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COMPACT METRIC SPACES AS EXPONENTIABLE METRIC COMPACT HAUSDORFF SPACES

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Abstract. The class of compact metric spaces (metric spaces which induce a compact topology), despite being an important class of metric spaces, is not closed under many natural constructions. For instance, infinite metric products of compact metric spaces will often fail to induce a compact topology, while on the other hand, the topological product of any family of compact metric spaces is always a compact topological space, evidencing a discrepancy between topological and metric products.

The notion of metric compact Hausdorff space – compact Hausdorff spaces equipped with a lower semi-continuous metric – provides a class of spaces which keep track of metric data and topological data separately, thereby forming a complete and cocomplete category, while including all compact metric spaces.

Our main contribution is that this inclusion can be characterised via universal property: the compact metric spaces are the exponentiable metric compact Hausdorff spaces with respect to a natural monoidal structure on the respective category.

Introduction

A fruitful theory of (topological) spaces should allow for constructing new spaces out of a family of old ones; it is also often desirable for two given constructions to commute with one other (see also [5]). Prominent examples of this phenomena are the so-called "convenient categories of spaces" [23], that is, categories of spaces which are Cartesian closed (in the sense that they do admit a canonical function space $[X, Y]$ for all spaces X and Y), for this property implies that "products commute with all quotients".

As it is well-known, many important categories of topology are *not* Cartesian closed, motivating the study of those spaces X – called *exponentiable* spaces – which do admit such a function space $[X, Y]$ for all Y; put more formally, those spaces X such that the endofunctor $- \times X$ has a right adjoint $[X, -]$. For instance, the exponentiable topological spaces are precisely the core-compact spaces [9], and the exponentiable metric spaces are the totally quasi-convex spaces [7]. In this context, it might seem rather disappointing that the exponentiable compact Hausdorff spaces are trivial: as shown in [6], a compact Hausdorff space is exponentiable in the category CH of compact Hausdorff spaces and continuous maps if and only if it is finite. The situation does not improve if we move to the category OrdCH of Nachbin's ordered compact Hausdorff spaces with monotone and continuous maps [19]: it is not difficult to prove that in this category the exponentiable objects are essentially the finite ones as well (see Corollary 2.11).

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Surprisingly, we have more luck considering *metric compact Hausdorff spaces* and appropriate morphisms. These spaces constitute a simultaneous generalisation of ordered compact Hausdorff spaces and of compact metric spaces; in a nutshell, a metric compact Hausdorff space is given by a set equipped with a metric and a compact Hausdorff topology (not necessarily induced by the metric) which are compatible in an appropriate sense. We stress that every compact metric space is also a metric compact Hausdorff space, and several arguments usually applied to compact metric space can be used in this context as well (see, for instance, Proposition 1.12). This fact makes us confident that metric compact Hausdorff spaces provide a flexible generalisation of compact metric spaces. Metric compact Hausdorff spaces and the corresponding category were introduced in [12] based on [24]. However, with different motivation, similar combinations of topology and metric were also investigated in [16, 3, 4, 20].

Our main contribution is that the inclusion of compact metric spaces into metric compact Hausdorff spaces can be characterised via universal property: the compact metric spaces are the exponentiable metric compact Hausdorff space with respect to a natural monoidal structure on the category of metric compact Hausdorff spaces and morphisms. We start by characterising those metric spaces which are exponentiable with respect to a suitable monoidal structure on the category of metric spaces. We obtain an easily checkable characterization (Theorem 2.5) which allows us to prove that exponentiability for certain Minkowski tensors is equivalent to total quasi-convexity (Corollary 2.6). Building on this, we obtain our main result (Theorem 2.19): the exponentiable (with respect to a suitable monoidal structure) separated metric compact Hausdorff spaces are precisely the compact metric spaces. Hence, upholding the importance of metric compact Hausdorff spaces, the main result of our paper shows that compact metric spaces arise naturally as an important subclass of all separated metric compact Hausdorff spaces.

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1. Preliminaries

In this paper we use the designation *metric* in the generalised sense of [14]; namely as a distance function

$$d\colon X \times X \longrightarrow [0,\infty]$$

such that $d(x,x) = 0$ and $d(x,y) + d(y,z) \geq d(x,z)$, for all $x, y, z \in X$. A *metric space* is a pair (X,d) consisting of a set X and a metric d on X, but we often write simply X instead of (X,d). A *metric map* $f\colon X \to Y$ between metric spaces consists of a function $f\colon X \to Y$ between the sets X and Y satisfying $d(x_1,x_2) \geq d(f(x_1),f(x_2))$, for all $x_1, x_2 \in X$. The category of metric spaces and metric maps is denoted by Met.

Example 1.1. Of particular importance is the metric space $[0,\infty]$ with the metric μ defined by $\mu(u,v) = \max\{v-u, 0\}$, for all $u, v \in [0,\infty]$.

Since our notion of metric is *asymmetric*; that is, we generally have $d(x,y) \neq d(y,x)$ for some pairs x, y in a metric space X, we have two distinct notions of open balls and closed balls, denoted by

$$B^{\triangleleft}(x,r) = \{y \mid d(y,x) < r\} \qquad B^{\triangleright}(x,r) = \{y \mid d(x,y) < r\}$$
$$\overline{B}^{\triangleleft}(x,r) = \{y \mid d(y,x) \leq r\} \qquad \overline{B}^{\triangleright}(x,r) = \{y \mid d(x,y) \leq r\}$$

which are respectively called the *lower* and *upper* open/closed balls with center $x \in X$ and radius $r \in [0,\infty]$. By definition, the lower and upper open balls form a sub-basis for the *topology on X induced by d*, hence, the collection of all *symmetric open balls*

$$B(x,r) = B^{\triangleleft}(x,r) \cap B^{\triangleright}(x,r) = \{y \mid d(x,y) < r \;\&\; d(y,x) < r\}$$

with center $x \in X$ and radius $r \in [0,\infty]$ form a basis for this topology. As usual, a metric space is called *compact* whenever this topology is compact. We also note that this topology is Hausdorff if and only if X is *separated*, that is,

$$(0 = d(x,y) \;\&\; 0 = d(y,x)) \implies x = y$$

for all $x, y \in X$.

The category Met has a terminal object, namely the one-element space 1, and for metric spaces X, Y, the product $X \times Y$ of X and Y is given by the Cartesian product $X \times Y$ of the underlying sets equipped with the *max-metric* defined by

$$d((x_1,y_1),(x_2,y_2)) = \max\{d(x_1,x_2), d(y_1,y_2)\},$$

for all $x_1, x_2 \in X$ and $y_1, y_2 \in Y$.

A metric space X is *exponentiable in* Met whenever the functor $- \times X \colon \mathsf{Met} \to \mathsf{Met}$ has a right adjoint, which we denote by $(-)^X$. The category Met is not Cartesian closed, and below (Theorem 1.4) we recall the characterisation of exponentiable metric spaces obtained in [7] (see also [8]) via a *quasi-convexity* condition:

Definition 1.2. Let X be a metric space.

(1) We say X is *totally quasi-convex* whenever, for all $x_1, x_2 \in X$ and all $u_1, u_2 \in [0,\infty]$ such that $d(x_1,x_2) = u_1 + u_2$, for all $\varepsilon > 0$, there exists $z \in X$ such that $d(x_1,z) \leq u_1 + \varepsilon$ and $d(z,x_2) \leq u_2 + \varepsilon$.
(2) We say X is *totally convex* if for all $x, y \in X$ and all $u_1.u_2 \in [0,\infty]$ with $d(x,y) = u_1 + u_2$, there exists $z \in X$ such that $d(x,z) \leq u_1$ and $d(z,y) \leq u_2$.

Remark 1.3. Note that, thanks to the triangle inequality, one necessarily has $d(x,z) = u_1$ and $d(z,y) = u_2$ when $u_1 + u_2 < \infty$ in the definition of a totally convex space. Clearly, every totally convex metric space is totally quasi-convex, and one easily verifies the converse for compact metric spaces.[1]

Theorem 1.4 ([7])**.** *The following assertions are equivalent, for a metric space X.*

(i) X is exponentiable in Met.
(ii) For all $x_1, x_2 \in X$ and all $u_1, u_2 \in [0,\infty]$,

$$\inf_{z \in X} \big(\max\{d(x_1,z), u_1\} + \max\{d(z,x_2), u_2\}\big) \leq \max\{d(x_1,x_2), u_1 + u_2\}.$$

(iii) X is totally quasi-convex.

[1] In [7] we stated erroneously that every complete totally quasi-convex metric space is totally convex, a mistake brought to our attention by Carlyle Stewart (University of Stellenbosch, South Africa).

For metric spaces X and Y, another important metric on the Cartesian product $X \times Y$ of the sets X and Y is the *plus-metric* defined by

$$d((x_1, y_1), (x_2, y_2)) \coloneqq d(x_1, x_2) + d(y_1, y_2),$$

for all $x_1, x_2 \in X$ and $y_1, y_2 \in Y$. We denote the resulting metric space by $X \oplus Y$. In contrast to the categorical product, this construction makes Met a closed symmetric monoidal category (see [14]); here, the right adjoint $[X, -]_+$ of $- \oplus X$ is given by

$$[X, Y]_+ = \{f \colon X \to Y \mid f \text{ is metric}\}, \quad d(f, g) = \sup_{x \in X} d(f(x), g(x))$$

for all metric spaces X and Y and metric maps $f, g \colon X \to Y$.

Below we investigate the interaction of exponentiability with two well-known constructions of metric spaces. Here we will use here the following notation: for a metric space X, $A, B \subseteq X$ and $\varepsilon > 0$,

$$C_{A,B,\varepsilon} = \{z \in X \mid \exists x \in A, y \in B \,.\, (d(x, z) \leq u_1 + \varepsilon \;\&\; d(z, y) \leq u_2 + \varepsilon)\}.$$

First, we consider the metric space $\mathcal{H}(X)$ of all subsets of X, with the *asymmetric Hausdorff metric* defined by

$$d(A, B) = \sup_{y \in B} \inf_{x \in A} d(x, y),$$

for all $A, B \subseteq X$ ([21, 11], but see also [15]. We note that, since $d(x, y) = d(\{x\}, \{y\})$ for all $x, y \in X$, the canonical map $X \to \mathcal{H}(X), x \mapsto \{x\}$ is an isometric embedding.

Proposition 1.5. *Let X be a metric space. Then X is totally quasi-convex if and only if $\mathcal{H}(X)$ is totally quasi-convex.*

Proof. Assume first that X is totally quasi-convex. Let $A, B \subseteq X$ and $u_1, u_2 \in [0, \infty]$ with $d(A, B) = u_1 + u_2$. Let $\varepsilon > 0$. One easily verifies that

$$d(A, C_{A,B,\varepsilon}) \leq u_1 + \varepsilon \quad \text{and} \quad d(C_{A,B,\varepsilon}, B) \leq u_2 + \varepsilon.$$

Assume now that $\mathcal{H}(X)$ is totally quasi-convex. Let $x_1, x_2 \in X$ and $u_1, u_2 \in [0, \infty]$ with $d(x_1, x_2) = u_1 + u_2$. Let $\varepsilon > 0$. By hypothesis, there exists a subset $C \subseteq X$ so that

$$u_1 + \varepsilon > u_1 + \frac{\varepsilon}{2} \geq d(\{x_1\}, C) = \sup_{z \in C} d(x_1, z)$$

and

$$u_2 + \varepsilon > u_2 + \frac{\varepsilon}{2} \geq d(C, \{x_2\}) = \inf_{z \in C} d(z, x_2).$$

By the second inequality, there is some $z_0 \in C$ with $d(z_0, x_2) \leq u_2 + \varepsilon$, by the first inequality, we also have $d(x_1, z_0) \leq u_1 + \varepsilon$. □

For a metric space X, the set $\mathcal{U}(X)$ of all ultrafilters on X becomes a metric space with distance

$$d(\mathfrak{x}, \mathfrak{y}) = \sup_{A \in \mathfrak{x}, B \in \mathfrak{y}} \inf_{x \in A, y \in B} d(x, y),$$

for all $\mathfrak{x}, \mathfrak{y} \in \mathcal{U}(X)$ (see [13, Section III.5]). In particular, for all $x, y \in X$, we have $d(\dot{x}, \dot{y}) = d(x, y)$, making the map $X \to \mathcal{U}(X)$, $x \mapsto \dot{x}$ an isometric embedding. We also note that the metric space $\mathcal{U}(X)$ is in general not compact: for instance, if X is discrete, then $\mathcal{U}(X)$ is discrete.

Proposition 1.6. *Let X be a metric space. Then the following assertions are equivalent.*

(i) X is exponentiable, i.e. totally quasi-convex.
(ii) $\mathcal{U}(X)$ is totally convex.
(iii) $\mathcal{U}(X)$ is totally quasi-convex.

Proof. Assume first that X is exponentiable. Let $\mathfrak{x}, \mathfrak{y} \in \mathcal{U}(X)$ with $d(\mathfrak{x}, \mathfrak{y}) = u_1 + u_2$. The collection

$$\{C_{A,B,\varepsilon} \mid A \in \mathfrak{x}, B \in \mathfrak{y}, \varepsilon > 0\}$$

is a filter base on X, which can be extended to an ultrafilter $\mathfrak{z} \in \mathcal{U}(X)$. For each $A \in \mathfrak{x}$, $C \in \mathfrak{z}$ and $\varepsilon > 0$, we have $C \cap C_{A,X,\varepsilon} \neq \varnothing$, hence, there are $x \in A$ and $z \in C$ with $d(x, z) \leq u_1 + \varepsilon$. This proves $d(\mathfrak{x}, \mathfrak{z}) \leq u_1 + \varepsilon$, for each $\varepsilon > 0$, and therefore $d(\mathfrak{x}, \mathfrak{z}) \leq u_1$. A similar argument shows that $d(\mathfrak{z}, \mathfrak{y}) \leq u_2$. Therefore $\mathcal{U}(X)$ is totally convex.

Clearly, if $\mathcal{U}(X)$ is totally convex, then $\mathcal{U}(X)$ is totally quasi-convex.

Assume now that $\mathcal{U}(X)$ is totally quasi-convex and let $x, y \in X$ with $d(x, y) = u_1 + u_2$ and let $\varepsilon > 0$. By hypothesis, there exists $\mathfrak{z} \in \mathcal{U}(X)$ such that $d(\dot{x}, \mathfrak{z}) \leq u_1 + \frac{\varepsilon}{2}$ and $d(\mathfrak{z}, \dot{y}) \leq u_2 + \frac{\varepsilon}{2}$. Therefore, for every $C \in \mathfrak{z}$,

$$\inf_{z \in C} d(x, z) < u_1 + \varepsilon \quad \text{and} \quad \inf_{z \in Z} d(z, y) < u_2 + \varepsilon.$$

Put $D = \{z \in Z \mid d(x, z) \leq u_1 + \varepsilon\}$. The first inequality above implies that $D \in \mathfrak{z}$, and then the second one entails that there exits some $z \in D$ with $d(z, y) \leq u_2 + \varepsilon$. We conclude that X is totally quasi-convex. □

In the remainder of this section we will consider the main objects of interest for this paper: metric spaces equipped with an additional compact Hausdorff topology.

Definition 1.7. A *metric compact Hausdorff space* X consists of a compact Hausdorff space X together with a lower semicontinuous metric

$$d\colon X \times X \longrightarrow [0, \infty],$$

that is, the metric d is continuous with respect to the topology on $[0, \infty]$ generated by the basic closed sets $[0, u]$ ($u \in [0, \infty]$).

Example 1.8. The metric space $[0, \infty]$ of Example 1.1 becomes a metric compact Hausdorff space with the *Euclidean topology* on $[0, \infty]$, that is, the topology generated by the basic closed sets $[0, u]$ and $[u, \infty]$ ($u \in [0, \infty]$). We note that this topology differs from the one induced by the metric μ, which is not compact.

Example 1.9. For each metric space X, the metric space $\mathcal{U}(X)$ becomes a metric compact Hausdorff space with the *Zariski topology* (also called *hull-kernel topology*, see [10]), that is, the topology with basic open sets

$$\{\mathfrak{x} \in \mathcal{U}(X) \mid A \in \mathfrak{x}\} \quad (A \subseteq X).$$

We find it worthwhile to point out that this topology only depends on the underlying set of X and not on its metric.

Example 1.10. Recall from [19] that an *ordered compact Hausdorff space* is a compact Hausdorff space X equipped with an order relation $\leq$ (i.e. a reflexive and transitive relation on X) such that

$$\{(x, x') \mid x \leq x'\} \subseteq X \times X$$

is closed in the product space $X \times X$.

Every ordered compact Hausdorff space can be viewed as a metric compact Hausdorff space via the metric d defined by

$$d(x,y) = \begin{cases} 0 & \text{if } x \leq y, \\ \infty & \text{if } x \not\leq y, \end{cases}$$

for all $x, y \in X$.

Remark 1.11. If X be a metric compact Hausdorff space, then the closed balls are closed but the open balls need not be open (see Proposition 1.14). Furthermore, if X is separated, then the metric topology refines the topology of X (see [3, Lemma 1.4]).

The following result is a slight generalisation of Remark 1.3.

Proposition 1.12. *Totally quasi-convex metric compact Hausdorff spaces are totally convex.*

Proof. Let X be a totally quasi-convex metric compact Hausdorff space, and let $x_1, x_2 \in X$ and $u_1, u_2 \in [0,\infty]$ such that $d(x_1,x_2) = u_1 + u_2$. Hence, for all $\varepsilon > 0$, there exists $z \in X$ such that $d(x_1,z) \leq u_1 + \varepsilon$ and $d(z,x_2) \leq u_2 + \varepsilon$. For every $\varepsilon > 0$, we consider the set

$$\begin{aligned} A_\varepsilon &= \{y \in X \mid d(x_1,y) \leq u_1 + \varepsilon \text{ and } d(y,x_2) \leq u_2 + \varepsilon\} \\ &= d(x_1,-)^{-1}([0,u_1+\varepsilon]) \cap d(-,x_2)^{-1}([0,u_2+\varepsilon]), \end{aligned}$$

since the metric $d\colon X \times X \to [0,\infty]$ is lower semicontinuous, A_ε is closed in X. By hypothesis, $A_\varepsilon \neq \varnothing$ for each $\varepsilon > 0$, and since $A_\varepsilon \subseteq A_\delta$ whenever $\varepsilon \leq \delta$, the collection of these sets is a filter basis. Thus, we may choose an ultrafilter $\mathfrak{x}$ which contains these sets.

Since X is compact Hausdorff, $\mathfrak{x}$ converges to (a unique) $z \in X$. Hence, for every $\varepsilon > 0$, we have that $z \in A_\varepsilon$, that is, $d(x_1,z) \leq u_1 + \varepsilon$ and $d(z,x_2) \leq u_2 + \varepsilon$. Thus, we obtain $d(x_1,z) \leq u_1$ and $d(z,x_2) \leq u_2$. □

Remark 1.13. As it is shown in [12], a metric compact Hausdorff space can be equivalently described as a metric space X, equipped with a compact Hausdorff topology, so that the ultrafilter convergence $\alpha\colon \mathcal{U}(X) \to X$ is a metric map. This description allows for a slightly shorter argument in the proof of Proposition 1.12, based on Proposition 1.6. If $d(x_1,x_2) = u_1 + u_2$, then there is some ultrafilter $\mathfrak{z}$ on X with $d(\dot{x_1},\mathfrak{z}) \leq u_1$ and $d(\mathfrak{z},\dot{x_2}) \leq u_2$. Therefore $d(x_1,\alpha(\mathfrak{z})) \leq u_1$ and $d(\alpha(\mathfrak{z}),x_2) \leq u_2$.

The following result follows from standard arguments in metric spaces.

Proposition 1.14. *Let X be a compact Hausdorff space equipped with a metric $d\colon X \times X \to [0,\infty]$. Then the following assertions are equivalent.*

- *(i) The upper/lower balls of the metric space X are open in the topology of X.*
- *(ii) The symmetric open balls of X are open in the topology of X.*
- *(iii) The map $d\colon X \times X \to [0,\infty]$ is continuous with respect to the Euclidean topology of $[0,\infty]$.*

We denote by MetCH the category of metric compact Hausdorff spaces and metric and continuous maps. Both the max-metric and the plus-metric endow MetCH with a symmetric monoidal structure; however, as we show in the next section,

MetCH not closed for either of these structures. For a metric space X, the function $d\colon X \times X \to [0,\infty]$ is continuous with respect to the topology on X induced by the metric and the Euclidean topology on $[0,\infty]$, therefore every compact separated metric space is a metric compact Hausdorff space when equipped with the topology induced by the metric. We denote the full subcategory of Met formed by all compact separated metric spaces by $\mathsf{CompMet}$, and therefore we have a full embedding

$$\mathsf{CompMet} \longrightarrow \mathsf{MetCH}.$$

From Proposition 1.14 and Remark 1.11 we obtain the following observation.

Lemma 1.15. *A separated metric compact Hausdorff space is compact metric if and only if any (and hence all) of the three assertions of Proposition 1.14 hold.*

We conclude this section with a metric version of Urysohn's Lemma essentially obtained in [16, Theorem 2.4] (see also [4, Theorem 1.6]).

Theorem 1.16. *Let X be a metric compact Hausdorff space and $A, B \subseteq X$ closed subsets. Put $r \coloneqq \inf\{d(x,y) \mid x \in A, y \in B\}$. Then there exists a continuous metric map $f\colon X \to [0,r]$ with $A \subseteq f^{-1}(0)$ and $B \subseteq f^{-1}(r)$.*

Corollary 1.17. *The set $\{[0,r] \mid 0 < r < \infty\}$ and the space $[0,\infty]$ are initially dense in MetCH.*

2. Main results

To treat various monoidal structures on Met and MetCH simultaneously, we consider here an arbitrary ordered commutative monoid structure $\otimes$ on $[0,\infty]$ with neutral element 0. Furthermore, we assume that $\otimes$ satisfies *Mulholland's inequality*[2] [17, 18] (a generalization of Minkowski's inequality (Example 2.2)):

$$(u_1 + v_1) \otimes (u_2 + v_2) \leq (u_1 \otimes u_2) + (v_1 \otimes v_2),$$

for all $u_1, u_2, v_1, v_2 \in [0,\infty]$.

Indeed, this inequality ensures that the function $d\colon (X \times Y) \times (X \times Y) \to [0,\infty]$ defined by

$$d\big((x,y),(x',y')\big) \coloneqq X(x,x') \otimes Y(y,y'),$$

for all $x, x' \in X$ and $y, y' \in Y$, is a metric on the product space $X \times Y$. Naturally, the reflexivity condition holds, and so does the triangle inequality: we have

$$\begin{aligned} d\big((x,y),(x',y')\big) &= X(x,x') \otimes Y(y,y') \\ &\leq \big(X(x,x'') + X(x'',x')\big) \otimes \big(Y(y,y'') + Y(y'',y')\big) \\ &\leq \big(X(x,x'') \otimes Y(y,y'')\big) + \big(X(x'',x') \otimes Y(y'',y')\big) \\ &= d\big((x,y),(x'',y'')\big) + d\big((x'',y''),(x',y')\big) \end{aligned}$$

for all $x, x', x'' \in X$ and $y, y', y'' \in Y$. We denote this metric space by $X \otimes Y$, and we note that this defines a symmetric monoidal structre on the category Met.

Lemma 2.1. *If $\otimes$ is an ordered commutative monoid structure on $[0,\infty]$ satisfying Mulholland's inequality, then:*

- *we have $a + b \geq a \otimes b \geq \max(a,b)$ for all $a, b \in [0,\infty]$,*
- *$\otimes$ defines a metric map $[0,\infty] \otimes [0,\infty] \to [0,\infty]$.*

[2]This condition simply states that $\otimes\colon \mathsf{Met} \times \mathsf{Met} \to \mathsf{Met}$ is a lax monoidal functor w.r.t to the tensor induced by the plus-metric. In [2], this means $+$ *dominates* $\otimes$.

Proof. For the inequalities, simply note that, for all $a, b \in [0, \infty]$,

$$a + b = (a \otimes 0) + (0 \otimes b) \geq (a + 0) \otimes (0 + b) = a \otimes b,$$

and that monotonicity of $\otimes$ guarantees $a \otimes b \geq a \otimes 0 = a$, and $a \otimes b \geq 0 \otimes b = b$.

The second reduces to proving that $\mu(a \otimes c, b \otimes d) \leq \mu(a, b) \otimes \mu(c, d)$[3] for all $a, b, c, d \in [0, \infty]$. This follows by monotonicity and Mulholland's inequality. □

This yields numerous consequences. For one, we have that the identity functions $A \oplus B \to A \otimes B \to A \times B$ are metric maps, and therefore the topology on $A \otimes B$ induced by the metric is the product topology. It also follows that $\otimes\colon [0, \infty] \times [0, \infty] \to [0, \infty]$ is a continuous map, and therefore this tensor product defines a symmetric monoidal structure on MetCH as well.

Additionally, we also find that the map $- \otimes u\colon [0, \infty] \to [0, \infty]$ preserves infima for all $u \in [0, \infty]$. Thus, $- \otimes u$ has a right adjoint with respect to the "greater or equal relation" on $[0, \infty]$, which we denote by $[u, -]_\otimes\colon [0, \infty] \to [0, \infty]$. In other words, Mulholland's inequality ensures $\otimes$ is a commutative quantale on $[0, \infty]$ with order relation $\geq$ and unit 0 (see [22]).

We call a metric (compact Hausdorff) space $\otimes$-*exponentiable* whenever $- \otimes X$ has a right adjoint, denoted by $[X, -]_\otimes$.

Example 2.2. (a) We may consider $\otimes = +$ and $\otimes = \max$.

(b) For $1 < p < \infty$, we may consider $u \otimes_p v = \sqrt[p]{u^p + v^p}$, thanks to *Minkowski's inequality*:

$$\sqrt[p]{(u_1 + v_1)^p + (u_2 + v_2)^p} \leq \sqrt[p]{u_1^p + u_2^p} + \sqrt[p]{v_1^p + v_2^p},$$

which additionally states that the equality holds if and only if $u_1 v_2 = u_2 v_1$.

We point out that $\otimes_1 = +$ and $\lim_{p \to \infty} \otimes_p = \max$.

(c) From [18, Theorem 1], we deduce that if $g\colon [0, \infty] \to [g(0), \infty]$ is a strictly increasing, continuous, convex and *geoconvex*[4] function, then the binary operation defined by $x \otimes y = g^{-1}\big(g(x) + g(y) - g(0)\big)$ is an ordered commutative monoid structure on $[0, \infty]$ satisfying Mulholland's inequality.

We have $g(x) = e^x$ as an example that is not a power function.

(d) [2, Table 2.6] lists several more examples[5], as one may check.

Remark 2.3. By general adjunction rules, there is a bijection $[X, Y]_\otimes \cong \mathsf{Met}(X, Y)$, and the diagrams

$$\begin{array}{ccc} [X,Y]_\otimes \otimes X & & \\ \cong\downarrow & \searrow^{\text{counit}} & \\ \mathsf{Met}(X,Y) \otimes X & \xrightarrow[\text{ev}]{} & Y, \end{array} \qquad \begin{array}{ccc} & & [X, Y \otimes X]_\otimes \\ & \nearrow^{\text{unit}} & \downarrow\cong \\ Y & \xrightarrow[\eta]{} & \mathsf{Met}(X, Y \otimes X) \end{array}$$

commute. Here, the canonical map $\eta\colon Y \to \mathsf{Met}(X, Y \otimes X)$ sends $y \in Y$ to the metric map defined by $x \mapsto (y, x)$. For an $\otimes$-exponentiable metric compact Hausdorff space, the same assertions hold, with $\mathsf{MetCH}(X, Y)$ in place of $\mathsf{Met}(X, Y)$. Henceforth, we take the underlying set of $[X, Y]_\otimes$ to be equal to $\mathsf{Met}(X, Y)$ respectively $\mathsf{MetCH}(X, Y)$, and the unit and the counit to be given by η and ev, respectively.

[3] Category theorists might recognise this as the *mate* of Mulholland's inequality.

[4] A real-valued function g is geoconvex if $g(\sqrt{xy}) \leq \sqrt{g(x)g(y)}$ for all x, y in the domain.

[5] Not all; examples can only be found among the strict copulas.

Below we give a slight generalisation of Theorem 1.4.

Theorem 2.4. *If X be a metric space, then the following are equivalent:*

(i) X is $\otimes$-exponentiable in Met.

(ii) For all $x_1, x_2 \in X$ and all $u_1, u_2 \in [0,\infty]$,

$$\inf_{z\in X}\big((d(x_1,z)\otimes u_1)+(d(z,x_2)\otimes u_2)\big) \leq d(x_1,x_2)\otimes(u_1+u_2).$$

If X is $\otimes$-exponentiable in Met*, then the metric of $[X,Y]_\otimes$ is given by*

$$d(f,g) = \sup_{x,x'\in X}[d(x,x'), d(f(x),g(x'))]_\otimes, . \tag{2.1}$$

Proof. For a $\otimes$-exponentiable metric space X, its counit $\mathrm{ev}\colon [X,Y]_\otimes \otimes X \to Y$ is a metric map, hence

$$d(f,g) \geq \underbrace{\sup_{x,x'\in X}[d(x,x'), d(f(x),g(x'))]_\otimes}_{=:u}.$$

Consider now the metric space $Z=\{0,1\}$ with $d(0,1)=u$ and $d(1,0)=\infty$. Then the map

$$X\otimes Z \longrightarrow Y, \quad (x,0)\longmapsto f(x),\ (x,1)\longmapsto g(x)$$

is metric, hence also the map

$$Z \longrightarrow [X,Y]_\otimes, \quad 0\longmapsto f,\ 1\longmapsto g$$

is metric, and therefore $u \geq d(f,g)$. We conclude that a metric space X is $\otimes$-exponentiable in Met if and only if the distance defined by (2.1) satisfies the triangle inequality, for all metric spaces Y. With the same proof as in [7] one verifies that this is indeed equivalent to the assertion (ii) above. In fact, an easy calculation shows that (ii) guarantees the d satisfies the triangle inequality, for all metric spaces Y.

Assume now that X is $\otimes$-exponentiable. We consider the maps

$$\begin{aligned}
&\varphi_1\colon X\longrightarrow[0,\infty],\ x\longmapsto d(x_1,x),\\
&\varphi_2\colon X\longrightarrow[0,\infty],\ x\longmapsto d(x_1,x)\otimes u_1,\\
&\varphi_3\colon X\longrightarrow[0,\infty],\ x\longmapsto \inf_{z\in X}\big((d(x_1,z)\otimes u_1)+(d(z,x)\otimes u_2)\big).
\end{aligned}$$

for each $x_1,x_2\in X$ and $u_1,u_2\in[0,\infty]$. Then $\varphi_1,\varphi_2,\varphi_3$ are in fact metric maps since, for all $x\in X$, all $u\in[0,\infty]$ and all sets I, the maps

$$d(x,-)\colon X\longrightarrow[0,\infty], \quad -\otimes u\colon [0,\infty]\longrightarrow[0,\infty], \quad \inf\colon [0,\infty]^I\longrightarrow[0,\infty]$$

are metric. For all metric maps $\varphi,\varphi'\colon X\to[0,\infty]$ and $u\in[0,\infty]$, we have $u\geq d(\varphi,\varphi')$ if and only if, for all $x,x'\in X$, $u\otimes d(x,x')\geq \mu(\varphi(x),\varphi'(x'))$, which in turn is equivalent to $(u\otimes d(x,x))+\varphi(x)\geq\varphi'(x')$, for all $x,x'\in X$. From that we conclude that $u_1\geq d(\varphi_1,\varphi_2)$ and $u_2\geq d(\varphi_2,\varphi_3)$. Since d satisfies the triangle inequality, $u_1+u_2\geq d(\varphi_1,\varphi_3)$. Therefore

$$\begin{aligned}
(u_1+u_2)\otimes d(x_1,x_2) &= ((u_1+u_2)\otimes d(x_1,x_2))+\varphi_1(x_1)\\
&\geq \varphi_3(x_2)=\inf_{z\in X}\big((d(x_1,z)\otimes u_1)+(d(z,x_2)\otimes u_2)\big).
\end{aligned}$$

□

We remark that it is even possible to characterise $\otimes$-exponentiability for a space X in terms of a more useful metric condition on X. Intuitively, X must simultaneously "almost" satisfy the converses of the triangle inequality and Mulholland's inequality.

To make this intuition precise, we fix the following notation for a subset $S \subseteq X$:

$$B_\varepsilon(S) = \{x \in X \mid \exists s \in S \ : d(x,s) < \varepsilon \text{ and } d(s,x) < \varepsilon\},$$
$$\overline{B}_\varepsilon(S) = \{x \in X \mid \exists s \in S \ : d(x,s) \leq \varepsilon \text{ and } d(s,x) \leq \varepsilon\}.$$

Additionally, we consider the continuous function

$$\Omega_{d,u_1,u_2} \colon [0,d] \longrightarrow [0,\infty],\ r \longmapsto r \otimes u_1 + (d-r) \otimes u_2$$

and note that we have $\Omega(r) \geq d \otimes (u_1 + u_2)$ for all r, by Mulholland's inequality. With this in mind, we define

$$M_{d,u_1,u_2} = \Omega^{-1}\big(d \otimes (u_1+u_2)\big) = \{r \in [0,d] \mid r \otimes u_1 + (d-r) \otimes u_2 = d \otimes (u_1+u_2)\}.$$

Note that $M_{0,u_1,u_2} = \{0\}$, $M_{\infty,u_1,u_2} = [0,\infty]$, $M_{d,0,0} = [0,d]$, and $M_{d,\infty,u_2} = M_{d,u_1,\infty} = [0,d]$. These are the cases $d = 0, d = \infty$, $u_1 + u_2 = 0$ and $u_1 + u_2 = \infty$, respectively. We call these the *boundary conditions*.

Theorem 2.5. *Assume that $a \otimes x < b \otimes x$ whenever $a < b$ and $x < \infty$.*[6] *The following are equivalent for a metric space X:*

(i) X is $\otimes$-exponentiable.
(ii) for all $x, y \in X$, all $u_1, u_2 \in [0,\infty]$ and all $\varepsilon > 0$, there exists $z \in X$ with $d(x,z) + d(z,y) \leq d(x,y) + \varepsilon$ and $d(x,z) \in \overline{B}_\varepsilon(M_{d(x,y),u_1,u_2})$.

Proof. We use Theorem 2.4. To prove (i) implies (ii), let $x, y \in X$, let $u_1, u_2 \in [0,\infty]$, and let $\varepsilon > 0$. Under the boundary conditions, there is nothing to show; hence, we assume $0 < d(x,y) < \infty$, and $0 < u_1 + u_2 < \infty$.

We write $M = M_{d(x,y),u_1,u_2}$; if $B_\varepsilon(M) = [0, d(x,y)]$, there is nothing to check; else, we take

$$\eta_1 = \min \Omega|_{[0,d(x,y)]\setminus B_\varepsilon(M)}, \qquad \eta_2 = \big(d(x,y) + \varepsilon\big) \otimes (u_1 + u_2),$$

highlighting that if $\Omega(a) \leq \eta_1$, then $a \in \overline{B}_\varepsilon(M)$.

Taking $\omega = \min(\eta_1, \eta_2) - d(x,y) \otimes (u_1 + u_2) > 0$, by $\otimes$-exponentiability we find $z \in X$ with

$$d(x,z) \otimes u_1 + d(z,y) \otimes u_2 \leq d(x,y) \otimes (u_1 + u_2) + \omega = \min(\eta_1, \eta_2).$$

We have $d(x,y) - d(x,z) \leq d(z,y)$, from which we deduce that $\Omega(d(x,z)) \leq \eta_1$, and hence $d(x,z) \in \overline{B}_\varepsilon(M)$. Moreover, we also have

$$\big(d(x,z) + d(z,y)\big) \otimes (u_1 + u_2) \leq d(x,z) \otimes u_1 + d(z,y) \otimes u_2 \leq \eta_2,$$

from which we deduce that $d(x,z) + d(z,y) \leq d(x,y) + \varepsilon$.

To show (ii) implies (i), let $x, y \in X$, $u_1, u_2 \in [0,\infty]$, and let $\eta > 0$. With nothing to prove under the boundary conditions, we assume $0 < d(x,y) < \infty$, and $0 < u_1 + u_2 < \infty$.

Let $\varepsilon_1 = \frac{\eta}{2} > 0$. Since Ω is a continuous function defined on a compact interval, there exists $\varepsilon_2 > 0$ such that if $r \in \overline{B}_{\varepsilon_2}(M)$, then $\Omega(r) < d(x,y) \otimes (u_1 + u_2) + \varepsilon_1$. Indeed, for all $x \in M$, there exists $\delta > 0$ such that for all y with $|x - y| < \delta$, we have $\Omega(y) - \Omega(x) < \varepsilon_1$. Now, by compactness, we find finitely many δ; take ε_2 to be the smallest.

So, with $\varepsilon = \min(\varepsilon_1, \varepsilon_2) > 0$, via (ii) we find $z \in X$ such that $d(x,z) \in \overline{B}_{\varepsilon_2}(M)$ and $d(x,z) + d(z,y) \leq d(x,y) + \varepsilon_1$.

[6] Such binary operations are called *strict* in [2].

Thus, we have that

$$\begin{aligned} d(x,z)\otimes u_1 + d(z,y)\otimes u_2 &\leq d(x,z)\otimes u_1 + \big(d(x,y) - d(x,z) + \tfrac{\eta}{2}\big)\otimes u_2 \\ &\leq \Omega\big(d(x,z)\big) + \tfrac{\eta}{2} \leq d(x,y)\otimes(u_1+u_2) + \eta, \end{aligned}$$

as desired. □

As a corollary, we characterise $\otimes_p$-exponentiability:

Corollary 2.6. *If $p > 1$, a metric space X is $\otimes_p$-exponentiable if and only if X is totally quasi-convex.*

Proof. As noted in Example 2.2(b), we have $M_{d,u_1,u_2} = \{d\cdot\frac{u_1}{u_1+u_2}\}$ for $0 < u_1+u_2 < \infty$ and $0 < d < \infty$, by Minkowski's inequality. One then immediately checks that Theorem 2.5(ii) is equivalent to total quasi-convexity. □

Corollary 2.7. *Under the hypothesis of Theorem 2.5, if $M_{d,u_1,u_2} = \varnothing$ outside the boundary conditions, then a metric space is $\otimes$-exponentiable if and only if it is discrete.*

Remark 2.8. We conjecture that $M_{d,u_1,u_2} = \varnothing$ outside the boundary conditions for all tensors not of the form $\otimes_p$.

In the remainder of this section we analyse $\otimes$-exponentiability for metric compact Hausdorff spaces; we start by providing a necessary condition.

Proposition 2.9. *Let X be a metric compact Hausdorff space. Then (1) $\Longrightarrow$ (2) $\Longrightarrow$ (3) below.*

(1) X is $\otimes$-exponentiable on MetCH.
(2) For every metric compact Hausdorff space Y, $\mathsf{MetCH}(X,Y)$ is a closed subspace of the product space $\prod_{x\in X} Y$.
(3) The lower/upper open balls are open in the topology of X.

Proof. Assume first (1). For every $x \in X$, $\mathrm{ev}_x \colon [X,Y]_\otimes \to Y$ is an arrow in MetCH, in particular it is continuous, hence we obtain a canonical injective and continuous map $\mathsf{MetCH}(X,Y) \to \prod_{x\in X} Y$. By compactness, this map is a closed embedding.

Assume now (2). Hence, for each $x \in X$, the map $\pi_x \colon \mathsf{MetCH}(X,[0,\infty]) \to [0,\infty]$, $f \mapsto f(x)$ is continuous. For every $\varepsilon < \varepsilon_0$ and $U \subseteq X$ open with $B^\triangleright(x_0,\varepsilon_0) \subseteq U$ we put

$$\mathcal{C}_{U,\varepsilon} \coloneqq \{h \colon X \to [0,\infty] \text{ in } \mathsf{MetCH} \mid h(x_0) = 0 \text{ and } h(z) \geq \varepsilon \text{ for all } z \in U^{\complement}\}.$$

By Theorem 1.16, $\mathcal{C}_{U,\varepsilon} \neq \varnothing$; moreover,

$$\mathcal{C}_{U,\varepsilon} \cap \mathcal{C}_{V,\delta} \supseteq \mathcal{C}_{U\cap V,\max\{\varepsilon,\delta\}}.$$

Therefore there exists an ultrafilter $\mathfrak{x}$ on $\mathsf{MetCH}(X,[0,\infty])$ such that $\mathcal{C}_{U,\varepsilon} \in \mathfrak{x}$ for all $\varepsilon < \varepsilon_0$ and $U \subseteq X$ open with $B^\triangleright(x_0,\varepsilon_0) \subseteq U$. By compactness, $\mathfrak{x}$ converges to some $f \in \mathsf{MetCH}(X,[0,\infty])$. We have the following facts.

(1) $f(x_0) = 0$. In fact, for every $\mathcal{A} \in \mathfrak{x}$, $\mathcal{A} \cap \mathcal{C}_{X,\frac{\varepsilon_0}{2}} \neq \varnothing$, that is, there is some $k \in \mathcal{A}$ with $k(x_0) = 0$. Therefore $\pi_{x_0}(\mathfrak{x})$ is the principal ultrafilter generated by 0. Since $\pi_{x_0}(\mathfrak{x}) \to f(x_0)$, we conclude that $f(x_0) = 0$.
(2) $f(x) < \varepsilon_0$ for every $x \in B^\triangleright(x_0,\varepsilon_0)$. In fact, since f is a metric map,

$$f(x) = f(x) - f(x_0) \leq X(x_0,x) < \varepsilon_0.$$

(3) $f(y) \geq \varepsilon_0$ for every $y \notin B^{\triangleright}(x_0, \varepsilon_0)$. To see this, note first that, by Theorem 1.16, for every $\varepsilon < \varepsilon_0$ there exists an arrow $h_\varepsilon \colon X \to [0,\infty]$ in MetCH with $h_\varepsilon(x_0) = 0$ and $h_\varepsilon(y) \geq \varepsilon$. Put

$$U = \bigcup_{\varepsilon < \varepsilon_0} h_\varepsilon^{-1}([0,\varepsilon[).$$

Then $U \subseteq X$ is open, $B^{\triangleright}(x_0,\varepsilon_0) \subseteq U$ and $y \notin U$.

Let now $\varepsilon < \varepsilon_0$. For every $\mathcal{A} \in \mathfrak{x}$, $\mathcal{A} \cap \mathcal{C}_{U,\varepsilon} \neq \varnothing$, that is, there is some $k \in \mathcal{A}$ with $k(y) \geq \varepsilon$. Therefore $[\varepsilon,\infty] \in \pi_y(\mathfrak{x})$, since $\pi_y(\mathfrak{x}) \to f(y)$, we conclude that $f(y) \geq \varepsilon$. Consequently, $f(y) \geq \varepsilon_0$.

Putting everything together, we have shown that $B^{\triangleright}(x_0,\varepsilon_0) = f^{-1}([0,\varepsilon_0[)$ is open in X. A similar argument shows that $B^{\triangleleft}(x_0,\varepsilon_0)$ is open in X. □

Using Lemma 1.15 and Proposition 2.9, we deduce immediately the following result.

Theorem 2.10. *Let X be a $\otimes$-exponentiable separated metric compact Hausdorff space. Then X is a compact metric space.*

Using Example 1.10, we conclude the following fact for ordered compact Hausdorff spaces.

Corollary 2.11. *Let X be an exponentiable ordered compact Hausdorff space. Then, for every $x \in X$, the sets $\downarrow x = \{y \in X \mid y \leq x\}$ and $\uparrow x = \{y \in X \mid x \leq y\}$ are open in X. In particular, if X is antisymmetric, then X is finite.*

Now that we have a better understanding of the topological structure of $\otimes$-exponentiable (separated) metric compact Hausdorff spaces, we expose their metric structure.

Proposition 2.12. *A $\otimes$-exponentiable metric compact Hausdorff space is also $\otimes$-exponentiable in* Met.

Proof. The underlying set of $[X,Y]_\otimes$ is necessarily given by $\mathsf{MetCH}(X,Y)$, and the same argument as in the proof of Theorem 2.4 shows that the distance of $[X,Y]_\otimes$ is given by (2.1), using that the space Z is finite. In fact, as in the proof of Theorem 2.4, we show that an exponentiable metric compact space satisfies Condition (ii) of Theorem 2.4. First observe that, since X is $\otimes$-exponentiable, the metric $d \colon X \times X \to [0,\infty]$ is even continuous (see Propositions 2.9 and 1.14). Since $\otimes \colon [0,\infty] \times [0,\infty] \to [0,\infty]$ is continuous, we obtain immediately that the metric functions $\varphi_1, \varphi_2 \colon X \to [0,\infty]$ defined in the proof of Theorem 2.4 are continuous. Regarding $\varphi_3 \colon X \to [0,\infty]$, observe first that the map

$$\psi \colon X \times X \longrightarrow [0,\infty],\ (z,x) \longmapsto (d(x_1,z) \otimes u_1) + (d(z,x) \otimes u_2)$$

is continuous. With $[0,\infty]^X$ denoting the exponential in the category of topological spaces and continuous maps (which can be taken since X is compact Hausdorff), its exponential mate

$$\overline{\psi} \colon X \longrightarrow [0,\infty]^X$$

is also continuous. Furthermore, $\inf \colon [0,\infty]^X \to [0,\infty]$ is continuous since X is compact (see the proof of [1, Proposition 4.1]), therefore $\varphi_3 = \inf \cdot \overline{\psi}$ is continuous. As in the proof of Theorem 2.4, we conclude that

$$(u_1 + u_2) \otimes d(x_1,x_2) \geq \inf_{z \in X} \big((d(x_1,z) \otimes u_1) + (d(z,x_2) \otimes u_2)\big)$$

for all $x_1, x_2 \in X$ and all $u_1, u_2 \in [0,\infty]$. □

We will now also see that the assertion (ii) of Theorem 2.4 implies $\otimes$-exponentiabilty in MetCH, for compact separated metric spaces.

Lemma 2.13. *Let X be a metric space and Y be a metric compact Hausdorff space. Then the subset $\mathsf{Met}(X,Y) \subseteq \prod_{x\in X} Y$ is closed in the product space $\prod_{x\in X} Y$.*

Proof. Let $f \in Y^{|X|}$ with $f \notin \mathsf{Met}(X,Y)$. Then there exist $x_1, x_2 \in X$ with $d(x_1,x_2) < d(f(x_1),f(x_2))$. Put $\varepsilon = d(x_1,x_2)$, then $d(f(x_1),f(x_2)) = \varepsilon + \eta$ with $\eta > 0$. By Theorem 1.16, there exists $h\colon X \to [0,\infty]$ in MetCH with $h(f(x_1)) = 0$ and $h(f(x_2)) = \varepsilon + \eta$. Put $U = h^{-1}([0, \frac{\eta}{2}[)$ and $V = h^{-1}(]\varepsilon + \frac{\eta}{2}, \infty])$, and note that U and V are open subsets of X with $f(x_1) \in U$ and $f(x_2) \in V$. Moreover, for all $y_1 \in U$ and $y_2 \in V$, $d(y_1,y_2) \geq h(y_2) - h(y_1) > \varepsilon$. Therefore $W = \pi_{x_1}^{-1}(U) \cap \pi_{x_2}^{-1}(V)$ is an open neighbourhood of f with $W \cap \mathsf{Met}(X,Y) = \varnothing$. We conclude that $\mathsf{Met}(X,Y)$ is closed in $Y^{|X|}$. □

Corollary 2.14. *For every metric space X and every metric compact Hausdorff space Y, we have that $\mathsf{Met}(X,Y)$ is a metric compact Hausdorff space with the structure inherited from $\prod_{x\in X} Y$.*

Lemma 2.15. *If X is a compact separated metric space and Y is a metric compact Hausdorff space, then the canonical map $\eta\colon Y \to \mathsf{Met}(X, Y \otimes X)$ is continuous.*

Lemma 2.16. *Let X and Y be compact separated metric spaces. Then the map $\mathrm{ev}\colon \mathsf{Met}(X,Y) \otimes X \to Y$ is continuous.*

Proof. Let $f\colon X \to Y$ be a metric map, $x \in X$ and $V \subseteq Y$ an open neighbourhood of $f(x)$. Since Y is a compact metric space, there is some $\varepsilon > 0$ so that $B(f(x), 2\varepsilon) \subseteq V$. Then $W \coloneqq \pi_x^{-1}(B(f(x),\varepsilon)) \times B(x,\varepsilon)$ is an open neighbourhood of (f,x) and, for all $(h,z) \in W$,

$$d(h(z), f(x)) \leq d(h(z),h(x)) + d(h(x),f(x)) \leq 2\varepsilon$$

and $d(f(x),h(z)) \leq 2\varepsilon$. Therefore $\mathrm{ev}(W) \subseteq V$. □

With the help of Corollary 1.17, we extend Lemma 2.16 to all metric compact Hausdorff spaces Y.

Lemma 2.17. *Let X be a compact separated metric space and Y be a metric compact Hausdorff space. Then the map $\mathrm{ev}\colon \mathsf{Met}(X,Y) \otimes X \to Y$ is continuous.*

Proof. Let $\varphi\colon Y \to [0,r]$ be a continuous and metric map. First observe that map

$$\mathsf{Met}(X,\varphi) \otimes 1_X\colon \mathsf{Met}(X,Y) \otimes X \longrightarrow \mathsf{Met}(X,[0,r]) \otimes X, \quad (f,x) \longmapsto (\varphi f, x)$$

is continuous (and metric). Secondly, the diagram

$$\begin{array}{ccc} \mathsf{Met}(X,Y) \otimes X & \xrightarrow{\mathrm{ev}} & Y \\ {\scriptstyle \mathsf{Met}(X,\varphi)\otimes 1_X}\downarrow & & \downarrow{\scriptstyle \varphi} \\ \mathsf{Met}(X,[0,r]) \otimes X & \xrightarrow[\mathrm{ev}]{} & [0,r] \end{array}$$

commutes. Since $[0,r]$ is a compact separated metric space, $\mathrm{ev}\colon \mathsf{Met}(X,[0,r]) \otimes X \to [0,r]$ is continuous (Lemma 2.16) and therefore also $\mathrm{ev} \cdot (\mathsf{Met}(X,\varphi) \otimes 1_X)$. Since the cone of all metric and continuous maps $Y \to [0,r]$ separates the points of Y (Corollary 1.17), $\mathrm{ev}\colon \mathsf{Met}(X,Y) \otimes X \to Y$ is also continuous. □

Lemma 2.18. *If X is a compact metric space, $\otimes$-exponentiable in* Met*, and Y is a metric compact Hausdorff space, then $[X,Y]_\otimes$ is a metric compact Hausdorff space.*

Proof. First note that, for all $x, x' \in X$, the composite

$$[X,Y]_\otimes \times [X,Y]_\otimes \xrightarrow{\mathrm{ev}_x \times \mathrm{ev}_{x'}} Y \times Y \xrightarrow{d} [0,\infty] \xrightarrow{[X(x,x'),-]_\otimes} [0,\infty]$$

is lower semicontinuous, and hence

$$[X,Y]_\otimes \times [X,Y]_\otimes \longrightarrow \prod_{x,x' \in X} [0,\infty] \xrightarrow[\sup]{} [0,\infty]$$

is lower semicontinuous as well. □

Putting everything together:

Theorem 2.19. *We consider a continuous commutative quantale structure $\otimes$ on $[0,\infty]$ with neutral element 0 satisfying Mulholland's inequality, that is,*

$$(u_1 + u_2) \otimes (v_1 + v_2) \leq (u_1 \otimes v_1) + (u_2 \otimes v_2), \quad \text{for all } u_1, u_2, v_1, v_2 \in [0,\infty].$$

Then, a separated metric compact Hausdorff space is $\otimes$-exponentiable in MetCH *if and only if it is compact metric and $\otimes$-exponentiable in* Met*.*

Corollary 2.20. *In particular, a separated metric compact Hausdorff space is*

- *$+$-exponentiable in* MetCH *if and only if it is compact metric.*
- *$\otimes_p$-exponentiable in* MetCH *if and only if it is compact metric and totally convex, for $1 < p \leq \infty$.*

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