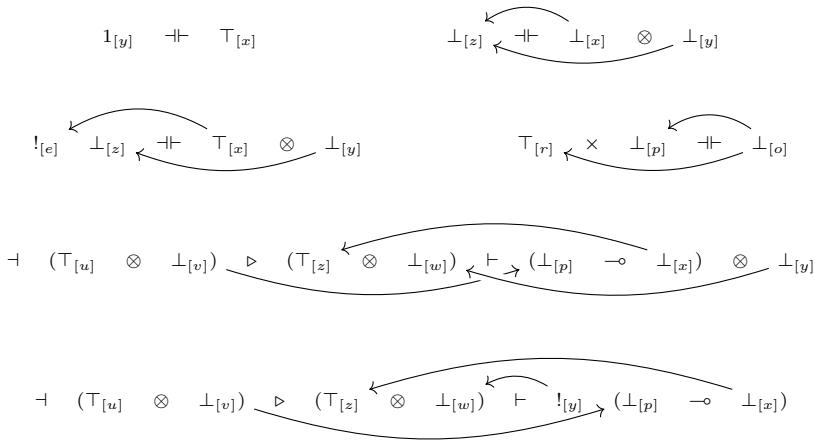
Notation. Given an $\ell^{\circ}_{\mathcal{B}}$-diagram $\mathfrak{S}$, let $\mathrm{End}^{\mathrm{nJkr}}_{\mathfrak{S}} := \mathrm{End}_{\mathfrak{S}} \setminus \mathrm{End}^{\mathrm{Jkr}}_{\mathfrak{S}}$ and $\mathrm{End}^{\mathrm{nIni}}_{\mathfrak{S}} := \mathrm{End}_{\mathfrak{S}} \setminus \mathrm{End}^{\mathrm{Ini}}_{\mathfrak{S}}$.

Definition 3.2.22 (correctness)**.** A pre-$\ell^{\circ}_{\mathcal{B}}$-net $\mathfrak{r} :: \mathfrak{S}$ is said to be *correct* if any shredding $\mathfrak{r}/^{\varphi}_{\vartheta} :: \mathfrak{S}$ of any slice $\mathfrak{r}/_{\vartheta} :: \mathfrak{S}$ of $\mathfrak{r} :: \mathfrak{S}$ satisfies:

(1) Each positive end of $\mathfrak{S}$ is covered by a valid sequence in $\mathfrak{r}/^{\varphi}_{\vartheta} :: \mathfrak{S}$;
(2) The relation $\mathrm{Fun}(\mathfrak{r} :: \mathfrak{S}) := \{\, (\boldsymbol{s}.\boldsymbol{x}, \boldsymbol{y}) \mid \boldsymbol{s}.\boldsymbol{x}.\boldsymbol{y} \in \mathrm{VS}(\mathfrak{r} :: \mathfrak{S})^{\mathrm{Even}}, \sharp\boldsymbol{x}, \sharp\boldsymbol{y} \in \mathrm{End}^{\mathrm{nJkr}}_{\mathfrak{S}} \,\}$ defines a (total) injection on non-joker cores in the sense that it satisfies
   (a) $|\{\, \sharp\boldsymbol{y} \mid (\boldsymbol{s}.\boldsymbol{x}, \boldsymbol{y}) \in \mathrm{Fun}(\mathfrak{r} :: \mathfrak{S}) \,\}| = 1$ for each $\boldsymbol{s}.\boldsymbol{x} \in \mathrm{VS}(\mathfrak{r} :: \mathfrak{S})^{\mathrm{Odd}}$ with $\sharp\boldsymbol{x} \in \mathrm{End}^{\mathrm{nJkr}}_{\mathfrak{S}}$;
   (b) If $(\boldsymbol{s}.\boldsymbol{x}, \boldsymbol{y}), (\boldsymbol{t}.\boldsymbol{u}, \boldsymbol{v}) \in \mathrm{Fun}(\mathfrak{r} :: \mathfrak{S})$ with $\sharp\boldsymbol{x}, \sharp\boldsymbol{u} \in \mathrm{End}^{\mathrm{nJkr}}_{\mathfrak{S}}$ and $\sharp\boldsymbol{x} \neq \sharp\boldsymbol{u}$, then $\sharp\boldsymbol{y} \neq \sharp\boldsymbol{v}$;
(3) The partial function $\mathrm{Fun}(\mathfrak{r} :: \mathfrak{S}) : \mathrm{MV}(\mathfrak{r} :: \mathfrak{S})^{\mathrm{Odd}} \rightharpoonup \mathrm{MV}(\mathfrak{S})^{+}$ that maps an odd-length valid sequence $\boldsymbol{s}$ in $\mathfrak{r} :: \mathfrak{S}$ to
   its extension $\boldsymbol{s}.\boldsymbol{x}$, if any, is a (total) injection;
(4) For a valid sequence $\boldsymbol{s}$ in $\mathfrak{r}/^{\varphi}_{\vartheta} :: \mathfrak{S}$ and a subtree $\mathfrak{G}(U_1 \times U_2)$ or $\mathfrak{G}!_{[e]}U$ of $\mathfrak{S}$, where U_1, U_2 and U range over positive $\ell^{\circ}_{\mathcal{B}}$-types and $\ell^{\circ}_{\mathcal{B}}$-cuts, at most $\mathfrak{G}U_1$ or $\mathfrak{G}U_2$ has an end in $\boldsymbol{s}$, and at most one of three forms, e, x or ex ($x \in \mathrm{End}_U$), is a move in $\boldsymbol{s}$;
(5) Given a negative and a positive $\ell^{\circ}_{\mathcal{B}}$-subtrees of $\mathfrak{S}$ of the forms $\mathfrak{G}(Q \otimes Q')$ (respectively, $\mathfrak{G}(!_{[e]}Q)$) and $\mathfrak{G}(N \triangleright P)$, respectively, if $\mathfrak{S}$ has two $\mathcal{B}$-arcs between $\mathfrak{G}P$ and $\mathfrak{G}Q$ and between $\mathfrak{G}P$ and $\mathfrak{G}Q'$ (respectively, between $\mathfrak{G}P$ and e and between $\mathfrak{G}P$ and $\mathfrak{G}Q$), then it has none between $\mathfrak{G}N$ and $\mathfrak{G}(Q \otimes Q')$ (respectively, between $\mathfrak{G}N$ and $\mathfrak{G}!_{[e]}Q$).

The four axioms are used in a proof of the geometric characterisation (Theorem 3.2.44). Their roles are illustrated by examples below. For the correctness of a pre-$\ell^{\circ}_{\mathcal{B}}$-net, we first slice and shred it since otherwise some of the last three axioms might fail even when it should not.

Example 3.2.23. Consider the four sliced, shredded pre-$\ell^{\circ}_{\mathcal{B}}$-nets

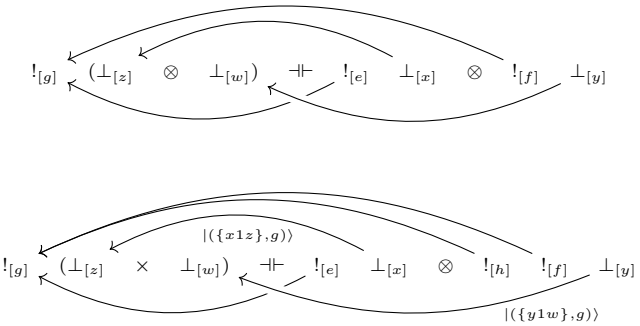


Neither of them is freely generated (as we will see by the geometric characterisation). The upmost two are incorrect due to the first and the second axioms in Definition 3.2.22, respectively, the next two due to the third axiom, and the remaining two due to the last axiom.

Example 3.2.24. All the pre-$\ell^{\bullet}_{\mathcal{B}}$-nets but the second from the left in Example 3.2.8 are correct; the second does not satisfy the first axiom in Definition 3.2.22. The right one in Example 3.2.9 is correct, but the left one is not since it does not satisfy the first axiom. All the pre-$\ell^{\bullet}_{\mathcal{B}}$-nets in Examples 3.2.10–3.2.12 are correct.

Definition 3.2.25 (injectivity)**.** A $\mathcal{B}$-arc in a pre-$\ell^{\circ}_{\mathcal{B}}$-net $\mathfrak{r} :: \mathfrak{S}$ between !-ends e and f is said to be *promotional* if $\mathfrak{r}$ has another $\mathcal{B}$-arc from an end x of $\mathfrak{S}^{e}$ such that $x \neq e$ to an end y of $\mathfrak{S}^{f}$. A pre-$\ell^{\circ}_{\mathcal{B}}$-net is said to be *injective* if any shredding of any slice of it has no pair of non-promotional $\mathcal{B}$-arcs sharing the same target unless their sources occur in the same valid sequence.

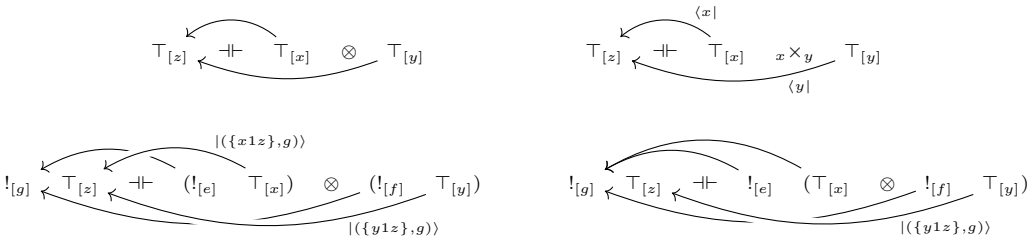
Example 3.2.26. Consider the two pre-$\ell^{\bullet}_{\mathcal{B}}$-nets



The upper one is correct yet not injective, while the lower one is correct and injective. We leave it as an exercise to check that the upper one is not finitely generated, but the lower one is.

We will show that injective, correct pre-$\ell^{\circ}_{\mathcal{B}}$-nets coincide, up to prime form, with freely generated ones (Theorem 3.2.44), where promotional $\mathcal{B}$-arcs are generated by the typing rule !R. Therefore, the exception for promotional ones in the injectivity of pre-$\ell^{\circ}_{\mathcal{B}}$-nets comes from this typing rule, as illustrated in the lower right pre-$\ell^{\circ}_{\mathcal{B}}$-net in Example 3.2.27 below.

Example 3.2.27. Consider the four pre-$\ell^{\circ}_{\mathcal{B}}$-nets



All of them are correct, while only the upper left one is not injective (but it is an $\ell^{\circ}_{\mathcal{B}}$-net as we will see). We leave it to the reader to show that all the pre-$\ell^{\circ}_{\mathcal{B}}$-nets except the upper left one are finitely generated, where their injectivity is protected against the typing rules $\times$R, !C and !R respectively by slicing, shredding and the exception for promotional $\mathcal{B}$-arcs in the injectivity.

Next, we henceforth focus on pre-$\ell^{\circ}_{\mathcal{B}}$-nets in *prime form*:

Notation. Given a $\mathcal{B}$-arc a, let

$$\underline{a} := \left(u \xrightarrow[\langle 1\rangle]{(\chi)\langle\kappa|\epsilon\{u1v/uiv\}\rangle} v\right) \quad \text{if } a = \left(u \xrightarrow[\langle i\rangle]{(\chi)\langle\kappa|\epsilon\rangle} v\right),$$

where the operator $_\{u1v/uiv\}$ replaces the element uiv in a colour, if any, with the one $u1v$, and it is lifted to a pair $(c, e) \in \mathrm{Col} \times \mathscr{V}$ by $(c, e)\{u1v/uiv\} := (c\{u1v/uiv\}, e)$ and further to a finite set ϵ of these pairs in the pointwise fashion.

Definition 3.2.28 (prime)**.** Among $\ell^{\circ}_{\mathcal{B}}$-diagrams isomorphic to a given $\ell^{\circ}_{\mathcal{B}}$-diagram $\mathfrak{S}$ in the category of labelled graphs and label-preserving graph morphisms, fix an arbitrary representative, written $\rho\mathfrak{S}$. An $\ell^{\circ}_{\mathcal{B}}$-diagram of the form $\rho\mathfrak{S}$ for some $\ell^{\circ}_{\mathcal{B}}$-diagram $\mathfrak{S}$ is said to be *prime*.

A pre-$\ell^{\circ}_{\mathcal{B}}$-net $\mathfrak{r} :: \mathfrak{S}$ is said to be *prime* if so is $\mathfrak{S}$, and moreover:

(1) For each $\mathfrak{B}$-arc $a = (u \xrightarrow[\langle i_0 \rangle]{(\chi)\langle\kappa|\epsilon\rangle} v)$ in $\mathfrak{r}$, if $\mathfrak{r}$ has exactly n ($\geqslant 1$) $\mathfrak{B}$-arcs a_i ($i \in \overline{n}$) with $\underline{a_i} = \underline{a}$ (*n.b.* a is one of a_i), then ϵ is of the form $\epsilon = \epsilon_0 \cup \{(\{ui_0v\} \cup c_0, e)\}$, and

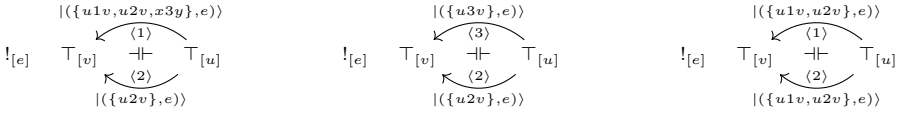
$$\{a_i\}_{i\in\overline{n}} = \left\{ u \xrightarrow[\langle i \rangle]{\langle\kappa|\epsilon_0\cup\{(\{uiv\}\cup c_0,e)\}\rangle} v \mid i \in \overline{n} \right\};$$

(2) For each valid sequence $\boldsymbol{s}$ in $\mathfrak{r} :: \mathfrak{S}$, all subset relations required by the fourth axiom in Definition 3.2.5 and the condition (3b) in Definition 3.2.6 on $\boldsymbol{s}$ are equality;
(3) For each positive !-end e of $\mathfrak{S}$, the finite family of colours c such that (c, e) occurs in the exponential label of a $\mathcal{B}$-arc in $\mathfrak{r}$, whose target is of $\mathfrak{S}^e$ yet not e, and source is not of $\mathfrak{S}^e$, forms a partition of $\mathrm{Col}(\{\, a \in \mathfrak{r} \mid \mathrm{tgt}(a) \in \mathrm{End}_{\mathfrak{S}^e} \setminus \{e\}, \mathrm{src}(a) \notin \mathrm{End}_{\mathfrak{S}^e} \,\})$;
(4) Among pre-$\ell^{\bullet}_{\mathcal{B}}$-nets on $\mathfrak{S}$ satisfying the above three conditions and sharing exactly the same valid sequences, $\mathfrak{r}$ is minimal in its *size* $\|\mathfrak{r}\| \in \mathbb{N}$ defined by

$$\|\mathfrak{r}\| := \sum_{a\in E_{\mathfrak{r}}} (|\mathrm{car}(a)| + |\mathrm{exp}(a)| + 1). \tag{10}$$

Example 3.2.29. Recall that not $\mathcal{B}$-realisers themselves but their completions are drawn in the following examples. Assume that the $\ell^{\circ}_{\mathcal{B}}$-diagrams of the pre-$\ell^{\circ}_{\mathcal{B}}$-nets given below are all prime.

The left pre-$\ell^{\bullet}_{\mathcal{B}}$-net in Figure 12 is prime, while the right one is not due to the second axiom in Definition 3.2.28. The right pre-$\ell^{\bullet}_{\mathcal{B}}$-net in Figure 13 is prime, but the left one is not due to the last axiom. The pre-$\ell_{\mathcal{B}}$-net in Example 3.2.3 is prime, while those



except the last one are not due to the first and the third axioms. All the pre-$\ell^{\circ}_{\mathcal{B}}$-nets mentioned above are correct, where only the last one is not *injective* (Definition 3.2.34).

Proposition 3.2.30 (uniqueness of prime pre-$\ell^{\circ}_{\mathcal{B}}$-nets)**.** *If prime pre-$\ell^{\circ}_{\mathcal{B}}$-nets* $\mathfrak{r}, \mathfrak{r}' :: \mathfrak{S}$ *satisfy* $\mathrm{VS}(\mathfrak{r} :: \mathfrak{S}) = \mathrm{VS}(\mathfrak{r}' :: \mathfrak{S})$*, then* $\mathfrak{r} = \mathfrak{r}'$.

Proof. First, by the last axiom in Definition 3.2.28, every $\mathcal{B}$-arc in $\mathfrak{r}$ or $\mathfrak{r}'$ is consumed in some valid sequence in $\mathrm{VS}(\mathfrak{r} :: \mathfrak{S})$. Next, the arrowy label of each (consumed) $\mathcal{B}$-arc in $\mathfrak{r}$ or $\mathfrak{r}'$, if any, is uniquely determined by the labels of valid sequences due to the second axiom in Definition 3.2.5. Moreover, by the first two axioms in Definition 3.2.28, the cartesian and the identifier labels of each $\mathcal{B}$-arc in $\mathfrak{r}$ or $\mathfrak{r}'$ are completely determined too. It remains to consider exponential labels.

Assume for a contradiction that $\mathfrak{r}$ and $\mathfrak{r}'$ contain $\mathcal{B}$-arcs of the forms

$$u \xrightarrow[\langle i \rangle]{(\chi)\langle\kappa|\epsilon\rangle} v, \qquad\qquad u \xrightarrow[\langle i \rangle]{(\chi)\langle\kappa|\epsilon'\rangle} v,$$

respectively, such that $\epsilon \neq \epsilon'$. At least ϵ or ϵ' must be nonempty; assume $(c, e) \in \epsilon$ without loss of generality. By the first axiom in Definition 3.1.8, some colour c' satisfies $(c', e) \in \epsilon'$, where $uiv \in c \cap c'$ by the second axiom. Thus, c and c' are not disjoint, so $c = c'$ by the third axiom in Definition 3.2.28. We have shown $\epsilon \subseteq \epsilon'$, and $\epsilon' \subseteq \epsilon$ is symmetric, deriving a contradiction. □

Example 3.2.31. Only the left pre-$\ell^{\circ}_{\mathcal{B}}$-net is prime in Figure 12, and only the right one is prime in Figure 13, where recall that in these figures $\mathcal{B}$-realisers are completed.

Notation. Abusing notation, let ρ be the operator that transforms a valid sequence $\boldsymbol{s}$ in a pre-$\ell^{\circ}_{\mathcal{B}}$-net $\mathfrak{r} :: \mathfrak{S}$ into another finite sequence $\rho\boldsymbol{s}$ by the renaming of ends corresponding to the mapping $\mathfrak{S} \mapsto \rho\mathfrak{S}$, and it is lifted to a set of valid sequences in $\mathfrak{r} :: \mathfrak{S}$ in the pointwise fashion.

Proposition 3.2.32 (existence of prime form)**.** *Every pre-$\ell^{\circ}_{\mathcal{B}}$-net $\mathfrak{r} :: \mathfrak{S}$ has a prime one $\mathfrak{r}' :: \rho\mathfrak{S}$ such that $\rho\mathrm{VS}(\mathfrak{r} :: \mathfrak{S}) = \mathrm{VS}(\mathfrak{r}' :: \rho\mathfrak{S})$, where $\mathfrak{r}' :: \rho\mathfrak{S}$ is injective (respectively, correct) if so is $\mathfrak{r} :: \mathfrak{S}$, and every prime one $\mathfrak{r}'' :: \rho\mathfrak{S}$ such that $\rho\mathrm{VS}(\mathfrak{r} :: \mathfrak{S}) = \mathrm{VS}(\mathfrak{r}'' :: \rho\mathfrak{S})$ satisfies $\mathfrak{r}' = \mathfrak{r}''$.*

Proof. We prove the first part by induction on the largest length of an element of $\mathrm{VS}(\mathfrak{r} :: \mathfrak{S})$, from which the second follows by Proposition 3.2.30. The base case is simple: take $\mathfrak{r}' := \emptyset$.

For the inductive step, assume that we have shown the claim for the length $n \in \mathbb{N}$, and the largest length of an element of $\mathrm{VS}(\mathfrak{r} :: \mathfrak{S})$ is $n + 1$. Take all $\mathcal{B}$-arcs in $\mathfrak{r}$ consumed in some valid sequences in $\mathfrak{r} :: \mathfrak{S}$ of length at most n, and define $\mathfrak{r}_n$ to be the subgraph of $\mathfrak{r}$ induced by these $\mathcal{B}$-arcs. Clearly, $\mathfrak{r}_n :: \mathfrak{S}$ is a pre-$\ell^{\circ}_{\mathcal{B}}$-net. Thus, by applying the induction hypothesis to $\mathfrak{r}_n :: \mathfrak{S}$, we obtain a prime one $\mathfrak{r}'_n :: \rho\mathfrak{S}$. Next, let $\tilde{\mathfrak{r}} :: \rho\mathfrak{S}$ be the pre-$\ell^{\circ}_{\mathcal{B}}$-net obtained from $\mathfrak{r} :: \mathfrak{S}$ by the replacement of ends that corresponds to the map $\mathfrak{S} \mapsto \rho\mathfrak{S}$, and on each $a \in \tilde{\mathfrak{r}}$ of length $n + 1$,

(1) If a is not consumed in any valid sequences in $\tilde{\mathfrak{r}} :: \rho\mathfrak{S}$, then exclude a from $\mathfrak{r}$ and delete $\mathrm{Col}(a)$ from colours in the exponential labels of $\mathcal{B}$-arcs in $\mathfrak{r}$;
(2) If a is not excluded in the previous step, by which assume that for some $k \geqslant 1$ there are k cartesian conditions for a in valid sequences in $\tilde{\mathfrak{r}} :: \rho\mathfrak{S}$, then duplicate a into k copies of it, corresponding to the cartesian conditions, and change the cartesian label of each copy into the one whose finite set of variables is the corresponding cartesian condition.

We then add to $\mathfrak{r}'_n$ these copies of consumed $\mathcal{B}$-arcs of length $n + 1$, whose labels are potentially modified, obtaining a $\mathcal{B}$-realiser $\mathfrak{r}'$ such that $\mathfrak{r}' :: \rho\mathfrak{S}$ is a prime pre-$\ell^{\circ}_{\mathcal{B}}$-net by construction. Note that the order of the choice $a \in \tilde{\mathfrak{r}}$ does not matter for this procedure. □

Definition 3.2.33 (prime form)**.** Proposition 3.2.32 transforms every pre-$\ell^{\circ}_{\mathcal{B}}$-net $\mathfrak{r} :: \mathfrak{S}$ into a unique prime one, called the *prime form* of $\mathfrak{r} :: \mathfrak{S}$ and written $\rho(\mathfrak{r} :: \mathfrak{S})$ or $\rho\mathfrak{r} :: \rho\mathfrak{S}$, such that $\rho\mathrm{VS}(\mathfrak{r} :: \mathfrak{S}) = \mathrm{VS}(\rho\mathfrak{r} :: \rho\mathfrak{S})$, where $\rho(\mathfrak{r} :: \mathfrak{S})$ is correct and injective if so is $\mathfrak{r} :: \mathfrak{S}$.

Remark. The prime form of a freely generated pre-$\ell^{\circ}_{\mathcal{B}}$-net may not be freely generated. For instance, the pre-$\ell_{\mathcal{B}}$-net



is freely generated, but its prime form, obtained from it by the deletion of the $\mathcal{B}$-arc, is not.

We finally introduce *saturation* as the last step towards $\ell^{\circ}_{\mathcal{B}}$-nets:

Definition 3.2.34 ($\ell^{\circ}_{\mathcal{B}}$-skeletons)**.** An *$\ell^{\bullet}_{\mathcal{B}}$-skeleton* is an injective, correct, prime pre-$\ell^{\bullet}_{\mathcal{B}}$-net, and it is called an *$\ell_{\mathcal{B}}$-skeleton* if it is a pre-$\ell_{\mathcal{B}}$-net.

Let $\sim$ be the relation between $\ell^{\circ}_{\mathcal{B}}$-skeletons $\mathfrak{s}, \mathfrak{s}' :: \mathfrak{S}$ such that $\mathfrak{s} \sim \mathfrak{s}' :: \mathfrak{S}$ holds if $\mathfrak{s}'$ is obtained from $\mathfrak{s}$ by the replacement of the source of a $\mathcal{B}$-arc in $\mathfrak{s}$ whose target is joker and the corresponding change of labels, if any, and $\approx$ the equivalence relation between $\ell^{\circ}_{\mathcal{B}}$-skeletons generated by $\sim$.

Remark. Any non-promotional $\mathcal{B}$-arc is not replaced with another $\mathcal{B}$-arc by the relation $\sim$ as the change of the source of a non-promotional $\mathcal{B}$-arc breaks the second axiom in Definition 3.2.22. A related remark is that the replacement of the source of a $\mathcal{B}$-arc in $\mathfrak{s}$ for obtaining $\mathfrak{s}'$ such that $\mathfrak{s} \sim \mathfrak{s}' :: \mathfrak{S}$ from $\mathfrak{s}$ may yield a $\mathcal{B}$-arc that already exists in $\mathfrak{s}$ since promotional $\mathcal{B}$-arcs are exempt from the injectivity axiom; this is why $\approx$ is not only the reflexive, transitive closure of $\sim$ but also the symmetric closure, where see the second half of Example 3.2.38.

We are now ready to introduce morphism in our free (pointed) linear category:

Definition 3.2.35 ($\ell^{\circ}_{\mathcal{B}}$-nets)**.** An *$\ell^{\bullet}_{\mathcal{B}}$-net* (respectively, an *$\ell_{\mathcal{B}}$-net*) is a pre-$\ell^{\bullet}_{\mathcal{B}}$-net (respectively, a pre-$\ell_{\mathcal{B}}$-net) $\mathfrak{r} :: \mathfrak{S}$ such that $\mathfrak{r}$ is *$\approx$-saturated* in the sense that the equation

$$\mathfrak{r} = (\mathfrak{s}_0)^{\approx}_{\mathfrak{S}} := \bigcup_{\mathfrak{s}_0 \approx \mathfrak{s} :: \mathfrak{S}} \mathfrak{s} \tag{11}$$

holds for an $\ell^{\bullet}_{\mathcal{B}}$-skeleton (respectively, an $\ell_{\mathcal{B}}$-skeleton) $\mathfrak{s}_0 :: \mathfrak{S}$, called a *$\approx$-representative* of $\mathfrak{r} :: \mathfrak{S}$.

Remark. It is easy to see that the union (11) is a $\mathcal{B}$-realiser on $\mathfrak{S}$ (not only a pseudo-$\mathcal{B}$-realiser).

Notation. Given an $\ell^{\circ}_{\mathcal{B}}$-skeleton $\mathfrak{s} :: \mathfrak{S}$, we abbreviate the $\approx$-saturation $\mathfrak{s}^{\approx}_{\mathfrak{S}}$ as $\mathfrak{s}^{\approx}$ when the $\ell^{\circ}_{\mathcal{B}}$-diagram $\mathfrak{S}$ is evident. For a pre-$\ell^{\circ}_{\mathcal{B}}$-term $r :: S$ such that $\rho\mathfrak{G}(r :: S)$ is an $\ell^{\circ}_{\mathcal{B}}$-skeleton, let

$$\rho\mathfrak{G}(r :: S)^{\approx} := (\rho\mathfrak{G}r)^{\approx} :: \rho\mathfrak{G}S,$$

and we call it the $\ell^{\circ}_{\mathcal{B}}$-net *represented* by $r :: S$.

Definition 3.2.36 ($\ell^{\circ}_{\mathcal{B}}$-terms)**.** A pre-$\ell^{\circ}_{\mathcal{B}}$-term $r :: S$ is called an *$\ell^{\circ}_{\mathcal{B}}$-term* if $\rho\mathfrak{G}(r :: S)$ is an $\ell^{\circ}_{\mathcal{B}}$-net, where, by abuse of terminology, a pre-$\ell^{\circ}_{\mathcal{B}}$-term $s :: S$ is called a *$\approx$-representative* of $r :: S$ if $\rho\mathfrak{G}(s :: S)$ is an $\ell^{\circ}_{\mathcal{B}}$-skeleton that is an $\approx$-representative of $\rho\mathfrak{G}(r :: S)$.

Proposition 3.2.37 (decidability I)**.** *Given an $\ell^{\circ}_{\mathcal{B}}$-diagram $\mathfrak{S}$,*

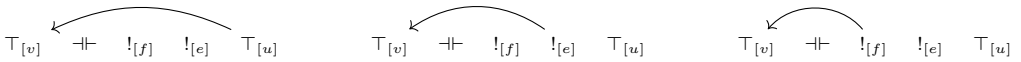
1. *It is decidable whether a given pre-$\ell^{\circ}_{\mathcal{B}}$-net $\mathfrak{s} :: \mathfrak{S}$ is an $\ell^{\circ}_{\mathcal{B}}$-skeleton;*
2. *It is decidable whether a given pre-$\ell^{\circ}_{\mathcal{B}}$-net $\mathfrak{r} :: \mathfrak{S}$ is an $\ell^{\circ}_{\mathcal{B}}$-net.*

Proof. The set $\mathrm{VS}(\mathfrak{s} :: \mathfrak{S})$ of all valid sequences in $\mathfrak{s} :: \mathfrak{S}$ is finite. From this, it is straightforward to see that the first clause holds: it is decidable whether $\mathfrak{s} :: \mathfrak{S}$ is injective, correct and prime.

The second clause follows from the first since $\mathfrak{r} :: \mathfrak{S}$ has only finitely many $\approx$-representatives: it suffices to obtain an arbitrary $\approx$-representative $\mathfrak{s}_0 :: \mathfrak{S}$ from $\mathfrak{r} :: \mathfrak{S}$ by pruning some of the $\mathcal{B}$-arcs in $\mathfrak{r}$ breaking injectivity, construct all $\ell^{\circ}_{\mathcal{B}}$-skeletons $\mathfrak{s}' :: \mathfrak{S}$ such that $\mathfrak{s}_0 \approx \mathfrak{s}' :: \mathfrak{S}$ from $\mathfrak{s}_0 :: \mathfrak{S}$ and finally check if the union $\bigcup_{\mathfrak{s}_0 \approx \mathfrak{s}' :: \mathfrak{S}} \mathfrak{s}'$ coincides with $\mathfrak{r}$. □

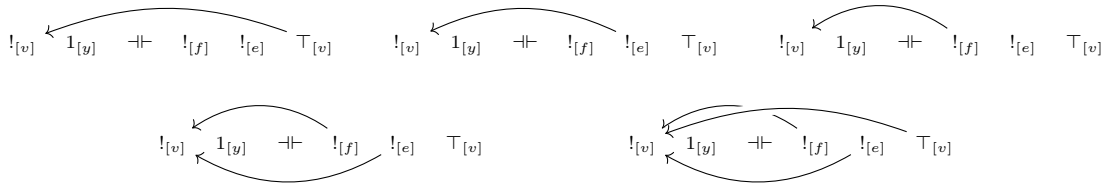
We will prove that a pre-$\ell^{\circ}_{\mathcal{B}}$-net is the prime form of a freely generated one if and only if it is an $\ell^{\circ}_{\mathcal{B}}$-skeleton, and the equality between $\ell^{\circ}_{\mathcal{B}}$-nets matches a free (pointed) linear category, in which morphisms are normalised $\ell^{\circ}_{\mathcal{B}}$-nets. In this setting, $\approx$-saturation is crucial for joker ends, as illustrated already in Figures 2–3, and further in Example 3.2.38 below.

Example 3.2.38. Every free linear category has just one morphism of the form $\top \to !!\top$, while there are three $\ell_{\mathcal{B}}$-skeletons on the $\ell_{\mathcal{B}}$-diagram $\mathfrak{G}(\top_{[v]} \dashv\vdash !_{[f]}!_{[e]}\top_{[u]})$:



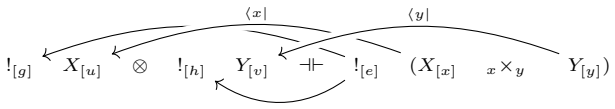
Nevertheless, these $\ell_{\mathcal{B}}$-skeletons are all identified under $\approx$-equivalence. Note that the corresponding $\ell_{\mathcal{B}}$-net is *not* an $\ell_{\mathcal{B}}$-skeleton since it does not satisfy the second axiom in Definition 3.2.22.

In contrast, the $\ell_{\mathcal{B}}$-diagram $\mathfrak{G}(!_{[v]}1_{[y]} \dashv\vdash !_{[f]}!_{[e]}\top_{[v]})$ has the five $\ell_{\mathcal{B}}$-skeletons



but they are all identified under $\approx$-equivalence. For the Seely isomorphism $!1 \cong \top$, this identification must hold. In the latter case, the *symmetric* closure of the relation $\sim$ for $\approx$-equivalence is crucial, illustrating its difference from the instances of Trimble rewiring in [MO03, Hug12].

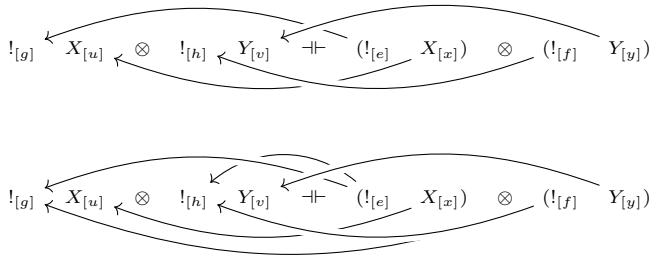
Example 3.2.39. The $\ell_{\mathcal{B}}$-diagram $\mathfrak{G}(!_{[g]}X_{[u]} \otimes !_{[h]}Y_{[v]} \dashv\vdash !_{[e]}(X_{[x]}{}_x\times_y Y_{[y]}))$ has just the $\ell_{\mathcal{B}}$-skeleton



which is also an $\ell_{\mathcal{B}}$-net. This $\ell_{\mathcal{B}}$-skeleton has the maximal valid sequences

$$x.u, \qquad y.v, \qquad e.g, \qquad e.h, \qquad ex.gu, \qquad ey.hv.$$

In contrast, the $\ell_{\mathcal{B}}$-diagram $\mathfrak{G}(!_{[g]}X_{[u]} \otimes !_{[h]}Y_{[v]} \nvdash !_{[e]}X_{[x]} \otimes !_{[f]}Y_{[y]})$ has the two



where the lower one is an $\ell_{\mathcal{B}}$-net, and the upper one is its $\approx$-representative. Thus, the last two $\ell_{\mathcal{B}}$-skeletons are identified by $\approx$-saturation. In light of the Seely isomorphism $!(A \times B) \cong !A \otimes !B$, this identification must hold. The last $\ell_{\mathcal{B}}$-net has the maximal valid sequences

$$x.u, \qquad y.v, \qquad e.g, \qquad e.h, \qquad f.g, \qquad f.h, \qquad ex.gu, \qquad ey.hv.$$

Let us now prove that $\ell^{\circ}_{\mathcal{B}}$-nets precisely characterise freely generated pre-$\ell^{\circ}_{\mathcal{B}}$-nets up to prime form. To this end, the following two technical lemmata are used:

Lemma 3.2.40 (acyclicity)**.** *An $\ell^{\circ}_{\mathcal{B}}$-tree is said to be* closed *if it is of the form $\mathfrak{G}(T \multimap T')$. Given an $\ell^{\circ}_{\mathcal{B}}$-skeleton $\mathfrak{s} :: \mathfrak{S}$, let*

$$\mathfrak{G}(N \multimap P) <_{\mathfrak{s}::\mathfrak{S}} \mathfrak{G}(N' \multimap P')$$

be the relation between closed positive $\ell^{\circ}_{\mathcal{B}}$-trees that are subtrees of $\mathfrak{S}$ that holds if

1. *The $\ell^{\circ}_{\mathcal{B}}$-trees $\mathfrak{G}(N \multimap P)$ and $\mathfrak{G}(N' \multimap P')$ are disjoint as graphs;*
2. *The $\ell^{\circ}_{\mathcal{B}}$-skeleton $\mathfrak{s} :: \mathfrak{S}$ has a nonempty valid sequence $\boldsymbol{x}_1.\boldsymbol{x}_2 \ldots \boldsymbol{x}_{2n}$ of even-length with the last two moves $\boldsymbol{x}_{2n-1}$ and $\boldsymbol{x}_{2n}$ initial in $\mathfrak{G}N'$ and $\mathfrak{G}P$, respectively.*

Then, the relation $<_{\mathfrak{r}::\mathfrak{S}}$, seen as a directed graph, is acyclic.

Proof. If $\mathfrak{G}(N \multimap P) <^{n}_{\mathfrak{s}::\mathfrak{S}} \mathfrak{G}(N \multimap P)$ for some positive integer n and closed positive $\ell^{\circ}_{\mathcal{B}}$-tree $\mathfrak{G}(N \multimap P)$ in $\mathfrak{S}$, then it contradicts the acyclicity of $\overline{\mathfrak{s}}_{\mathfrak{S}}$ (at some initial end of $\mathfrak{G}N$). □

Lemma 3.2.41 (division I)**.** *Let $\mathfrak{s} :: \mathfrak{S}$ be a sliced, shredded $\ell^{\circ}_{\mathcal{B}}$-skeleton, where $\mathfrak{S}$ has a closed positive $\ell^{\circ}_{\mathcal{B}}$-tree $\mathfrak{G}(N \multimap P)$ and an initial one of the form $\mathfrak{G}(Q \otimes Q')$ (respectively, $\mathfrak{G}(!_{[e]}Q)$), and initial ends of the latter justify those of the former in $\mathfrak{S}$. If $\mathfrak{s} :: \mathfrak{S}$ has two valid sequences starting respectively with ends of $\mathfrak{G}Q$ and $\mathfrak{G}Q'$ (respectively, e and an end of $\mathfrak{G}Q$), with both having ends of $\mathfrak{G}P$, then $\mathfrak{s}$ has none containing ends of $\mathfrak{G}N$ and $\mathfrak{G}(Q \otimes Q')$ (respectively, $\mathfrak{G}(!_{[e]}Q)$).*

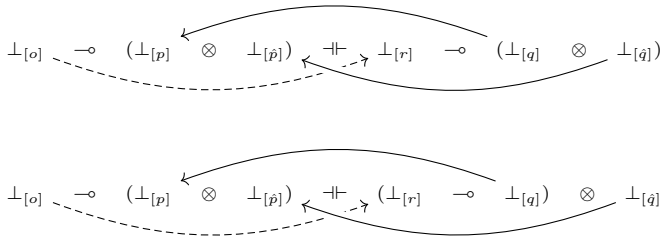
Proof. We focus on the case where the initial $\ell^{\circ}_{\mathcal{B}}$-tree in $\mathfrak{S}$ is $\mathfrak{G}(Q \otimes Q')$ since the other case is analogous. Under the assumption of the claim, suppose for a contradiction that $\mathfrak{s} :: \mathfrak{S}$ has a valid sequence containing ends of $\mathfrak{G}N$ and $\mathfrak{G}Q$; the other case in which $\mathfrak{s} :: \mathfrak{S}$ has a valid sequence containing ends of $\mathfrak{G}N$ and $\mathfrak{G}Q'$ is symmetric. Then, since $\mathfrak{s} :: \mathfrak{S}$ is sliced and shredded, it also has a valid sequence starting with an end of $\mathfrak{G}Q'$, containing ends of $\mathfrak{G}N$ and $\mathfrak{G}Q$, contradicting its local alternation with respect to $\mathfrak{G}(Q \otimes Q')$.

□

Lemma 3.2.42 (division II)**.** *Let $\mathfrak{s} :: \mathfrak{S}$ be a sliced, shredded $\ell^{\circ}_{\mathcal{B}}$-skeleton, where $\mathfrak{S}$ has positive $\ell^{\circ}_{\mathcal{B}}$-trees $\mathfrak{G}(N \multimap P)$ and $\mathfrak{G}(!_{[v]}R)$. If $\mathfrak{s}$ has two $\mathcal{B}$-arcs whose targets are v and an end of $\mathfrak{G}(R)$, respectively, then all $\mathcal{B}$-arcs in $\mathfrak{s}$ between $\mathfrak{G}(N \multimap P)$ and $\mathfrak{G}(!_{[v]}R)$ go through only either $\mathfrak{G}(N)$ or $\mathfrak{G}(P)$, not both.*

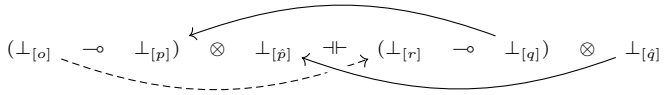
Proof. By a straightforward case analysis, for which the first condition $|\boldsymbol{v}_{2i-1}| \geqslant |\boldsymbol{v}_{2i}|$ of the third axiom in Definition 3.2.6 as well as the injectivity of the $\ell^{\circ}_{\mathcal{B}}$-skeleton and the local alternation of valid sequences are crucial. □

Example 3.2.43. Consider the two pre-$\ell^{\bullet}_{\mathcal{B}}$-nets



The upper one is an $\ell^{\bullet}_{\mathcal{B}}$-skeleton (and also an $\ell^{\bullet}_{\mathcal{B}}$-net), while the lower one is not since due to the dashed $\mathcal{B}$-arc it is not prime (and even without the dashed one it is not correct either). By Lemma 3.2.41, the dashed one cannot exist unless the pre-$\ell^{\bullet}_{\mathcal{B}}$-net is not an $\ell^{\bullet}_{\mathcal{B}}$-skeleton.

Meanwhile, the pre-$\ell^{\bullet}_{\mathcal{B}}$-net



is an $\ell^{\bullet}_{\mathcal{B}}$-skeleton, which does not satisfy the assumption of Lemma 3.2.41.

We are now ready to prove:

Theorem 3.2.44 (graphical full completeness)**.** *Every $\ell^{\circ}_{\mathcal{B}}$-skeleton $\mathfrak{s} :: \mathfrak{S}$ has a freely generated pre-$\ell^{\circ}_{\mathcal{B}}$-net $s :: S$ that satisfies $\rho\mathfrak{G}(S) = \mathfrak{S}$ and $\rho\mathfrak{G}(s) = \mathfrak{s}$.*

Proof. We proceed by induction on the size (or number of edges) of $\mathfrak{S}$. First, assume that $\mathfrak{s} :: \mathfrak{S}$ is sliced and shredded; we later remove this restriction. Under this assumption, we focus on factors of the domain of $\mathfrak{S}$ of the forms

$$\mathfrak{G}\top_{[v]}, \qquad \mathfrak{G}1_{[v]}, \qquad \mathfrak{G}\bot_{[v]}, \qquad \mathfrak{G}X_{[v]}, \qquad \mathfrak{G}(N \multimap P), \qquad \mathfrak{G}(!_{[v]}P),$$

where $v \in \mathcal{V}$ and $X \in \mathrm{Obj}(\mathcal{B})$, those of the codomain of the forms

$$\mathfrak{G}(N \otimes Q), \qquad \mathfrak{G}\top_{[v]}, \qquad \mathfrak{G}X_{[v]}, \qquad \mathfrak{G}(!_{[v]}N),$$

and those of the centre of the forms

$$\mathfrak{G}\top_{[v]}, \qquad \mathfrak{G}(N \triangleright P), \qquad \mathfrak{G}(!_{[v]}M)$$

because the other factors are *trivial* in the sense that in the induction they are straightforward to handle by typing rules. For instance, we deal with those of the form $\mathfrak{G}(P \times R)$ in the domain by $\times$L thanks to the third axiom in Definition 3.2.22, If these trivial factors are in $\mathfrak{S}$, then we deal with them in some order. Thus, in the following, we focus on nontrivial factors.

We next handle factors of the form $\mathfrak{G}(N \multimap P)$ in the domain. Among the minimal ones with respect to the order $<_{\mathfrak{s}::\mathfrak{S}}$ in Lemma 3.2.40, we choose an arbitrary one $\mathfrak{G}(N_0 \multimap P_0)$. We want to apply $\multimap$L to $\mathfrak{G}(N_0 \multimap P_0)$, and we proceed as follows:

(1) If $\mathfrak{S}$ has a factor of the form $\mathfrak{G}(!_{[v]}R)$ in the domain, then by the third axiom in Definition 3.2.22 only one of the three forms of moves of $\mathfrak{G}(!_{[v]}R)$ may occur in a valid sequence in $\mathfrak{s} :: \mathfrak{S}$. If all these moves are of length one, then we may apply !D or !W to $\mathfrak{G}(!_{[v]}R)$; thus, in the following, we assume the remaining form of the moves (of length two), so that we may apply Lemma 3.2.42.

(2) If the codomain is of the form $\mathfrak{G}\top_{[v]}$ or $\mathfrak{G}X_{[v]}$, then we may apply $\multimap$L to $\mathfrak{G}(N_0 \multimap P_0)$.
(3) If the codomain is of the form $\mathfrak{G}(N \otimes N')$ or $\mathfrak{G}(!_{[v]}N)$, then apply Lemma 3.2.41, so that we may apply $\multimap$L to $\mathfrak{G}(N_0 \multimap P_0)$.

By the last axiom in Definition 3.2.22, we can similarly handle nontrivial cases on factors of the centre of the form $\mathfrak{G}(N \triangleright P)$ and apply Cut to it. Besides the trivial factors listed above, we henceforth skip those $\mathfrak{G}(N \multimap P)$ and $\mathfrak{G}(N \triangleright P)$ below too, by which the centre is empty.

For the remaining cases, we proceed by case analysis on the factor of the codomain:

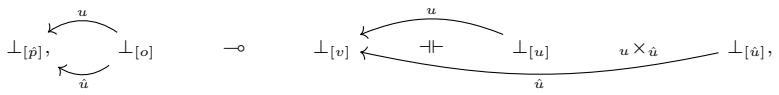
- Assume that the factor is $\mathfrak{G}\bot_{[v_0]}$. Then, $\mathfrak{s}$ has a $\mathcal{B}$-arc from v_0 to a factor of the domain of the form $\mathfrak{G}\bot_{[v]}$ or $\mathfrak{G}(!_{[e]}P)$, where the domain has no other factor by the first axiom in Definition 3.2.22. We can apply $\bot$L to $\mathfrak{G}\bot_{[v]}$, and !D to $\mathfrak{G}(!_{[e]}P)$: $\mathfrak{s}$ has a $\mathcal{B}$-arc from v_0 to $\mathfrak{G}P$, but none to e by the third axiom in Definition 3.2.22.
- Assume that the factor is of the form $\mathfrak{G}(\top_{[v]})$. Then, each factor of the domain, if any, is of the form $\mathfrak{G}(!_{[e]}P)$, where those of $\mathfrak{G}(X_{[w]})$, $\mathfrak{G}(1_{[w]})$ or $\mathfrak{G}(\bot_{[w]})$ are excluded since otherwise w would not be covered. If the domain has no factor, then we apply $\top$R to $\mathfrak{G}(\top_{[v]})$. If it has a factor of the form $\mathfrak{G}(!_{[e]}P)$, then $\mathfrak{s}$ has a $\mathcal{B}$-arc from v to e, but none to $\mathfrak{G}P$ by the third axiom in Definition 3.2.22, so we can apply !W to $\mathfrak{G}(!_{[e]}P)$.
- Assume that the factor is of the form $\mathfrak{G}(N \otimes Q)$. Then, each factor of the domain is of the form $\mathfrak{G}(!_{[e]}P)$ with a $\mathcal{B}$-arc in $\mathfrak{s}$ from some f to e and another from some o to an end p of $\mathfrak{G}P$. The two $\mathcal{B}$-arcs are both from either $\mathfrak{G}N$ or $\mathfrak{G}Q$ by Definition 3.2.6: if ep is in a valid sequence $\boldsymbol{s}$ in $\mathfrak{s} :: \mathfrak{S}$, then so is fo. Thus, we can apply $\otimes$R to $\mathfrak{G}(N \otimes Q)$.
- Assume that the factor is of the form $\mathfrak{G}(!_{[e]}N)$. Then, $\mathfrak{s}$ has a $\mathcal{B}$-arc from e to an factor of the domain of the form $\mathfrak{G}(!_{[f]}P)$. If it is to $\mathfrak{G}P$, then every $\mathcal{B}$-arc in $\mathfrak{s}$ to $\mathfrak{G}(!_{[f]}P)$ is to $\mathfrak{G}P$ by the third axiom in Definition 3.2.22, so apply !D to $\mathfrak{G}(!_{[f]}P)$. Thus, we may assume that all $\mathcal{B}$-arcs in $\mathfrak{s}$ from e are promotional, so apply !R to $\mathfrak{G}(!_{[e]}N)$.

We have completed the case where $\mathfrak{s} :: \mathfrak{S}$ is sliced and expanded. It remains to handle the other case. We then proceed by induction on the sum $|\mathfrak{s} :: \mathfrak{S}|^{+!}_{-\times}$ of the number of negative $\times$-vertices of $\mathfrak{S}$ and that of different exponential labels of $\mathcal{B}$-arcs in $\mathfrak{s}$. Because we have handled the case of $|\mathfrak{r} :: \mathfrak{S}|^{+!}_{-\times} = 0$ (*i.e.* $\mathfrak{s} :: \mathfrak{S}$ is sliced and expanded), we now assume $|\mathfrak{s} :: \mathfrak{S}|^{+!}_{-\times} > 0$.

If $\mathfrak{S}$ has negative $\times$-vertices, then take the highest ones and choose the leftmost one v among them. Let $\mathfrak{G}(O_x \times_y Q)$ be the subtree of $\mathfrak{S}$ rooted at v. We then take the two slices of $\mathfrak{s} :: \mathfrak{S}$ with respect to v, say $\mathfrak{s}_1 :: \mathfrak{S}_1$ and $\mathfrak{s}_2 :: \mathfrak{S}_2$. By the induction hypothesis (with respect to $|\mathfrak{s} :: \mathfrak{S}|^{+!}_{-\times}$), we obtain freely generated pre-$\ell^{\bullet}_{(B)}$-terms, $s_1 :: S_1$ and $s_2 :: S_2$, such that $\mathfrak{G}(S_i) = \mathfrak{S}_i$ and $\rho\mathfrak{G}(s_i) = \mathfrak{s}_i$ for all $i \in \overline{2}$. Because of the nature of our (nested) induction, the free generations of $s_1 :: S_1$ and $s_2 :: S_2$ may differ only in typing rules regarding O and Q. As a result, we can apply $\times$R to suitable components of the free generations, with the help of the last axiom in Definition 3.2.22, and obtain a free generation of a pre-$\ell^{\bullet}_{(B)}$-term $s :: S$ such that $\mathfrak{G}(S) = \mathfrak{S}$ and $\rho\mathfrak{G}(s) = \mathfrak{s}$. We require the application of $\times$R in the free generation to be earliest possible.

In a similar vein, if $\mathfrak{s}$ has $\mathcal{B}$-arcs with nontrivial exponential labels, then take the expansion $\mathrm{Exp}(\mathfrak{s}) :: \mathrm{Exp}(\mathfrak{S})$ and apply the induction hypothesis to it, obtaining freely generation of a pre-$\ell^{\bullet}_{(B)}$-term $s' :: S'$ such that $\mathfrak{G}(S') = \mathrm{Exp}(\mathfrak{S})$ and $\rho\mathfrak{G}(s') = \mathrm{Exp}(\mathfrak{s})$. In the free generation of $s' :: S'$, we insert !C that reverses the expansion at the earliest possible points in the free generation, yielding a free generation of a pre-$\ell^{\bullet}_{(B)}$-term $s :: S$ such that $\mathfrak{G}(S) = \mathfrak{S}$ and $\rho\mathfrak{G}(s) = \mathfrak{s}$. □

Example 3.2.45. For the $\ell^{\bullet}_{\mathcal{B}}$-skeleton



the proof of Theorem 3.2.44 yields a free generation of the pre-$\ell^{\bullet}_{\mathcal{B}}$-term

$$\{o \to p\} : \bot_{[p]}, \{u \xrightarrow{u} v, \hat{u} \xrightarrow{\hat{u}} v\} : \bot_{[o]} \multimap \bot_{[v]} \dashv\vdash \emptyset : \bot_{[u]\,u} \times_{\hat{u}} \bot_{[\hat{u}]}.$$

This pre-$\ell^{\bullet}_{\mathcal{B}}$-term does not coincide with the $\ell^{\bullet}_{\mathcal{B}}$-skeleton, but it does *up to prime form*.

At the end of this section, we establish a useful technical lemma:

Lemma 3.2.46 (congruence)**.** *Let $r :: S$ be a pre-$\ell^{\circ}_{\mathfrak{B}}$-net obtained from a family $\{r_i :: S_i\}_i$ of freely generated ones by a typing rule, and similarly $r' :: S'$ from $\{r'_i :: S'_i\}_i$ by the same rule. If $\mathfrak{G}(S_i) = \mathfrak{G}(S'_i)$ and $\rho\mathfrak{G}(r_i) = \rho\mathfrak{G}(r'_i)$ for all i, then $\mathfrak{G}(S) = \mathfrak{G}(S')$ and $\rho\mathfrak{G}(r) = \rho\mathfrak{G}(r')$.*

Proof. The lemma holds intuitively because the only point is that a typing rule may increase the number of joker ends that witness the relation $\sim$ (*e.g.* consider the one Cut or $\otimes$R), but the equivalence $\approx$ absorbs this. It is formally by induction on the free generation of $r :: S$. □

3.3. **Graphical reduction.** We finally define *$\ell^{\circ}_{\mathcal{B}}$-reduction*. For presenting its generic definition precisely and concisely, we employ pre-$\ell^{\circ}_{\mathcal{B}}$-terms in the definition as a compact notation for pre-$\ell^{\circ}_{\mathcal{B}}$-nets. After the definition is given, we describe examples of $\ell^{\circ}_{\mathcal{B}}$-reduction directly in terms of pre-$\ell^{\circ}_{\mathcal{B}}$-nets since they are intuitive and easy to manipulate (as already illustrated before).

Remark. In some of the examples of $\ell^{\circ}_{\mathcal{B}}$-reduction given in previous sections, for simplicity some multiples steps in $\ell^{\circ}_{\mathcal{B}}$-reduction defined below are abbreviated into a single step.

Notation. Given a pre-$\ell^{\circ}_{\mathcal{B}}$-term $r :: S$, a $\mathcal{B}$-web $\mathscr{A}$, a variable x, an $\ell^{\circ}_{\mathcal{B}}$-type T and finite sets $\kappa_0 \in \wp_{\mathrm{fin}}(\mathcal{V})$ and $\epsilon_0 \in \wp_{\mathrm{fin}}(\mathrm{Col} \times \mathcal{V})$, we define

- $r\restriction^x := \{\, a \in r \mid \mathrm{tgt}(a) = x \,\}$, $r\restriction^T := \bigcup_{y \in \mathrm{End}_T} r\restriction^y$, $r\downharpoonright_x := r \setminus r\restriction^x$ and $r\downharpoonright_T := r \setminus r\restriction^T$;
- $r\restriction_x := \{\, a \in r \mid \mathrm{src}(a) = x \,\}$, $r\restriction_T := \bigcup_{y \in \mathrm{End}_T} r\restriction_y$, $r\downharpoonright^x := r \setminus r\restriction_x$ and $r\downharpoonright^T := r \setminus r\restriction_T$;
- $r\restriction\langle\epsilon_0\rangle := \{\, a \in r \mid \exp(a) \subseteq \epsilon \,\}$ and $r\downharpoonright\epsilon_0 := r \setminus r\restriction\epsilon_0$;
- We write $S[M]$ for an $\ell^{\circ}_{\mathcal{B}}$-sequent S together with the information that it contains an occurrence of an $\ell^{\circ}_{\mathcal{B}}$-cut M as a subexpression, and, after $S[M]$ is fixed, $S[M']$ for the $\ell^{\circ}_{\mathcal{B}}$-sequent obtained from S by the replacement of the occurrence of M with that of an $\ell^{\circ}_{\mathcal{B}}$-cut M' (used exclusively in an expression of the form $S[M] \rightsquigarrow_j S[M']$ below);[13]
- If e is an !-end, then $\mathrm{Col}_{\mathscr{A}}[e] := \{\, c \in \mathrm{Col} \mid a \in \mathscr{A}, (c, e) \in \exp(a) \,\}$.

Definition 3.3.1 ($\ell^{\circ}_{\mathcal{B}}$-reduction)**.** Let *$\ell^{\bullet}_{\mathcal{B}}$-reduction* (respectively, *$\ell_{\mathcal{B}}$-reduction*) be the union

$$\rightsquigarrow := \bigcup_{j=1}^{6} \rightsquigarrow_j$$

of the relations $\rightsquigarrow_j$ between pre-$\ell^{\bullet}_{\mathcal{B}}$-terms (respectively, pre-$\ell_{\mathcal{B}}$-terms) in Figure 18, where in $\rightsquigarrow_5$ the $\ell^{\circ}_{\mathcal{B}}$-types $!_{[e_i]}N_i$ and $!_{[\hat{e}_i]}P_i$ are arbitrarily chosen and satisfying $\underline{!_{[e_i]}N_i = !_{[e]}N = !_{[\hat{e}_i]}P_i}$, and it is lifted to a relation, also written $\rightsquigarrow$, between pre-$\ell^{\circ}_{\mathcal{B}}$-nets *via* the interpretation $\mathfrak{G}$.

Table 2 lists the roles of the components of $\ell^{\circ}_{\mathcal{B}}$-reduction together with relevant constructs.

Case	$\rightsquigarrow_1$	$\rightsquigarrow_j$ $(2 \leqslant j \leqslant 4)$	$\rightsquigarrow_5$	$\rightsquigarrow_6$
Construct	$X/\top/1/\bot$	$\otimes/\,\&\,/\,\multimap$	!W/!D/!C	All
Role	Yanking	Splitting	Yanking/splitting/Copying	Minimisation

Table 2. Components of $\ell^{\circ}_{\mathcal{B}}$-reduction

[13]For clarity, it is better to write $S[M]\{M'/M\}$ or $S\{M'/M\}$ for the latter, instead of $S[M']$. Yet, M and M' are often gigantic, so we adopt the more compact one $S[M']$. This choice does not bring confusion in practice.

$$r :: S[I_{[v]} \triangleright I_{[\hat{v}]}] \rightsquigarrow_1 r{\downarrow^v}_{\hat{v}} \cup \left\{ x \xrightarrow[\langle 1 \rangle]{\langle \hat{\kappa} | \hat{\epsilon} \rangle} y \mid \left(x \xrightarrow[\langle 1 \rangle]{\chi \langle \hat{\kappa} | \hat{\epsilon} \rangle} \hat{v} \right) \in r{\uparrow^{\hat{v}}} \right\}$$
$$\cup \left\{ x \xrightarrow[\langle i \rangle]{\zeta \circ \chi \langle \hat{\kappa} | (\hat{\epsilon} \cup \epsilon) \setminus (\hat{\epsilon} \cap \epsilon) \rangle} y \mid \left(x \xrightarrow[\langle 1 \rangle]{\chi \langle \hat{\kappa} | \hat{\epsilon} \rangle} \hat{v} \right), \left(v \xrightarrow[\langle i \rangle]{\zeta \langle \kappa | \epsilon \rangle} y \right) \in r \right\} :: S[\top_{[\hat{v}]}]$$

$$r :: S[I_{[v]} \triangleright I_{[\hat{v}]}] \rightsquigarrow_1 r{\downarrow^v}_{\hat{v}} \cup \left\{ x \xrightarrow[\langle 1 \rangle]{\langle \hat{\kappa} | \hat{\epsilon} \rangle} y \mid \left(x \xrightarrow[\langle 1 \rangle]{(\chi) \langle \hat{\kappa} | \hat{\epsilon} \rangle} \hat{v} \right) \in r{\uparrow^{\hat{v}}} \right\}$$
$$\cup \left\{ x \xrightarrow[\langle i \rangle]{\langle \hat{\kappa} | (\hat{\epsilon} \cup \epsilon) \setminus (\hat{\epsilon} \cap \epsilon) \rangle} y \mid \left(x \xrightarrow[\langle 1 \rangle]{(\chi) \langle \hat{\kappa} | \hat{\epsilon} \rangle} \hat{v} \right), \left(v \xrightarrow[\langle i \rangle]{\langle \kappa | \epsilon \rangle} y \right) \in r \right\} :: S[\top_{[\hat{v}]}] \quad (\text{if } I \in \mathrm{Obj}(\mathcal{B}) \cup \{\top, 1, \bot\})$$

$$r :: S[(N \otimes Q) \triangleright (P \otimes R)] \rightsquigarrow_2 r :: S[(N \triangleright P) \otimes (Q \triangleright R)]$$

$$r :: S[(N_x \times_y Q) \triangleright (P \times R)] \rightsquigarrow_3 r - \langle \{x, y\} | :: S[(N \triangleright P) \times (Q \triangleright R)]$$

$$r :: S[(P \multimap Q) \triangleright (N \multimap R)] \rightsquigarrow_4 r :: S[(N \triangleright P) \otimes (Q \triangleright R)]$$

$$r :: S[!_{[e]} N \triangleright !_{[\hat{e}]} P] \rightsquigarrow_5 r{\downarrow^e}_{\hat{e}} \cup \left\{ f \xrightarrow[\langle i \rangle]{\langle \hat{\kappa} | (\hat{\epsilon} \cup \epsilon) \setminus (\hat{\epsilon} \cap \epsilon) \rangle} g \mid \left(f \xrightarrow[\langle 1 \rangle]{\langle \hat{\kappa} | \hat{\epsilon} \rangle} \hat{e} \right), \left(e \xrightarrow[\langle i \rangle]{\langle \kappa | \epsilon \rangle} g \right) \in r \right\} :: S[!_{[\hat{e}]} (N \triangleright P)] \quad (\text{if } \mathrm{Col}_{\mathfrak{S} r}[\hat{e}] = \emptyset)$$

$$r :: S[!_{[e]} N \triangleright !_{[\hat{e}]} P] \rightsquigarrow_5 r{\downarrow^{!_{[e]} N}}_{!_{[e]} N}{\downarrow_{\hat{e}}} \cup \left\{ u \xrightarrow[\langle i \rangle]{(\chi) \langle \kappa | \epsilon \cup \{(\hat{c}, \hat{e})\} \rangle} v \mid \left(u \xrightarrow[\langle i \rangle]{(\chi) \langle \kappa | \epsilon \rangle} v \right) \in r{\uparrow_N} \cup r{\uparrow^N} \right\}$$
$$\cup \left\{ f \xrightarrow[\langle i \rangle]{\langle \hat{\kappa} | (\hat{\epsilon} \cup \epsilon) \setminus (\hat{\epsilon} \cap \epsilon) \rangle} g \mid \left(f \xrightarrow[\langle 1 \rangle]{\langle \hat{\kappa} | \hat{\epsilon} \rangle} \hat{e} \right), \left(e \xrightarrow[\langle i \rangle]{\langle \kappa | \epsilon \rangle} g \right) \in r \right\} :: S[!_{[\hat{e}]} (N \triangleright P)] \quad (\text{if } \mathrm{Col}_{\mathfrak{S} r}[\hat{e}] = \{\hat{c}\})$$

$$r :: S[!_{[e]} N \triangleright !_{[\hat{e}]} P] \rightsquigarrow_5 \rho \left(\bigcup_{i=1}^{n} r{\downarrow}\{ (\hat{c}_j, \hat{e}) \mid j \in \overline{n}, j \neq i \}\{!_{[e_i]} N_i / !_{[e]} N\}\{!_{[\hat{e}_i]} P_i / !_{[\hat{e}]} P\} :: S\left[\bigotimes_{i=1}^{n} (!_{[e_i]} N_i \triangleright !_{[\hat{e}_i]} P_i) \right] \right)$$
$$(\text{if } n := |\mathrm{Col}_{\mathfrak{S} r}[\hat{e}]| > 1 \text{ and } \mathrm{Col}_{\mathfrak{S} r}[\hat{e}] = \{\hat{c}_i\}_{i=1}^{n})$$

$$r :: \Gamma \dashv \Pi \vdash \Phi \rightsquigarrow_6 r{\downarrow_\Pi} :: \Gamma \dashv\vdash \Phi \quad (\text{if } \Pi \text{ is nonempty, and } \triangleright \text{ does not occur in } \Pi)$$

FIGURE 18. $\ell^\circ_{\mathcal{B}}$-reduction

Example 3.3.2. The examples of $\ell^\circ_{\mathcal{B}}$-reduction in previous sections conform to Definition 3.3.1.

Example 3.3.3. Figure 19 draws the exact version of $\ell^\circ_{\mathcal{B}}$-reduction sketched in Figure 5, where we write the binary operation $\otimes$ in $\ell^\circ_{\mathcal{B}}$-ports vertically. Note in this example how the rule $\rightsquigarrow_6$ implements copying (including the adjustment of exponential labels).

We next verify basic properties of $\ell^\circ_{\mathcal{B}}$-reduction: *subject reduction*, *strong normalisation* and *confluence*. They together imply that *normalisation* of $\ell^\circ_{\mathcal{B}}$-nets by $\ell^\circ_{\mathcal{B}}$-reduction is well-defined.

Theorem 3.3.4 (subject reduction)**.** *If an $\ell^\circ_{\mathcal{B}}$-net $\mathfrak{r} :: \mathfrak{S}$ with $\mathfrak{S} = \mathfrak{G}(\Gamma \dashv \Pi \vdash \Phi)$ has $\mathfrak{r} :: \mathfrak{S} \rightsquigarrow \tilde{\mathfrak{r}} :: \tilde{\mathfrak{S}}$, then $\tilde{\mathfrak{r}} :: \tilde{\mathfrak{S}}$ is also an $\ell^\circ_{\mathcal{B}}$-net with $\tilde{\mathfrak{S}} = \mathfrak{G}(\Gamma \dashv \tilde{\Pi} \vdash \Phi)$ for some centre $\tilde{\Pi}$.*

Proof. It is tedious yet straightforward to see by case analysis that $\ell^\circ_{\mathcal{B}}$-reduction preserves the correctness and the prime form of pre-$\ell^\circ_{\mathcal{B}}$-nets. We focus on the cases of $\rightsquigarrow_1$ and $\rightsquigarrow_5$ since the other cases are analogous or simpler. For the case of $\rightsquigarrow_1$, the only nontrivial point is the second axiom in Definition 3.2.22, but $\rightsquigarrow_1$ preserves the axiom because x (respectively, v) is non-joker if so is $\hat{v}$ (respectively, y) by the second axiom in Definition 3.2.5. For the case of $\rightsquigarrow_6$, the only nontrivial point is the first axiom in Definition 3.2.22, but $\rightsquigarrow_6$ trivially retains the axiom since the axiom is on the expansion of a (sliced) pre-$\ell^\circ_{\mathcal{B}}$-net, not the pre-$\ell^\circ_{\mathcal{B}}$-net itself.

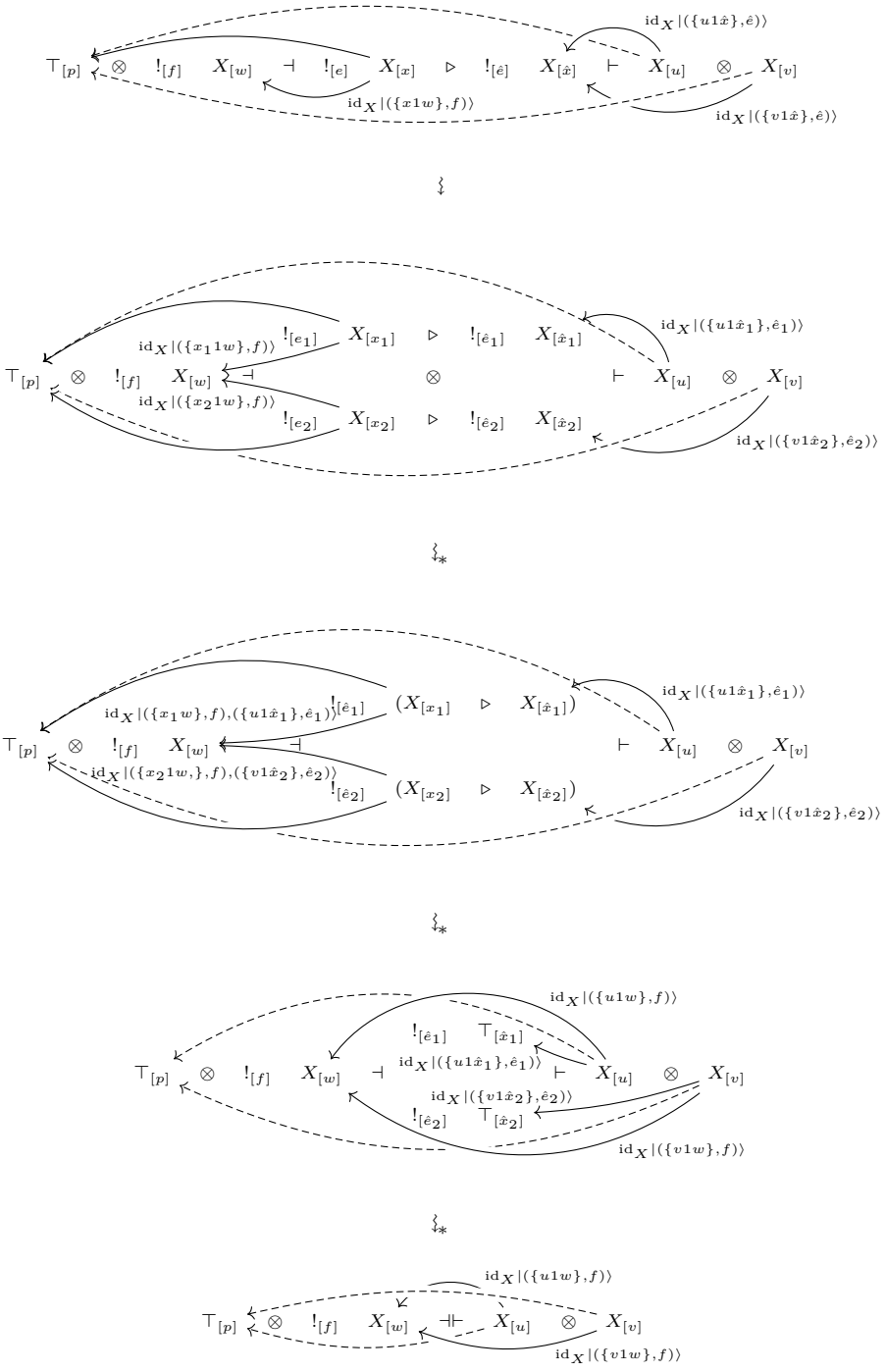


FIGURE 19. Revisiting normalisation by copying and yanking

We next show that, when $\ell^\circ_{\mathcal{B}}$-reduction is applied to an $\ell^\circ_{\mathcal{B}}$-skeleton, it yields a union of $\ell^\circ_{\mathcal{B}}$-skeletons pairwise $\approx$-equivalent. In this regard, nontrivial cases are $\rightsquigarrow_1$ and $\rightsquigarrow_5$. Each step of them may increase the number of witnesses of $\sim$, but as in the proof of (iii) and (iv) in [MO03, Theorem 43] it preserves $\approx$ as it is not $\sim$ itself but its *transitive*, reflexive, symmetric closure.

It remains to prove that $\ell^\circ_{\mathcal{B}}$-reduction preserves $\approx$-saturation. Again, nontrivial cases are $\rightsquigarrow_1$ and $\rightsquigarrow_5$, but it is easy to handle these cases by the second axiom in Definition 3.2.5. □

Theorem 3.3.5 (strong normalisation)**.** *No pre-$\ell^\circ_{\mathcal{B}}$-net $\mathfrak{r} :: \mathfrak{S}$ has a countably infinite sequence $(\mathfrak{r}_i :: \mathfrak{S}_i \rightsquigarrow \mathfrak{r}_{i+1} :: \mathfrak{S}_{i+1})_{i\in\mathbb{N}}$ of $\ell^\circ_{\mathcal{B}}$-reduction such that $\mathfrak{S}_0 = \mathfrak{S}$ and $\mathfrak{r}_0 = \mathfrak{r}$.*

Proof. We first define the *magnitude* $\|T\| \in \mathbb{N}_+$ of each $\ell^\circ_{\mathcal{B}}$-type T inductively by

$$\|I_{[v]}\| := 1 \quad (I \in \mathrm{Obj}(\mathcal{B}) \cup \{\top, 1, \bot\}), \qquad \|!_{[e]}T\| := \|T\| + 1,$$

$$\|T \spadesuit T'\| := \|T\| + \|T'\| + 1 \quad (\spadesuit \in \{\otimes, \times, {}_x\times_y, \multimap\}),$$

and lift it to that $\|M\| \in \mathbb{N}_+$ of each $\ell^\circ_{\mathcal{B}}$-port M by

$$\|\top_{[v]}\| := 1, \qquad \|N \triangleright P\| := 2 \times \|N\| = 2 \times \|P\|,$$

$$\|M \clubsuit M'\| := \|M\| + \|M'\| + 1 \quad (\clubsuit \in \{\otimes, \times\}), \qquad \|!_{[e]}M\| := \|M\| + 1,$$

and to that $\|\mathscr{M} : M\| \in \mathbb{N}_+$ of a pair $\mathscr{M} : M$ of a $\mathcal{B}$-web $\mathscr{M}$ and an $\ell^\circ_{\mathcal{B}}$-cut M by

$$\|\mathscr{M} : M\| := (|\mathscr{M}| + 1) \times \Big(\sum_{e \in \mathrm{End}^!_M} (|\mathrm{Col}_{\mathscr{M}}[e]| + 1)^2 \Big) \times (\|M\| + 1).$$

We then define the *rank* of a pre-$\ell^\circ_{\mathcal{B}}$-term $r :: S$, written $\mathrm{Rank}(r :: S)$, where $S = \Gamma \dashv (\mathscr{M}_j : M_j)_{j=1}^m \vdash N$, by

$$\mathrm{Rank}(r :: S) := \sum_{j=1}^m \|\mathscr{M}_j : M_j\| \quad \text{if } m = 0, \qquad \mathrm{Rank}(r :: S) := \sum_{j=1}^m \|\mathscr{M}_j : M_j\| \quad \text{if } m > 0,$$

and finally lift it to a pre-$\ell^\circ_{\mathcal{B}}$-net $\mathfrak{r} :: \mathfrak{S}$ by

$$\mathrm{Rank}(\mathfrak{r} :: \mathfrak{S}) := \mathrm{Rank}(\mathfrak{G}^{-1}(\mathfrak{r} :: \mathfrak{S}))$$

It is then straightforward to see by case analysis that $\ell^\circ_{\mathcal{B}}$-reduction strictly decreases the rank of a pre-$\ell^\circ_{\mathcal{B}}$-net: if $\mathfrak{r} :: \mathfrak{S} \rightsquigarrow \mathfrak{r}' :: \mathfrak{S}'$, then $\mathrm{Rank}(\mathfrak{r} :: \mathfrak{S}) > \mathrm{Rank}(\mathfrak{r}' :: \mathfrak{S}')$.

Finally, we assume for a contradiction that there is a countably infinite sequence

$$(\mathfrak{r}_i :: \mathfrak{S}_i \rightsquigarrow \mathfrak{r}_{i+1} :: \mathfrak{S}_{i+1})_{i\in\mathbb{N}}$$

of $\ell^\circ_{\mathcal{B}}$-reduction. Then, we have the infinite chain

$$\mathrm{Rank}(\mathfrak{r}_0 :: \mathfrak{S}_0) > \mathrm{Rank}(\mathfrak{r}_1 :: \mathfrak{S}_1) > \mathrm{Rank}(\mathfrak{r}_2 :: \mathfrak{S}_2) > \cdots$$

of ranks, which contradicts the finiteness and the non-negativity of a rank. □

Theorem 3.3.6 (confluence)**.** *If $\mathfrak{r} :: \mathfrak{S}$ is an $\ell^\circ_{\mathcal{B}}$-net, $\mathfrak{r} :: \mathfrak{S} \rightsquigarrow \mathfrak{r}' :: \mathfrak{S}'$ and $\mathfrak{r} :: \mathfrak{S} \rightsquigarrow \mathfrak{r}'' :: \mathfrak{S}''$, then $\mathfrak{r}' :: \mathfrak{S}' \rightsquigarrow^* \mathfrak{r}''' :: \mathfrak{S}'''$ and $\mathfrak{r}'' :: \mathfrak{S}'' \rightsquigarrow^* \mathfrak{r}''' :: \mathfrak{S}'''$ for some pre-$\ell^\circ_{\mathcal{B}}$-net $\mathfrak{r}''' :: \mathfrak{S}'''$.*

Proof. By straightforward case analysis on $\mathfrak{r} :: \mathfrak{S} \rightsquigarrow \mathfrak{r}' :: \mathfrak{S}'$ and $\mathfrak{r} :: \mathfrak{S} \rightsquigarrow \mathfrak{r}'' :: \mathfrak{S}''$. An overlap of cases may occur due the one $\rightsquigarrow_5$, but it can be handled by finite sequences of $\ell^\circ_{\mathcal{B}}$-reduction. □

Remark. If the case $\rightsquigarrow_5$ were excluded, then $\ell^\circ_{\mathcal{B}}$-reduction would enjoy not only confluence but also *diamond property*: if $\mathfrak{r} :: \mathfrak{S}$ is an $\ell^\circ_{\mathcal{B}}$-net, $\mathfrak{r} :: \mathfrak{S} \rightsquigarrow \mathfrak{r}' :: \mathfrak{S}'$ and $\mathfrak{r} :: \mathfrak{S} \rightsquigarrow \mathfrak{r}'' :: \mathfrak{S}''$, then $\mathfrak{r}' :: \mathfrak{S}' \rightsquigarrow \mathfrak{r}''' :: \mathfrak{S}'''$ and $\mathfrak{r}'' :: \mathfrak{S}'' \rightsquigarrow \mathfrak{r}''' :: \mathfrak{S}'''$ for some pre-$\ell^\circ_{\mathcal{B}}$-net $\mathfrak{r}''' :: \mathfrak{S}'''$.

As a result, we obtain the following:

Corollary 3.3.7 (correctness)**.** *Every $\ell^\circ_{\mathcal{B}}$-net $\mathfrak{r}_0 : \mathfrak{S}_0$ has a maximal finite sequence*

$$(\mathfrak{r}_{i-1} :: \mathfrak{S}_{i-1} \rightsquigarrow \mathfrak{r}_i :: \mathfrak{S}_i)_{i=1}^{n}$$

of $\ell^\circ_{\mathcal{B}}$-reduction,[14] *and any of these maximal finite sequences satisfies:*

1. *$\mathfrak{r}_n :: \mathfrak{S}_n$ is normalised, where $\mathfrak{G}^{-1}\mathfrak{S}_n = \Gamma \dashv\vdash \Phi$ if $\mathfrak{G}^{-1}\mathfrak{S}_0 = \Gamma \dashv \Sigma \vdash \Phi$;*
2. *$\mathfrak{r}_n :: \mathfrak{S}_n$ is unique: if $(\mathfrak{r}'_{i-1} :: \mathfrak{S}'_{i-1} \rightsquigarrow \mathfrak{r}'_i :: \mathfrak{S}'_i)_{i=1}^{n'}$ is a finite sequence of $\ell^\circ_{\mathcal{B}}$-reduction with $\mathfrak{r}'_{n'} :: \mathfrak{S}'_{n'}$ normalised and $(\mathfrak{r}'_0 :: \mathfrak{S}'_0) = (\mathfrak{r}_0 :: \mathfrak{S}_0)$, then $(\mathfrak{r}'_{n'} :: \mathfrak{S}'_{n'}) = (\mathfrak{r}_n :: \mathfrak{S}_n)$.*

Proof. By Theorems 3.3.4–3.3.6, where the uniqueness of normalisation is a consequence of the confluence and the strong normalisation of $\ell^\circ_{\mathcal{B}}$-reduction by Newman's lemma [New42]. □

We call the normalised one $\mathfrak{r}_n :: \mathfrak{S}_n$ in the corollary the *normal form* of $\mathfrak{r}_0 :: \mathfrak{S}_0$ and write $\eta(\mathfrak{r}_0 :: \mathfrak{S}_0)$ or $\eta\mathfrak{r}_0 :: \eta\mathfrak{S}_0$ for it. The corollary guarantees that the following is well-defined:

Definition 3.3.8 ($\ell^\circ_{\mathcal{B}}$-equivalence)**.** Two $\ell^\circ_{\mathcal{B}}$-nets $\mathfrak{r} :: \mathfrak{S}$ and $\mathfrak{r}' :: \mathfrak{S}'$ are said to be *$\ell^\circ_{\mathcal{B}}$-equivalent* and written $\mathfrak{r} :: \mathfrak{S} \simeq_{\ell^\circ_{\mathcal{B}}} \mathfrak{r}' :: \mathfrak{S}'$ if their normal forms coincide. The equivalence relation $\simeq_{\ell^\circ_{\mathcal{B}}}$ is called *$\ell^\circ_{\mathcal{B}}$-equivalence*, and the equivalence class of an ℓ-net $\mathfrak{r} :: \mathfrak{S}$ with respect to $\ell^\circ_{\mathcal{B}}$-equivalence is called the *$\ell^\circ_{\mathcal{B}}$-equivalence class* of $\mathfrak{r} :: \mathfrak{S}$ and written $[\mathfrak{r} :: \mathfrak{S}]_{\ell^\circ_{\mathcal{B}}}$.

4. Graphical free linear categories

In the present section, we study the $\ell^\circ_{\mathcal{B}}$-calculus through category theory [ML13], specifically in terms of *(pointed) linear categories* [Ben94, Bie95, MMDPR05]. We first recall the notion of (strict) (pointed) linear categories and show that the $\ell^\circ_{\mathcal{B}}$-calculus forms a strict (pointed) linear category $\tilde{\mathrm{Cl}}(\ell^\circ_{\mathcal{B}})$ in §4.1. We then establish a main theorem in §4.2: *freeness* of $\tilde{\mathrm{Cl}}(\ell^\circ_{\mathcal{B}})$ over $\mathcal{B}$.

4.1. Linear categories.

We first recall *(pointed) linear categories* [BBDPH93, Definition 3]. They are equivalent to *new Seely categories* [Bie95, MMDPR05] (also known as *monoidal storage categories* [BCS09]) when they have finite products, as shown by Bierman [Bie95, §3]. Strictly speaking, we adopt the definition of new Seely categories for its simplicity and elegance, but by abuse of terminology we call them 'linear categories' since the name signifies our subject.

Definition 4.1.1 (linear categories [Bie95, MMDPR05])**.** A *linear category* is a symmetric monoidal closed category $\mathcal{L}$ with finite products, equipped with a comonad ! on itself, called a *(linear) exponential (comonad)*, such that the canonical adjunction

(12)
$$\mathcal{L}_! \underset{G}{\overset{F}{\rightleftarrows}} \mathcal{L} \quad (\perp)$$

between $\mathcal{L}$ and the (necessarily cartesian closed [Bie95]) coKleisli category $\mathcal{L}_!$ is monoidal. A linear category is said to be *pointed* if it comes with a distinguished object $\perp$, called *bottom*.

Bierman [Bie95, §2] proved that every linear category $\mathcal{L} = (\mathcal{L}, \top, \otimes, !)$ contains the *Seely isomorphisms* [See89, Bie95]

$$!1 \cong \top, \qquad !(A \times B) \cong !A \otimes !B \quad (A, B \in \mathcal{L}),$$

and the exponential ! forms a symmetric monoidal comonad. In the sense of the isomorphisms, the exponential *mediates* the symmetric monoidal and the cartesian structures of $\mathcal{L}$.

[14] When $n = 0$, this finite sequence is *empty*.

Example 4.1.2. Categories arising in algebra, homological algebra and geometry provide examples of linear categories through a general construction as follows.

Porst [Por08, §2.7] proved that every locally presentable symmetric monoidal closed category admits cofree cocommutative comonoids, hence forming a *Lafont category* [Laf88], a particular type of linear category. Since it also has finite products, it gives rise to a linear category.

This construction applies, in particular, to the following categories:

- The category Mod_R of modules over a commutative unital ring R and R-linear maps. Its tensor product is the usual tensor product of modules over R, its tensor unit is R, and its internal hom is given by the R-modules of R-linear maps with pointwise addition and scalar multiplication. Finite products are given by the zero module and direct sums. This example includes the categories of vector spaces over a field and of abelian groups.
- The category $\mathrm{Mod}(\mathcal{O}_X)$ of sheaves of modules on a commutative ringed space $\mathcal{O}_X = (X, \mathcal{O}_X)$ and morphisms of $\mathcal{O}_X$-modules. Its tensor product is the sheaf tensor product, its tensor unit is $\mathcal{O}_X$, and its internal hom $\mathrm{Mod}(\mathcal{O}_X)(\mathcal{F}, \mathcal{G})$ is the sheaf whose sections over an open subset $U \subseteq X$ are the morphisms $\mathcal{F}|_U \to \mathcal{G}|_U$ of $\mathcal{O}_X|_U$-modules [The26, Tags 01CA and 01CN]. This includes examples on schemes, complex manifolds and smooth manifolds, using their respective structure sheaves. In the smooth case, the sheaves of sections of finite-rank vector bundles are precisely the finite locally free C^∞_X-modules, though this subcategory need not be closed under the exponential.
- The category $\mathrm{Ch}(R)$ of unbounded chain complexes of R-modules and chain maps, with the usual tensor product of complexes, Koszul symmetry and internal hom complex; see Hovey [Hov99, Proposition 4.2.13]. Finite products are computed degreewisely.
- The category $\mathrm{Rep}_k(G)$ of representations of a fixed discrete group G on vector spaces over a field k, with G-equivariant linear maps as morphisms. Its tensor product is the usual tensor product over k, equipped with the diagonal action
$$g \cdot (v \otimes w) = (g \cdot v) \otimes (g \cdot w),$$
and its tensor unit is k with the trivial action. The internal hom $\mathrm{Rep}_k(G)(V, W)$ is the vector space of all linear maps $V \to W$, equipped with the action
$$(g \cdot f)(v) = g \cdot f(g^{-1} \cdot v).$$
These operations make $\mathrm{Rep}_k(G)$ symmetric monoidal closed. Its underlying category is equivalent to the category of modules over the group algebra $k[G]$, and thus locally presentable. Here, representations are allowed to be infinite-dimensional.

These categories are Grothendieck abelian and hence locally presentable [Bek00, Proposition 3.10]. For sheaves of modules, see [The26, Tag 079Q]; for chain complexes, see [Bek00, Proposition 3.13]. The case of group representations follows from the equivalence with $\mathrm{Mod}_{k[G]}$. Thus Porst's theorem applies to each of the symmetric monoidal closed structures, yielding a linear category whose exponential is given by the cofree cocommutative comonoid construction.

Example 4.1.3. The category Rel of sets and binary relations gives rise to a linear category [Mel09, MTT18]. Its tensor product is the cartesian product $\times$ of sets, with a singleton set $\{*\}$ as tensor unit. It is compact closed, with each object self-dual; its internal hom from X to Y is $X \times Y$. Its categorical finite products are given by the empty set $\emptyset$ and disjoint union $\uplus$.

Given a set X, the exponential $!X$ is the set of all finite multisets of elements of X. For a relation $r \subseteq X \times Y$, the relation $!r$ relates two multisets precisely when their occurrences can be paired so that each pair belongs to r. Splitting a multiset on a disjoint union into its two components gives the Seely isomorphisms
$$!\emptyset \cong \{*\}, \qquad\qquad !(X \uplus Y) \cong !X \times !Y.$$

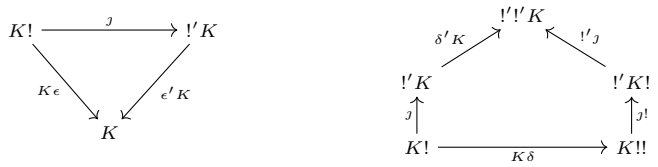


FIGURE 20. Compatibility of a linear functor with counits and comultiplications

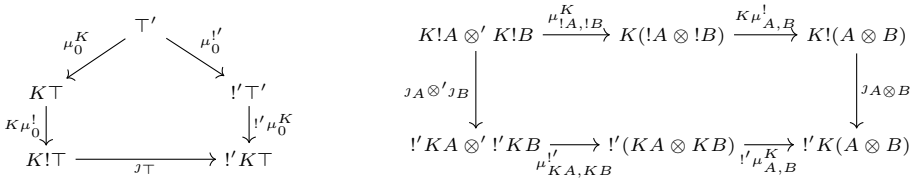


FIGURE 21. Compatibility of a linear functor with lax structures

We next define a suitable notion of *morphisms between (pointed) linear categories*, which are functors that preserve the structure of a (pointed) linear category:

Notation. Given a monoidal functor $F : \mathcal{M} \to \mathcal{M}'$ between monoidal categories $\mathcal{M} = (\mathcal{M}, \top, \otimes)$ and $\mathcal{M}' = (\mathcal{M}', \top', \otimes')$, we write

$$\mu_0^F : \top' \to F\top, \qquad \mu_{A,B}^F : FA \otimes' FB \to F(A \otimes B) \quad (A, B \in \mathcal{M})$$

for its lax coherence morphisms.

Definition 4.1.4 (linear functors)**.** A *linear functor* between linear categories $\mathcal{L} = (\mathcal{L}, \top, \otimes, !)$ and $\mathcal{L}' = (\mathcal{L}', \top', \otimes', !')$ is a strong symmetric monoidal closed functor $K : \mathcal{L} \to \mathcal{L}'$ that preserves finite products, equipped with a natural isomorphism

$$\jmath : K! \xrightarrow{\sim} !'K : \mathcal{L} \to \mathcal{L}' \tag{13}$$

that is

(1) Compatible with the counits and the comultiplications of the exponentials $! = (!, \epsilon, \delta)$ and $!' = (!', \epsilon', \delta')$ in the sense that the diagrams in Figure 20 commute;
(2) Monoidal with respect to the symmetric monoidal functors $K!$ and $!'K$, *i.e.* compatible with their lax structures in the sense that the diagrams in Figure 21 commute.

A linear functor between pointed linear categories is said to be *pointed* if it preserves bottoms up to isomorphisms.

Recall that a strong symmetric monoidal *closed* functor between symmetric monoidal closed categories is a strong symmetric monoidal functor that preserves closure up to natural isomorphisms. This requirement is reasonable for a (pointed) linear functor, or a morphism between (pointed) linear categories, since closure is a part of a (pointed) linear category. In contrast, the functor part of the exponential ! of a (pointed) linear category is in general *not* closed. In fact, the instance of our graphical (pointed) linear category is not closed (Example 4.1.10).

The natural isomorphism $\jmath : K! \xrightarrow{\sim} !'K$ (13) and the first axiom in Definition 4.1.4 come from the standard (2-)category of (co)monads [Str72]. The second axiom means that $\jmath$ is a *monoidal* natural transformation between the symmetric monoidal functors $K!, !'K : \mathcal{L} \to \mathcal{L}'$.

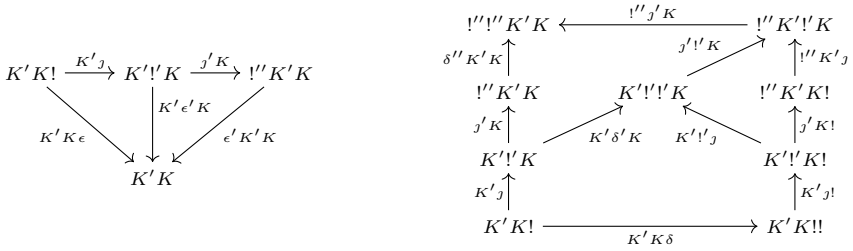


FIGURE 22. Commutative diagrams for an axiom on the composition of linear functors

In the present work, we henceforth focus on *strict* (pointed) linear categories and (pointed) linear functors in the following sense:

Definition 4.1.5 (strictness)**.** A *strict* (pointed) linear category is a (pointed) linear category with chosen finite products and closure (as well as bottom). A *strict* (pointed) linear functor is a (pointed) linear functor between strict (pointed) linear categories, whose symmetric monoidal closed functor and natural isomorphism (13) are both strict, that strictly preserves the chosen finite products (as well as bottom).

Then, our main goal is to prove that the $\ell^{\circ}_{\mathcal{B}}$-calculus forms a *free* strict small (pointed) linear category over $\mathcal{B}$ in the following sense:

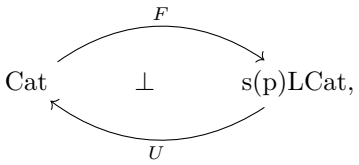
Definition 4.1.6 (categories of linear categories)**.** Let LCat be the category in which

- An object is a small linear category;
- A morphism $\mathcal{L} \to \mathcal{L}'$ is a linear functor between $\mathcal{L}$ and $\mathcal{L}'$,

and pLCat the subcategory of LCat that consists of small pointed linear categories and pointed linear functors between them. Similarly, let sLCat be the category of small strict linear categories and strict linear functors, and spLCat the subcategory of sLCat that consists of small strict pointed linear categories and strict pointed linear functors.

Strong symmetric monoidal closed functors are closed under composition [Kel82], and it is straightforward to verify that the axioms on (pointed) linear functors are preserved under composition. For example, given linear functors $K = (K, \jmath) : \mathcal{L} \to \mathcal{L}'$ and $K' = (K', \jmath') : \mathcal{L}' \to \mathcal{L}''$, the composition $K'K : \mathcal{L} \to \mathcal{L}''$ satisfies the first axiom in Definition 4.1.4 by the commutative diagrams in Figure 22, where the exponentials of $\mathcal{L}$, $\mathcal{L}'$ and $\mathcal{L}''$ are written $! = (!, \epsilon, \delta)$, $!' = (!', \epsilon', \delta')$ and $!'' = (!'', \epsilon'', \delta'')$, respectively. Using the naturality of the coherence (iso)morphisms, we may similarly show that $K'K$ satisfies the second axiom; we leave the details to the reader. It then follows that (p)LCat is a well-defined category. Similarly, s(p)LCat is well-defined too.

Definition 4.1.7 (freeness)**.** Let U : s(p)LCat $\to$ Cat be the functor that maps small strict (pointed) linear categories to their underlying small categories, and strict (pointed) linear functors to their underlying functors. If there is a left adjoint F : Cat $\to$ s(p)LCat to U, *i.e.*

(14)


then the value $F\mathcal{C} \in$ Obj(s(p)LCat) for each $\mathcal{C} \in$ Obj(Cat) is said to be *free* over $\mathcal{C}$.

We shall prove that our graphical (pointed) linear category is free in this sense. We choose the category s(p)LCat, not the other (p)LCat, since otherwise the freeness would hold not as defined above but only up to suitable 2-cells, for which we need to digresses from the main line of the present work and develop some 2-categorical structure. We make that choice in order to focus on our main contributions, though the other choice is certainly interesting to study too.

Having fixed our categorical structure, we next build a small strict (pointed) linear category $\tilde{\mathrm{Cl}}(\ell^{\circ}_{\mathcal{B}})$ from the $\ell^{\circ}_{\mathcal{B}}$-calculus and show that the map $\mathcal{B} \mapsto \tilde{\mathrm{Cl}}(\ell^{\circ}_{\mathcal{B}})$ extends to the left adjoint F to the functor U in (14). To this end, in contrast to type-theoretic categories [LS88, Pit01, Jac99, MMDPR05], a problem in our category is *identities*: an identity $A \to A$ in the former is simply (an equivalence class of) the variable λ-term $x : A \vdash x : A$, but it does not fit the latter since

(1) Such a variable λ-term does not exist in the $\ell^{\circ}_{\mathcal{B}}$-calculus;
(2) If we formally add variable λ-terms to the $\ell^{\circ}_{\mathcal{B}}$-calculus, then we would need formal equations or rewritings for the extension to match (pointed) linear categories.

Then, we instead *derive* a class of freely generated pre-$\ell^{\circ}_{\mathcal{B}}$-nets that yield identities. Again, for ease of notation, we describe those pre-$\ell^{\circ}_{\mathcal{B}}$-nets through the corresponding pre-$\ell^{\circ}_{\mathcal{B}}$-terms.

Notation. A *formula* in *intuitionistic linear logic (ILL)* [Gir87, GL87] over $\mathcal{B}$ is an expression A defined by

$$A := X \mid \top \mid 1 \mid A \otimes A' \mid A \times A' \mid A \multimap A' \mid !A,$$

where $X \in \mathrm{Obj}(\mathcal{B})$, and a one B in *intuitionistic linear logic with bottom (ILB)* by

$$B := X \mid \top \mid 1 \mid \bot \mid B \otimes B' \mid B \times B' \mid B \multimap B' \mid !B,$$

where $\top$ is called *top*, 1 *one*, $\bot$ *bottom*, $\otimes$ *tensor*, $\times$ *product*, $\multimap$ *linear implication*, and $!$ *of-course* (or *exponential*). Let $\mathfrak{G}A$ and $\mathfrak{G}B$ denote the parse trees of A and B, called *ILL-* and *ILB-trees* over $\mathcal{B}$, respectively. We apply the names of A and B to $\mathfrak{G}A$ and $\mathfrak{G}B$, respectively, too.

These formulae are objects in the formal free (pointed) linear category over $\mathcal{B}$. Let us assign a unique (up to renamings of variables) $\ell^{\circ}_{\mathcal{B}}$-type, written T_A, to each formula A in ILL or ILB and distinguish between different copies of T_A by auxiliary symbols, *e.g.* T'_A, $\hat{T}_A$, and so on.

Remark. If two $\ell^{\circ}_{\mathcal{B}}$-types are written without such a subscript $(_)_A$, *e.g.* T and T', then they may be completely unrelated $\ell^{\circ}_{\mathcal{B}}$-types; the above convention is only for the assigned ones T_A.

Definition 4.1.8 ($\ell^{\circ}_{\mathcal{B}}$-copy-cats)**.** The *$\ell^{\bullet}_{\mathcal{B}}$-copy-cat*[15] on a formula A in ILB, written cp_A, is the normalised pre-$\ell^{\bullet}_{\mathcal{B}}$-term

$$\mathscr{D}_{\check{T}_A,\hat{T}_A} : \check{T}_A \dashv\vdash \mathscr{C}_{\check{T}_A,\hat{T}_A} : \hat{T}_A,$$

where the $\mathcal{B}$-webs $\mathscr{D}_{\check{T}_A,\hat{T}_A}$ and $\mathscr{C}_{\check{T}_A,\hat{T}_A}$ are abbreviated respectively as $\check{\mathscr{D}}_A$ and $\hat{\mathscr{C}}_A$ (where the auxiliary symbols on T_A and $\mathscr{C}_A/\mathscr{D}_A$ are synchronised), defined inductively by

(1) $\mathrm{cp}_X := \{v \xrightarrow{\mathrm{id}_X} w\} : X_{[w]} \dashv\vdash \emptyset : X_{[v]}$, where $X \in \mathrm{Obj}(\mathcal{B})$;
(2) $\mathrm{cp}_I := \{v \to w\} : I_{[w]} \dashv\vdash \emptyset : I_{[v]}$, where $I \in \{\top, \bot\}$;
(3) $\mathrm{cp}_1 := \emptyset : 1_{[w]} \dashv\vdash \emptyset : 1_{[v]}$;
(4) $\mathrm{cp}_{A\otimes B} := \check{\mathscr{D}}_A \cup \check{\mathscr{D}}_B : \check{T}_A \otimes \check{T}_B \dashv\vdash \hat{\mathscr{C}}_A \cup \hat{\mathscr{C}}_B : \hat{T}_A \otimes \hat{T}_B$;
(5) $\mathrm{cp}_{A\times B} := (\check{\mathscr{D}}_A + \langle x|) \cup (\check{\mathscr{D}}_B + \langle y|) : \check{T}_A \times \check{T}_B \dashv\vdash (\hat{\mathscr{C}}_A + \langle x|) \cup (\hat{\mathscr{C}}_B + \langle y|) : \hat{T}_{Ax}\times_y \hat{T}_B$;
(6) $\mathrm{cp}_{A\multimap B} := \hat{\mathscr{C}}_A \cup \check{\mathscr{D}}_B : \hat{T}_A \multimap \check{T}_B \dashv\vdash \check{\mathscr{D}}_A \cup \hat{\mathscr{C}}_B : \check{T}_A \multimap \hat{T}_B$;
(7) $\mathrm{cp}_{!A} := (\check{\mathscr{D}}_A + |(\mathrm{Col}(\check{\mathscr{D}}_A), \check{e})\rangle) \cup \{\hat{e} \to \check{e}\} : !_{[\check{e}]}\check{T}_A \dashv\vdash \hat{\mathscr{C}}_A + |(\mathrm{Col}(\check{\mathscr{D}}_A), \check{e})\rangle : !_{[\hat{e}]}\hat{T}_A$,

where the variables in cp_A are arbitrarily chosen and pairwise distinct, and it is called an *$\ell_{\mathcal{B}}$-copy-cat* if A is a formula in ILL.

[15]This terminology comes from game semantics [Hyl97, A+97], where identities in a category of games and strategies are called *copy-cat* strategies since they compute simply by 'copy-catting' previous moves in a play.

Notation. The informal, auxiliary symbols $\check{(_)}$ and $\hat{(_)}$ used for $\ell^{\circ}_{\mathcal{B}}$-copy-cats may vary; *e.g.* cp_A is also written

$$\mathscr{D}_A : T_A \dashv\Vdash \mathscr{C}'_A : T'_A, \qquad \mathscr{D}'_A : T'_A \dashv\Vdash \mathscr{C}''_A : T''_A, \qquad \mathscr{D}''_A : T''_A \dashv\Vdash \mathscr{C}'''_A : T'''_A,$$

and so forth, where again the symbols on each pair of a $\mathcal{B}$-web and an $\ell^{\circ}_{\mathcal{B}}$-type are synchronised.

It is easy to see by induction that $\ell^{\circ}_{\mathcal{B}}$-copy-cats are freely generated pre-$\ell^{\circ}_{\mathcal{B}}$-terms containing no exponential label whose set is of cardinality more than one. Moreover, they satisfy:

Proposition 4.1.9 (generalised unit laws)**.** *Any $\ell^{\bullet}_{\mathcal{B}}$-net of the form*

$$\rho\mathfrak{G}(\gamma : \Gamma, \mathscr{P} : T_A, \delta : \Delta \dashv \sigma : \Sigma \vdash \mathscr{B} : T_B)^{\approx}, \tag{15}$$

where by Theorem 3.2.44 the pre-$\ell^{\bullet}_{\mathcal{B}}$-term $\gamma : \Gamma, \mathscr{P} : T_A, \delta : \Delta \dashv \sigma : \Sigma \vdash \mathscr{B} : T_B$ is assumed to be freely generated without loss of generality, is $\ell^{\bullet}_{\mathcal{B}}$-equivalent to those

1. $\rho\mathfrak{G}(\gamma : \Gamma, \check{\mathscr{D}}_A : \check{T}_A, \delta : \Delta \dashv \hat{\mathscr{C}}_A \cup \mathscr{P} : \hat{T}_A \triangleright T_A, \sigma : \Sigma \vdash \mathscr{B} : T_B)^{\approx}$,
2. $\rho\mathfrak{G}(\gamma : \Gamma, \mathscr{P} : T_A, \delta : \Delta \dashv \sigma : \Sigma, \mathscr{B} \cup \check{\mathscr{D}}_B : T_B \triangleright \check{T}_B \vdash \hat{\mathscr{C}}_B : \hat{T}_B)^{\approx}$

obtained from the one (15) and $\ell^{\bullet}_{\mathcal{B}}$-copy-cats by the typing rules Cut *and* XL via *the map $\rho\mathfrak{G}$.*

Proof. By Theorem 3.2.44, we may proceed by induction on a derivation of a freely generated pre-$\ell^{\bullet}_{\mathcal{B}}$-term $s :: S$ representing the $\ell^{\bullet}_{\mathcal{B}}$-net $\mathfrak{r} :: \mathfrak{S}$ (15), *i.e.* $\rho\mathfrak{G}(s :: S)^{\approx} = \mathfrak{r} :: \mathfrak{S}$. In the following, we only pick up some key cases in the induction since the other cases are simpler or analogous.

First, for the inductive step on the first clause (1) with respect to the typing rule $\otimes$L, assume the derivation

$$(\textsc{Cut}, \mathrm{XL}^*)\ \dfrac{\dfrac{}{\mathrm{cp}_{A\otimes B}} \qquad (\otimes\mathrm{L}, \mathrm{XL}^*)\ \dfrac{\dfrac{}{\gamma : \Gamma, \mathscr{P} : T_A, \mathscr{Q} : T_B, \delta : \Delta \dashv \sigma : \Sigma \vdash \mathscr{C} : T_C\ (\Box)}}{\gamma : \Gamma, \mathscr{P} \cup \mathscr{Q} : T_{A\otimes B}, \delta : \Delta \dashv \sigma : \Sigma \vdash \mathscr{C} : T_C\ (*)}}{\gamma : \Gamma, \check{\mathscr{D}}_{A\otimes B} : \check{T}_{A\otimes B}, \delta : \Delta \dashv \hat{\mathscr{C}}_{A\otimes B} \cup \mathscr{P} \cup \mathscr{Q} : \hat{T}_{A\otimes B} \triangleright T_{A\otimes B}, \sigma : \Sigma \vdash \mathscr{C} : T_C\ (\star)}$$

of a freely generated pre-$\ell^{\bullet}_{\mathcal{B}}$-term $(\star)$, where the one $(*)$ represents the $\ell^{\bullet}_{\mathcal{B}}$-net (15) in this case. The $\ell^{\bullet}_{\mathcal{B}}$-net $\rho\mathfrak{G}(\star)^{\approx}$ is $\ell^{\bullet}_{\mathcal{B}}$-reduced to the one represented by the freely generated pre-$\ell^{\bullet}_{\mathcal{B}}$-term

$$\gamma : \Gamma, \check{\mathscr{D}}_A \cup \check{\mathscr{D}}_B : \check{T}_A \otimes \check{T}_B, \delta : \Delta \dashv \hat{\mathscr{C}}_A \cup \mathscr{P} \cup \hat{\mathscr{C}}_B \cup \mathscr{Q} : (\hat{T}_A \triangleright T_A) \otimes (\hat{T}_B \triangleright T_B), \sigma : \Sigma \vdash \mathscr{C} : T_C,$$

which can be derived as

$$\dfrac{\dfrac{\dfrac{\dfrac{}{\mathrm{cp}_B} \qquad (\textsc{Cut}, \mathrm{XL}^*)\ \dfrac{\dfrac{}{\mathrm{cp}_A} \qquad \dfrac{}{\gamma : \Gamma, \mathscr{P} : T_A, \mathscr{Q} : T_B, \delta : \Delta \dashv \sigma : \Sigma \vdash \mathscr{C} : T_C\ (\Box)}}{\gamma : \Gamma, \check{\mathscr{D}}_A : \check{T}_A, \mathscr{Q} : T_B, \delta : \Delta \dashv \hat{\mathscr{C}}_A \cup \mathscr{P} : \hat{T}_A \triangleright T_A, \sigma : \Sigma \vdash \mathscr{C} : T_C\ (\heartsuit)}}{(\textsc{Cut}, \mathrm{XL}^*)\ \gamma : \Gamma, \check{\mathscr{D}}_A : \check{T}_A, \check{\mathscr{D}}_B : \check{T}_B, \delta : \Delta \dashv \hat{\mathscr{C}}_A \cup \mathscr{P} : \hat{T}_A \triangleright T_A, \hat{\mathscr{C}}_B \cup \mathscr{Q} : \hat{T}_B \triangleright T_B, \sigma : \Sigma \vdash \mathscr{C} : T_C\ (\diamondsuit)}}{(\otimes\mathrm{L}, \mathrm{XL}^*)\ \gamma : \Gamma, \check{\mathscr{D}}_A \cup \check{\mathscr{D}}_B : \check{T}_{A\otimes B}, \delta : \Delta \dashv \hat{\mathscr{C}}_A \cup \mathscr{P} : \hat{T}_A \triangleright T_A, \hat{\mathscr{C}}_B \cup \mathscr{Q} : \hat{T}_B \triangleright T_B, \sigma : \Sigma \vdash \mathscr{C} : T_C}}{(\otimes\mathrm{M}, \mathrm{XM}^*)\ \gamma : \Gamma, \check{\mathscr{D}}_A \cup \check{\mathscr{D}}_B : \check{T}_{A\otimes B}, \delta : \Delta \dashv \hat{\mathscr{C}}_A \cup \mathscr{P} \cup \hat{\mathscr{C}}_B \cup \mathscr{Q} : (\hat{T}_A \triangleright T_A) \otimes (\hat{T}_B \triangleright T_B), \sigma : \Sigma \vdash \mathscr{C} : T_C}$$

By the induction hypothesis, we see that the $\ell^{\bullet}_{\mathcal{B}}$-net $\rho\mathfrak{G}(\Box)^{\approx}$ is $\ell^{\bullet}_{\mathcal{B}}$-equivalent to the one $\ell\mathfrak{G}(\heartsuit)^{\approx}$, which is similarly $\ell^{\bullet}_{\mathcal{B}}$-equivalent to the one $\rho\mathfrak{G}(\diamondsuit)^{\approx}$. From this observation, we conclude that the $\ell^{\bullet}_{\mathcal{B}}$-equivalence $\rho\mathfrak{G}(\star)^{\approx} \simeq_{\ell^{\bullet}_{\mathcal{B}}} \rho\mathfrak{G}(*)^{\approx}$ holds by the definition of $\ell^{\bullet}_{\mathcal{B}}$-reduction (and Theorem 3.3.6) since the $\ell^{\bullet}_{\mathcal{B}}$-nets $\rho\mathfrak{G}(\star)^{\approx}$ and $\rho\mathfrak{G}(\diamondsuit)^{\approx}$ are different only in a way insignificant for $\ell^{\bullet}_{\mathcal{B}}$-reduction. This completes the main case of the induction step.

The minor cases follow more readily from the induction hypotheses. The second clause (2) in the same induction step is similar to the minor cases.

The base cases and the other inductive steps of the induction are simpler or analogous. □

Remark. The (ordinary) unit laws are obtained from the generalised ones by the restriction of the $\ell^{\bullet}_{\mathcal{B}}$-net (15) to the one $\rho\mathfrak{G}(\mathscr{P} : T_A \dashv \sigma : \Sigma \vdash \mathscr{B} : T_B)^{\approx}$, but the induction of the proof does not work for the unit laws (as seen in the inductive step on $\otimes$L given above). Another remark is that for the inductive steps on $\multimap$L and $\multimap$R the induction proceeds on the two $\ell^{\bullet}_{\mathcal{B}}$-equivalences (1) and (2) respectively and simultaneously.

Example 4.1.10. Let us take the $\ell^{\bullet}_{\mathcal{B}}$-copy-cat $\mathrm{cp}_{!(\bot\multimap\bot)}$, or

$$\{\hat{e} \to \check{e}, \hat{x} \xrightarrow{|(\{\hat{x}\check{x}\},\check{e})\rangle} \check{x}\} : !_{\check{e}}(\bot_{[\check{y}]} \multimap \bot_{[\check{x}]}) \dashv\vdash \{\check{y} \xrightarrow{|(\{\hat{x}\check{x}\},\check{e})\rangle} \hat{y}\} : !_{\hat{e}}(\bot_{[\hat{y}]} \multimap \bot_{[\hat{x}]}),$$

as the one representing the $\ell^{\bullet}_{\mathcal{B}}$-net (15), for which the two clauses of Proposition 4.1.9 coincide. Figure 23 shows $\ell^{\bullet}_{\mathcal{B}}$-reduction that demonstrates the $\ell^{\bullet}_{\mathcal{B}}$-equivalence given by the proposition.

Example 4.1.11. Consider the freely generated pre-$\ell^{\bullet}_{\mathcal{B}}$-term $(\star)$ derived as

$$(\textsc{Cut})\ \frac{\dfrac{}{\{w \xrightarrow{\langle \hat{x}|} v, w \xrightarrow{\langle \hat{y}|} u\} : \bot_{[u]} \times \bot_{[v]}, \{\hat{x} \xrightarrow{\langle \hat{x}|} z, \hat{y} \xrightarrow{\langle \hat{y}|} z\} : \bot_{[w]} \multimap \bot_{[z]} \dashv\vdash \emptyset : \bot_{[\hat{x}]\hat{x}\times_{\hat{y}}\bot_{[\hat{y}]}}\ (*)} \qquad \dfrac{}{\mathrm{cp}_{\bot\times\bot}}}{\begin{array}{c}\{w \xrightarrow{\langle \hat{x}|} v, w \xrightarrow{\langle \hat{y}|} u\} : \bot_{[u]} \times \bot_{[v]}, \{\hat{x} \xrightarrow{\langle \hat{x}|} z, \hat{y} \xrightarrow{\langle \hat{y}|} z\} : \bot_{[w]} \multimap \bot_{[z]} \dashv \\ \{\tilde{x} \xrightarrow{\langle \tilde{x}|} x, \tilde{y} \xrightarrow{\langle \tilde{y}|} y\} : \bot_{[\tilde{x}]\tilde{x}\times_{\tilde{y}}\bot_{[\tilde{y}]}} \triangleright \bot_{[x]} \times \bot_{[y]} \vdash \emptyset : \bot_{[\tilde{x}]\tilde{x}\times_{\tilde{y}}\bot_{[\tilde{y}]}}\ (\star)\end{array}}$$

The $\ell^{\bullet}_{\mathcal{B}}$-net $\rho\mathfrak{G}(\star)^{\approx}$ is $\ell^{\bullet}_{\mathcal{B}}$-reduced in Figure 24, whose result is the one $\rho\mathfrak{G}(*)^{\approx}$ by the generalised unit laws. The adjustment of cartesian labels for the prime of $\rho\mathfrak{G}(\star)^{\approx}$ made before the execution of $\ell^{\bullet}_{\mathcal{B}}$-reduction is vital since without it the second $\ell^{\bullet}_{\mathcal{B}}$-net would not be an $\ell^{\bullet}_{\mathcal{B}}$-net.

We now define our graphical categories. For convenience, we first present their preliminary version, in which we take the quotient of $\ell^{\bullet}_{\mathcal{B}}$-nets with respect to $\ell^{\bullet}_{\mathcal{B}}$-equivalence, and later their official version in which we dispense with the quotient by focusing on normalised $\ell^{\bullet}_{\mathcal{B}}$-nets.

Definition 4.1.12 (graphical categories)**.** Let $\mathrm{Cl}(\ell^{\bullet}_{\mathcal{B}})$ be the category in which

- An object is an ILB-tree $\mathfrak{G}A$, where A is a formula in ILB;
- A morphism $\mathfrak{G}A \to \mathfrak{G}B$ is the $\ell^{\bullet}_{\mathcal{B}}$-equivalence class $[\mathfrak{r} :: \mathfrak{S}]_{\ell^{\bullet}_{\mathcal{B}}}$ of an $\ell^{\bullet}_{\mathcal{B}}$-net $\mathfrak{r} :: \mathfrak{S}$ with $\mathfrak{S} = \mathfrak{G}(T_A \dashv \Sigma \vdash T_B)$ for some finite sequence Σ of $\ell^{\bullet}_{\mathcal{B}}$-cuts, where for convenience $\mathfrak{r} :: \mathfrak{S}$ is written without loss of generality in the form
$$\rho\mathfrak{G}(\mathscr{P} : T_A \dashv \sigma : \Sigma \vdash \mathscr{B} : T_B)^{\approx},$$
where $\mathscr{P} : T_A \dashv \sigma : \Sigma \vdash \mathscr{B} : T_B$ is a pre-$\ell^{\bullet}_{\mathcal{B}}$-term freely generated by Theorem 3.2.44;
- The composition of morphisms $[\rho\mathfrak{G}(\mathscr{P} : T_A \dashv \sigma : \Sigma \vdash \mathscr{B} : T_B)^{\approx}]_{\ell^{\bullet}_{\mathcal{B}}} : \mathfrak{G}A \to \mathfrak{G}B$ and $[\rho\mathfrak{G}(\mathscr{Q} : \hat{T}_B \dashv \pi : \Pi \vdash \mathscr{C} : T_C)^{\approx}]_{\ell^{\bullet}_{\mathcal{B}}} : \mathfrak{G}B \to \mathfrak{G}C$ is given by arbitrarily renaming vertices of $\mathfrak{G}(T_A \dashv \Sigma \vdash T_B)$ and $\mathfrak{G}(\hat{T}_B \dashv \Pi \vdash T_C)$ in such a way that their vertex sets become disjoint, where let $\mathfrak{G}(\mathscr{P}' : T'_A \dashv \sigma' : \Sigma' \vdash \mathscr{B}' : T'_B)^{\approx}$ and $\mathfrak{G}(\mathscr{Q}' : \hat{T}'_B \dashv \pi' : \Pi' \vdash \mathscr{C}' : T'_C)^{\approx}$ be the resulting pre-$\ell^{\bullet}_{\mathcal{B}}$-nets, and defining the composition to be
$$[\rho\mathfrak{G}(\mathscr{P}' : T'_A \dashv \sigma' : \Sigma', \mathscr{B}' \cup \mathscr{Q}' : T'_B \triangleright \hat{T}'_B, \pi' : \Pi' \vdash \mathscr{C}' : T'_C)^{\approx}]_{\ell^{\bullet}_{\mathcal{B}}} : \mathfrak{G}A \to \mathfrak{G}C;$$
- The identity $\mathrm{id}_A : A \to A$ is
$$[\rho\mathfrak{G}(\mathrm{cp}_A)^{\approx}]_{\ell^{\bullet}_{\mathcal{B}}} : \mathfrak{G}A \to \mathfrak{G}A.$$

In addition, the full subcategory $\mathrm{Cl}(\ell_{\mathcal{B}}) \hookrightarrow \mathrm{Cl}(\ell^{\bullet}_{\mathcal{B}})$ consist of ILL-trees as objects.

Notation. We abbreviate objects $\mathfrak{G}A$ in the above categories as A unless it brings confusion.

We will soon focus on *normalised* $\ell^{\circ}_{\mathcal{B}}$-nets and build the wide subcategory $\tilde{\mathrm{Cl}}(\ell^{\circ}_{\mathcal{B}}) \hookrightarrow \mathrm{Cl}(\ell^{\circ}_{\mathcal{B}})$ of normalised ones (Definition 4.1.17), which is isomorphic to $\mathrm{Cl}(\ell^{\circ}_{\mathcal{B}})$ yet *free from* $\ell^{\circ}_{\mathcal{B}}$*-equivalence*. While $\tilde{\mathrm{Cl}}(\ell^{\circ}_{\mathcal{B}})$ is simpler than $\mathrm{Cl}(\ell^{\circ}_{\mathcal{B}})$ and quotient-free, $\mathrm{Cl}(\ell^{\circ}_{\mathcal{B}})$ analyses our graphical constructs more deeply than $\tilde{\mathrm{Cl}}(\ell^{\circ}_{\mathcal{B}})$ *via* categorical structures on non-normalised $\ell^{\circ}_{\mathcal{B}}$-nets. For example, the composition in $\mathrm{Cl}(\ell^{\circ}_{\mathcal{B}})$ is well-defined for all (composable) $\ell^{\circ}_{\mathcal{B}}$-nets, not only for normalised ones. In other words, we conduct a strict screening on the graphical calculus by temporarily choosing $\mathrm{Cl}(\ell^{\circ}_{\mathcal{B}})$ over $\tilde{\mathrm{Cl}}(\ell^{\circ}_{\mathcal{B}})$. In addition, $\mathrm{Cl}(\ell^{\circ}_{\mathcal{B}})$ is richer in structures; *e.g.* it extends to a *bicategorical* generalisation of a (pointed) linear category just by the discard of $\ell^{\circ}_{\mathcal{B}}$-equivalence, as in Yamada

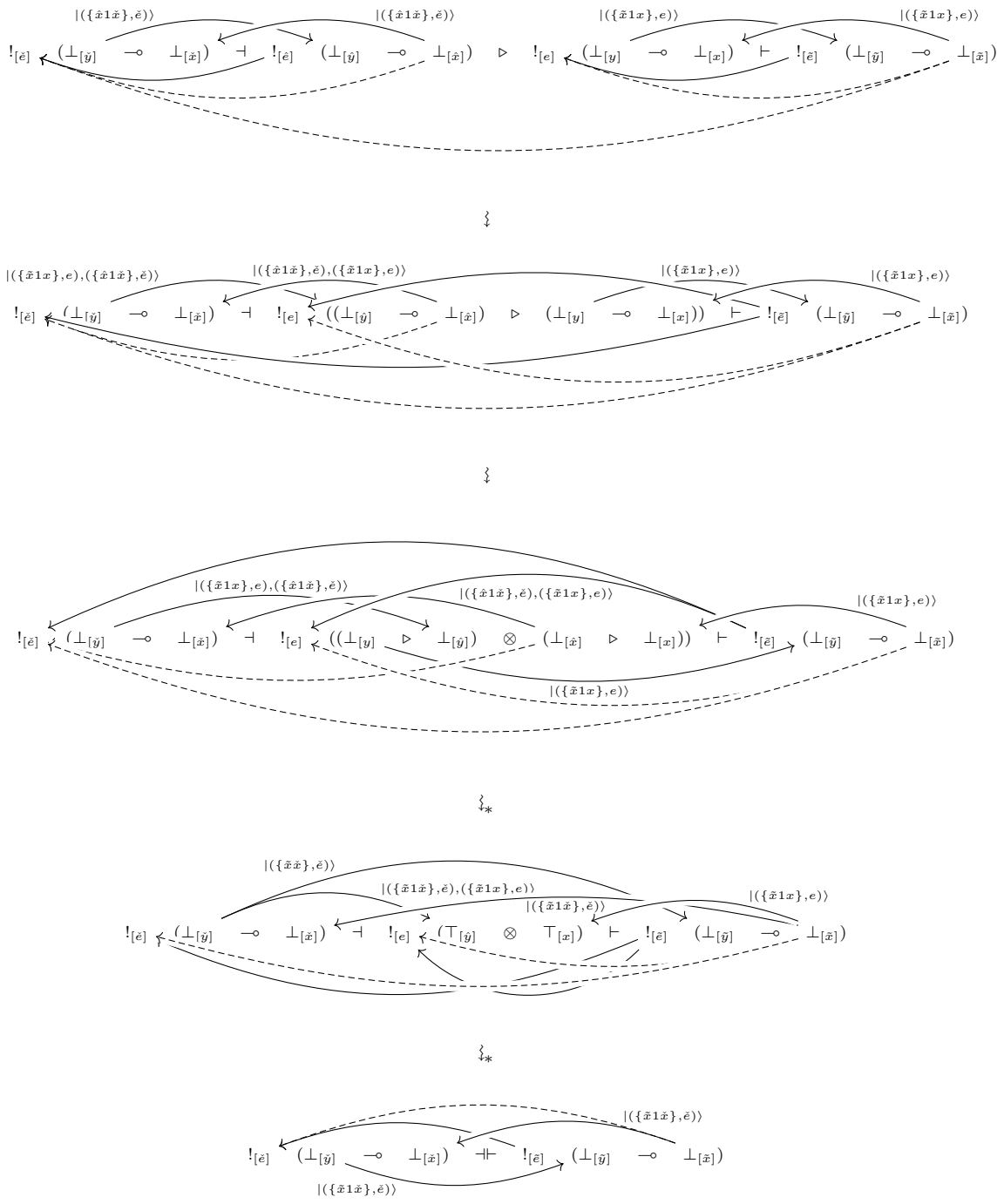


FIGURE 23. $\ell^{\bullet}_{\mathcal{B}}$-reduction witnessing the generalised unit law

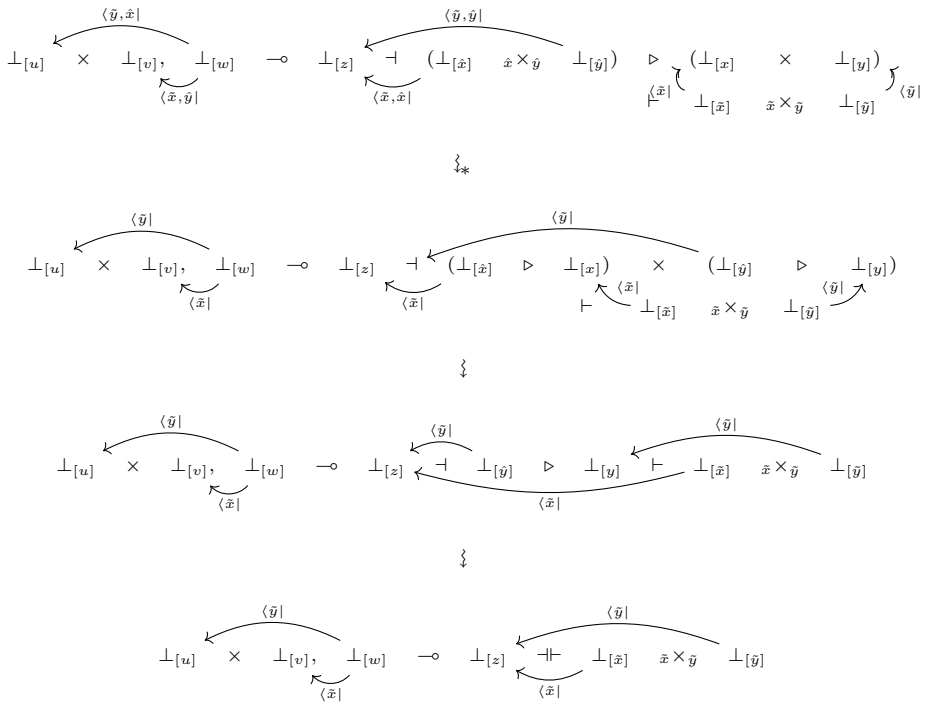


FIGURE 24. $\ell^{\bullet}_{\mathcal{B}}$-reduction on cartesian product

and Abramsky [YA20],[16] and this is the first step towards the study of links between the calculus and Fiore *et al.* [FGH26] mentioned in §1.3. Yet, our eventual focus in this work is $\tilde{\mathrm{Cl}}(\ell^{\circ}_{\mathcal{B}})$.

The next technical device is useful for the rest of this work. In a free generation of a pre-$\ell^{\circ}_{\mathcal{B}}$-term, a rooted subtree of the free generation induced by a vertex can be seen as a 'leaf', called a *generalised leaf*. Abusing notation,[17] we write $\underline{n}$ ($n \in \mathbb{N}$) for an arbitrarily fixed expression for n such that $\underline{n} \neq \underline{m}$ if $n \neq m$, and $\mathcal{N}$ for the discrete category of the numerals.

Definition 4.1.13 (uniform families)**.** A family $f = \{f_{\boldsymbol{A}}\}_{\boldsymbol{A} \in \mathrm{Obj}(\mathrm{Cl}(\ell^{\circ}_{\mathcal{B}}))^n}$ of morphisms $f_{\boldsymbol{A}}$ in $\mathrm{Cl}(\ell^{\circ}_{\mathcal{B}})$, where $n \in \mathbb{N}$ is called its *arity*, is said to be *uniform* if, for any $\boldsymbol{A} = (A_i)_{i=1}^n, \boldsymbol{B} = (B_i)_{i=1}^n \in \mathrm{Obj}(\mathrm{Cl}(\ell^{\circ}_{\mathcal{B}}))^n$, there are pre-$\ell^{\circ}_{\mathcal{B}}$-terms $u_{\boldsymbol{A}} :: S_{\boldsymbol{A}}$ and $u_{\boldsymbol{B}} :: S_{\boldsymbol{B}}$ that are

(1) Both free from arrowy labels of morphisms in $\mathcal{B}$ except those of identities;
(2) Freely generated by the same derivation from the $\ell^{\circ}_{\mathcal{B}}$-copy-cats cp_{A_i} and cp_{B_i}, serving as generalised leaves, respectively, in the order of the indexing i;
(3) $f_{\boldsymbol{A}} = [\rho\mathfrak{G}(u_{\boldsymbol{A}} :: S_{\boldsymbol{A}})^{\approx}]_{\ell^{\bullet}_{\mathcal{B}}}$ and $f_{\boldsymbol{B}} = [\rho\mathfrak{G}(u_{\boldsymbol{B}} :: S_{\boldsymbol{B}})^{\approx}]_{\ell^{\bullet}_{\mathcal{B}}}$.

A uniform family of morphisms is *uniform* in the sense that its elements are constructible by the same free generation from $\ell^{\circ}_{\mathcal{B}}$-copy-cats, where only the $\ell^{\circ}_{\mathcal{B}}$-copy-cats may vary. Two uniform families $f = \{f_{\boldsymbol{A}}\}_{\boldsymbol{A} \in \mathrm{Obj}(\mathrm{Cl}(\ell^{\circ}_{\mathcal{B}}))^n}$ and $f' = \{f'_{\boldsymbol{A}}\}_{\boldsymbol{A} \in \mathrm{Obj}(\mathrm{Cl}(\ell^{\circ}_{\mathcal{B}}))^{n'}}$ of morphisms are said to be *equal* and written $f = f'$ if $n = n'$ and $f_{\boldsymbol{A}} = f'_{\boldsymbol{A}}$ for all $\boldsymbol{A} \in \mathrm{Obj}(\mathrm{Cl}(\ell^{\circ}_{\mathcal{B}}))^n$. It is easy to see that,

[16] This direction is straightforward to pursue since the bicategorical generalisation is obtained from (pointed) linear categories just by the replacement of equality between morphisms with equivalence relations as in [YA20]. Yet, since it diverges from our focus to spell out the details, we leave the generalisation to future work.

[17] While the following notation for a numeral conflicts with the one $\underline{\boldsymbol{s}}$ for a finite sequence $\boldsymbol{s}$, it does not bring confusion in practice. The notation is to make it easy to distinguish the numeral $\underline{1}$ from the formula 1.

given two uniform families $f = \{f_{\boldsymbol{A}}\}_{\boldsymbol{A} \in \mathrm{Obj}(\mathrm{Cl}(\ell^\circ_{\mathcal{B}}))^n}$ and $g = \{g_{\boldsymbol{A}}\}_{\boldsymbol{A} \in \mathrm{Obj}(\mathrm{Cl}(\ell^\circ_{\mathcal{B}}))^n}$, $f_{\boldsymbol{A}}$ and $g_{\boldsymbol{A}}$ are composable for some $\boldsymbol{A} \in \mathrm{Obj}(\mathrm{Cl}(\ell^\circ_{\mathcal{B}}))^n$ if and only if so are they for all $\boldsymbol{A} \in \mathrm{Obj}(\mathrm{Cl}(\ell^\circ_{\mathcal{B}}))^n$. In the positive case, the *composition* of f and g is defined to be the uniform family

$$g \circ f := \{g_{\boldsymbol{A}} \circ f_{\boldsymbol{A}}\}_{\boldsymbol{A} \in \mathrm{Obj}(\mathrm{Cl}(\ell^\circ_{\mathcal{B}}))^n}.$$

We shall be interested in a class of uniform families that are certain natural transformations.

We then translate each uniform family $f = \{f_{\boldsymbol{A}}\}_{\boldsymbol{A}}$ of morphisms into a *single* morphism $\mathrm{dep}(f)$. The idea is to represent the indices $\boldsymbol{A}$ by natural numbers (just like variables).

Definition 4.1.14 (deputy)**.** The *deputy* of a uniform family $f = \{f_{\boldsymbol{A}}\}_{\boldsymbol{A} \in \mathrm{Obj}(\mathrm{Cl}(\ell^\circ_{\mathcal{B}}))^n}$ of morphisms in $\mathrm{Cl}(\ell^\circ_{\mathcal{B}})$ is the morphism $\mathrm{dep}(f)$ in $\mathrm{Cl}(\ell^\circ_{\mathcal{N}})$ given by the following procedure:

(1) Pick any member $f_{\boldsymbol{A}}$ of f, where assume $\boldsymbol{A} = (A_i)_{i=1}^n$ and $f_{\boldsymbol{A}} = [\rho\mathfrak{G}(u_{\boldsymbol{A}} :: S_{\boldsymbol{A}})^{\approx}]_{\ell^\bullet_{\mathcal{B}}}$ for a pre-$\ell^\circ_{\mathcal{B}}$-term $u_{\boldsymbol{A}} :: S_{\boldsymbol{A}}$ freely generated from the $\ell^\circ_{\mathcal{B}}$-copy-cats cp_{A_i};
(2) Replace each generalised leaf cp_{A_i} in the derivation of $u_{\boldsymbol{A}} :: S_{\boldsymbol{A}}$ with the one $\mathrm{cp}_{\underline{i}}$ (which is possible by the first axiom in Definition 4.1.13), freely generating a pre-$\ell^\circ_{\mathbb{N}}$-term $u :: S$;
(3) Define $\mathrm{dep}(f) := [\rho\mathfrak{G}(u :: S)^{\approx}]_{\ell^\bullet_{\mathcal{B}}}$.

Clearly, the deputy $\mathrm{dep}(f)$ of any uniform family of morphisms does not depend on the choice of an element $f_{\boldsymbol{A}}$ of f or its derivation in the first step. Besides, it is easy to see that, if f and g are composable families of morphisms, then

$$\mathrm{dep}(g \circ f) = \mathrm{dep}(g) \circ \mathrm{dep}(f).$$

In this sense, the operation dep is *compositional*. Moreover, it is *faithful*:

Theorem 4.1.15 (faithfulness of deputy)**.** *Two uniform families f and f' of morphisms in $\mathrm{Cl}(\ell^\circ_{\mathcal{B}})$ of the same arity satisfy $f = f'$ if and only if $\mathrm{dep}(f) = \mathrm{dep}(f')$.*

Proof. The necessity ($f = f' \Rightarrow \mathrm{dep}(f) = \mathrm{dep}(f')$) is obvious, and the sufficiency ($f = f' \Leftarrow \mathrm{dep}(f) = \mathrm{dep}(f')$) holds as every $\ell^\circ_{\mathcal{B}}$-copy-cat consists of $\mathcal{B}$-arcs between pairs of corresponding ends, as templated by those $\mathrm{cp}_{\underline{i}}$, where Proposition 4.1.9 ensures that $\ell^\circ_{\mathcal{B}}$-reduction with any $\ell^\circ_{\mathcal{B}}$-copy-cat is also templated by that with cp_i, and the ends $\underline{i}$ are not joker. □

This theorem reduces the infinitary equation $f = f'$ to the finitary one $\mathrm{dep}(f) = \mathrm{dep}(f')$. We are now ready to prove the first main result of the present section:

Theorem 4.1.16 (graphical linear categories)**.** *The structure $\mathrm{Cl}(\ell^\bullet_{\mathcal{B}})$ gives rise to a strict pointed linear category, and the one $\mathrm{Cl}(\ell_{\mathcal{B}})$ a strict linear category.*

Proof. We focus on the case $\mathrm{Cl}(\ell^\bullet_{\mathcal{B}})$ since the other is simpler. We first show that $\mathrm{Cl}(\ell^\bullet_{\mathcal{B}})$ is a well-defined category. By the typing rule Cut, the only nontrivial point for the composition to be well-defined is whether it is independent of a choice of $\approx$-representatives of the component morphisms, and it is by Corollary 3.3.7. The other operations on morphisms introduced below are shown to be well-defined by similar arguments, and we henceforth skip this issue. It then follows that $\mathrm{Cl}(\ell^\bullet_{\mathcal{B}})$ is a well-defined category, which satisfies the associativity of the composition by construction, and the unit laws by Proposition 4.1.9.

We next show a symmetric monoidal closed structure on $\mathrm{Cl}(\ell^\bullet_{\mathcal{B}})$. The bifunctor is given by tensor $\otimes$, whose arrow-map transforms a pair of morphisms

$$[\rho\mathfrak{G}(\mathscr{P} : T_A \dashv \sigma : \Sigma \vdash \mathscr{C} : T_C)^{\approx}]_{\ell^\bullet_{\mathcal{B}}} : A \to C, \qquad [\rho\mathfrak{G}(\mathscr{Q} : T_B \dashv \pi : \Pi \vdash \mathscr{D} : T_D)^{\approx}]_{\ell^\bullet_{\mathcal{B}}} : B \to D,$$

where without loss of generality we assume that their vertex sets are disjoint and keep using this assumption without mentioning it further in analogous situations in the remainder of this article, into the one (with the help of the typing rules $\otimes$R and $\otimes$L)

$$[\rho\mathfrak{G}(\mathscr{P} \cup \mathscr{Q} : T_A \otimes T_B \dashv \sigma : \Sigma, \pi : \Pi \vdash \mathscr{C} \cup \mathscr{D} : T_C \otimes T_D)^{\approx}]_{\ell^\bullet_{\mathcal{B}}} : A \otimes B \to C \otimes D.$$

It is straightforward to see its functoriality: it preserves identities by the definition of $\ell^{\bullet}_{\mathcal{B}}$-copycats, and composition by the definition of $\ell^{\bullet}_{\mathcal{B}}$-reduction (and Corollary 3.3.7).

The unit is given by top $\top$. Fixing an arbitrary end $u' \in \mathrm{End}^{\mathrm{Ini}}_{T'_A}$, the natural isomorphisms

$$A \otimes \top \cong A \cong \top \otimes A$$

are given by the morphisms (with the help of the typing rules on $\top$ and $\otimes$)[18]

- $[\rho\mathfrak{G}(\mathscr{D}_A \cup \{u' \to v\} : T_A \otimes \top_{[v]} \dashv\vdash \mathscr{C}'_A : T'_A)^{\approx}]_{\ell^{\bullet}_{\mathcal{B}}} : A \otimes \top \to A$,
- $[\rho\mathfrak{G}(\mathscr{D}_A : T_A \dashv\vdash \mathscr{C}'_A : T'_A \otimes \top_{[v]})^{\approx}]_{\ell^{\bullet}_{\mathcal{B}}} : A \to A \otimes \top$,
- $[\rho\mathfrak{G}(\mathscr{D}_A \cup \{u' \to v\} : \top_{[v]} \otimes T_A \dashv\vdash \mathscr{C}'_A : T'_A)^{\approx}]_{\ell^{\bullet}_{\mathcal{B}}} : \top \otimes A \to A$,
- $[\rho\mathfrak{G}(\mathscr{D}_A : T_A \dashv\vdash \mathscr{C}'_A : \top_{[v]} \otimes T'_A)^{\approx}]_{\ell^{\bullet}_{\mathcal{B}}} : A \to \top \otimes A$.

The natural isomorphisms

$$A \otimes (B \otimes C) \cong (A \otimes B) \otimes C, \qquad A \otimes B \cong B \otimes A$$

are the evident analogues of the identities on $A \otimes (B \otimes C)$ and $A \otimes B$, respectively. For some of their invertibility, the $\approx$-saturation of $\ell^{\bullet}_{\mathcal{B}}$-nets is vital, as illustrated in Figures 2–3. Similarly to the proof of Proposition 4.1.9, they satisfy the naturality and the coherence axioms.

The closure is given by linear implication $\multimap$, the evaluation $(A \multimap B) \otimes A \to B$ by the typing rules $\multimap$L, XL and $\otimes$L applied to cp_A and cp_B, and the currying by the one $\multimap$R. Again, similarly to the proof of Proposition 4.1.9, we see that the axioms on this closure hold.

Next, we define a comonad on $\mathrm{Cl}(\ell^{\bullet}_{\mathcal{B}})$. The functor is given by of-course !, whose arrow-map transforms a given morphism

$$[\rho\mathfrak{G}(\mathscr{P} : T_A \dashv (\mathscr{M}_j : M_j)_j \vdash \mathscr{B} : T_B)^{\approx}]_{\ell^{\bullet}_{\mathcal{B}}} : A \to B$$

into the one (with the help of the typing rules !D, !DM and !R)

$$[\rho\mathfrak{G}(\mathscr{P}' \cup \{x \to z\} : !_{[z]}T_A \dashv (\mathscr{M}'_j \cup \{x \to y_j\} : !_{[y_j]}M_j)_j \vdash \mathscr{B}' : !_{[x]}T_B)^{\approx}]_{\ell^{\bullet}_{\mathcal{B}}} : !A \to !B,$$

where

$$\mathscr{P}' := \mathscr{P} + |(\mathrm{Col}(\mathscr{P}), z)\rangle, \qquad \mathscr{M}'_j := \mathscr{M}_j + |(\mathrm{Col}(\mathscr{M}_j), y_j)\rangle,$$

$$\mathscr{B}' := (\mathscr{B}\restriction_{T_A} + |(\mathrm{Col}(\mathscr{P}), z)\rangle) \cup \bigcup_j (\mathscr{B}\restriction_{M_j} + |(\mathrm{Col}(\mathscr{M}_j), y_j)\rangle).$$

Its functoriality is shown in a way similar to the case of the bifunctor $\otimes$. The components of the counit ϵ and the comultiplication μ of the comonad at each object $A \in \mathrm{Cl}(\ell^{\bullet}_{\mathcal{B}})$ are the morphisms (given with the help of the typing rules !D and !R)

- $[\rho\mathfrak{G}(\mathscr{D}_A + |(\mathrm{Col}(\mathscr{D}_A), z)\rangle : !_{[z]}T_A \dashv\vdash \mathscr{C}'_A + |(\mathrm{Col}(\mathscr{D}_A), z)\rangle : T'_A)^{\approx}]_{\ell^{\bullet}_{\mathcal{B}}} : !A \to A$,
- $[\rho\mathfrak{G}((\mathscr{D}_A + |(\mathrm{Col}(\mathscr{D}_A), z)\rangle) \cup \{x \to z, y \to z\} : !_{[z]}T_A \dashv\vdash \mathscr{C}'_A + |(\mathrm{Col}(\mathscr{D}_A), z)\rangle : !_{[x]}!_{[y]}T'_A)^{\approx}]_{\ell^{\bullet}_{\mathcal{B}}} : !A \to !!A$,

respectively. Note that the counit and the comultiplication are both uniform families of morphisms. Because the operation dep is compositional, faithful, and satisfying $\mathrm{dep}(!f) = !\mathrm{dep}(f)$ for a uniform family $f = \{f_{\boldsymbol{A}}\}_{\boldsymbol{A} \in \mathrm{Obj}(\mathrm{Cl}(\ell^{\circ}_{\mathcal{B}}))^n}$ of morphisms, where $!f := \{!f_{\boldsymbol{A}}\}_{\boldsymbol{A}}$, it suffices to check the axioms of a comonad on the deputies of the counit and the comultiplication:

- $[\rho\mathfrak{G}(\{v' \xrightarrow{\mathrm{id}_{\underline{1}}|(\{v'v\},z)\rangle} v\} : !_{[z]}\underline{1}_{[v]} \dashv\vdash \emptyset : \underline{1}_{[v']})^{\approx}]_{\ell^{\bullet}_{\mathcal{B}}} : !\underline{1} \to \underline{1}$,
- $[\rho\mathfrak{G}(\{v' \xrightarrow{\mathrm{id}_{\underline{1}}|(\{v'v\},z)\rangle} v\} \cup \{x \to z, y \to z\} : !_{[z]}\underline{1}_{[v]} \dashv\vdash \emptyset : !_{[x]}!_{[y]}\underline{1}_{[v']})^{\approx}]_{\ell^{\bullet}_{\mathcal{B}}} : !\underline{1} \to !!\underline{1}$.

[18] It is a nontrivial problem to design a calculus for (pointed) linear categories having these natural isomorphisms without recourse to a quotient. For instance, the composition $A \otimes \top \xrightarrow{\sim} A \xrightarrow{\sim} A \otimes \top$ would not coincide with the identity $\mathrm{id}_{A \otimes \top}$ if we define the typing rules on $\top$ in the same way as those on $\bot$ in Figure 15.

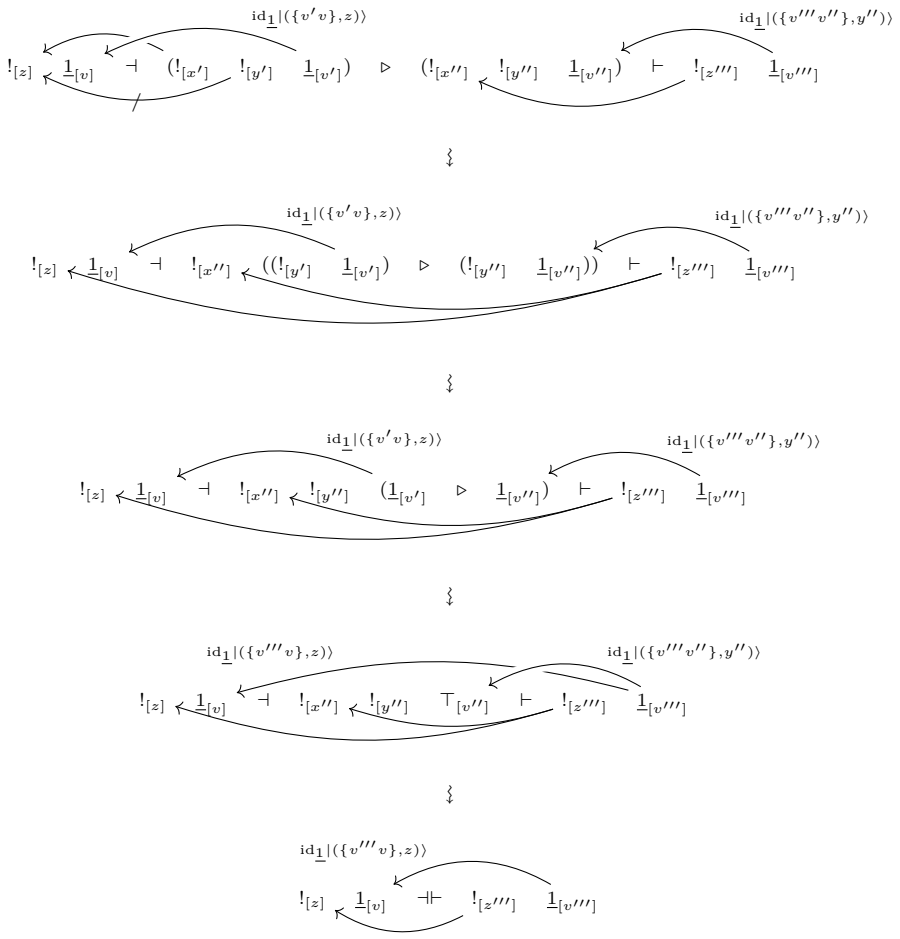


FIGURE 25. $\ell^{\bullet}_{\mathcal{B}}$-reduction for a comonad

Then, for instance, the axiom $!\epsilon \circ \delta = \mathrm{id}_!$ is verified by the $\ell^{\bullet}_{\mathcal{B}}$-reduction in Figure 25, where we add slash / to an edge to be deleted during the conversion into the prime form. The other axioms are verified in a similar vein, and we leave the details to the reader.

Remark. In this way, we *reduce* the axioms of a comonad to geometry of quotient-free labelled finite graphs. In contrast, the standard string diagrams [JS91], extended to a comonad [BCLS20], *translate* the axioms into formal equations between string diagrams [BCLS20, Formula (1)].

It remains to show that the adjunction (12), where $\mathcal{L} = \mathrm{Cl}(\ell^{\bullet}_{\mathcal{B}})$, is monoidal. To this end, we first check the required axioms on the mediating morphisms for the lax monoidal functor F:

- $[\rho\mathfrak{G}(\{e \to v\} : \top_{[v]} \dashv\vdash \emptyset : !_{[e]}1_{[u]})^{\approx}]_{\ell^{\bullet}_{\mathcal{B}}} : \top \to !1$;
- $[\rho\mathfrak{G}((\mathscr{D}_A + \langle \tilde{x}|) \cup (\mathscr{D}_B + \langle \tilde{y}|) \cup \{\tilde{g} \to e, \tilde{g} \to f\} : !_{[e]}T_A \otimes !_{[f]}T_B \dashv\vdash (\tilde{\mathscr{C}}_A + \langle \tilde{x}|) \cup (\tilde{\mathscr{C}}_B + \langle \tilde{y}|) : !_{[\tilde{g}]}(\tilde{T}_{A\tilde{x}} \times_{\tilde{y}} \tilde{T}_B)^{\approx}]_{\ell^{\bullet}_{\mathcal{B}}} : !A \otimes !B \to !(A \times B)$, which is natural in $A, B \in \mathrm{Cl}(\ell^{\bullet}_{\mathcal{B}})$.

Again, since the axioms are only concerned with interactions between the mediating morphisms and coherence natural isomorphisms (which are represented by freely generated pre-$\ell^{\bullet}_{\mathcal{B}}$-terms),

we may replace the second mediating morphism with its deputy

$$[\rho\mathfrak{G}(\{\tilde{x} \to x, \tilde{y} \to y, \tilde{g} \to e, \tilde{g} \to f\} : !_{[e]}\underline{1}_{[x]} \otimes !_{[f]}\underline{2}_{[y]} \dashv\vdash \emptyset : !_{[\tilde{g}]}(\underline{1}_{[\tilde{x}]\tilde{x}} \times_{\tilde{y}} \underline{2}_{[\tilde{y}]})^{\approx}]_{\ell^{\bullet}_{\mathcal{B}}}.$$

The first axiom is the compatibility of the mediating morphisms with the coherence natural isomorphisms for unit laws; in our setting, it is the commutativity of the diagrams

$$\begin{array}{ccc} !1 \otimes !Y & \xrightarrow{\mu_{1,Y}} & !(1 \times Y) \\ {\scriptstyle \mu_0 \otimes \mathrm{id}_{!Y}}\uparrow & & \downarrow{\scriptstyle !\lambda_Y} \\ \top \otimes !Y & \xrightarrow[\lambda_{!Y}]{} & !Y \end{array} \qquad \begin{array}{ccc} !Y \otimes !1 & \xrightarrow{\mu_{Y,1}} & !(Y \times 1) \\ {\scriptstyle \mathrm{id}_{!Y} \otimes \mu_0}\uparrow & & \downarrow{\scriptstyle !\gamma_Y} \\ !Y \otimes \top & \xrightarrow[\gamma_{!Y}]{} & !Y \end{array}$$

in $\mathrm{Cl}(\ell_{\bullet})$. We focus on the first diagram because the second is analogous. Then, $\ell^{\bullet}_{\mathcal{B}}$-reduction in Figure 26 verifies the commutativity of the first diagram. Similarly, we can also prove the compatibility of the mediating morphisms with the coherence natural isomorphisms for associativity. It is simpler to verify the axioms on G. We leave those details to the reader.

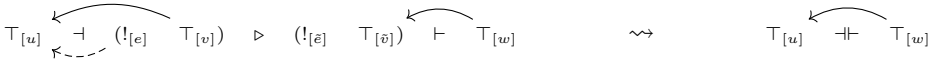
It remains to show that the (co)unit the adjunction (12) is monoidal. We focus on the case of the counit and leave that of the unit to the reader. We have to show that the diagrams

$$\begin{array}{ccc} & \top & \\ {\scriptstyle \mu_0}\swarrow & & \searrow{\scriptstyle \mathrm{id}_{\top}} \\ !\top & \xrightarrow[\epsilon_{\top}]{} & \top \end{array} \qquad \begin{array}{ccc} !X \otimes !Y & \xrightarrow{\epsilon_X \otimes \epsilon_Y} & X \otimes Y \\ {\scriptstyle \mu_{X,Y}}\downarrow & & \downarrow{\scriptstyle \mathrm{id}_{X \otimes Y}} \\ !(X \otimes Y) & \xrightarrow[\epsilon_{X \otimes Y}]{} & X \otimes Y \end{array}$$

in $\mathrm{Cl}(\ell^{\bullet}_{\mathcal{B}})$ commute. Again, by Theorem 4.1.15, it suffices to reason about the deputy

$$\rho\mathfrak{G}(\{\tilde{x} \to x\} : !_{[e]}X_{[x]} \dashv\vdash \emptyset : X_{[\tilde{x}]})^{\approx}.$$

of the counit. Then, the first commutativity is verified by



and the second as in Figure 27. □

Remark. There is no $\ell^{\circ}_{\mathcal{B}}$-net on $\mathfrak{G}(X_{[y]} \otimes 1_{[v]} \dashv\vdash X_{[x]})$, where $X \in \mathcal{B}$, because only $x \to y$ is a possible $\mathcal{B}$-arc, but it does not cover v. For this reason, one 1 is *not* the unit for tensor $\otimes$.

Definition 4.1.17 (quotient-free graphical categories)**.** Let $\tilde{\mathrm{Cl}}(\ell^{\circ}_{\mathcal{B}})$ be the category obtained from the wide subcategory of $\mathrm{Cl}(\ell^{\circ}_{\mathcal{B}})$ whose morphisms are normalised $\ell^{\circ}_{\mathcal{B}}$-nets by the discard of $\ell^{\circ}_{\mathcal{B}}$-equivalence (since $\ell^{\circ}_{\mathcal{B}}$-equivalence is vacant for the normalised ones).

By construction, we have the isomorphism $\tilde{\mathrm{Cl}}(\ell^{\circ}_{\mathcal{B}}) \cong \mathrm{Cl}(\ell^{\circ}_{\mathcal{B}})$ of categories. Since we can easily apply the proof of Theorem 4.1.16 to $\tilde{\mathrm{Cl}}(\ell^{\circ}_{\mathcal{B}})$, we obtain:

Corollary 4.1.18 (quotient-free graphical linear categories)**.** *The structure* $\tilde{\mathrm{Cl}}(\ell^{\bullet}_{\mathcal{B}})$ *forms a strict pointed linear category, and the one* $\tilde{\mathrm{Cl}}(\ell_{\mathcal{B}})$ *a strict linear category.*

We have seen that every categorical construction of the (pointed) linear category $\mathrm{Cl}(\ell^{\bullet}_{\mathcal{B}})$ or $\tilde{\mathrm{Cl}}(\ell^{\bullet}_{\mathcal{B}})$ is realised in terms of typing rules. We also see in the next section that conversely every typing rule corresponds to one of the categorical constructions. This result justifies typing rules as constructs for the free generation.

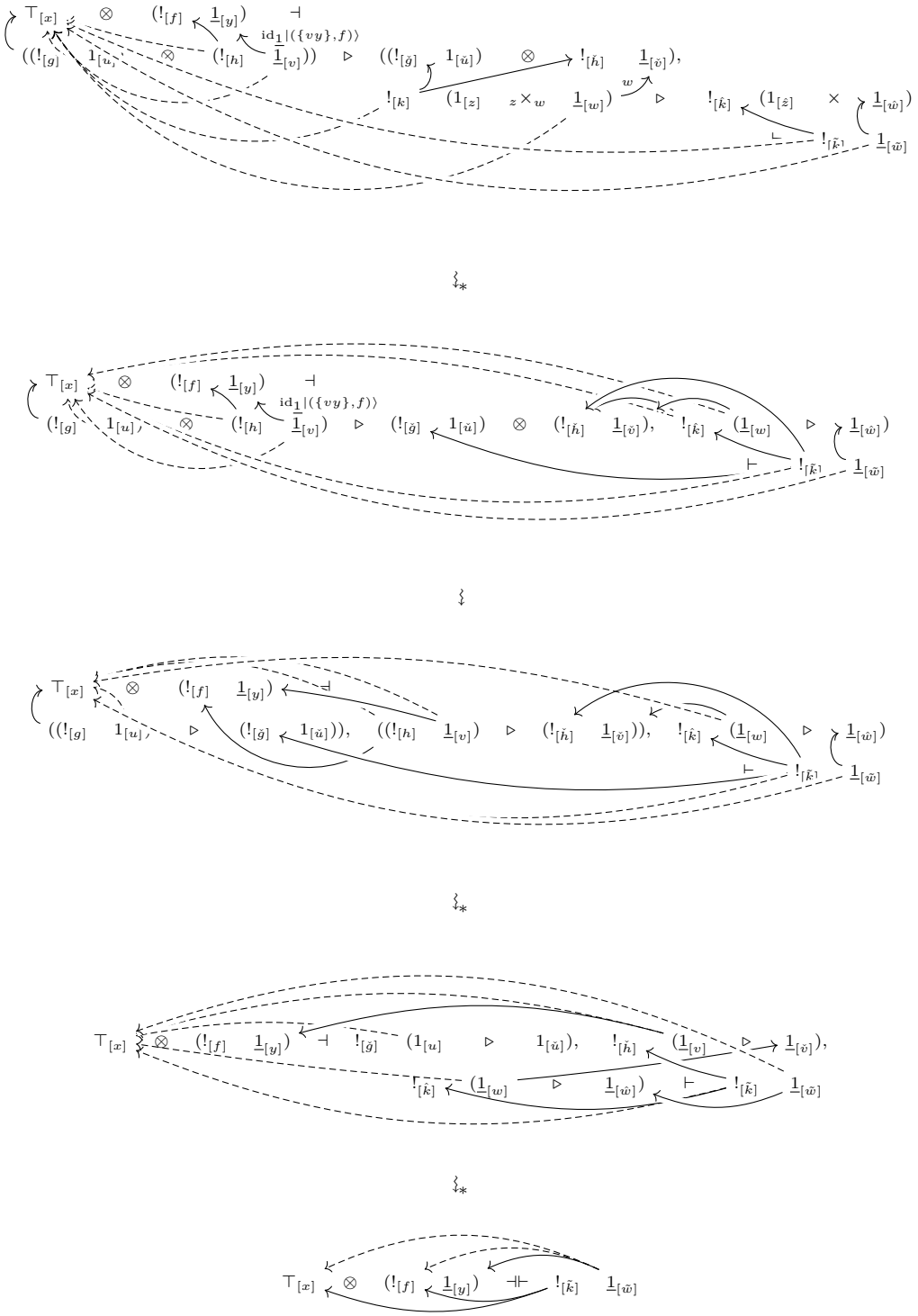


FIGURE 26. $\ell^{\bullet}_{\mathcal{B}}$-reduction for exponential

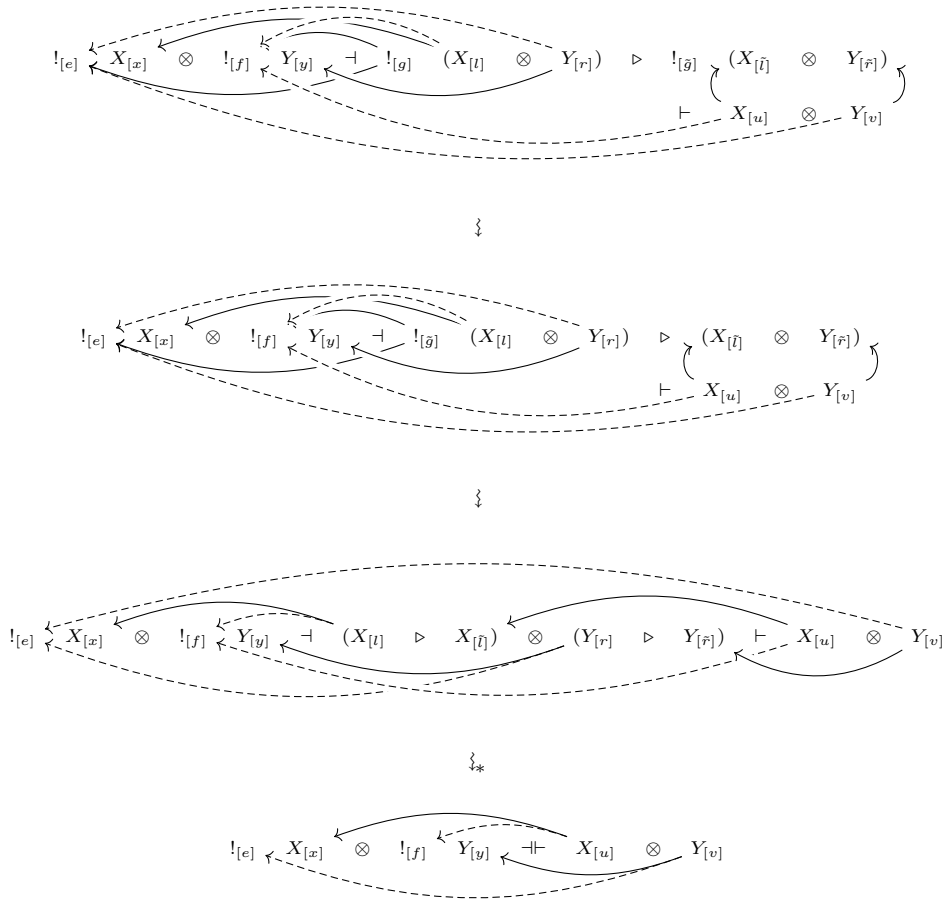


FIGURE 27. Another ℓ-reduction for exponential

4.2. **Soundness, completeness and freeness.** Let us next interpret the $\ell^{\circ}_{\mathcal{B}}$-calculus in strict (pointed) linear categories and show its *soundness* and *completeness*. We then lift this result to *freeness* of the strict (pointed) linear category $\tilde{\mathrm{Cl}}(\ell^{\circ}_{\mathcal{B}})$ over $\mathcal{B}$ in the sense of Definition 4.1.7.

The interpretation assigns to $\ell^{\circ}_{\mathcal{B}}$-trees objects by induction on the underlying $\ell^{\circ}_{\mathcal{B}}$-types. The denotations of $\ell^{\circ}_{\mathcal{B}}$-nets are not straightforward: *a priori* they cannot be defined by induction on free generations of pre-$\ell^{\circ}_{\mathcal{B}}$-nets (since a free generation of a pre-$\ell^{\circ}_{\mathcal{B}}$-net may not be unique), and the free generations are on $\ell^{\circ}_{\mathcal{B}}$-skeletons, not on $\ell^{\circ}_{\mathcal{B}}$-nets (*via* Theorem 3.2.44), in the first place.

For this problem, we first define an interpretation of free generations of pre-$\ell^{\circ}_{\mathcal{B}}$-nets and show that if $\ell^{\circ}_{\mathcal{B}}$-skeletons $\mathfrak{s}, \mathfrak{s}' :: \mathfrak{S}$ satisfy $\mathfrak{s} \approx \mathfrak{s}' :: \mathfrak{S}$, then the denotations of *any* free generations of pre-$\ell^{\circ}_{\mathcal{B}}$-nets that are equal to $\mathfrak{s}, \mathfrak{s}' :: \mathfrak{S}$ up to prime form, respectively, coincide. By this result, a choice of an $\approx$-representative of an $\ell^{\circ}_{\mathcal{B}}$-net or its free generation does not matter, and hence we can lift the interpretation of free generations to $\ell^{\circ}_{\mathcal{B}}$-nets in the evident way.

For ease of notation, we first describe the interpretation in terms of (typing rules of) pre-$\ell^{\circ}_{\mathcal{B}}$-terms, as it can be presented concisely, and then lift it to pre-$\ell^{\circ}_{\mathcal{B}}$-nets through the map $\mathfrak{G}$.

Notation. Let $\mathcal{D} \mid s :: S$ denote a free generation $\mathcal{D}$ of a pre-$\ell^{\circ}_{\mathcal{B}}$-term $s :: S$, where $\mathcal{D}$, or a proof tree with the root $s :: S$ [TS00], is represented compactly as follows:

(1) $\mathcal{D}$ is written Ax if $\mathcal{D}$ is a singleton rooted tree of an axiom (*i.e.* a typing rule that does not have any hypotheses) Ax in Figure 15;
(2) $\mathcal{D}$ is written $\mathrm{TR}(\mathcal{D}_1, \mathcal{D}_2, \ldots, \mathcal{D}_n)$ if $\mathcal{D}$ is given from free generations $\mathcal{D}_i$ for $1 \leqslant i \leqslant n$, where $1 \leqslant n \leqslant 2$, by a typing rule TR with n hypotheses in Figure 15 (*n.b.*, by allowing n to be 0, the first clause can be seen as the second for $n = 0$).

Definition 4.2.1 (semantics of $\ell^{\circ}_{\mathcal{B}}$-types and free generations)**.** The *interpretation* of $\ell^{\bullet}_{\mathcal{B}}$-types and free generations of pre-$\ell^{\bullet}_{\mathcal{B}}$-terms in a strict pointed linear category $\mathcal{L} = (\mathcal{L}, \top, \otimes, \multimap, 1, \times, !, \bot)$ over a functor $\nu : \mathcal{B} \to U\mathcal{L}$ (where recall $U : \mathrm{spLCat} \to \mathrm{Cat}$ in Definition 4.1.7) consists of:

(1) The assignment $[\![_]\!]^{\nu}_{\mathcal{L}}$ of an object $[\![A]\!]^{\nu}_{\mathcal{L}}$ in $\mathcal{L}$ to each formula A in ILB over $\mathcal{B}$ defined inductively by

$$[\![X]\!]^{\nu}_{\mathcal{L}} := \nu(X) \quad (X \in \mathrm{Obj}(\mathcal{B})), \qquad [\![I]\!]^{\nu}_{\mathcal{L}} := I \quad (I \in \{\top, 1, \bot\}), \qquad [\![!A]\!]^{\nu}_{\mathcal{L}} := ![\![A]\!]^{\nu}_{\mathcal{L}},$$

$$[\![A \spadesuit B]\!]^{\nu}_{\mathcal{L}} := [\![A]\!]^{\nu}_{\mathcal{L}} \spadesuit [\![B]\!]^{\nu}_{\mathcal{L}} \quad (\spadesuit \in \{\otimes, \multimap, \times\}),$$

and lifted to each $\ell^{\bullet}_{\mathcal{B}}$-type T and to each finite sequence $(T_i)_{i=1}^n$ of them by

$$[\![T]\!]^{\nu}_{\mathcal{L}} := [\![\underline{T}]\!]^{\nu}_{\mathcal{L}}, \qquad [\![(T_i)_{i=1}^n]\!]^{\nu}_{\mathcal{L}} := \bigotimes_{i=1}^{n} [\![T_i]\!]^{\nu}_{\mathcal{L}};$$

(2) The assignment $[\![_]\!]^{\nu}_{\mathcal{L}}$ of a morphism $[\![\, \mathcal{D} \mid s :: S \,]\!]^{\nu}_{\mathcal{L}} : [\![\Gamma]\!]^{\nu}_{\mathcal{L}} \to [\![N]\!]^{\nu}_{\mathcal{L}}$ in $\mathcal{L}$ to each free generation $\mathcal{D} \mid s :: S$ of a pre-$\ell^{\bullet}_{\mathcal{B}}$-term $s :: S$, where $S = \Gamma \dashv \Pi \vdash N$ for some finite sequence Π of $\ell^{\bullet}_{\mathcal{B}}$-cuts, defined inductively in Figure 28, where
- $\alpha_{A,B,C} : A \otimes (B \otimes C) \xrightarrow{\sim} (A \otimes B) \otimes C$ is the natural isomorphism for associativity;
- $\varsigma_{A,B} : A \otimes B \xrightarrow{\sim} B \otimes A$ is that for symmetry;
- $\lambda_A : \top \otimes A \xrightarrow{\sim} A$ and $\gamma_A : A \otimes \top \xrightarrow{\sim} A$ are those for unit laws;
- $\omega_A : !A \to \top$ is the morphism for weakening [Bie95, MMDPR05];
- $\epsilon_A : !A \to A$ is the natural transformation for dereliction [Bie95, MMDPR05];
- $\vartheta_A : !A \to !A \otimes !A$ is that for contraction [Bie95, MMDPR05];
- $\delta_A : !A \to !!A$ is that for digging [Bie95, MMDPR05];
- $\mu_{(A_i)_{i=1}^n} : \bigotimes_{i=1}^n !A_i \to !(\bigotimes_{i=1}^n A_i)$ is the natural transformation constructed from the mediating one $\mu_{A,B} : !A \otimes !B \to !(A \otimes B)$;
- $\tau_A : A \to 1$ is the unique morphism to the terminal object 1;
- $\pi^{(1)}_{A,B} : A \times B \to A$ and $\pi^{(2)}_{A,B} : A \times B \to B$ are the projections of a product $A \times B$;
- $\langle f, g \rangle : C \to A \times B$ is the pairing of morphisms $f : C \to A$ and $g : C \to B$;
- $\mathrm{ev}_{A,B} : (A \multimap B) \otimes A \to B$ is the evaluation on objects A and B;
- $\Lambda_{A,B,C} : \mathcal{L}(A \otimes B, C) \xrightarrow{\sim} \mathcal{L}(A, B \multimap C)$ is the currying on objects A, B and C,

and we mostly suppress the associativity isomorphism in the interpretation of the domain Γ (*n.b.* tensor product $\bigotimes$ in the interpretation of the domain Γ is left associative).

By restriction, these assignments induce those $[\![_]\!]^{\nu'}_{\mathcal{L}'}$, constituting the *interpretation* over a functor $\nu' : \mathcal{B} \to U\mathcal{L}'$, of objects and morphisms in a strict linear category $\mathcal{L}'$ respectively to $\ell_{\mathcal{B}}$-types and free generations of pre-$\ell_{\mathcal{B}}$-terms.

This interpretation is *invariant* under the choice of a free generation or an $\approx$-representative:

Proposition 4.2.2 (invariance of semantics)**.** *If $\ell^{\circ}_{\mathcal{B}}$-skeletons $\mathfrak{s}, \mathfrak{s}' :: \mathfrak{S}$ satisfy $\mathfrak{s} \approx \mathfrak{s}' :: \mathfrak{S}$, then, for any free generation $\mathcal{D}$ of a pre-$\ell^{\circ}_{\mathcal{B}}$-net $\mathfrak{r} :: \mathfrak{R}$ and that $\mathcal{D}'$ of a pre-$\ell^{\circ}_{\mathcal{B}}$-net $\mathfrak{r}' :: \mathfrak{R}'$ such that $\rho(\mathfrak{r} :: \mathfrak{R}) = \mathfrak{s} :: \mathfrak{S}$ and $\rho(\mathfrak{r}' :: \mathfrak{R}') = \mathfrak{s}' :: \mathfrak{S}$, we have*

$$[\![\, \mathcal{D} \mid \mathfrak{r} :: \mathfrak{R} \,]\!]^{\nu}_{\mathcal{L}} = [\![\, \mathcal{D}' \mid \mathfrak{r}' :: \mathfrak{R}' \,]\!]^{\nu}_{\mathcal{L}}$$

for any strict (pointed) linear category $\mathcal{L}$ and a functor $\nu : \mathcal{B} \to U\mathcal{L}$.

Proof. By induction on the free generation $\mathcal{D}$ followed by case analysis on the other $\mathcal{D}'$. In the following, we focus on key cases of the induction since the other cases are simpler or similar.

First, consider the case where $\mathcal{D}$ is the singleton derivation of a pre-$\ell^\circ_{\mathcal{B}}$-net $\mathfrak{r} :: \mathfrak{R} = \mathfrak{G}((\emptyset : P_i)_i \vdash \emptyset : 1_{[v]})$ that consists of 1R. The denotation $[\![\, \mathcal{D} \mid \mathfrak{r} :: \mathfrak{R} \,]\!]^\nu_{\mathcal{L}}$ is the unique morphism to the terminal object. Note that $\mathcal{D}'$ can be a different derivation, but, whatever $\mathcal{D}'$ is, the denotation $[\![\, \mathcal{D}' \mid \mathfrak{r}' :: \mathfrak{R}' \,]\!]^\nu_{\mathcal{L}}$ must be the unique morphism to the terminal object.

Next, take the case where $\mathcal{D}$ is a derivation of $\mathfrak{r} :: \mathfrak{R} = \mathfrak{G}(\gamma : \Gamma, \{u \to v\} : \top_{[v]} \vdash \mathscr{N} : N)$ that ends with $\top$L. In this case, $\mathfrak{R}'$ is of the form $\mathfrak{G}(\Gamma', \top_{[v']} \vdash N')$ (since $\rho\mathfrak{R} = \rho\mathfrak{R}'$). By induction on $\mathcal{D}'$, we yield a derivation $\mathcal{D}''$ of some $\mathfrak{r}'' :: \mathfrak{R}''$ that ends with $\top$L with $\rho(\mathfrak{r}'' :: \mathfrak{R}'') \approx \mathfrak{s}' :: \mathfrak{S}$ and $[\![\, \mathcal{D}' \mid \mathfrak{r}' :: \mathfrak{R}' \,]\!]^\nu_{\mathcal{L}} = [\![\, \mathcal{D}'' \mid \mathfrak{r}'' :: \mathfrak{R}'' \,]\!]^\nu_{\mathcal{L}}$ (*n.b.* $\rho(\mathfrak{r}'' :: \mathfrak{R}'') \approx \mathfrak{s}' :: \mathfrak{S}$ is possible, but in general $\rho(\mathfrak{r}'' :: \mathfrak{R}'') = \mathfrak{s}' :: \mathfrak{S}$ is not since $\mathcal{D}''$ ends with $\top$L). For some cases of the induction on $\mathcal{D}'$, $\rho(\mathfrak{r} :: \mathfrak{R}) \approx \rho(\mathfrak{r}' :: \mathfrak{R}')$ is vital. It then suffices to apply the induction hypothesis to the derivations $\tilde{\mathcal{D}}$ and $\tilde{\mathcal{D}}''$ obtained from $\mathcal{D}$ and $\mathcal{D}''$, respectively, by the deletion of the last occurrences of $\top$L.

As the last key case, assume that $\mathcal{D}$ is a derivation of a pre-$\ell^\circ_{\mathcal{B}}$-net $\mathfrak{r} :: \mathfrak{R}$ that ends with $\otimes$R. Again, we see by induction on $\mathcal{D}'$ that there is a derivation $\mathcal{D}''$ of some $\mathfrak{r}'' :: \mathfrak{R}''$ that ends with $\otimes$R, whose hypotheses are the same up to prime form as those of the last occurrence of $\otimes$R in $\mathcal{D}$, and satisfies $\rho(\mathfrak{r}'' :: \mathfrak{R}'') \approx \mathfrak{s}' :: \mathfrak{S}$ and $[\![\, \mathcal{D}' \mid \mathfrak{r}' :: \mathfrak{R}' \,]\!]^\nu_{\mathcal{L}} = [\![\, \mathcal{D}'' \mid \mathfrak{r}'' :: \mathfrak{R}'' \,]\!]^\nu_{\mathcal{L}}$. Again, for some cases of the induction on $\mathcal{D}'$, $\rho(\mathfrak{r} :: \mathfrak{R}) \approx \rho(\mathfrak{r}' :: \mathfrak{R}')$ is crucial. Thus, similarly to the last case, we can apply the induction hypotheses to the respective pairs of the derivations obtained from $\mathcal{D}$ and $\mathcal{D}''$, respectively, by the deletion of the last occurrences of $\otimes$R. □

This proposition ensures that the following is well-defined:

Definition 4.2.3 (semantics of $\ell^\circ_{\mathcal{B}}$-nets)**.** The *interpretation* $[\![\mathfrak{G}A]\!]^\nu_{\mathcal{L}}$ of each ILB- or ILL-tree $\mathfrak{G}A$ in a strict (pointed) linear category $\mathcal{L}$ over a functor $\nu : \mathcal{B} \to U\mathcal{L}$ is defined by

$$[\![\mathfrak{G}A]\!]^\nu_{\mathcal{L}} := [\![A]\!]^\nu_{\mathcal{L}},$$

and that $[\![\mathfrak{r} :: \mathfrak{S}]\!]^\nu_{\mathcal{L}}$ of each $\ell^\circ_{\mathcal{B}}$-net $\mathfrak{r} :: \mathfrak{S}$, where let $\mathfrak{p} :: \mathfrak{R}$ be any pre-$\ell^\circ_{\mathcal{B}}$-net freely generated by any derivation $\mathcal{D}$ such that $\rho(\mathfrak{p} :: \mathfrak{R})^{\approx} = \mathfrak{r} :: \mathfrak{S}$, is defined by

$$[\![\mathfrak{r} :: \mathfrak{S}]\!]^\nu_{\mathcal{L}} := [\![\, \mathcal{D} \mid \mathfrak{p} :: \mathfrak{R} \,]\!]^\nu_{\mathcal{L}}.$$

Remark. By Theorem 3.2.44 and Proposition 4.2.2, we can and will use induction on free generations of pre-$\ell^\circ_{\mathcal{B}}$-nets for reasoning about the interpretation of $\ell^\circ_{\mathcal{B}}$-nets.

Then, equational *soundness* follows by construction:

Corollary 4.2.4 (soundness)**.** *Let $\mathfrak{r}, \mathfrak{r}' :: \mathfrak{S}$ be $\ell^\circ_{\mathcal{B}}$-nets, $\mathcal{L}$ a strict (pointed) linear category, and $\nu : \mathcal{B} \to U\mathcal{L}$ a functor. If $\mathfrak{r} = \mathfrak{r}' :: \mathfrak{S}$, then $[\![\mathfrak{r} :: \mathfrak{S}]\!]^\nu_{\mathcal{L}} = [\![\mathfrak{r}' :: \mathfrak{S}]\!]^\nu_{\mathcal{L}}$.*

Proof. Immediate from Proposition 4.2.2. □

The interpretation is also equationally *complete*:

Corollary 4.2.5 (completeness)**.** *Two normalised $\ell^\circ_{\mathcal{B}}$-nets $\mathfrak{r}, \mathfrak{r}' :: \mathfrak{S}$ are equal if the equation $[\![\mathfrak{r} :: \mathfrak{S}]\!]^\nu_{\mathcal{L}} = [\![\mathfrak{r}' :: \mathfrak{S}]\!]^\nu_{\mathcal{L}}$ holds for every strict (pointed) linear category $\mathcal{L}$ and functor $\nu : \mathcal{B} \to U\mathcal{L}$.*

Proof. Let $\nu_0 : \mathcal{B} \to U\tilde{\mathrm{Cl}}(\ell^\circ_{\mathcal{B}})$ be the functor that maps

$$X \mapsto \mathfrak{G}X, \qquad (\chi : X \to Y) \mapsto \rho\mathfrak{G}(\{y \xrightarrow{\chi} x\} : X_{[x]} \vdash \emptyset : Y_{[y]}),$$

and take the interpretation $[\![_]\!]^{\nu_0}_{\tilde{\mathrm{Cl}}(\ell^\circ_{\mathcal{B}})}$ in $\tilde{\mathrm{Cl}}(\ell^\circ_{\mathcal{B}})$. We see by induction on free generations of pre-$\ell^\circ_{\mathcal{B}}$-nets that every free generation $\mathcal{D}$ of a normalised pre-$\ell^\circ_{\mathcal{B}}$-net $\mathfrak{p} :: \mathfrak{R}$ enjoys $[\![\, \mathcal{D} \mid \mathfrak{p} :: \mathfrak{R} \,]\!]^{\nu_0}_{\tilde{\mathrm{Cl}}(\ell^\circ_{\mathcal{B}})} =$

$\rho(\mathfrak{p} :: \mathfrak{R})^{\approx}$, and thus the interpretation $[\![_]\!]^{\nu_0}_{\tilde{\mathrm{Cl}}(\ell^\circ_{\mathcal{B}})}$ is the identity map on normalised $\ell^\circ_{\mathcal{B}}$-nets. Thus, the equation $\mathfrak{r} = \mathfrak{r}' :: \mathfrak{S}$ follows from the one $[\![\mathfrak{r} :: \mathfrak{S}]\!]^{\nu_0}_{\tilde{\mathrm{Cl}}(\ell^\circ_{\mathcal{B}})} = [\![\mathfrak{r}' :: \mathfrak{S}]\!]^{\nu_0}_{\tilde{\mathrm{Cl}}(\ell^\circ_{\mathcal{B}})}$. □

Finally, we lift these soundness and completeness to *freeness* of $\tilde{\mathrm{Cl}}(\ell^\circ_{\mathcal{B}})$ in the sense defined in Definition 4.1.7:

Theorem 4.2.6 (freeness)**.** *The strict (pointed) linear category $\tilde{\mathrm{Cl}}(\ell^\circ_{\mathcal{B}})$ is free over $\mathcal{B}$.*

Proof. Let $\mathcal{L}$ be a small strict (pointed) linear category, and $\nu : \mathcal{B} \to U\mathcal{L}$ a functor. We first construct a morphism $\tilde{\mathrm{Cl}}(\ell^\circ_{\mathcal{B}}) \to \mathcal{L}$ in s(p)LCat. By Corollary 4.2.4, the interpretation $[\![_]\!]^{\nu}_{\mathcal{L}}$ yields the desired morphism $[\![_]\!]^{\nu}_{\mathcal{B},\mathcal{L}} : \tilde{\mathrm{Cl}}(\ell^\circ_{\mathcal{B}}) \to \mathcal{L}$ since by construction it strictly preserves all constructs of a free (pointed) linear category, except that the preservation of identities is proven by induction on objects. For convenience, we also write $[\![_]\!]^{\nu}_{\mathcal{B},\mathcal{L}}$ for $[\![_]\!]^{\nu}_{\mathcal{L}}$.

Conversely, every morphism $L : \tilde{\mathrm{Cl}}(\ell^\circ_{\mathcal{B}}) \to \mathcal{L}$ in s(p)LCat induces a functor $|L| : \mathcal{B} \to U\mathcal{L}$ by

$$|L|(X) := L(\mathfrak{G}X), \qquad |L|(\chi : X \to Y) := L(\rho\mathfrak{G}(\{y \xrightarrow{\chi} x\} : X_{[x]} \dashv\vdash \emptyset : Y_{[y]})),$$

where it is easy to see its functoriality. For convenience, we also write $|L|_{\mathcal{B},\mathcal{L}}$ for $|L|$.

These constructions $[\![_]\!]_{\mathcal{B},\mathcal{L}}$ and $|_|_{\mathcal{B},\mathcal{L}}$ are inverses to each other: $|[\![_]\!]^{\nu}_{\mathcal{B},\mathcal{L}}| = \nu$ holds by construction, and $[\![_]\!]^{|L|}_{\mathcal{B},\mathcal{L}} = L$ holds as L is completely determined by its action on the base objects $\mathfrak{G}(X)$ and morphisms $\rho\mathfrak{G}(\{y \xrightarrow{\chi} x\} : X_{[x]} \dashv\vdash \emptyset : Y_{[y]})$, where note that all objects and morphisms in $\tilde{\mathrm{Cl}}(\ell^\circ_{\mathcal{B}})$ are freely generated from those in $\mathcal{B}$ by constructs of a free (pointed) linear category. The two equations hold since we focus on the *strict* structures in the sense of Definition 4.1.7.

We next lift the map $\tilde{\mathrm{Cl}}(\ell^\circ_{_}) : \mathcal{B} \mapsto \tilde{\mathrm{Cl}}(\ell^\circ_{\mathcal{B}})$ to a functor $\tilde{\mathrm{Cl}}(\ell^\circ_{_}) : \mathrm{Cat} \to \mathrm{s(p)LCat}$: for a functor $F : \mathcal{C} \to \mathcal{B}$, the morphism $\tilde{\mathrm{Cl}}(\ell^\circ_{F}) : \tilde{\mathrm{Cl}}(\ell^\circ_{\mathcal{C}}) \to \tilde{\mathrm{Cl}}(\ell^\circ_{\mathcal{B}})$ in s(p)LCat is the unique one that maps

$$\mathfrak{G}(X) \mapsto \mathfrak{G}(FX), \qquad \rho\mathfrak{G}(\{y \xrightarrow{\chi} x\} : X_{[x]} \dashv\vdash \emptyset : Y_{[y]}) \mapsto \rho\mathfrak{G}(\{y \xrightarrow{F\chi} x\} : FX_{[x]} \dashv\vdash \emptyset : FY_{[y]}).$$

Finally, the naturality of the adjunction

$$\mathrm{Cat} \underset{U}{\overset{\tilde{\mathrm{Cl}}(\ell^\circ_{_})}{\rightleftarrows}} \mathrm{s(p)LCat}, \qquad (\perp)$$

i.e. the equation

$$|K \circ L \circ \tilde{\mathrm{Cl}}(\ell^\circ_{F})| = UK \circ |L| \circ F : \mathcal{C} \to UL'$$

for any functor $F : \mathcal{C} \to \mathcal{B}$ and morphism $K : \mathcal{L} \to \mathcal{L}'$ in s(p)LCat, holds by construction. □

Corollary 4.2.7 (decidability II)**.** *If the equality between morphisms in $\mathcal{B}$ is decidable, then so is the one in any free (pointed) linear category over $\mathcal{B}$.*

Proof. Assume that the equality between morphisms in $\mathcal{B}$ is decidable. Since $\tilde{\mathrm{Cl}}(\ell^\circ_{\mathcal{B}})$ is free over $\mathcal{B}$ by Theorem 4.2.6, it suffices to show the decidability of the equality between morphisms in $\tilde{\mathrm{Cl}}(\ell^\circ_{\mathcal{B}})$. The decidability holds since every $\ell^\circ_{\mathcal{B}}$-net is a finite (labelled) graph, except components of $\mathcal{B}$ (but it is not a problem since the equality between morphisms in $\mathcal{B}$ is decidable). □

5. Application to coherence

We lastly provide a new graphical proof of *coherence* on symmetric monoidal closed categories [Mac63, KM71, MS07] and extend this result to (pointed) linear categories. In this section, we extensively use the natural transformations/isomorphisms listed in Definition 4.2.1.

5.1. **Graphical coherence.** Recall that the coherence is concerned with *allowable* natural transformations [KM71, §2]. For convenience, we reformulate the allowability in a slightly different way and call it *SMC-allowability*. We then extend it to *linear-allowability* in the expected way.

Definition 5.1.1 (SMC-allowability [KM71])**.** A morphism in a free symmetric monoidal closed category $\mathcal{M}$ over $\mathcal{B}$ is said to be *SMC-allowable* if

(1) It is between objects both of which are constructed inductively from objects in $\mathcal{B}$ and (monoidal) unit by tensor product and closure;
(2) It is constructed inductively from identities, the isomorphisms

$$\alpha_{A,B,C}, \qquad \lambda_A, \qquad \gamma_A, \qquad \varsigma_{A,B} \quad (A, B, C \in \mathrm{Obj}(\mathcal{B}))$$

in Definition 4.2.1 and their inverses, the evaluation $\mathrm{ev}_{A,B} : (A \multimap B) \otimes A \to B$ and the currying $\Lambda(\mathrm{id}_{A\otimes B}) : A \to (A \multimap A \otimes B)$ by composition, tensor product and closure,

and a natural transformations between symmetric monoidal closed functors from $\mathcal{M}$ to itself is said to be *SMC-allowable* in $\mathcal{M}$ if so are all its components.

A symmetric monoidal closed category $\mathcal{M}$ is said to be *coherent* if any two parallel allowable natural transformations in $\mathcal{M}$ coincide. A symmetric monoidal closed category is *not* (unconditionally) coherent [KM71, §1]. Kelly and Mac Lane then established a sufficient criterion for the coherence [KM71, Theorem 2.4]. For this classic result, they defined a graphical representation of allowable natural transformations, called *Eilenberg–Kelly–Mac Lane graphs* [EK66, KM71].

Voreadou [Vor77] tried to strengthen the sufficient criterion into a necessary, sufficient one, but one of her lemmata was false. Based on her idea, however, Mehats and Soloviev [MS07] finally established a necessary, sufficient criterion for the coherence in terms of a sequent calculus.

Definition 5.1.2 (linear-allowability)**.** A morphism in a free linear category $\mathcal{L}$ over $\mathcal{B}$ is said to be *linear-allowable* if

(1) It is between objects both of which are constructed inductively from objects in $\mathcal{B}$ and monoidal and cartesian units by tensor and cartesian products, closure and exponential;
(2) It is constructed inductively from identities, the isomorphisms

$$\alpha_{A,B,C}, \qquad \lambda_A, \qquad \gamma_A, \qquad \varsigma_{A,B} \quad (A, B, C \in \mathrm{Obj}(\mathcal{B}))$$

and their inverses, the evaluation $\mathrm{ev}_{A,B}$, the currying $\Lambda(\mathrm{id}_{A\otimes B})$, and the morphisms

$$\omega_A, \qquad \delta_A, \qquad \vartheta_A, \qquad \mu_{A,B}, \qquad \xi_A, \qquad \tau_A, \qquad \pi^{(1)}_{A,B}, \qquad \pi^{(2)}_{A,B}$$

in Definition 4.2.1 by composition, tensor product, closure and exponential,

and a natural transformations between linear functors between $\mathcal{L}$ and itself is said to be *linear-allowable* in $\mathcal{L}$ if so are all its components.

SMC-allowable morphisms in the symmetric monoidal closed part of $\tilde{\mathrm{Cl}}(\ell_{\mathcal{N}})$ can be seen as an equivalent analogue of Eilenberg–Kelly–Mac Lane graphs, and they are extended to linear-allowable morphisms in $\tilde{\mathrm{Cl}}(\ell_{\mathcal{N}})$. We make use of these morphisms *via* the following:

Proposition 5.1.3 (allowability of deputy)**.** *The deputy* $\mathrm{dep}(\eta)$ *of any SMC- (respectively, linear-)allowable natural transformation* η *in* $\tilde{\mathrm{Cl}}(\ell_{\mathcal{B}})$ *is SMC- (respectively, linear-)allowable.*

Proof. By induction on the construction of η. □

Note that the operation dep of taking deputy is analogous to the functor from the category of formal functors and allowable natural transformations to the category of formal shapes and Eilenberg–Kelly–Mac Lane graphs [KM71, p. 105].

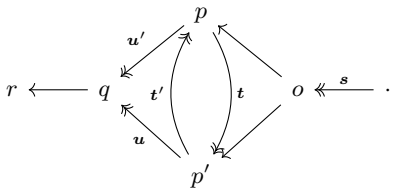
Convention. A valid sequence is said to be *bare* if it is a finite sequence of ends (*i.e.* moves of length one). An end y is *hereditarily justified* by another x in an $\ell^{\circ}_{\mathcal{B}}$-diagram $\mathfrak{S}$ if there is a finite sequence $(x_i)_{i=1}^{n}$ of ends such that $x_1 = x$, $x_n = y$, and x_i justifies x_{i+1} in $\mathfrak{S}$ when $1 \leqslant i < n$.

Theorem 5.1.4 (graphical coherence)**.** *Two linear-allowable morphisms $f, f' : A \rightrightarrows B$ in $\mathrm{Cl}(\ell_{\mathcal{N}})$ differ if and only if some $\approx$-representatives $\mathfrak{s}, \mathfrak{s}' :: \mathfrak{S}$ of the shreddings of the slices of f and f' at the same slice assignment, respectively, have bare valid sequences $\boldsymbol{s}.o.p.\boldsymbol{t}.p'.\boldsymbol{u}.q.r \in \mathrm{VS}(\mathfrak{s} :: \mathfrak{S})$ and $\boldsymbol{s}.o.p'.\boldsymbol{t}'.p.\boldsymbol{u}'.q.r \in \mathrm{VS}(\mathfrak{s}' :: \mathfrak{S})$ of even length, where $p \neq p'$ and $q \neq q'$, such that*

1. *Every valid path from o to p of length more than one goes through p';*
2. *The ends q and r are decorated with the same numeral such that every hereditary justification of q (respectively, r) contains p (respectively, p').*

i.e. *the two valid sequences diverge at $o \to p$ and $o \to p'$ yet converge at $q \to r$, or graphically*

(16)



where the double-head arrow $\twoheadrightarrow$ abbreviates a finite sequence of edges.

Proof. We first show that the conjunction of the graphical conditions (1)–(2) implies that $\mathfrak{s}$ and $\mathfrak{s}'$ are not $\approx$-equivalent. Assume for a contradiction that they are $\approx$-equivalent. There is a finite sequence of $\ell_{\mathcal{N}}$-skeletons $(\mathfrak{s}_i :: \mathfrak{S})_{i=1}^{n}$ such that $\mathfrak{s}_i \sim \mathfrak{s}_{i+1} :: \mathfrak{S}$ if $1 \leqslant i < n$, $\mathfrak{s}_1 = \mathfrak{s}$ and $\mathfrak{s}_n = \mathfrak{s}'$. By the first condition above and the first axiom in Definition 3.2.22, some $\mathfrak{s}_{i_0}$ $(1 < i_0 < n)$ must contain both $o \to p$ and $o \to p'$ as well as $q \to r$. Then, $\mathfrak{s}_{i_0} :: \mathfrak{S}$ has a valid sequence of the form $\boldsymbol{s}.o.p.\boldsymbol{u}''.q.r$, where r is not justified yet, a contradiction.

For the other direction, assume that some $\approx$-representatives $\mathfrak{s}, \mathfrak{s}' :: \mathfrak{S}$ of the shreddings of the slices of f and f' at the same slice assignment, respectively, are not $\approx$-equivalent. By the definition of valid sequences, the failure of $\approx$-equivalence must come from the conjunction of the two graphical conditions. □

The previous, proof-theoretic method [MS07] has to deal with superfluous syntactic details in formal derivations such as permutations of logical rules in a formal derivation. In contrast, our graphical approach completely dispenses with those details. When we focus on the symmetric monoidal closed part of the initial (pointed) linear category, Theorem 5.1.4 provides a novel graphical characterisation of coherence on the free symmetric monoidal closed category.

In addition, our graphical coherence immediately derives the following classic result by Kelly and Mac Lane [KM71]. They [KM71, p. 106] defined *constant* formal free symmetric monoidal closed functors to be those not containing an identity functor, and *proper* ones inductively by

1. $\mathrm{id}_{\top}$ and id_X $(X \in \mathcal{B})$ are proper;
2. $F \otimes G$ is proper if so are F and G;
3. $F \multimap G$ is proper if so are F and G unless only G is constant.

Then, they provide the following sufficient criterion for coherence:

Corollary 5.1.5 (Kelly–Mac Lane sufficient condition [KM71])**.** *Let $\mathcal{M}$ be a symmetric monoidal closed category, and $\eta, \eta' : F \rightrightarrows G$ be SMC-allowable natural transformations in $\mathcal{M}$. If the formal free symmetric monoidal closed functors functors F and G are both proper, then $\eta = \eta'$.*

Proof. We derive the corollary by a new graphical argument: the corollary follows immediately from Theorem 5.1.4 because the sufficient condition makes the graphical condition (16) false. □

Acknowledgments: This work is financed by National Funds through the Portuguese Foundation for Science and Technology within the project IBEX, with reference 10.54499/PTDC/CCI-COM/4280/2021, and the Centre for Mathematics of the University of Coimbra for the projects UIDB/00324/2020 and 2022.06122.CEECIND.

References

[A+97] Samson Abramsky et al., *Semantics of interaction: An introduction to game semantics*, Semantics and Logics of Computation **14** (1997), 1–31.

[APGSZ22] Mario Alvarez-Picallo, Dan Ghica, David Sprunger, and Fabio Zanasi, *Rewriting for monoidal closed categories*, 7th International Conference on Formal Structures for Computation and Deduction (FSCD 2022), Schloss Dagstuhl–Leibniz-Zentrum für Informatik, 2022, pp. 29–1.

[Bac59] John W Backus, *The syntax and the semantics of the proposed international algebraic language of the zurich acm-gamm conference*, ICIP Proceedings, 1959, pp. 125–132.

[Bar06] Michael Barr, **-autonomous categories*, vol. 752, Springer, 2006.

[BBDPH93] Nick Benton, Gavin Bierman, Valeria De Paiva, and Martin Hyland, *Linear λ-calculus and categorical models revisited*, Computer Science Logic: 6th Workshop, CSL'92 San Miniato, Italy, September 28–October 2, 1992 Selected Papers 6, Springer, 1993, pp. 61–84.

[BCLS20] Richard F Blute, J Robin B Cockett, J-S P Lemay, and Robert AG Seely, *Differential categories revisited*, Applied Categorical Structures **28** (2020), 171–235.

[BCS06] Richard F Blute, J Robin B Cockett, and Robert AG Seely, *Differential categories*, Mathematical structures in computer science **16** (2006), no. 6, 1049–1083.

[BCS09] ———, *Cartesian differential categories*, Theory and Applications of Categories **22** (2009), no. 23, 622–672.

[BCST96] Richard F Blute, J Robin B Cockett, Robert AG Seely, and Todd H Trimble, *Natural deduction and coherence for weakly distributive categories*, Journal of Pure and Applied Algebra **113** (1996), no. 3, 229–296.

[Bek00] Tibor Beke, *Sheafifiable homotopy model categories*, Mathematical Proceedings of the Cambridge Philosophical Society **129** (2000), no. 3, 447–475.

[Ben94] P Nick Benton, *A mixed linear and non-linear logic: Proofs, terms and models*, International Workshop on Computer Science Logic, Springer, 1994, pp. 121–135.

[BG20] Patrick Baillot and Alexis Ghyselen, *Combining linear logic and size types for implicit complexity*, Theoretical Computer Science **813** (2020), 70–99.

[Bie94] Gavin M. Bierman, *On intuitionistic linear logic*, Ph.D. thesis, 1994.

[Bie95] Gavin M Bierman, *What is a categorical model of intuitionistic linear logic?*, International Conference on Typed Lambda Calculi and Applications, Springer, 1995, pp. 78–93.

[BPS94] Richard F Blute, Prakash Panangaden, and Robert AG Seely, *Holomorphic models of exponential types in linear logic*, Mathematical Foundations of Programming Semantics: 9th International Conference New Orleans, LA, USA, April 7–10, 1993 Proceedings 9, Springer, 1994, pp. 474–512.

[BS10] John Baez and Mike Stay, *Physics, topology, logic and computation: a rosetta stone*, New structures for physics, Springer, 2010, pp. 95–172.

[Chu40] Alonzo Church, *A formulation of the simple theory of types*, The journal of symbolic logic **5** (1940), no. 2, 56–68.

[CS99] J Robin B Cockett and Robert AG Seely, *Linearly distributive functors*, Journal of Pure and Applied Algebra **143** (1999), no. 1-3, 155–203.

[DJ03] Vincent Danos and Jean-Baptiste Joinet, *Linear logic and elementary time*, Information and Computation **183** (2003), no. 1, 123–137.

[EK66] Samuel Eilenberg and Gregory Maxwell Kelly, *A generalization of the functorial calculus*, Journal of Algebra **3** (1966), no. 3, 366–375.

[FGH26] Marcelo Fiore, Nicola Gambino, and Martin Hyland, *Monoidal bicategories, differential linear logic, and analytic functors*.

[Gen35] Gerhard Gentzen, *Untersuchungen über das logische schließen. i*, Mathematische zeitschrift **39** (1935), no. 1, 176–210.

[Gir87] Jean-Yves Girard, *Linear logic*, Theoretical Computer Science **50** (1987), no. 1, 1–101.

[Gir89] ———, *Towards a geometry of interaction*, Contemporary Mathematics **92** (1989), no. 69-108, 6.

[Gir98] Jean-Yves Girard, *Light linear logic*, Information and Computation **143** (1998), no. 2, 175–204.

[Gir99] Jean-Yves Girard, *Coherent banach spaces: a continuous denotational semantics*, Theoretical Computer Science **227** (1999), no. 1-2, 275–297.

[GL87] Jean-Yves Girard and Yves Lafont, *Linear logic and lazy computation*, International Joint Conference on Theory and Practice of Software Development, Springer, 1987, pp. 52–66.

[GTL89] Jean-Yves Girard, Paul Taylor, and Yves Lafont, *Proofs and types*, vol. 7, Cambridge university press Cambridge, 1989.

[Has20] Ryu Hasegawa, *A categorical reduction system for linear logic*, Theory and Applications of Categories **35** (2020), no. 50, 1833–1870.

[Hof03] Martin Hofmann, *Linear types and non-size-increasing polynomial time computation*, Information and Computation **183** (2003), no. 1, 57–85, International Workshop on Implicit Computational Complexity (ICC'99).

[Hov99] Mark Hovey, *Model categories*, Mathematical Surveys and Monographs, vol. 63, American Mathematical Society, Providence, RI, 1999.

[Hug06] Dominic JD Hughes, *Proofs without syntax*, Annals of Mathematics (2006), 1065–1076.

[Hug12] ———, *Simple free star-autonomous categories and full coherence*, Journal of Pure and Applied Algebra **216** (2012), no. 11, 2386–2410.

[Hyl97] Martin Hyland, *Game semantics*, Semantics and Logics of Computation, vol. 14, Cambridge University Press, New York, 1997, p. 131.

[Jac99] Bart Jacobs, *Categorical logic and type theory*, vol. 141, Elsevier, 1999.

[JS91] André Joyal and Ross Street, *The geometry of tensor calculus, i*, Advances in mathematics **88** (1991), no. 1, 55–112.

[Kel82] Gregory Maxwell Kelly, *Basic concepts of enriched category theory*, vol. 64, CUP Archive, 1982.

[KKR96] Heinrich Kleisli, Hans-Peter Künzi, and Jiří Rosickỳ, *A topological banach space model of linear logic*, Categorical Topology: Proceedings of the L'Aquila Conference (1994), Springer, 1996, pp. 155–162.

[KM71] Gregory Maxwell Kelly and Saunders MacLane, *Coherence in closed categories*, Journal of Pure and Applied Algebra **1** (1971), no. 1, 97–140.

[KO99] Thong-Wei Koh and CH Luke Ong, *Internal languages for autonomous and star-autonomous categories*, Electronic Notes in Theoretical Computer Science **29** (1999), 151.

[Laf88] Yves Lafont, *Logiques, catégories et machines*, Ph.D. thesis, PhD thesis, Université Paris 7, 1988.

[Laf04] ———, *Soft linear logic and polynomial time*, Theoretical computer science **318** (2004), no. 1, 163–180.

[Lam08] François Lamarche, *Proof nets for intuitionistic linear logic: Essential nets*.

[LS88] Joachim Lambek and Philip J Scott, *Introduction to higher-order categorical logic*, vol. 7, Cambridge University Press, 1988.

[LS06] François Lamarche and Lutz Straßburger, *From proof nets to the free star-autonomous category*, Logical Methods in Computer Science **2** (2006).

[Mac63] Saunders MacLane, *Natural associativity and commutativity*.

[Mel06] Paul-André Melliès, *Functorial boxes in string diagrams*, International workshop on computer science logic, Springer, 2006, pp. 1–30.

[Mel09] Paul-André Mellies, *Categorical semantics of linear logic*, Panoramas et syntheses **27** (2009), 15–215.

[Mel22] Paul-André Melliès, *A functorial excursion between algebraic geometry and linear logic*, Proceedings of the 37th Annual ACM/IEEE Symposium on Logic in Computer Science, 2022, pp. 1–13.

[ML13] Saunders Mac Lane, *Categories for the working mathematician*, vol. 5, Springer Science & Business Media, 2013.

[MMDPR05] Maria Emilia Maietti, Paola Maneggia, Valeria De Paiva, and Eike Ritter, *Relating categorical semantics for intuitionistic linear logic*, Applied categorical structures **13** (2005), 1–36.

[MO03] Andrzej S. Murawski and C-HL Ong, *Exhausting strategies, joker games and full completeness for imll with unit*, Theoretical Computer Science **294** (2003), no. 1-2, 269–305.

[MS07] L Mehats and Sergei Soloviev, *Coherence in smccs and equivalences on derivations in imll with unit*, Annals of Pure and Applied Logic **147** (2007), no. 3, 127–179.

[MTT18] Paul-André Melliès, Nicolas Tabareau, and Christine Tasson, *An explicit formula for the free exponential modality of linear logic*, Mathematical Structures in Computer Science **28** (2018), no. 7, 1253–1286.

[New42] Maxwell Herman Alexander Newman, *On theories with a combinatorial definition of" equivalence"*, Annals of mathematics **43** (1942), no. 2, 223–243.

[P+71] Roger Penrose et al., *Applications of negative dimensional tensors*, Combinatorial mathematics and its applications **1** (1971), no. 221-244, 3.

[Pen71] Roger Penrose, *Angular momentum: an approach to combinatorial space-time*, Quantum theory and beyond **151** (1971).

[Pit01] Andrew M Pitts, *Categorical logic*, Handbook of logic in computer science, Oxford University Press, 2001, pp. 39–123.

[Por08] Hans-E Porst, *On categories of monoids, comonoids, and bimonoids*, Quaestiones Mathematicae **31** (2008), no. 2, 127–139.

[RDG26] Callum Reader and Alessandro Di Giorgio, *String Diagrams for Closed Symmetric Monoidal Categories*, 34th EACSL Annual Conference on Computer Science Logic (CSL 2026) (Dagstuhl, Germany) (Stefano Guerrini and Barbara König, eds.), Leibniz International Proceedings in Informatics (LIPIcs), vol. 363, Schloss Dagstuhl – Leibniz-Zentrum für Informatik, 2026, pp. 12:1–12:23.

[Ril22] Mitchell Riley, *A bunched homotopy type theory for synthetic stable homotopy theory*, Ph.D. thesis, Wesleyan University, 2022.

[See89] Robert AG Seely, *Linear logic, star-autonomous categories and cofree coalgebras*, Categories in computer science and logic **92** (1989), 371–382.

[Sel11] Peter Selinger, *A survey of graphical languages for monoidal categories*, New structures for physics (2011), 289–355.

[Sol91] SV Soloviev, *On the conditions of full coherence in closed categories*, Journal of Pure and Applied Algebra **69** (1991), no. 3, 301–329.

[Sol97] Sergei Soloviev, *Proof of a conjecture of s. mac lane*, Annals of Pure and Applied Logic **90** (1997), no. 1-3, 101–162.

[SS23] Hisham Sati and Urs Schreiber, *The quantum monadology*, arXiv preprint arXiv:2310.15735 (2023).

[Str72] Ross Street, *The formal theory of monads*, Journal of Pure and Applied Algebra **2** (1972), no. 2, 149–168.

[The26] The Stacks Project Authors, *The stacks project*, `https://stacks.math.columbia.edu`, 2026.

[Tri94] Todd Hampton Trimble, *Linear logic, bimodules, and full coherence for autonomous categories*, Rutgers The State University of New Jersey, School of Graduate Studies, 1994.

[TRO92] AS TROELSTRA, *Lectures on linear logic*, CSLI Lecture Notes 29 **3** (1992).

[TS00] Anne Sjerp Troelstra and Helmut Schwichtenberg, *Basic proof theory*, no. 43, Cambridge University Press, 2000.

[Vor77] Rodiani Voreadou, *Coherence and non-commutative diagrams in closed categories*, vol. 182, American Mathematical Soc., 1977.

[VZ14] Benoît Valiron and Steve Zdancewic, *Finite vector spaces as model of simply-typed lambda-calculi*, International Colloquium on Theoretical Aspects of Computing, Springer, 2014, pp. 442–459.

[WGZ24] Paul Wilson, Dan Ghica, and Fabio Zanasi, *String diagrams for strictification and coherence*, Logical Methods in Computer Science **20** (2024).

[YA20] Norihiro Yamada and Samson Abramsky, *Dynamic game semantics*, Mathematical Structures in Computer Science **30** (2020), no. 8, 892–951.

Centre for Mathematics, University of Coimbra (CMUC), 3000-143 Coimbra, Portugal
Email address: `norihiro@mat.uc.pt`

$$(\mathrm{XL})\ \frac{f = [\![\, \mathcal{F} \mid \gamma : \Gamma, \mathscr{P} : P, \mathscr{R} : R, \delta : \Delta \dashv \sigma : \Sigma \vdash \mathscr{N} : N \,]\!]^{\nu}_{\mathcal{L}}}{[\![\, \mathrm{XL}(\mathcal{F}) \mid \gamma : \Gamma, \mathscr{R} : R, \mathscr{P} : P, \delta : \Delta \dashv \sigma : \Sigma \vdash \mathscr{N} : N \,]\!]^{\nu}_{\mathcal{L}} := f \circ (\mathrm{id}_{[\![\Gamma]\!]^{\nu}_{\mathcal{L}}} \otimes \varsigma_{[\![R]\!]^{\nu}_{\mathcal{L}}, [\![P]\!]^{\nu}_{\mathcal{L}}} \otimes \mathrm{id}_{[\![\Delta]\!]^{\nu}_{\mathcal{L}}})}$$

$$(\mathrm{XM})\ \frac{f = [\![\, \mathcal{F} \mid \gamma : \Gamma \dashv \sigma : \Sigma, \mathscr{M} : M, \mathscr{L} : L, \pi : \Pi \vdash \mathscr{N} : N \,]\!]^{\nu}_{\mathcal{L}}}{[\![\, \mathrm{XM}(\mathcal{F}) \mid \gamma : \Gamma \dashv \sigma : \Sigma, \mathscr{L} : L, \mathscr{M} : M, \pi : \Pi \vdash \mathscr{N} : N \,]\!]^{\nu}_{\mathcal{L}} := f}$$

$$(\textsc{Cut})\ \frac{f = [\![\, \mathcal{F} \mid \gamma : \Gamma \dashv \sigma : \Sigma \vdash \mathscr{N} : N \,]\!]^{\nu}_{\mathcal{L}} \qquad g = [\![\, \mathcal{G} \mid \delta : \Delta, \mathscr{P} : P \dashv \pi : \Pi \vdash \mathscr{Q} : Q \,]\!]^{\nu}_{\mathcal{L}}}{[\![\, \textsc{Cut}(\mathcal{F}, \mathcal{G}) \mid \gamma : \Gamma, \delta : \Delta \dashv \sigma : \Sigma, \mathscr{N} \cup \mathscr{P} : N \triangleright P, \pi : \Pi \vdash \mathscr{Q} : Q \,]\!]^{\nu}_{\mathcal{L}} := g \circ (\mathrm{id}_{[\![\Delta]\!]^{\nu}_{\mathcal{L}}} \otimes f) \circ \varsigma_{[\![\Gamma]\!]^{\nu}_{\mathcal{L}}, [\![\Delta]\!]^{\nu}_{\mathcal{L}}}} \ (\underline{N} = \underline{P})$$

$$(\mathrm{BC})\ \frac{}{[\![\, \mathrm{BC} \mid \{u \xrightarrow{\chi} v\} : X_{[v]} \dashv\vdash \emptyset : Y_{[u]} \,]\!]^{\nu}_{\mathcal{L}} := \nu(\chi)} \ (\chi \in \mathcal{B}(X, Y))$$

$$(!\mathrm{W})\ \frac{f = [\![\, \mathcal{F} \mid \gamma : \Gamma \dashv \sigma : \Sigma \vdash \mathscr{N} : N \,]\!]^{\nu}_{\mathcal{L}}}{[\![\, !\mathrm{W}(\mathcal{F}) \mid \gamma : \Gamma, \{v \to w\} : !_{[w]} P \dashv \sigma : \Sigma \vdash \mathscr{N} : N \,]\!]^{\nu}_{\mathcal{L}} := f \circ \gamma_{[\![\Gamma]\!]^{\nu}_{\mathcal{L}}} \circ (\mathrm{id}_{[\![\Gamma]\!]^{\nu}_{\mathcal{L}}} \otimes \omega_{[\![P]\!]^{\nu}_{\mathcal{L}}})} \ (v \in \mathrm{End}^{\mathrm{Ini}}_{N})$$

$$(!\mathrm{D})\ \frac{f = [\![\, \mathcal{F} \mid t :: (\Gamma, P \dashv \Sigma \vdash N) \,]\!]^{\nu}_{\mathcal{L}}}{[\![\, !\mathrm{D}(\mathcal{F}) \mid (r \setminus (r \updownarrow P)) \cup (\{\, a + |(\mathrm{Col}(r \uparrow P), w)\rangle \mid a \in (r \updownarrow P) \,\}) :: (\Gamma, !_{[w]} P \dashv \Sigma \vdash N) \,]\!]^{\nu}_{\mathcal{L}} := f \circ (\mathrm{id}_{[\![\Gamma]\!]^{\nu}_{\mathcal{L}}} \otimes \epsilon_{[\![P]\!]^{\nu}_{\mathcal{L}}})}$$

$$(!\mathrm{DM})\ \frac{f = [\![\, \mathcal{F} \mid r : (\Gamma \dashv \Sigma, M \vdash N) \,]\!]^{\nu}_{\mathcal{L}}}{[\![\, !\mathrm{DM}(\mathcal{F}) \mid (r \setminus (r \updownarrow M)) \cup (\{\, a + |(\mathrm{Col}(r \uparrow M), w)\rangle \mid a \in (r \updownarrow M) \,\}) :: \Gamma \dashv \Sigma, !_{[w]} M \vdash N \,]\!]^{\nu}_{\mathcal{L}} := f}$$

$$(!\mathrm{C})\ \frac{f = [\![\, \mathcal{F} \mid \gamma : \Gamma, \mathscr{P} : !_{[w]} P, \hat{\mathscr{P}} : !_{[\hat{w}]} \hat{P} \dashv \sigma : \Sigma \vdash \mathscr{N} : N \,]\!]^{\nu}_{\mathcal{L}}}{\begin{array}{l}[\![\, !\mathrm{C}(\mathcal{F}) \mid (\gamma\{!_{[w]} P / !_{[\hat{w}]} \hat{P}\} : \Gamma, \mathscr{P} \cup \hat{\mathscr{P}}\{!_{[w]} P / !_{[\hat{w}]} \hat{P}\} : !_{[w]} P \dashv \sigma\{!_{[w]} P / !_{[\hat{w}]} \hat{P}\} : \Sigma \vdash \mathscr{N}\{!_{[w]} P / !_{[\hat{w}]} \hat{P}\} : N) \,]\!]^{\nu}_{\mathcal{L}} \\ \qquad := f \circ (\mathrm{id}_{[\![\Gamma]\!]^{\nu}_{\mathcal{L}}} \otimes \vartheta_{[\![P]\!]^{\nu}_{\mathcal{L}}})\end{array}} \ (\underline{P} = \underline{\hat{P}})$$

$$(!\mathrm{R})\ \frac{f = [\![\, \mathcal{F} \mid (\mathscr{P}_i : !_{[w_i]} P_i)^n_{i=1} \dashv (\mathscr{M}_j : !_{[v_j]} M_j)^m_{j=1} \vdash \mathscr{N} : N \,]\!]^{\nu}_{\mathcal{L}}}{[\![\, !\mathrm{R}(\mathcal{F}) \mid (\mathscr{P}_i \cup \{u \to w_i\} : !_{[w_i]} P_i)^n_{i=1} \dashv (\mathscr{M}_j \cup \{u \to v_j\} : !_{[v_j]} M_j)^m_{j=1} \vdash \mathscr{N} : !_{[u]} N \,]\!]_{\mathcal{L}} := !f \circ \mu_{(![\![P_i]\!]^{\nu}_{\mathcal{L}})_i} \circ \bigotimes\nolimits^n_{i=1} \delta_{[\![P_i]\!]^{\nu}_{\mathcal{L}}}}$$

$$(1\mathrm{R})\ \frac{}{[\![(\emptyset : P_i)^n_{i=1} \dashv\vdash \emptyset : 1_{[u]}]\!]^{\nu}_{\mathcal{L}} := \tau_{\bigotimes^n_{i=1} [\![P_i]\!]^{\nu}_{\mathcal{L}}}}$$

$$(\top\mathrm{L})\ \frac{f = [\![\, \mathcal{F} \mid \gamma : \Gamma \dashv \sigma : \Sigma \vdash \mathscr{N} : N \,]\!]^{\nu}_{\mathcal{L}}}{[\![\, \top\mathrm{L}(\mathcal{F}) \mid \gamma : \Gamma, \{u \to v\} : \top_{[v]} \dashv \sigma : \Sigma \vdash \mathscr{N} : N \,]\!]^{\nu}_{\mathcal{L}} := f \circ \gamma_{[\![\Gamma]\!]^{\nu}_{\mathcal{L}}}} \ (v \in \mathrm{End}^{\mathrm{Ini}}_{N})$$

$$(\top\mathrm{M})\ \frac{f = [\![\, \mathcal{F} \mid \gamma : \Gamma \dashv \sigma : \Sigma \vdash \mathscr{N} : N \,]\!]^{\nu}_{\mathcal{L}}}{[\![\, \top\mathrm{M}(\mathcal{F}) \mid \gamma : \Gamma \dashv \sigma : \Sigma, \{u \to v\} : \top_{[v]} \vdash \mathscr{N} : N \,]\!]^{\nu}_{\mathcal{L}} := f} \ (v \in \mathrm{End}^{\mathrm{Ini}}_{N}) \qquad (\top\mathrm{R})\ \frac{}{[\![\, \top\mathrm{R} \mid \dashv\vdash \emptyset : \top_{[v]} \,]\!]^{\nu}_{\mathcal{L}} := \mathrm{id}_{\top}}$$

$$(\bot\mathrm{L})\ \frac{}{[\![\, \bot\mathrm{L} \mid \{v_0 \mapsto v\} : \bot_{[v]} \dashv\vdash \,]\!]^{\nu}_{\mathcal{L}} := \mathrm{id}_{\bot}} \qquad (\bot\mathrm{R})\ \frac{f = [\![\, \mathcal{F} \mid \gamma : \Gamma \dashv \sigma : \Sigma \vdash \,]\!]^{\nu}_{\mathcal{L}}}{[\![\, \bot\mathrm{R}(\mathcal{F}) \mid \gamma\{v/v_0\} : \Gamma \dashv \sigma\{v/v_0\} : \Sigma \vdash \emptyset : \bot_{[v]} \,]\!]^{\nu}_{\mathcal{L}} := f}$$

$$(\otimes\mathrm{L})\ \frac{f = [\![\, \mathcal{F} \mid \gamma : \Gamma, \mathscr{P} : P, \mathscr{R} : R \dashv \sigma : \Sigma \vdash \mathscr{N} : N \,]\!]^{\nu}_{\mathcal{L}}}{[\![\, \otimes\mathrm{L}(\mathcal{F}) \mid \gamma : \Gamma, \mathscr{P} \cup \mathscr{R} : P \otimes R \dashv \sigma : \Sigma \vdash \mathscr{N} : N \,]\!]^{\nu}_{\mathcal{L}} := f \circ \alpha_{[\![\Gamma]\!]^{\nu}_{\mathcal{L}}, [\![P]\!]^{\nu}_{\mathcal{L}}, [\![R]\!]^{\nu}_{\mathcal{L}}}}$$

$$(\otimes\mathrm{M})\ \frac{f = [\![\, \mathcal{F} \mid \gamma : \Gamma \dashv \sigma : \Sigma, \mathscr{L} : L, \mathscr{M} : M \vdash \mathscr{N} : N \,]\!]^{\nu}_{\mathcal{L}}}{[\![\, \otimes\mathrm{M}(\mathcal{F}) \mid \gamma : \Gamma \dashv \sigma : \Sigma, \mathscr{L} \cup \mathscr{M} : L \otimes M \vdash \mathscr{N} : N \,]\!]^{\nu}_{\mathcal{L}} := f}$$

$$(\otimes\mathrm{R})\ \frac{f = [\![\, \mathcal{F} \mid \gamma : \Gamma \dashv \sigma : \Sigma \vdash \mathscr{N} : N \,]\!]^{\nu}_{\mathcal{L}} \qquad g = [\![\, \mathcal{G} \mid \delta : \Delta \dashv \pi : \Pi \vdash \mathscr{Q} : Q \,]\!]^{\nu}_{\mathcal{L}}}{[\![\, \otimes\mathrm{R}(\mathcal{F}, \mathcal{G}) \mid \gamma : \Gamma, \delta : \Delta \dashv \sigma : \Sigma, \pi : \Pi \vdash \mathscr{N} \cup \mathscr{Q} : N \otimes Q \,]\!]^{\nu}_{\mathcal{L}} := f \otimes g}$$

$$(\times\mathrm{L})\ \frac{f = [\![\, \mathcal{F} \mid \gamma : \Gamma, \mathscr{P} : P_i \dashv \sigma : \Sigma \vdash \mathscr{N} : N \,]\!]^{\nu}_{\mathcal{L}}}{[\![\, \times\mathrm{L}(\mathcal{F}) \mid \gamma : \Gamma, \mathscr{P} : P_1 \times P_2 \dashv \sigma : \Sigma \vdash \mathscr{N} : N \,]\!]^{\nu}_{\mathcal{L}} := f \circ (\mathrm{id}_{[\![\Gamma\,]\!]^{\nu}_{\mathcal{L}}} \otimes \pi^{(i)}_{[\![P_1]\!]^{\nu}_{\mathcal{L}} \times [\![P_2]\!]^{\nu}_{\mathcal{L}}})} \ (i \in \overline{2})$$

$$(\times\mathrm{M})\ \frac{f = [\![\, \mathcal{F} \mid \gamma : \Gamma \dashv \sigma : \Sigma, \mathscr{M} : M_i \vdash \mathscr{N} : N \,]\!]^{\nu}_{\mathcal{L}}}{[\![\, \times\mathrm{M}(\mathcal{F}) \mid \gamma : \Gamma \dashv \sigma : \Sigma, \mathscr{M} : M_1 \times M_2 \vdash \mathscr{N} : N \,]\!]^{\nu}_{\mathcal{L}} := f} \ (i \in \overline{2})$$

$$(\times\mathrm{R})\ \frac{f = [\![\, \mathcal{F} \mid (\mathscr{P}_i : P_i)^n_{i=1} \dashv \sigma : \Sigma \vdash \mathscr{N} : N \,]\!]^{\nu}_{\mathcal{L}} \qquad g = [\![\, \mathcal{G} \mid (\hat{\mathscr{P}}_i : P_i)^n_{i=1} \dashv \hat{\sigma} : \Sigma \vdash \mathscr{Q} : Q \,]\!]^{\nu}_{\mathcal{L}}}{[\![\, \times\mathrm{R}(\mathcal{F}, \mathcal{G}) \mid ((\mathscr{P}_i + \langle x|) \cup (\hat{\mathscr{P}}_i + \langle y|) : P_i)^n_{i=1} \dashv (\sigma + \langle x|) \cup (\hat{\sigma} + \langle y|) : \Sigma \vdash (\mathscr{N} + \langle x|) \cup (\mathscr{Q} + \langle y|) : N_x \times_y Q \,]\!]^{\nu}_{\mathcal{L}} := \langle f, g \rangle}$$

$$(\multimap\mathrm{L})\ \frac{f = [\![\, \mathcal{F} \mid \gamma : \Gamma \dashv \sigma : \Sigma \vdash \mathscr{N} : N \,]\!]^{\nu}_{\mathcal{L}} \qquad g = [\![\, \mathcal{G} \mid \delta : \Delta, \mathscr{P} : P \dashv \pi : \Pi \vdash \mathscr{Q} : Q \,]\!]^{\nu}_{\mathcal{L}}}{\begin{array}{l}[\![\, \multimap\mathrm{L}(\mathcal{F}, \mathcal{G}) \mid \gamma : \Gamma, \delta : \Delta, \mathscr{N} \cup \mathscr{P} : N \multimap P \dashv \sigma : \Sigma, \pi : \Pi \vdash \mathscr{Q} : Q \,]\!]^{\nu}_{\mathcal{L}} \\ \qquad := g \circ (\mathrm{id}_{[\![\Delta]\!]^{\nu}_{\mathcal{L}}} \otimes \mathrm{ev}_{[\![N]\!]^{\nu}_{\mathcal{L}}, [\![P]\!]^{\nu}_{\mathcal{L}}}) \circ \varsigma_{[\![N]\!]^{\nu}_{\mathcal{L}}, [\![\Delta, N \multimap P]\!]^{\nu}_{\mathcal{L}}} \circ (f \otimes \mathrm{id}_{[\![\Delta, N \multimap P]\!]^{\nu}_{\mathcal{L}}})\end{array}}$$

$$(\multimap\mathrm{R})\ \frac{f = [\![\, \mathcal{F} \mid \gamma : \Gamma, \mathscr{P} : P \dashv \sigma : \Sigma \vdash \mathscr{N} : N \,]\!]^{\nu}_{\mathcal{L}}}{[\![\, \multimap\mathrm{R}(\mathcal{F}) \mid \gamma : \Gamma \dashv \sigma : \Sigma \vdash \mathscr{P} \cup \mathscr{N} : P \multimap N \,]\!]^{\nu}_{\mathcal{L}} := \Lambda_{[\![\Gamma]\!]^{\nu}_{\mathcal{L}}, [\![P]\!]^{\nu}_{\mathcal{L}}, [\![N]\!]^{\nu}_{\mathcal{L}}}(f)}$$

FIGURE 28. The interpretation of free generations of pre-$\ell^{\bullet}_{\mathcal{B}}$-terms