A central limit theorem for associated variables

Paulo Eduardo Oliveira*
Univ. Coimbra, Dep Matemática
Apartado 3008, 3000 Coimbra, Portugal

Abstract

Using a coupling technique we prove a Central Limit Theorem for associated random variables supposing only the existence of moments of second order, and assumptions that imply some sort of weak stationarity. Supposing the existence of absolute moments of order 3 and without any stationarity condition, we derive a convergence rate, based on a convenient version of the classical Berry-Esséen inequality.

1 Introduction

The rate of convergence on the Central Limit Theorem as been a subject of wide interest and for which there exists an extensive literature. An account of results and references may be found in Hall [6] or Rachev [8]. Many of the results obtained refer to the uniform distance and use the Berry-Esséen inequality as a tool to derive the appropriate bounds. A modification of the classical procedure was proposed by Wallace [13] by considering formal expansions of the characteristic functions with respect to some reference distribution function, typically some gaussian distribution function. This procedure, called in the literature as Edgeworth expansion, provide bounds that are asymptotically better than those attainable via the classical Berry-Eséen inequality, although requiring the existence of moments of order 4 or 5, depending on the way the errors are controlled. For independent identically distributed random variables it is known that the best attainable rate based on the Berry-Esséen inequality is of order $n^{-1/2}$, when only requiring the existence of third order absolute moments. If there exists moments of order bigger than 3, then this convergence rate may be improved. Some results on the convergence rate where also obtained using distances other than the uniform. A discussion about some distances and their relations may be found in Maejima, Rachev [4]. Dropping the identically distributed assumption, it is still possible to retain essentially the same convergence rate although the normalization required becomes different. Our interest is to look at this sort of results replacing the independence by an association assumption. Results for this dependence structure were obtained by Wood [14] with respect to the uniform distance keeping a stationarity assumption, and by Suquet [11] with respect to an weighted L^2 distance which metrises convergence in distribution. We will restrict ourselves to bounds based on the Berry-Esséen inequality as these still provide, for reasonable sized samples, better bounds than those obtained via Edgeworth expansions. As mentioned before, these produce rates which are asymptotically better, as the error becomes of order $n^{-3/2}$, but the constants involved seem to be so large, that Edgeworth expansion based inequalities only become better than their Berry-Esséen

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counterparts for very large samples. Results by Seoh, Hallin [9] show that for the Wilcoxon signed rank statistic the sample size should be at least 128 098 in order to improve the Berry-Esséen bounds. Besides, once the result for independent variables is established, the corresponding one for associated variables is derived easily reproducing exactly the same arguments as in the proof of our Theorem 3.5 below. The bounds for independent variables being classical, we include here a version of the Berry-Esséen Theorem for sake of completeness.

We shall now define our framework and make more precise some of the mentioned results. Let $X_1, \ldots, X_n$ be centered random variables, $S_n = X_1 + \cdots + X_n$ and $s_n^2 = \mathrm{E} S_n^2$. If the variables are independent and identically distributed with finite third order moments, then the classical Berry-Esséen Theorem states that

$$\sup_x |F_n^*(x) - N_1(x)| \le c \frac{\beta}{\sigma^3 \sqrt{n}} \tag{1}$$

where F_n^* is the distribution function of $s_n^{-1} S_n$, N_a the distribution function of a gaussian variable with mean 0 and variance a, $\beta = \mathrm{E}|X_1|^3$ and $\sigma^2 = \mathrm{E} X_1^2$. The constant c, as proved by Shiganov [10], may be taken less or equal than 0.7915. If the variables are not identically distributed the upper bound in (1) my be replaced by $c s_n^{-3} \sum_{j=1}^n \beta_j$, where $\beta_j = \mathrm{E}|X_j|^3$ (see, for example, Galambos [3]). The main tool for proving these upper bounds is the following inequality proved independently by Berry [1] and Esséen [2].

Lemma 1.1 *Let F_1 and F_2 be distribution functions with characteristic functions φ_1 and φ_2, respectively, and assume that F_2 is differentiable with $f_2 = F_2'$. Then, for every $U > 0$,*

$$\sup |F_1(x) - F_2(x)| \le \frac{1}{\pi} \int_{-U}^{U} \left| \frac{\varphi_1(t) - \varphi_2(t)}{t} \right| dt + \frac{24B}{\pi U} \tag{2}$$

where $B = \sup |f_2(x)|$.

This will be used here to derive a convenient version of the Berry-Esséen Theorem, referring to a different normalization of the sum S_n. In what follows F_n denotes the distribution function of $n^{-1/2} S_n$ and we will suppose that there exists a constant $c_0 > 0$ such that

$$\inf_n \frac{s_n^2}{n} \ge c_0. \tag{3}$$

2 A convergence rate for independent variables

Following the proof of Theorem 10 in Galambos [3] we derive a Berry-Esséen result, which will be used later when studying the convergence rate for associated random variables. We will denote by φ_Z the characteristic function of the random variable Z.

Theorem 2.1 *Let $X_1, \ldots, X_n$ be independent centered random variables with finite third order absolute moments $\beta_j = \mathrm{E}|X_j|^3$. If (3) holds then*

$$\sup_x \left| F_n(x) - N_{n^{-1} s_n^2}(x) \right| \le \frac{24 \sum_j \beta_j}{\pi s_n^2 n^{1/2}} + \frac{96 \sum_j \beta_j}{c_0 \pi \sqrt{2\pi} s_n n}. \tag{4}$$

Proof : The proof will consist on deriving convenient bounds for $\left|\varphi_{S_n}(tn^{-1/2}) - \exp(-\frac{t^2 s_n^2}{2n})\right|$ and then apply (2). Suppose first that

$$\frac{n^{1/2}}{2(\sum_j \beta_j)^{1/3}} \le |t| \le \frac{c_0 n^{3/2}}{4\sum_j \beta_j}.$$

As $\left|\varphi_{X_j}\right|^2$ is the characteristic function of $X_j - Y_j$ where Y_j has the same distribution as X_j and is independent of X_j, a Taylor expansion gives, for some $\theta \in (-1,1)$,

$$\left|\varphi_{X_j}(\frac{t}{\sqrt{n}})\right|^2 = 1 - \frac{\sigma_j^2 t^2}{n} + \theta \frac{t^3 \mathrm{E}(X_j - Y_j)^3}{6n^{3/2}} \le 1 - \frac{\sigma_j^2 t^2}{n} + \theta \frac{4\,|t|^3\,\beta_j}{3n^{3/2}} \le$$

$$\le \exp\left(-\frac{\sigma_j^2 t^2}{n} + \frac{4\,|t|^3\,\beta_j}{3n^{3/2}}\right)$$

as $\mathrm{E}(X_j - Y_j) = 0$, $\mathrm{Var}(X_j - Y_j) = 2\sigma_j^2 = 2\mathrm{E}X_j^2$ and $\mathrm{E}\,|X_j - Y_j|^3 \le \mathrm{E}(|X_j| + |Y_j|)^3 \le 2\mathrm{E}\,|X_j|^3 + 6\mathrm{E}\,|X_j|\,\mathrm{E}Y_j^2 \le 8\beta_j$. Now

$$\left|\varphi_{S_n}(\frac{t}{\sqrt{n}})\right|^2 = \prod_{j=1}^{n} \left|\varphi_{X_j}(\frac{t}{\sqrt{n}})\right|^2 \le \exp\left(-\frac{t^2}{n}\sum_j \sigma_j^2 + \frac{4\,|t|^3}{3n^{3/2}}\sum_j \beta_j\right) \le$$

$$\le \exp\left(-\frac{t^2 s_n^2}{n} + \frac{t^2 s_n^2}{3n}\right) = \exp\left(-\frac{2t^2 s_n^2}{3n}\right)$$

taking account of the upper bound for t and (3). Finally, for t in the stated interval,

$$\left|\varphi_{S_n}(\frac{t}{\sqrt{n}}) - \exp(-\frac{t^2 s_n^2}{2n})\right| \le \exp\left(-\frac{2t^2 s_n^2}{3n}\right) + \exp\left(-\frac{t^2 s_n^2}{2n}\right) \le 2\exp\left(-\frac{t^2 s_n^2}{3n}\right).$$

Suppose now that $|t| \le \frac{n^{1/2}}{2(\sum_j \beta_j)^{1/3}}$. From the Taylor expansion

$$\varphi_{X_j}(\frac{t}{\sqrt{n}}) - 1 = -\frac{\sigma_j^2 t^2}{2n} + \theta \frac{t^3 \beta_j}{6n^{3/2}},$$

for some $\theta \in (-1,1)$, it follows

$$\left|\varphi_{X_j}(\frac{t}{\sqrt{n}}) - 1\right| \le \frac{\sigma_j^2 t^2}{2n} + \frac{|t|^3\,\beta_j}{6n^{3/2}} \le \frac{\sigma_j^2}{8(\sum_j \beta_j)^{2/3}} + \frac{\beta_j}{48\sum_j \beta_j} \le 7/48,$$

using Hölder's inequality and the fact that the β_j are non negative. It follows that in the interval $|t| \le \frac{n^{1/2}}{2(\sum_j \beta_j)^{1/3}}$ the characteristic function $\varphi_{X_j}(tn^{-1/2})$ is bounded away from zero. On the other hand

$$\left|\varphi_{X_j}(\frac{t}{\sqrt{n}}) - 1\right|^2 \le 2\frac{\sigma_j^4 t^4}{4n^2} + 2\frac{t^6 \beta_j^2}{36n^3} \le |t|^3\,\beta_j \left(\frac{\sigma_j^4}{4n^{3/2}\beta_j(\sum_j \beta_j)^{1/3}} + \frac{\beta_j}{144n^{3/2}\sum_j \beta_j}\right) \le$$

$$\le |t|^3\,\beta_j \frac{37}{144n^{3/2}},$$

from which follows that

$$\log\,\varphi_{X_j}(\frac{t}{\sqrt{n}}) = -\frac{\sigma_j^2 t^2}{2n} + \theta\frac{|t|^3\beta_j}{6n^{3/2}} + \gamma\frac{37\,|t|^3\beta_j}{144n^{3/2}} = -\frac{\sigma_j^2 t^2}{2n} + \eta_j\frac{|t|^3\beta_j}{2n^{3/2}}$$

where $\gamma\in(-1,1)$ and $\eta_j \le \frac{\theta}{3}+\frac{37\gamma}{72}\in(-1,1)$. Thus we find the expansion

$$\log\,\varphi_{S_n}(\frac{t}{\sqrt{n}}) = \sum_{j=1}^{n}\log\,\varphi_{X_j}(\frac{t}{\sqrt{n}}) = -\frac{t^2}{2n}\sum_j\sigma_j^2 + \eta\frac{|t|^3}{2n^{3/2}}\sum_j\beta_j,$$

from what follows, remembering that $s_n^2=\sum_j\sigma_j^2$,

$$\left|\varphi_{S_n}(\frac{t}{\sqrt{n}}) - \exp(-\frac{t^2 s_n^2}{2n})\right| \le \exp(-\frac{t^2 s_n^2}{2n})\left|\exp\left(\eta\frac{|t|^3}{2n^{3/2}}\sum_j\beta_j\right)-1\right| \le$$

$$\le \exp(-\frac{t^2 s_n^2}{2n})\frac{|t|^3}{2n^{3/2}}\sum_j\beta_j\exp\left(\frac{|t|^3}{2n^{3/2}}\sum_j\beta_j\right) \le$$

$$\le \frac{e^{1/16}}{2}\exp(-\frac{t^2 s_n^2}{2n})\frac{|t|^3}{n^{3/2}}\sum_j\beta_j.$$

From the two upper bounds derived in each interval for t we deduce that

$$\left|\varphi_{S_n}(\frac{t}{\sqrt{n}}) - \exp(-\frac{t^2 s_n^2}{2n})\right| \le 16\frac{|t|^3}{n^{3/2}}\sum_j\beta_j\ \exp(-\frac{t^2 s_n^2}{3n})$$

for every $|t|\le\frac{c_0 n^{3/2}}{4\sum_j\beta_j}$.

To finish the proof just use (2) with $U=\frac{c_0 n^{3/2}}{4\sum_j\beta_j}$ to find

$$\sup\left|F_n(x)-N_{n^{-1}s_n^2}(x)\right| \le \frac{16\sum_j\beta_j}{\pi n^{3/2}}\int_{-U}^{U}t^2\exp(-\frac{t^2 s_n^2}{3n})\,dt + \frac{96\sqrt{n}\sum_j\beta_j}{c_0\pi\sqrt{2\pi}s_n n^{3/2}} \le$$

$$\le \frac{16\sum_j\beta_j}{\pi n^{3/2}}\int_{-\infty}^{\infty}t^2\exp(-\frac{t^2 s_n^2}{3n})\,dt + \frac{96\sum_j\beta_j}{c_0\pi\sqrt{2\pi}s_n n} \le$$

$$\le \frac{24\sum_j\ \beta_j}{\pi s_n^2 n^{1/2}} + \frac{96\sum_j\beta_j}{c_0\pi\sqrt{2\pi}s_n n}. \qquad\blacksquare$$

Inequality (4) in the case of independent and identically distributed variables gives the $n^{-1/2}$ rate as usual.

Corollary 2.2 *Let $X_1,\ldots,X_n$ be independent and identically distributed centered random variables with finite third order absolute moments $\beta=\mathrm{E}\,|X_1|^3$. Then*

$$\sup\left|F_n(x)-N_{n^{-1}s_n^2}(x)\right| \le \frac{24\beta}{\pi\sigma^2 n^{1/2}} + \frac{96\beta}{\pi\sqrt{2\pi}\sigma^3 n^{1/2}}. \tag{5}$$

where $\sigma^2=\mathrm{E}X_1^2$.

3 A CLT for associated variables

In this section we will present a Central Limit Theorem and a convergence rate for associated random variables. We include here the definition and a basic inequality which is essential to our proofs. For a better account on association of random variables we refer the reader to Newman [5].

Definition 3.1 *The random variables $X_1, \ldots, X_n$, are associated if, for every f, g coordinatewise increasing real valued functions defined on $\mathbb{R}^n$,*

$$\mathrm{Cov}(f(X_1, \ldots, X_n),\, g(X_1, \ldots, X_n)) \geq 0.$$

The random variables X_n, $n \in \mathbb{N}$, are associated if, for every $n \in \mathbb{N}$, the variables $X_1, \ldots, X_n$ are associated.

A nice tool for proving Central Limit Theorems is the Newman's inequality, which provides a control on the characteristic functions by the covariance structure of the variables.

Theorem 3.2 (Newman [5]) *Let $X_1, \ldots, X_n$ be associated random variables with finite second order moments. Then*

$$\left| \mathrm{E} \exp(i \sum_{j=1}^{n} t_j X_j) - \prod_{j=1}^{n} \mathrm{E} \exp(i t_j X_j) \right| \leq \frac{1}{2} \sum_{\substack{j,k=1 \\ j \neq k}}^{n} |t_j t_k| \,\mathrm{Cov}(X_j,\, X_k). \tag{6}$$

According to this inequality, after some adequate control on the covariances, the variables may be replaced by another ones with the same distributions but independent. This coupling will be the basis of the results we will prove for associated variables. We begin with a Central Limit Theorem for which the proof is an adaptation to a somewhat simpler framework of the technique that was used in the proof of Theorem 9 in Oliveira, Suquet [7].

Theorem 3.3 *Let X_n, $n \in \mathbb{N}$, be associated centered random variables. For each $k \in \mathbb{N}$, let m be the largest integer less or equal to n/k and define $Y_{k,j} = \sum_{i=(j-1)k+1}^{jk} X_i$, $i = 1, \ldots, m$. Suppose that the following conditions are verified*

$$\lim_{n \to +\infty} \frac{s_n^2}{n} = \lim_{n \to +\infty} \frac{1}{n} \mathrm{E} S_n^2 = \sigma^2 > 0 \tag{7}$$

$$\lim_{m \to +\infty} \frac{1}{m} \sum_{j=1}^{m} \mathrm{E} Y_{k,j}^2 = a_k \qquad \text{and} \qquad \lim_{k \to +\infty} \frac{a_k}{k} = \sigma^2 \tag{8}$$

$$\forall \delta > 0,\ \frac{1}{n} \sum_{i=1}^{n} \int_{\{|X_i| > \delta n^{1/2}\}} X_i^2 \, d\mathrm{P} \longrightarrow 0. \tag{9}$$

Then $n^{-1/2} S_n$ converges in distribution to a gaussian random variable with mean 0 and variance σ^2.

Proof : The blocks $Y_{k,1}, \ldots, Y_{k,m}$ being increasing functions of the X_n, $n \in \mathbb{N}$, are associated, so the method of proof will consist on approximating the sum S_n by the sum of these blocks and reason as if these blocks were independent, based on (6). First we shall look at

$$\left|\varphi_{S_n}(\frac{t}{\sqrt{n}}) - \varphi_{S_{mk}}(\frac{t}{\sqrt{mk}})\right| \le |t| \operatorname{Var}^{1/2}(S_n - S_{mk}) =$$

$$\le |t| \left[\sum_{i,j=1}^{mk} \left(\frac{1}{\sqrt{n}} - \frac{1}{\sqrt{mk}}\right)^2 \mathrm{E}(X_i X_j) + \frac{1}{n} \sum_{i,j=mk+1}^{n} \mathrm{E}(X_i X_j) + \right.$$

$$\left. + \frac{1}{\sqrt{n}} \sum_{i=1}^{mk} \sum_{j=mk+1}^{n} \left(\frac{1}{\sqrt{n}} - \frac{1}{\sqrt{mk}}\right) \mathrm{E}(X_i X_j) \right]^{1/2} \le$$

$$\le |t| \left[\frac{k}{nm(\sqrt{mk}+\sqrt{n})^2} \sum_{i,j=1}^{mk} \mathrm{E}(X_i X_j) + \frac{n-mk}{n} \frac{1}{n-mk} \sum_{i,j=mk+1}^{n} \mathrm{E}(X_i X_j) + \right.$$

$$\left. + \frac{\sqrt{k}}{n\sqrt{m}(\sqrt{mk}+\sqrt{n})} \sum_{i=1}^{mk} \sum_{j=mk+1}^{n} \mathrm{E}(X_i X_j) \right]^{1/2}$$

which converges to zero according to (7). Next, we approach $\varphi_{S_{mk}}$ by the product of the characteristic functions of the blocks $Y_{k,j}$, using (6). According to (7) and (8), there exists a constant $c > 0$ such that for m (thus n) large enough,

$$\left|\varphi_{S_{mk}}(\frac{t}{\sqrt{mk}}) - \prod_{j=1}^{m} \varphi_{X_j}(\frac{t}{\sqrt{mk}})\right| \le \frac{1}{2} \sum_{\substack{i,j=1 \\ i \ne j}}^{n} \frac{t^2}{mk} \mathrm{E}(Y_{k,i} Y_{k,j}) \le ct^2(\sigma^2 - \frac{a_k}{k}).$$

From this point on we may reason as if the blocks $Y_{k,1}, \ldots, Y_{k,m}$ were independent (to be completely accurate we should introduce a new set of random variables with the required properties, but we will not do so to keep the notation simpler). That is, we must now check that these blocks verify a Central Limit Theorem, what we will accomplish by verifying the Lindeberg condition. Put $r_n^2 = (mk)^{-1} \sum_{j=1}^{m} \mathrm{E} Y_{k,j}^2$. From (8) and Lemma 4 in Oliveira, Suquet [7], it follows $r_n^2 \longrightarrow \frac{a_k}{k}$, so the Lindeberg condition for the blocks $Y_{k,1}, \ldots, Y_{k,m}$ writes

$$\sum_{j=1}^{m} \int_{\{|Y_{k,j}| > \varepsilon r_n \sqrt{mk}\}} \frac{1}{mk} Y_{k,j}^2 \, d\mathrm{P} \longrightarrow 0$$

where $\varepsilon > 0$ is fixed. From Lemma 4 in Utev [12] this integral is less or equal than

$$\frac{1}{m} \sum_{j=1}^{m} \sum_{i=(j-1)k+1}^{jk} \int_{\left\{|X_i| > \frac{\varepsilon}{2}\sqrt{\frac{a_k}{k}}\sqrt{\frac{m}{k}}\right\}} X_i^2 \, d\mathrm{P} \le \frac{1}{m} \sum_{j=1}^{mk} \int_{\left\{|X_i| > \frac{\varepsilon}{2k}\sqrt{\frac{a_k}{k}}\sqrt{mk}\right\}} X_i^2 \, d\mathrm{P}$$

which converges to zero when $n \longrightarrow +\infty$ (k fixed) according to (9). Summing up the inequalities above we have, for each k fixed, that there exits some constant $c > 0$ such that

$$\limsup_{n \to +\infty} \left|\varphi_{S_n}(\frac{t}{\sqrt{n}}) - e^{-\frac{t^2\sigma^2}{2}}\right| \le ct^2(\sigma^2 - \frac{a_k}{k}).$$

Finally, letting $k \longrightarrow +\infty$ and taking account of (8) it follows $\varphi_{S_n}(\frac{t}{\sqrt{n}}) \longrightarrow e^{-\frac{t^2\sigma^2}{2}}$, which proves the theorem. ■

We may state an immediate corollary for the case of stationary random variables.

Corollary 3.4 *Let X_n, $n \in \mathbb{N}$, be associated, weakly stationary centered random variables verifying (7) and (9). Then $n^{-1/2} S_n$ converges in distribution to a gaussian random variable with mean 0 and variance σ^2.*

Based on the bounds derived in the previous section and in inequality (6) we may prove a convergence rate for the Central Limit Theorem for associated variables. A result on this setting has been proved by Wood [14] who, supposing that the variables are stationary, obtained for $n = m \cdot k$,

$$\sup_x |F_n(x) - N_a(x)| \leq \frac{16 s_n^4 m (a^2 - s_n^2)}{9\pi \mathrm{E}^2 |S_n|^3} + \frac{3\mathrm{E}|S_n|^3}{s_n^3 m^{1/2}}$$

where $a^2 = \mathrm{E}X_1^2 + 2\sum_{k=2}^{\infty} \mathrm{Cov}(X_1, X_k)$. The stationarity assumption was dropped by Suquet [11] considering an weighted L^2 distance instead of the supremum distance.

Theorem 3.5 *Let X_n, $n \in \mathbb{N}$, be associated centered random variables. For each $k \in \mathbb{N}$ let m be the largest integer less or equal to n/k. Define $Y_{k,j} = \sum_{i=(j-1)k+1}^{jk} X_i$, $i = 1, \ldots, m$, $\sigma_{k,j}^2 = \mathrm{E}Y_{k,j}^2$ and $\tau_{k,j} = \mathrm{E}|Y_{k,j}|^3$. Suppose that $\inf_{m \in \mathbb{N}} \frac{1}{m} \sum_{j=1}^m \sigma_{k,j}^2 \geq c_0 > 0$. Then, for $n = m \cdot k$,*

$$\sup_x \left| F_n(x) - N_{n^{-1}s_n^2}(x) \right| \leq \frac{c_0^2 m^3}{(\sum_j \tau_{k,j})^2} \left| \frac{s_n^2}{n} - \frac{1}{m} \sum_j \sigma_{k,j}^2 \right| + \frac{24 \sum_j \tau_{k,j}}{\pi m^{1/2} \sum_j \sigma_{k,j}^2} + \frac{96 \sum_j \tau_{k,j}}{c_0 \pi \sqrt{2\pi} m (\sum_j \sigma_{k,j}^2)^{1/2}} \tag{10}$$

Proof : Using (2) we have

$$\left| F_n(x) - N_{n^{-1}s_n^2}(x) \right| \leq \frac{1}{\pi} \int_{-U}^{U} \left| \frac{\varphi_{S_n}(\frac{t}{\sqrt{n}}) - e^{-\frac{t^2 s_n^2}{2n}}}{t} \right| d\mathrm{P} + \frac{24\sqrt{n}}{\pi \sqrt{2\pi} s_n U}.$$

Note that $\frac{1}{\sqrt{s_n}} S_n = \frac{1}{\sqrt{m}} \sum_{j=1}^m Y_{k,j}$, so the integral may be decomposed in the sum

$$I_1 + I_2 + I_3 = \int_{-U}^{U} \left| \frac{\mathrm{E}\exp(i\frac{t}{\sqrt{m}} \sum_j Y_{k,j}) - \prod_{j=1}^m \mathrm{E}\exp(i\frac{t}{\sqrt{m}} Y_{k,j})}{t} \right| dt +$$

$$+ \int_{-U}^{U} \left| \frac{\prod_{j=1}^m \mathrm{E}\exp(i\frac{t}{\sqrt{m}} Y_{k,j}) - \exp(-\frac{t^2}{2m} \sum_j \sigma_{k,j}^2)}{t} \right| dt +$$

$$+ \int_{-U}^{U} \left| \frac{\exp(-\frac{t^2}{2m} \sum_j \sigma_{k,j}^2) - e^{-\frac{t^2 s_n^2}{2n}}}{t} \right| dt.$$

The third integral is bounded by using the inequality $|e^{-t} - e^{-s}| \leq |t - s|$, so

$$I_3 \leq \frac{1}{2} \left| \frac{s_n^2}{n} - \frac{1}{m} \sum_j \sigma_{k,j}^2 \right| \int_{-U}^{U} |t| \; dt = \frac{U^2}{2} \left| \frac{s_n^2}{n} - \frac{1}{m} \sum_j \sigma_{k,j}^2 \right|.$$

The integral I_1 is bounded using (6),

$$I_1 \leq \frac{1}{2m} \sum_{\substack{j,l=1 \\ j \neq l}}^{m} \mathrm{Cov}(Y_{k,j}, Y_{k,l}) \int_{-U}^{U} |t| \, dt = \frac{U^2}{2} \left| \frac{s_n^2}{n} - \frac{1}{m} \sum_j \sigma_{k,j}^2 \right|.$$

where we may choose $U = \frac{c_0 m^{3/2}}{4 \sum_j \tau_{k,j}}$ according to the proof of Theorem 2.1. Finally, to bound I_2 we use (4) to find

$$I_2 \leq \frac{24 \sum_j \tau_{k,j}}{\pi m^{1/2} \sum_j \sigma_{k,j}^2} + \frac{96 \sum_j \tau_{k,j}}{c_0 \pi \sqrt{2\pi} m (\sum_j \sigma_{k,j}^2)^{1/2}},$$

and we get (10) by summing up these bounds. ■

Note that the conditions of Theorem 3.3 imply both that $\inf_m \frac{1}{m} \sum_j \sigma_{k,j}^2 \geq c_0$ for some constant $c_0 > 0$ and that $\frac{s_n^2}{n} - \frac{1}{m} \sum_j \sigma_{k,j}^2 \longrightarrow 0$ according to (7) and (8), respectively. The extra term appearing in (10) when comparing to what appears in the independent case was already present in the stationary version of Wood [14] who also provided an example showing that this term can not be avoided, in general, and is responsible for a possibly arbitrarily slow convergence rate.

If we wish to derive an Edgeworth expansion based inequality, we need only to replace the bound of I_2 by the convenient one, as the integral I_1 remains the same, and I_3 is modified in a way that does not affect its final treatment, as in this case instead of $N_{n^{-1}s_n^2}(x)$ we will have this function plus a convenient perturbation.

Finally note that Newman's inequality (6) holds in a somewhat more general setting, as association is not really required. Indeed, it is easy to check that the proof of this inequality only requires that the variables are linear positive quadrant dependent (LPQD), thus as this inequality was the main tool to prove our results, they also hold for LPQD random variables. We refer the reader to Newman [5] for the definition of LPQD and the proof of (6).

The upper bounds (4), (5) and (10) derived here seem to be of interest only for large samples. Indeed, even in the independent and identically distributed case the upper bound (5) is, for reasonable values of n, larger than 1, thus rendering the bound useless. Some numerical evaluations suggest that there is a big gap to be filled by optimizing the constants involved. This question was not addressed here. The lack of optimization evidently reflects also on the bounds for associated variables, as these are derived by adding a convenient term to the independent version. The following tables, containing simulated values, confirm these comments. The first table reports simulated results for independent variables with distribution $\frac{5}{9}\delta_{-\frac{8}{9}} + \frac{4}{9}\delta_{\frac{10}{9}}$ compared with the upper bound (5), and to what we find when the variables are "slightly" associated. By "slightly" associated we mean that the variables are associated but X_i and X_k are in fact independent whenever $|k-i| \geq 2$, thus leaving us to deal only with the covariances of X_i and X_{i+1}. We will denote this by saying that the variables are associated(2).

n	upper bound (5)	independent	associated(2)
100	2.015192	0.093087	0.042001
200	1.424956	0.062520	0.058189
300	1.163472	0.036359	0.047569
500	0.901221	0.038086	0.042813

It is interesting to note that the observed upper bounds for associated variables seem to behave not very differently from the ones observed for independent variables. This seems to be connected to the fact that the variables are only "slightly" associated. If we call the variables associated(m) whenever they are associated and X_i, X_k are in fact independent if $|k-i| \geq m$, the effect of m is illustrated on the table below for variables constructed from variables with the distribution mentioned above, for the choice $n = 100$,

m	3	4	5	10	20
associated(m)	0.041783	0.102225	0.069066	0.376153	0.495995

The last values of this table show the effect of the correction term for associated variables appearing in (10).

The same effect is observed if we use absolutely continuous variables. The next table reports simulated values for exponentially distributed variables with parameter λ and $n = 100$,

$\lambda = 0.25$			$\lambda = 1$		
m	independent	associated (m)	m	independent	associated (m)
2	0.047228	0.047049	2	0.043543	0.024927
3	0.064108	0.069020	3	0.037238	0.045861
4	0.045430	0.099881	4	0.036112	0.059330
5	0.031737	0.093499	5	0.038310	0.107544
10	0.029662	0.162102	10	0.026067	0.145102

Note that the bound (5) is 10.32186 when $\lambda = 0.25$ and 4.788111 when $\lambda = 1$ due to the fact that the absolute third order moments are quite large.

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