DEPARTAMENTO DE MATEMÁTICA

98-05

THE CHANGE OF THE KRONECKER STRUCTURE OF A COMPLEX MATRIX PENCIL UNDER SMALL PERTURBATIONS

E. Marques de Sá

PRÉ-PUBLICAÇÕES

Fevereiro 1998

Departamento de Matemática
Universidade de Coimbra
Apartado 3008
3000 Coimbra - Portugal
Tel. (39)7003150 - Fax (39)32568

The Change of the Kronecker Structure of a Complex Matrix Pencil Under Small Perturbations

Eduardo Marques de Sá *
Departamento de Matemática
Universidade de Coimbra
3000 Coimbra, Portugal

Abstract. In this paper we give a complete solution to the following problem: given a complex matrix pencil, with known Kronecker normal form, describe the possible Kronecker structures of pencils which are small perturbations of the given pencil.

Short title: Pencils under small perturbations

1. Introduction

The problem stated in the abstract will henceforth be called the *perturbation problem*. The concept of 'Kronecker structure', as it appears in the statement, calls for a precise definition. For this purpose, we shall introduce the concept of *skeleton* of a pencil, which, roughly speaking, is what we obtain from the Kronecker normal form by disregarding the actual values of the pencil's eigenvalues.

* Centro de Matemática da Universidade de Coimbra/FCT. Partially supported successively by Junta Nacional de Investigação Científica e Tecnológica, Project Praxis XXI Mat/485/94, and Project 574/94 of Fundação Luso-Americana para o Desenvolvimento.

Given a pencil $\mathcal{A}$, we are interested in the skeletons Σ such that, in any neighbourhood of $\mathcal{A}$, Σ occurs as the skeleton of a pencil in that neighbourhood. The set of all such skeletons is denoted by $S_{\mathcal{A}}$. The solution given to the perturbation problem implies a characterization of $S_{\mathcal{A}}$.

In the last few years this and related questions have been studied in the literature. For example, in [9] a complete description is given for the Jordan structures of a perturbed square complex matrix; as we shall see, this essentially is a complete solution to the perturbation problem, when the given pencil is nonsingular. In [11] some perturbation results for complex pencils are proved, together with a complete characterization of the closure of the equivalence orbit of a complex pencil. In [6] the perturbation problem is completely solved for those special pencils arising in Linear Control Theory.

In February 1987, during a visit to Bilbao (Spain), I pointed out to the authors of [6] the relevance of the results of [11] to the problem under discussion. As a matter of fact, the complete description of the closure of orbits given by A. Pokrziwa is so intimately and obviously related to the perturbation problem, that it is only natural to suspect that a small addition to his paper would completely solve our problem. Some weeks after that visit I confirmed those suspitions. It turns out out that the elements of $S_{\mathcal{A}}$ are precisely the skeletons of equivalence orbits in whose closure $\mathcal{A}$ lies. In 1990, I. Hoyos sent me a copy of her thesis [8] where, among other results, a key perturbation theorem is obtained without direct use of [11]; that was one of the motivations to write down this paper, as it seems desirable to recognize the depth and importance of the results of [11].

The present paper describes how a complete solution to the perturbation problem may be obtained from [11], combined with results of [2, 9]. In section 3 we state and prove the main perturbation theorem. In section 4 we consider an inverse perturbation problem; Theorem 4.1 shows, again as a consequence of [11], that the perturbation theorem of section 3 is the best possible in a precise given sense. Theorem 5.3 is a quantitative refinement of Theorem 4.1.

This work has been presented to the Portuguese Mathematical Society Jubilee Conference held in Lisbon 1990.

Integer sequences. $\mathbf{N}$ is the set of positive integers. A sequence, say $(k_i : i \in \mathbf{N})$, where the k_i 's are integers, will be denoted by the corresponding upper case latin letter: K in the current example. If $M = (m_i : i \in \mathbf{N})$ is another such sequence, we denote by $K + M$ the componentwise sum. We write $K \prec\!\!\prec M$ whenever $k_1 + \cdots + k_w \leq m_1 + \cdots + m_w$, for all $w \in \mathbf{N}$. All sequences to be considered below are monotonic: some are decreasing, others are increasing (here, *decreasing* and *increasing* are meant in the weak sense). Therefore $\prec\!\!\prec$ will always be a *majorization relation* in the sense of (e.g.) [10], where the symbols $\prec_w$ and $\prec^w$ are used.

Now let $(K_\tau)_{\tau \in T}$ be a family of sequences, indexed by a set T. Assuming $K_\tau = (k_{\tau i} : i \in \mathbf{N})$ is decreasing for all $\tau \in T$, and $\{k_{\tau 1} : \tau \in T\}$ is bounded from above, the *union* of the family, denoted by

$$\cup(K_\tau : \tau \in T) \quad \text{or} \quad \bigcup_{\tau \in T} K_\tau ,$$

is the sequence $(u_1, u_2, \ldots)$, where u_j is the j-th greatest element of the family $(k_{\tau i} : \tau \in T,\ i \in \mathbf{N})$. The reader may easily check that

$$u_1 + \cdots + u_w = \sup \left\{ \sum_{\tau \in T} \sum_{i=1}^{s_\tau} k_{\tau i} \ :\ s_\tau \geq 0 \ \text{ and } \ \sum_{\tau \in T} s_\tau = w \right\} .$$

With the help of this formula, one may prove the following result, that will be extensively used in the sequel.

Lemma 1.1. *Let K_τ and M_τ be decreasing sequences such that $M_\tau \prec\!\!\prec K_\tau$, for each $\tau \in T$. Then*

$$\bigcup_{\tau \in T} M_\tau \prec\!\!\prec \bigcup_{\tau \in T} K_\tau .$$

■

For $c \in \mathbf{Z}$ the *constant sequence* $(c, c, c, \ldots)$ will be denoted by $[c]$. Note that if K is decreasing and $k_1 \leq c$, then $K \cup [c] = [c]$.

We say that K is a *partition* if K is a decreasing sequence of nonnegative integers, and k_i is positive for only a finite number of i 's. For partitions K and M we write $K \prec M$ whenever $K \prec\!\!\prec M$ and the sum of all k_i 's equals the sum of all m_i 's (cf [10]). As usual, the *conjugate* of a partition K, denoted by

$\tilde{K} = (\tilde{k}_s : s \in \mathbf{N})$, is defined by $\tilde{k}_s := \#\{i : k_i \geq s\}$. Then $\tilde{K}$ is a partition as well. We shall use, with no further comments, the following properties of partitions:

$$\tilde{\tilde{K}} = K\,, \quad (K+M)\tilde{} = \tilde{K} \cup \tilde{M}\,, \quad \text{and} \quad K \prec M \Leftrightarrow \tilde{M} \prec \tilde{K}\,.$$

2. Preliminary Results

We assume the reader is familiar with the Weirstrass-Kronecker theory of pencils (see [5, ch.12]). Our discussion involves only complex pencils. So an $m \times n$ pencil is a polynomial matrix, $\mathcal{A} = \lambda A_1 + \mu A_2$, where λ and μ are independent variables and A_1 and A_2 are $m \times n$ complex matrices. Sometimes it will be convenient to represent $\mathcal{A}$ by $\mathcal{A}(\lambda, \mu)$.

For any pair $(\alpha, \beta) \in \mathbf{C}^2$ such that $(\alpha, \beta) \neq (0, 0)$, define

$$\langle \alpha, \beta \rangle := \{(z\alpha, z\beta) : \ z \in \mathbf{C}\,, \ z \neq 0\}\,.$$

The set of all $\langle \alpha, \beta \rangle$ is the projective complex line, that we denote by $\mathbf{P}(\mathbf{C})$. We identify $\mathbf{P}(\mathbf{C})$ with $\mathbf{C}^\infty := \mathbf{C} \cup \{\infty\}$ in the usual way: a given $\langle \alpha, \beta \rangle \in \mathbf{P}(\mathbf{C})$ is identified with α/β; here, of course, we convention $\alpha/0 = \infty$, for any nonzero α. (Cf [1, §4.2,§4.3].)

We say $\langle \alpha, \beta \rangle \in \mathbf{P}(\mathbf{C})$, or $\alpha/\beta \in \mathbf{C}^\infty$, is an *eigenvalue* of $\mathcal{A} = \mathcal{A}(\lambda, \mu)$, whenever rank $\mathcal{A}(\alpha, \beta) <$ rank $\mathcal{A}$. Define the following pencils:

$$\mathcal{J}_s(\lambda, \mu) := \lambda I_s + \mu N_s \quad \text{and} \quad \mathcal{R}_s(\lambda, \mu) := \lambda[I_s \ 0] + \mu[0 \ I_s]\,,$$

where I_s is the $s \times s$ identity matrix, N_s is the usual upper triangular, nilpotent Jordan block of order s, and the 0's denote zero columns [thus $\mathcal{R}_s$ is $s \times (s+1)$]. $\mathcal{R}_s$ and its transposed, which we denote by s, are called *right* and *left Kronecker pencils*, respectively. Pencils of the form $\mathcal{J}_s(\lambda - \alpha\mu, \mu)$, with $\alpha \in \mathbf{C}$, or $\mathcal{J}_s(\mu, \lambda)$ are called *Jordan pencils*; note that α is the unique eigenvalue of the former pencil and ∞ is the unique eigenvalue of the latter.

According to a well-known theorem of Weirstrass and Kronecker (see, e.g., [5, Vol.2]), any pencil $\mathcal{A}$ is strictly equivalent to a direct sum of Jordan pencils and Kronecker pencils, which are called *Jordan and Kronecker components of* $\mathcal{A}$; such direct sum, known as *Kronecker normal form of* $\mathcal{A}$, is unique up to the order of the components. Here we are using the following convention: the Kronecker pencil $\mathcal{R}_0$ [$\mathcal{L}_0$] is said *to occur p times in $\mathcal{A}$* iff the Kronecker normal form of $\mathcal{A}$ has p zero columns [rows]. To simplify forthcoming formulas, we encode the Kronecker structure of a pencil differently from [11].

Definition 2.1. Given a pencil $\mathcal{A}$ and $\alpha \in \mathbf{C}$ we let $M(\alpha, \mathcal{A}) = (m_j(\alpha, \mathcal{A}) : j \in \mathbf{N})$, $H(\mathcal{A}) = (h_j(\mathcal{A}) : j \in \mathbf{N})$, and $V(\mathcal{A}) = (v_j(\mathcal{A}) : j \in \mathbf{N})$ be the sequences defined as follows: $m_j(\alpha, \mathcal{A})$ is the number of Jordan components of $\mathcal{A}$, with eigenvalue α, having at least j rows; $h_j(\mathcal{A})$ [$v_j(\mathcal{A})$] is the number of Kronecker components of $\mathcal{A}$ of type $\mathcal{R}_s$ [$\mathcal{L}_s$], such that $s < j$.

We denote by $h(\mathcal{A})$ [$v(\mathcal{A})$] the total number of right [left] Kronecker components of $\mathcal{A}$.

For any subset X of $\mathbf{C}^\infty$ we define

$$M(X, \mathcal{A}) := \bigcup_{\alpha \in X} M(\alpha, \mathcal{A}). \tag{1}$$

The *skeleton of $\mathcal{A}$* is the sequence of integer sequences

$$\mathrm{SK}(\mathcal{A}) := \big(M(\alpha_1, \mathcal{A}), \ldots, M(\alpha_p, \mathcal{A}), H(\mathcal{A}), V(\mathcal{A})\big),$$

where $\alpha_1, \ldots, \alpha_p$ are the distinct eigenvalues of $\mathcal{A}$, ordered in such a way that the p partitions $M(\alpha_i, \mathcal{A})$ occur in (say) increasing lexicographical order.

Remarks. (a) Letters 'h' and 'v' hold for '*horizontal*' and '*vertical*' blocks. $h(\mathcal{A})$ is the maximum $h_j(\mathcal{A})$, and $v(\mathcal{A})$ is the maximum $v_j(\mathcal{A})$. As $v(\mathcal{A}) = h(\mathcal{A}) + m - n$, we have chosen not to use $v(\mathcal{A})$.

(b) The union in (1) is essentially extended to the finite set of eigenvalues of $\mathcal{A}$ inside X; therefore $M(X, \mathcal{A})$ is a partition.

(c) Of course we have $p = 0$ if and only if $\mathcal{A}$ has no eigenvalue; in that case $\mathrm{SK}(\mathcal{A}) = (H(\mathcal{A}), V(\mathcal{A}))$.

Examples.

(1) For a 2×2 Jordan pencil, $\mathrm{SK}(\mathcal{J}_2) = ((1, 1, 0, 0, 0, \ldots), [0], [0])$.

(2) A 2×3 zero pencil has skeleton $([3], [2])$.

(3) $\mathrm{SK}(\mathcal{R}_2) = ((0, 0, 1, 1, 1, \ldots), [0])$.

Lemma 2.2.

(a) *An $s \times s$ pencil $\mathcal{A}$ is strictly equivalent to the $s \times s$ Jordan pencil with eigenvalue $\langle \alpha, \beta \rangle \in \mathbf{P}(\mathbf{C})$ iff $\mathcal{A}$ has only one eigenvalue,* rank $\mathcal{A} = s$ *and* rank $\mathcal{A}(\alpha, \beta) = s - 1$.

(b) *For $s > 0$, an $s \times (s+1)$ pencil $\mathcal{A}$ is strictly equivalent to the Kronecker pencil $\mathcal{R}_s$ iff* rank $\mathcal{A}(\alpha, \beta) = s$ *for all $\langle \alpha, \beta \rangle \in \mathbf{P}(\mathbf{C})$; that is, iff $\mathcal{A}$ has maximum rank and no eigenvalue.* ■

We now fix four complex numbers, a, b, c, d, such that $ad \neq bc$. Given $\mathcal{A}$ we define a new pencil $\tilde{\mathcal{A}}$ by changing variables in $\mathcal{A}$:

$$\tilde{\mathcal{A}}(\lambda, \mu) := \mathcal{A}(a\lambda + b\mu, c\lambda + d\mu) . \tag{2}$$

The Möbius (linear fractional) transformation, given by

$$\psi(z) := (az + b)/(cz + d) , \quad \text{for } z \in \mathbf{C}^{\infty} , \tag{3}$$

then naturaly arises, because α is an eigenvalue of $\tilde{\mathcal{A}}$ iff $\psi(\alpha)$ is an eigenvalue of $\mathcal{A}$. We stress the fact that ψ is a topological automorphism of $\mathbf{C}^{\infty}$. (See [1, §18.10] and [3, pp.43 ff] for details).

The following theorem is surely known but we did not find it in the literature. So we give a sketchy proof.

Theorem 2.3. *With the above notation the pencils $\mathcal{A}$ and $\tilde{\mathcal{A}}$ have the same skeleton and $M(\alpha, \tilde{\mathcal{A}}) = M(\psi(\alpha), \mathcal{A})$ for all α in $\mathbf{C}^{\infty}$.*

Proof. Without loss of generality we assume that $\mathcal{A}$ is in Kronecker normal form. Thus $\tilde{\mathcal{A}}$ also splits into a direct sum of pencils each of which is obtained from a component of $\mathcal{A}$ by the same change of variables. Moreover the number of zero rows [columns] is the same in $\mathcal{A}$ and $\tilde{\mathcal{A}}$. Therefore we may work out each block separately, that is, we only need to prove our theorem in the following two special cases: (a) $\mathcal{A}$ is a Jordan pencil; (b) $\mathcal{A}$ is a Kronecker pencil. In each case we may apply Lemma 2.2 to obtain the desired result. ■

The following result comes out as a consequence of [9, Th.5].

Theorem 2.4. *Let $\mathcal{T}$ be an $n \times n$ nonsingular complex pencil and let U be an open subset of $\mathbf{C}^\infty$, such that no eigenvalue of $\mathcal{T}$ lies on the boundary of U. Moreover let W be a closed subset of $\mathbf{C}^\infty$. Then for all pencils $\mathcal{S}$ sufficiently close to $\mathcal{T}$*

$$M(U,\mathcal{S}) \prec M(U,\mathcal{T}) \quad \text{and} \quad M(W,\mathcal{S}) \lessdot M(W,\mathcal{T}). \tag{4}$$

Proof. Note that the statement involving U implies the statement involving W because, given a closed W, there is U satisfying the hypotheses of the theorem, such that $U \supset W$ and no eigenvalue of $\mathcal{T}$ lies on $U \backslash W$.

The statement concerning U is (in essence) a particular case of [9, Th.5]. In fact, each eigenvalue τ of $\mathcal{T}(\lambda,\mu)$ contained in U may be individually treated: if τ is finite, apply [9, Th.5] to the holomorphic function $\mathcal{T}(\lambda,1)$ in a small disk centered at τ; if $\tau=\infty$, apply [9, Th.5] to the holomorphic function $\mathcal{T}(1,\mu)$ in a small disk centered at the origin. ■

Remark 2.5. It is easy to give an independent elementary proof of the above result based on [2] or [9, Th.1]. First we change variables in $\mathcal{T}$, according to (2)-(3), in such a way that, in the new pencil, $\tilde{\mathcal{T}} = \lambda \tilde{T}_1 + \mu \tilde{T}_2$, the matrix $\tilde{T}_1$ is nonsingular; then we apply [2] or [9, Th.1] to $\lambda I + \tilde{T}_2 \tilde{T}_1^{-1}$.

To close this section we point out that, in the notation of Definition 2.1, Theorem 3 of [11] has the following formulation: *a pencil $\mathcal{A}$ lies in the closure of the equivalence orbit of a pencil $\mathcal{S}$ iff the following relations hold:*

$$H(\mathcal{S}) \lessdot H(\mathcal{A})\ , \quad V(\mathcal{S}) \lessdot V(\mathcal{A}) \tag{5}$$

$$[h(\mathcal{S})] + M(\alpha,\mathcal{S}) \lessdot [h(\mathcal{A})] + M(\alpha,\mathcal{A})\,, \tag{6}$$

for all $\alpha \in \mathbf{C}^\infty$. Here, nothing essentially changes if we substitute 'v' for 'h' throughout. Note that condition (6) is equivalent to

$$[h(\mathcal{S})] + M(W,\mathcal{S}) \lessdot [h(\mathcal{A})] + M(W,\mathcal{A})\,, \quad \text{for all } W \subset \mathbf{C}^\infty\,. \tag{7}$$

In fact, for any pencil $\mathcal{P}$, the union of all sequences $[h(\mathcal{P})] + M(\alpha,\mathcal{P})$ (for $\alpha \in W$) is $[h(\mathcal{P})] + M(W,\mathcal{P})$. So (7) follows from (6) by a straightforward application of Lemma 1.1.

3. The Main Perturbation Result

The following theorem is obtained as a combination of the results of [11] and [9, Th.5]. The reader should recall the notations of Definition 2.1.

Theorem 3.1. *Let $\mathcal{A}$ be a pencil and let W be a closed subset of $\mathbf{C}^\infty$. There exists a neighborhood Ω of $\mathcal{A}$ such that any $\mathcal{S}$ in Ω satisfies*

$$H(\mathcal{S}) \prec\!\!\prec H(\mathcal{A}) \ , \quad V(\mathcal{S}) \prec\!\!\prec V(\mathcal{A}) \tag{8}$$

$$[h(\mathcal{S})] + M(W, \mathcal{S}) \prec\!\!\prec [h(\mathcal{A})] + M(W, \mathcal{A}) \, . \tag{9}$$

Conditions (8) were proved by A. Pokrziwa [11, Th.1 (4)-(5)]. Note that (9) is the same majorization occurring in (7), but the quantification on W is different. Here, I will offer a proof of (9) that I obtained (but not published) in 1987. For that purpose a lemma is first stated and proved. This lemma has also been obtained by I. Hoyos, in her PhD thesis, by a different method.

Lemma 3.2. *For a fixed $\alpha \in \mathbf{C}^\infty$ let U_α be a neighbourhood of α such that $\overline{U_\alpha} \backslash \{\alpha\}$ contains no eigenvalue of $\mathcal{A}$. Given a sequence of pencils $(\mathcal{S}_k : k \in \mathbf{N})$ converging to $\mathcal{A}$, for sufficiently large k we have*

$$[h(\mathcal{S}_k)] + M(U_\alpha, \mathcal{S}_k) \prec\!\!\prec [h(\mathcal{A})] + M(U_\alpha, \mathcal{A}) \, . \tag{10}$$

Proof. The pencils under consideration are $m \times n$, with m and n fixed. So we only have a finite number of possible skeletons and a finite number of possibilities for the left hand side of (10). Therefore, we only need to prove the Lemma when all the $\mathcal{S}_k$ have the same skeleton and $M(U_\alpha, \mathcal{S}_k)$ is independent of k; henceforth we assume these conditions hold. So we have only to prove the *existence* of k satisfying (10). For each k, there exist unitary matrices Q_k and P_k such that $\mathcal{S}_k = Q_k \mathcal{T}_k P_k$, where $\mathcal{T}_k$ has the block triangular form

$$\mathcal{T}_k = \begin{bmatrix} \mathcal{F}_k & * \\ 0 & \mathcal{G}_k \end{bmatrix} , \quad \text{with } \mathcal{F}_k := \begin{bmatrix} \mathcal{F}_{1k} & \cdots & * \\ & \ddots & \vdots \\ \text{O} & & \mathcal{F}_{sk} \end{bmatrix} ;$$

here, we assume that: (i) $\mathcal{F}_k$ is an upper triangular block matrix with s blocks, $\mathcal{F}_{1k}, \ldots, \mathcal{F}_{sk}$, along the diagonal; (ii) the integer s and the dimensions of the $\mathcal{F}_{ik}$

and $\mathcal{G}_k$ do not depend on k; (iii) each $\mathcal{F}_{ik}$ is a square, nonsingular pencil, with only one eigenvalue, denoted ξ_{ik}, and $\xi_{1k}, \ldots, \xi_{sk}$ are the distinct eigenvalues of $\mathcal{S}_k$; (iv) $\mathcal{G}_k$ has no eigenvalue, $H(\mathcal{G}_k) = H(\mathcal{S}_k)$ and $V(\mathcal{G}_k) = V(\mathcal{S}_k)$ (cf. [13, 14]). These conditions imply that $\mathcal{S}_k$ is strictly equivalent to $\mathcal{F}_{1k} \oplus \cdots \oplus \mathcal{F}_{sk} \oplus \mathcal{G}_k$ (see [5, 12]). We may assume that (Q_k), (P_k) and $(\xi_{ik} : k \in \mathbf{N})$ are convergent, and denote by ξ_i the limit of $(\xi_{ik} : k \in \mathbf{N})$, for $i = 1, \ldots, s$. Without loss of generality we assume: $\xi_{ik} \neq \infty$ if $\xi_i \neq \infty$, and $\xi_{ik} \neq 0$ if $\xi_i \neq 0$. (So far, we followed the idea of [11, pp.107-108]).

Let V_α be a closed neighborhood of α, small enough so that $V_\alpha \subset U_\alpha$ and no ξ_i lies in $V_\alpha \backslash \{\alpha\}$. Define, for $\omega \in [0, 1]$:

$$\mathcal{F}_{ik}^{\omega} = \mathcal{F}_{ik}^{\omega}(\lambda, \mu) := \begin{cases} \mathcal{F}_{ik}(\lambda + \mu\omega[\xi_i - \xi_{ik}]\,,\, \mu) & \text{if } \xi_i \neq \infty \\ \\ \mathcal{F}_{ik}(\lambda\,,\, \mu + \lambda\omega/\xi_{ik}) & \text{if } \xi_i = \infty\,. \end{cases}$$

Let $\mathcal{S}_k^{\omega} := Q_k \mathcal{T}_k^{\omega} P_k$, where $\mathcal{T}_k^{\omega}$ is the pencil obtained from $\mathcal{T}_k$ by replacing $\mathcal{F}_{1k}, \ldots, \mathcal{F}_{sk}$ with $\mathcal{F}_{1k}^{\omega}, \ldots, \mathcal{F}_{sk}^{\omega}$, respectively, and leaving the other blocks invariant.

We point out that the eigenvalue of $\mathcal{F}_{ik}^{\omega}$ lies on the *straight line segment* $[\xi_{ik}, \xi_i]$ (we put $[\xi, \infty] := \{\xi/\omega : 0 \leq \omega \leq 1\}$). We have one of two situations: either (I) $\alpha = \xi_t$, for some $t \in \{1, \ldots, s\}$, or (II) α is none of the ξ_i's. For large enough k we have: in case (I), $[\xi_{tk}, \xi_t]$ is a subset of V_α and $[\xi_{jk}, \xi_j] \cap V_\alpha$ is empty if $j \neq t$; in case (II), no $[\xi_{ik}, \xi_i]$ intersects V_α. So, in both cases, $M(V_\alpha, \mathcal{F}_{ik}^{\omega}) = M(V_\alpha, \mathcal{F}_{ik})$, and therefore $M(V_\alpha, \mathcal{S}_k^{\omega}) = M(V_\alpha, \mathcal{S}_k)$. Combined with Theorem 2.4 this yields, for large enough k,

$$M(V_\alpha, \mathcal{S}_k) \prec M(\alpha, \mathcal{S}_k^1)\,. \tag{11}$$

All $\mathcal{S}_k^1$ have the same spectrum, namely $\{\xi_1, \ldots, \xi_s\}$, and we have only a finite number of possible s-tuples $(M(\xi_1, \mathcal{S}_k^1), \ldots, M(\xi_s, \mathcal{S}_k^1))$. Therefore, there exists a subsequence of $(\mathcal{S}_k^1)$ with all entries in the same orbit. So we may assume, without loss of generality, that all pencils $\mathcal{S}_k^1$ lie in the same orbit. By [11], (6) holds with $\mathcal{S} = \mathcal{S}_k^1$; taking into account (11) and $h(\mathcal{S}_k^1) = h(\mathcal{S}_k)$ we obtain

$$M(V_\alpha, \mathcal{S}_k) \prec\!\!\prec M(\alpha, \mathcal{A}) + [h(\mathcal{A}) - h(\mathcal{S}_k)]\,, \tag{12}$$

for large enough k. By [11, Th.1], $M(\gamma, \mathcal{S}_k) \prec\!\!\prec [h(\mathcal{A}) - h(\mathcal{S}_k)]$ for all $\gamma \in U_\alpha \backslash V_\alpha$ and large enough k. Combining the last relation with (12) and Lemma 1.1, we obtain (10). ■

Proof of (9). For each $\alpha \in W$, let U_α be an open neighborhood of α such that $\bar{U}_\alpha \backslash \{\alpha\}$ contains no eigenvalue of $\mathcal{A}$. As W is compact, there exists $\alpha_1, \ldots, \alpha_r$ in W such that $U := U_{\alpha_1} \cup \cdots \cup U_{\alpha_r}$ contains W. By Lemma 3.2, (10) holds for all $\mathcal{S}_k$ sufficiently close to $\mathcal{A}$ and all $\alpha \in \{\alpha_1, \ldots, \alpha_r\}$. Applying Lemma 1.1 we get

$$[h(\mathcal{S})] + M(U, \mathcal{S}) \prec\!\!\!\prec [h(\mathcal{A})] + M(U, \mathcal{A}),$$

for all $\mathcal{S}_k$ sufficiently close to $\mathcal{A}$. As $M(W, \mathcal{S}) \prec\!\!\!\prec M(U, \mathcal{S})$ and $M(W, \mathcal{A}) = M(U, \mathcal{A})$, our claim (9) follows easily. ■

Theorem 3.3. *Let $\alpha_1, \ldots, \alpha_p$ be the distinct eigenvalues of $\mathcal{A}$ and, for each t, let U_t be a neighbourhood of α_t whose closure contains no other eigenvalue of $\mathcal{A}$. Define $U_0 := \mathbf{C}^\infty \backslash (U_1 \cup \cdots \cup U_p)$. There exists a neighborhood Ω of $\mathcal{A}$ such that any $\mathcal{S}$ in Ω satisfies* (8) *and*

$$[h(\mathcal{S})] + M(U_t, \mathcal{S}) \prec\!\!\!\prec [h(\mathcal{A})] + M(\alpha_t, \mathcal{A}), \quad \textit{for } t = 1, \ldots, p \tag{13}$$

$$\max M(U_0, \mathcal{S}) \leq h(\mathcal{A}) - h(\mathcal{S}). \tag{14}$$

Proof. Applying (9) to the closed sets $\bar{U}_1, \ldots, \bar{U}_p$, we obtain (13). Applying (9) to $\bar{U}_0$, we get $[h(\mathcal{S})] + M(U_0, \mathcal{S}) \prec\!\!\!\prec [h(\mathcal{A})]$, which is equivalent to (14). ■

Note that we may easily obtain (9) from (13)-(14). Thus Theorem 3.3 is an alternative formulation of Theorem 3.1.

4. A Converse Result

The following theorem shows that Theorem 3.1 is, in a certain sense, a best possible result. The reader should compare (15) with (8), and (16)-(17) with (13)-(14). As a corollary, we characterize the set of skeletons, $S_{\mathcal{A}}$, defined in the introduction.

Theorem 4.1. *Let p and q be nonnegative integers, $(M_1, \ldots, M_q, H, V)$ the skeleton of an $m \times n$ pencil, and $\mathcal{A}$ an $m \times n$ pencil having p (distinct) eigenvalues, $\alpha_1, \ldots, \alpha_p$. We are also given $p+1$ pairwise disjoint sets $X_0, \ldots, X_p$ such that $X_0 \cup \cdots \cup X_p = \{1, \ldots, q\}$. Denote by h the maximum of $H = (h_1, h_2, \ldots)$.*

We assume these entities satisfy

$$H \prec\!\!\prec H(\mathcal{A})\,, \quad V \prec\!\!\prec V(\mathcal{A}) \tag{15}$$

$$[h] + \bigcup_{s\in X_t} M_s \prec\!\!\prec [h(\mathcal{A})] + M(\alpha_t, \mathcal{A})\,, \quad \textit{for } t = 1,\ldots,p \tag{16}$$

$$\max \bigcup_{s\in X_0} M_s \leq h(\mathcal{A}) - h\,. \tag{17}$$

Finally, let U_t be a neighbourhood of α_t whose closure contains no eigenvalue of $\mathcal{A}$ other than α_t, and let $\{\tau_s : s \in X_0\}$ be a set of pairwise distinct elements of $\mathbf{C}^\infty\backslash(U_1 \cup \cdots \cup U_p)$ indexed by X_0.

For any neighbourhood Ω of $\mathcal{A}$ there exists a pencil $\mathcal{S}$ in Ω with q (distinct) eigenvalues, $\sigma_1, \ldots, \sigma_q$, satisfying the following conditions:

$$\left.\begin{array}{ll} \sigma_s = \tau_s\,, & \textit{for } s \in X_0 \\ \sigma_s \in U_t\,, & \textit{for } s \in X_t \textit{ and } t \in \{1,\ldots,p\} \end{array}\right\} \tag{18}$$

$$H(\mathcal{S}) = H \quad \textit{and} \quad V(\mathcal{S}) = V \tag{19}$$

$$M(\sigma_s, \mathcal{S}) = M_s\,, \quad \textit{for } s \in \{1,\ldots,q\}\,. \tag{20}$$

Proof. Let $\mathcal{N} = \mathcal{F} \oplus \mathcal{G}$ be an $m \times n$ pencil, in Kronecker normal form, such that $H(\mathcal{G}) = H$, $V(\mathcal{G}) = V$, $M(\alpha_t, \mathcal{F}) = \cup\{M_s : s \in X_t\}$, for $t = 1,\ldots,p$, and $M(\tau_s, \mathcal{F}) = M_s$, for $s \in X_0$. By [11], (15)-(17) mean that $\mathcal{A}$ lies in the closure of the orbit of $\mathcal{N}$. Therefore there exist invertible complex matrices, P and Q, such that $P\mathcal{N}Q$ lies in Ω.

Let σ_s, for $s \in X_1 \cup \cdots \cup X_p$, be any pairwise distinct complex numbers satisfying (18). For a fixed $t \in \{1,\ldots,p\}$ let us denote by J_{ti} the i-th greatest Jordan component of $\mathcal{N}$ with eigenvalue α_t. Clearly J_{ti} has order $\tilde{m}_i(\alpha_t, \mathcal{N})$ (recall: $\tilde{m}_i(\alpha_t, \mathcal{N})$ is the i-th coordinate of $\tilde{M}(\gamma_t, \mathcal{N})$, the conjugate of $M(\gamma_t, \mathcal{N})$). Assume α_t is finite (the case $\alpha_t = \infty$ is similar). We now perturb the diagonal of J_{ti} to obtain the following pencil

$$J'_{ti} := J_{ti} + \left[\bigoplus_{s\in X_t} (\sigma_s - \alpha_t)\mu I_{s,i}\right]\,,$$

where $I_{s,i}$ is the $\tilde{m}_{s,i} \times \tilde{m}_{s,i}$ identity matrix, with the notation $(\tilde{m}_{s,i} : i \in \mathbf{N})$ for the conjugate of M_s. Note that J_{ti} and $I_{s,i}$ have the same order, because

$\tilde{M}(\alpha_t, \mathcal{N}) = \Sigma\{\tilde{M}_s : s \in X_t\}$. J'_{ti} has spectrum $\{\sigma_s : s \in X_t\}$ with precisely one Jordan component with eigenvalue σ_s, and this component has order $\tilde{m}_{s,i}$.

Let us perturb in the way just described all the Jordan blocks occurring in $\mathcal{N}$, and let $\mathcal{N}'$ be the resulting pencil. Clearly, $\mathcal{S} := P\mathcal{N}'Q$ satisfies (19)-(20). Moreover, by appropriate choice of the σ_s ($s \notin X_0$) our perturbation may be made small enough so that $P\mathcal{N}'Q$ still lies in Ω. So the theorem holds with $\mathcal{S} := P\mathcal{N}'Q$. ■

Corollary 4.2. *The skeleton* $\Sigma = (M_1, \dots, M_q, H, V)$ *lies in* $S_{\mathcal{A}}$ *iff there exist pairwise disjoint sets* $X_0, \dots, X_p$*, satisfying* $X_0 \cup \dots \cup X_p = \{1, \dots, q\}$*, and the conditions* (15)-(17)*. That is,* Σ *belongs to* $S_{\mathcal{A}}$ *iff* $\mathcal{A}$ *lies in the closure of an equivalence orbit with skeleton* Σ*.* ■

Remark 4.3. The sequence of sets $X_0, \dots, X_p$ represents the way we wish to locate the eigenvalues of the perturbed pencil $\mathcal{S}$ with respect to those of $\mathcal{A}$: $\mathcal{S}$ is supposed to have a cluster of $|X_t|$ distinct eigenvalues around α_t, for each $t \in \{1, \dots, p\}$; the sum of the multiplicities of the eigenvalues in that cluster may be (i) less than, (ii) equal to, or (iii) greater than the multiplicity of α_t. In case (i) we are, in a certain sense, "deflating" α_t by a perturbation. If X_t is empty, that means we decided to rule out the possibility of $\mathcal{S}$ having an eigenvalue nearby α_t: in this extreme case, we may say that α_t has been *banished* from the eigenvalue list.

The eigenvalues σ_s, for $s \in X_0$ are unrelated to the spectrum of $\mathcal{A}$; we call them *alien eigenvalues* by obvious reasons: they may blossom in almost every place of $\mathbf{C}^{\infty}$ for convenient choices of arbitrarily small perturbations of $\mathcal{A}$. Alien eigenvalues are bound by the condition that no one of them is allowed to coincide with one α_t, otherwise *that* eigenvalue would have been labeled as a member of α_t's cluster. The other restriction on alien eigenvalues, namely (17) [or (14)], involves the maximum of $\cup\{M_s : s \in X_0\}$, which is nothing but the maximum number of Jordan blocks with the same alien eigenvalue. So the number of Jordan blocks of the perturbed pencil $\mathcal{S}$ corresponding to any given alien eigenvalue is at most $h(\mathcal{A}) - h(\mathcal{S})$ [clearly this equals $v(\mathcal{A}) - v(\mathcal{S})$]. Thus, roughly speaking, alien eigenvalues may be brought into existence at the cost of destruction of Kronecker blocks.

5. A Quantitative Refinement

In our final result we obtain a quantitative refinement of Theorem 4.1. Roughly speaking, we give a relation between positive real numbers ϵ and κ, such that, given a pencil $\mathcal{A}$, we may always find a pencil $\mathcal{S}$ inside a neighbourhood of $\mathcal{A}$ of radius ϵ, satisfying (19)-(20), with each cluster $\{\sigma_s : s \in X_t\}$ of nonalien eigenvalues *arbitrarily prescribed* in a neighbourhood of α_t of radius κ. It turns out that κ and ϵ have the same order of magnitude, for appropriate choices of the metrics in the space of pencils and in $\mathbf{C}^\infty$.

For pencils, distances are afforded by the following norm: if $\mathcal{P} = \lambda L + \mu M$ is an $m \times n$ pencil the *norm of* $\mathcal{P}$ is defined as the greatest singular value of the $m \times 2n$ complex matrix $[L\,,\,M]$.

To measure distances between eigenvalues we shall use the *chordal metric* of $\mathbf{C}^\infty$. Recall that $\mathbf{C}^\infty$ is naturally identified with a *Riemann sphere*, say $RS := \{(\xi, \eta, \zeta) : \xi^2 + \eta^2 + \zeta^2 = \zeta\}$, by means of the *stereographic projection* (see e.g. [4, pp.50-52] for details). The *chordal distance* between two points $z, w \in \mathbf{C}$, denoted $d^*(z, w)$, is the distance between the stereographic images of z and w in RS. We have

$$d^*(z,w) = |z-w| \left\{(1+|z|^2)(1+|w|^2)\right\}^{-1/2}. \tag{21}$$

If $\langle z_1, z_2 \rangle\,,\,\langle z_1, z_2 \rangle \in \mathbf{P}(\mathbf{C})$ satisfy $z_1/z_2 = z$ and $w_1/w_2 = w$ then (21) may be written in a nice symmetric way

$$d^*(z,w) = |z_1 w_2 - z_2 w_1| \left\{(1+|z_1|^2+|z_2|^2)(1+|w_1|^2+|w_2|^2)\right\}^{-1/2}. \tag{22}$$

This identity also holds in case $z = \infty$ or $w = \infty$. As $1/z$ is representable by z_2/z_1 and formula (22) is invariant under the interchanging of the subindices $1, 2$, it follows that $d^*(1/z, 1/w) = d^*(z, w)$.

Lemma 5.1. *Let $\sigma, \sigma' \in \mathbf{C}^\infty$ satisfy $d^*(\sigma, \sigma') < \rho$, where ρ is any positive real number such that $\rho < \sqrt{2}/2$. There exist $b, c \in \mathbf{C}$ such that $bc = 0$, $\sigma' = (\sigma + b)/(c\sigma + 1)$ and* $\max\{|b|, |c|\} < 2\rho/(1 - \sqrt{2}\,\rho)$.

Proof. First assume that $|\sigma| \leq 1$, i.e., the stereographic image of σ lies in the south hemisphere of RS. As $d^*(\sigma, \sigma') < \sqrt{2}/2$, we have $\sigma' \neq \infty$. Let us put $c := 0$

and $b := \sigma' - \sigma$. We easily get

$$(1+|\sigma|^2)(1+|\sigma'|^2) \leq 2[1+(|b|+|\sigma|)^2] \leq 2[2+2|b|+|b|^2] \leq 4(1+|b|/\sqrt{2})^2 .$$

Combining this with $d^*(\sigma,\sigma') < \rho$ and (21) we obtain $|b|/(1+|b|/\sqrt{2}) \leq 4\rho$, that is, $|b| \leq 2\rho/(1-\sqrt{2}\,\rho)$. The proof is complete in case $|\sigma| \leq 1$.

Now let $|\sigma| \geq 1$ (this includes the case $\sigma = \infty$). As $d^*(1/\sigma, 1/\sigma') < \sqrt{2}/2$, we have $1/\sigma' \neq \infty$, etc, etc. The proof is the same as before, taking the inverses of σ and σ' and reversing the roles of b and c. ∎

Lemma 5.2. *Let $\mathcal{P}'$ be a nonsingular pencil with exactly one eigenvalue $\sigma' \in \mathbf{C}^\infty$. If $\sigma \in \mathbf{C}^\infty$ satisfies $d^*(\sigma,\sigma') < \rho$, where $\rho < \sqrt{2}/2$, then there exists a pencil $\mathcal{P}$ with the same skeleton as $\mathcal{P}'$, having σ as (unique) eigenvalue and satisfying $\|\mathcal{P}-\mathcal{P}'\| < \|\mathcal{P}'\| 2\rho/(1-\sqrt{2}\,\rho)$.*

Proof. Write $\mathcal{P}' = \lambda L + \mu M$ and define $\mathcal{P} := (\lambda + b\mu)L + (c\lambda + \mu)M$, where b and c are as given in Lemma 5.1. Clearly $\mathcal{P}$ has the same skeleton as $\mathcal{P}'$ and has eigenvalue σ, because $\sigma' = (\sigma+b)/(c\sigma+1)$. Moreover

$$\|\mathcal{P}-\mathcal{P}'\| = \|b\lambda L + c\mu M\| = \|[bL\,,\,cM]\| = \|[L\,,\,M](bI \oplus cI)\| \leq \|\mathcal{P}'\| \max\{|b|,|c|\}.$$

Therefore, by Lemma 5.1, $\|\mathcal{P}-\mathcal{P}'\| < \|\mathcal{P}'\| 2\rho/(1-\sqrt{2}\,\rho)$. ∎

Theorem 5.3. *For a nonzero pencil $\mathcal{A}$, let us adopt all notations and assumptions of* Theorem 4.1 *up to* (17) *inclusive. Moreover, let us be given a real κ satisfying $0 < \kappa < \sqrt{2}/2$, and a set $\{\sigma_1,\ldots,\sigma_q\}$ of q distinct elements of $\mathbf{C}^\infty$ such that, for all $t \in \{1,\ldots,q\}$,*

$$\sigma_v \neq \alpha_t \ \ \textit{for}\ v \in X_0\,, \quad \textit{and} \quad d^*(\sigma_s,\alpha_t) < \kappa \ \ \textit{for}\ s \in X_t\,. \tag{23}$$

Then there exists a pencil $\mathcal{S}$ satisfying $\|\mathcal{A}-\mathcal{S}\| < \|\mathcal{A}\| 2\kappa/(1-\sqrt{2}\,\kappa)$ and the conditions (19) *and* (20).

Proof. Given $\sigma_1,\ldots,\sigma_q$, let δ be a positive real number satisfying $\delta < d^*(z,w)/10$, for all distinct z, w in $\{\alpha_1,\ldots,\alpha_p\} \cup \{\sigma_s : s \in X_0\}$. We shall apply Theorem 4.1 with $U_t := \{z : d^*(z,\alpha_t) < \delta\}$ and $\tau_s := \sigma_s$, for $s \in X_0$.

Choose a real ρ so that $0 < \rho < \kappa$ and $d^*(\sigma_s,\alpha_t) < \delta$ for all s in $X_1 \cup \cdots \cup X_p$, and choose a positive η satisfying

$$\eta + (\|\mathcal{A}\| + \eta)2\rho/(1-\sqrt{2}\,\rho) < \|\mathcal{A}\| 2\kappa/(1-\sqrt{2}\,\kappa)\,. \tag{24}$$

Such η exists because (24) is equivalent to $\eta(1 + f_\rho) < (f_\kappa - f_\rho)\|\mathcal{A}\|$ [here, f_x denotes $2x/(1-\sqrt{2}x)$], $\mathcal{A}$ is nonzero, and $f_\kappa > f_\rho$. By Theorem 4.1, there exists an $m \times n$ pencil $\mathcal{S}'$ such that $\|\mathcal{A} - \mathcal{S}'\| < \eta$, with distinct eigenvalues $\sigma'_1, \ldots, \sigma'_q$ satisfying (18)-(20) (with $\mathcal{S}$ and σ_s replaced by $\mathcal{S}'$ and σ'_s). A closer look at the proof of Theorem 4.1 shows that we may further assume $d^*(\sigma_s, \sigma'_s) \leq d^*(\sigma_s, \alpha_t)$; therefore, by (24),

$$d^*(\sigma_s, \sigma'_s) \leq \rho\,, \quad \text{for all } s \in \{1, \ldots, q\}\,. \tag{25}$$

Let P and Q be complex unitary matrices such that

$$\mathcal{S}' = P \begin{bmatrix} \mathcal{F} & * \\ 0 & \mathcal{G} \end{bmatrix} Q\,, \quad \text{with } \mathcal{F} = \begin{bmatrix} \mathcal{F}_1 & \cdots & * \\ & \ddots & \vdots \\ \mathrm{O} & & \mathcal{F}_q \end{bmatrix}, \tag{26}$$

where $\mathcal{G}$ has no eigenvalues, $H(\mathcal{G}) = H(\mathcal{S}')$, $V(\mathcal{G}) = V(\mathcal{S}')$ and each $\mathcal{F}_s$ is square nonsingular with σ'_s as unique eigenvalue, for $s = 1, \ldots, q$ (cf. [13, 14]). According to (25), Lemma 5.2 yields, for each s, a pencil $\mathcal{E}_s$ with the same skeleton as $\mathcal{F}_s$, having eigenvalue σ_s and satisfying $\|\mathcal{E}_s - \mathcal{F}_s\| \leq \|\mathcal{S}'\| 2\rho/(1 - \sqrt{2}\,\rho)$, for $s = 1, \ldots, q$.

Let $\mathcal{S}$ be the pencil obtained from $\mathcal{S}'$ by replacing (in the representation (26)) each block $\mathcal{F}_s$ by $\mathcal{E}_s$ and leaving all other blocks invariant. Clearly $\mathcal{S}$ satisfies (19)-(20). Moreover

$$\|\mathcal{S} - \mathcal{S}'\| = \sup\{\|\mathcal{E}_s - \mathcal{F}_s\| : s = 1, \ldots, q\} \leq \|\mathcal{S}'\| 2\rho/(1 - \sqrt{2}\,\rho)\,.$$

As $\|\mathcal{A} - \mathcal{S}'\| < \eta$ we have $\|\mathcal{S}'\| < \|\mathcal{A}\| + \eta$. We thus obtain the following sequence of inequalities, the last of which is (24):

$$\begin{aligned} \|\mathcal{A} - \mathcal{S}\| &\leq \|\mathcal{A} - \mathcal{S}'\| + \|\mathcal{S} - \mathcal{S}'\| < \eta + \|\mathcal{S}'\| 2\rho/(1 - \sqrt{2}\,\rho) \\ &< \eta + (\|\mathcal{A}\| + \eta) 2\rho/(1 - \sqrt{2}\,\rho) < \|\mathcal{A}\| 2\kappa/(1 - \sqrt{2}\,\kappa)\,. \end{aligned}$$

The theorem is proved.

References

[1] M. Berger, *Géométrie*, Cédic, Fernand Nathan, Paris 1979 (2^{nd} Ed.).

[2] H. Den Boor and G. Thijsse, Semi-Stability of Sums of Partial Multiplicities Under Additive Perturbations, *Int. Equat. Oper. Th.* 3:23-42 (1980)

[3] J. Dettman, *Applied Complex Variables*, The MacMillan Company, New York 1966

[4] J. Duncan, *The Elements of compex Analysis*, J. Wiley & Sons, London 1968

[5] F.R. Gantmacher, *The Theory of Matrices*, Chelsea Publishing Company, New York 1960

[6] J. Gracia, I. Hoyos and I. Zaballa, Perturbations of Linear Control Systems, *Linear Algebra and its Applications* 121:353-383 (1989).

[7] G. Hardy, J. Littlewood and G. Polya, *Inequalities*, Cambridge Univ. Press, London 1952.

[8] I. Hoyos, PhD Thesis, Vitoria 1990.

[9] A. Markus and E. Parillis, The Change of the Jordan Structure of a Matrix Under Small, *Linear Algebra and its Applications* 54:139-152 (1983) [translated from *Mat. Issled.* 54:98-109 (1980)].

[10] A. Marshall and I. Olkin, *Inequalities: Theory of Majorization and its Applications*, Academic Press, New York 1972.

[11] A. Pokrziwa, On Perturbations and the Equivalence Orbit of a Matrix Pencil, *Linear Algebra and its Applications* 82:99-121 (1986).

[12] W. Roth, The Equations $AX - YB = C$ and $AX - XB = C$ in Matrices, *Proc. Amer. Math. Soc.* 3:392-396 (1952).

[13] G. Stewart, On the Sensitivity of the Eigenvalue Problem $Ax = \lambda Bx$, *SIAM J. Numer. Anal.* 9:669-686 (1972).

[14] P. van Dooren, The Computation of Kronecker's Canonical Form of a Singular Pencil, *Linear Algebra and its Applications* 27:103-140 (1979).

Departamento de Matemática
Universidade de Coimbra

PRÉ-PUBLICAÇÕES

98-04 Andrew R. Conn, Luís N. Vicente, Chandu Visweswariah, *Two-step algorithms for nonlinear optimization with structured applications.*

98-03 E. Marques de Sá e Yulin Zhang, *The number of Kronecker indices of a special kind of square pencils.*

98-02 E. Marques de Sá e Yulin Zhang, *Nonsingular submatrices and the off diagonal indices of a square matrix.*

98-01 Paulo Eduardo Oliveira, *A central limit theorem for associated variables.*

97-26 Alexander Kovačec, *Semidefiniteness on cones and games on lattices.*

97-25 Pierre Jacob e Paulo Eduardo Oliveira, *Using non embedded partitions in histogram estimation.*

97-24 José Francisco Rodrigues e José Miguel Urbano, *On the Mathematical Analysis of a Valley Glacier Model.*

97-23 J. A. Ferreira e R. D. Grigorieff, *On the supraconvergence of elliptic finite difference schemes.*

97-22 Paula de Oliveira, Fernanda Patrício e João Santos, *Positive solutions with multistep methods for convection-diffusion-reaction equations.*

97-21 A. M. d'Azevedo Breda e F. J. Craveiro de Carvalho, *Products, covering spaces and group actions for locally Sierpinski spaces.*

97-20 E. Gonçalves e N. Mendes-Lopes, *Exponential decay of the convergence rate of a non-classical test for an AR model.*

97-19 Jiří Adámek e Lurdes Sousa, *Constructions of solid hulls.*

97-18 Jaime Carvalho e Silva, *O Ensino da Matemática na Universidade de Coimbra na segunda metade do séc. XIX.*

97-17 Ana Paula Santana, João Filipe Queiró e Eduardo Marques de Sá, *Group Representations and Matrix Spectral Problems.*

97-16 Maria Manuel Clementino, Eraldo Giuli e Walter Tholen, *What is a quotient map with respect to a closure operator?.*

97-15 Carlos Tenreiro, *Asymptotic normality of local polynomial estimators of regression function and its derivatives for time series.*

97-14 Luís Nunes Vicente, *An analysis of Newton's method for equivalent Karush-Kuhn-Tucker systems.*

97-13 F. J. Craveiro de Carvalho e S. A. Robertson, *The parallel group of a plane curve.*

97-12 João Filipe Queiró, *Matemática e Matemáticos em Portugal.*

97-11 Joana Margarida Nunes da Costa, *Variétés de Jacobi compatibles. Géométrie et réduction.*

97-10 J. A. Ferreira, *On moving finite difference methods for partial differential equations on two-dimensional domains.*

97-09 C.-M. Marle e J. M. Nunes da Costa, *Master symmetries and bi-Hamiltonian structures for the relativistic Toda lattice.*

97-08 C. R. Johnson e António Leal Duarte, *The maximum multiplicity of an eigenvalue in a matrix whose graph is a tree.*

97-07 António Leal Duarte, Jaime Carvalho e Silva, João Filipe Queiró, *Some notes on the History of Mathematics in Portugal.*

97-06 J. A. Ferreira, *The negative norms on the supraconvergence of FDM's for two-dimensional domains.*

97-05 E. Gonçalves e N. Mendes Lopes, *A consistent test for a vectorial AR(1) model with a non-independent error process.*

97-04 C. M. Martins, *On the autocorrelation structure of a bilinear model with non-independent shocks.*

97-03 C. M. Martins, *A note on the third-order moment structure of a bilinear model with non-independent shocks.*

97-02 F. Silva Leite e P. Saraiva, *A remarkable example of simple r-Lie algebras.*

97-01 Jorge Picado, *Efremovič's Theorem in pointfree context.*

96-18 Luís Nunes Vicente, *On interior-point Newton algorithms for discretized optimal control problems with state constraints.*

96-17 C. Gouriéroux e C. Tenreiro, *Local power properties of kernel based goodness of fit tests.*

96-16 Olga Azenhas, *The admissible interval for the invariant factors of a product of matrices.*

96-15 Alexander Kovačec, *A note on the Bruhat order of* S_n*, integer matrices, and the lattice* $\mathbb{N}^2$.

96-14 Jorge Picado, *Structured frames by Weil entourages.*

96-13 F. Silva Leite, G. Kun e P. Crouch, *Generalized De Casteljau algorithm for polynomial curves on Lie groups and symmetric spaces.*

96-12 F. Marcellán e J. Petronilho, *Orthogonal polynomials and quadratic transformations.*

96-11 E. Gonçalves, P. Jacob e N. Mendes Lopes, *A consistent test for bilinear time series.*

96-10 Cristina Caldeira, *Critical pairs of matrices for the degree of the minimal polynomial of the Kronecker sum.*

96-09 Esmeralda Gonçalves e Nazaré Mendes Lopes, *Statistical results on autoregressive heteroscedastic models.*

96-08 Isabel Figueiredo e Carlos Leal, *Ellipticity of the Koiter and Naghdi models for nonhomogeneous anisotropic shells.*

96-07 Maria Manuel Clementino e Walter Tholen, *Separated and connected maps.*

96-06 Lurdes Sousa, *Note on multisolid categories.*

96-05 J.M. Nunes da Costa, *On the reduction of Poisson-Nijenhuis structures by the Flaschka transformation.*

96-04 A.I. Aptekarev, A. Branquinho e F. Marcellán, *Toda-type differential equations for the recurrence coefficients of orthogonal polynomials and Freud transformation.*