Another Approach to Topological Descent Theory

Manuela Sobral*

Abstract

In the category $\mathcal{T}op$ of topological spaces and continuous functions, we prove that descent morphisms with respect to the class $\mathbb{E}$ of continuous bijections are exactly the descent morphisms, providing a new characterization of the latter in terms of subfibrations $\mathbb{E}(X)$ of the basic fibration given by $\mathcal{T}op/X$ which are, essentially, complete lattices. Also effective descent morphisms are characterized in terms of effective morphisms with respect to continuous bijections. For classes $\mathbb{E}$ satisfying suitable conditions, we show that the class of effective descent morphisms coincides with the one of effective $\mathbb{E}$-descent morphisms.

Key words: Descent data, (effective) descent map, monad, monadic functor, universal regular epimorphism, (effective) étale-descent.

AMS Subj. Class.: 54B30, 18A30, 18C20, 18A40, 54C10.

1 Introduction

Let $\mathcal{U}, \mathcal{R}$ and $\mathcal{E}$ denote the classes of universal regular epimorphisms, regular epimorphisms and effective descent morphisms, respectively.

In $\mathcal{T}op$, descent morphisms are exactly the universal regular epimorphisms ([6], 2.2) and one has the following inclusions $\mathcal{E} \subseteq \mathcal{U} \subseteq \mathcal{R}$.

It is well-known that the second inclusion is strict. In [9], J. Reiterman and W. Tholen gave a filter-theoretic characterization of effective descent maps as well as an example to show that $\mathcal{E}$ is properly contained in $\mathcal{U}$. Aiming to understand better the first we looked for an easier example of a descent map which is not effective for descent.

Using a criterion presented in [10], we give a very simple example involving bijective bundles over finite and quite small spaces. We also describe a way to define non-effective descent morphisms.

The subfibration given by the bijective maps over some space X of the basic fibration given by $\mathcal{T}op/X$ is a complete lattice: it is a small complete category with at most one morphism between each pair of objects. Such categories are called thin in [1].

In this context, these are relevant subcategories. Indeed, descent morphisms with respect to bijective maps (bijective-descent) are exactly the descent morphisms. Also effective descent maps can be characterized in terms of effective maps for bijective-descent.

Furthermore, for regular epimorphisms $p : E \to B$, bijective bundles occur in a natural way in the category $Des(p)$ of bundles over E equipped with descent data and morphisms compatible with it: for each object $(C, \gamma; \xi)$, the morphism γ from (C, γ, ξ) to the terminal object $(E, 1_E; p_1)$ has a

*Partial financial support by CMUC/FCT, PRAXIS Projects 2/2.1/MAT/46/94 and PCEX/P/MAT/46/96 is gratefully acknowledged.

(Bijective, $\mathcal{M}$)-factorization, which plays an important rôle here. It is the factorization induced in Des(p) by the comparison adjunction, as defined in Theorem 3.3 of [4].

A closer look to the meaning of descent data suggests a formulation of effective-global descent in terms of effective descent with respect to surjective maps (surjective-descent). We prove that, not only for surjective maps but also for classes $\mathbb{E}$ containing these maps and satisfying suitable conditions, the (effective) descent morphism are exactly the (effective) $\mathbb{E}$-descent maps.

I gratefully acknowledge very useful discussions with George Janelidze and Walter Tholen on the subject of this paper.

I am also grateful to Maria Cristina Pedicchio for inviting me to visit the University of Trieste, where the conversations with George Janelidze took place.

2 Notations and definitions

For a continuous map $p : E \to B$, let $\mathbb{T}=$ (T, η, μ) be the monad induced in $\mathcal{T}op/E$ by the adjunction

$$p! \dashv p^* : \mathcal{T}op/B \to \mathcal{T}op/E,$$

where p^* and $p!$ are defined by pulling back along p and by composition on the left with p, respectively.

Descent data for an object (C, γ), with respect to p, is given by a $\mathbb{T}$-structure map

$$\xi : (E \times_B C, \pi_1) \to (C, \gamma),$$

where $(E \times_B C, \pi_1, \pi_2)$ is the pullback of $(p, p\gamma)$. Indeed, the category $Des(p)$ of bundles over the space E equipped with descent data and maps compatible with it is, up to isomorphism, the Eilenberg-Moore category $(\mathcal{T}op/E)^{\mathbb{T}}$ of $\mathbb{T}$-algebras([3] and Beck (unpublished)).

If $\mathbb{E}$ is a class of morphisms in $\mathcal{T}op$ which is stable under pullback along p, the restriction of p^* to the full subcategory of $\mathcal{T}op/B$ with objects all $\mathbb{E}$-bundles over the space B is a functor $p^* : \mathbb{E}(B) \to \mathbb{E}(E)$. In the commutative diagram

$$\begin{tikzcd}
\mathcal{T}op/B \arrow[rr, "\Phi^p"] \arrow[dr] & & Des(p) \cong (\mathcal{T}op/E)^{\mathbb{T}} \arrow[dl] \\
& \mathcal{T}op/E & \\
\mathbb{E}(B) \arrow[uu] \arrow[rr, "\Phi^p_{\mathbf{E}}"] \arrow[dr, "p^*"'] & & Des_{\mathbf{E}}(p) \arrow[uu] \arrow[dl, "U^p"] \\
& \mathbb{E}(E) \arrow[uu] &
\end{tikzcd}$$

the vertical arrows are full embeddings, Φ^p is the comparison functor and $Des_{\mathbf{E}}(p)$ is the full subcategory of $Des(p)$ with objects all $\mathbb{T}$-algebras $(C, \gamma; \xi)$ such that $\gamma \in \mathbb{E}$.

A map p is $\mathbb{E}$-*descent* if $\Phi^p_{\mathbb{E}}$ is full and faithful and p is *effective* $\mathbb{E}$-*descent* if $\Phi^p_{\mathbb{E}}$ is an equivalence.

In case $\mathbb{E}$ is the class of all continuous maps, the prefix $\mathbb{E}$ is dropped. However, for emphasis, we sometimes use the terminology of [7] and speak of (effective) global-descent. We also speak about open-descent, surjective-descent and bijective-descent when $\mathbb{E}$ is the class of open embeddings, surjective and bijective maps, respectively.

3 Effective descent versus descent

In a category $\mathcal{X}$ with pullbacks, a *universal regular epimorphism* is a morphism whose pullback along any morphism is a regular epimorphism.

Universal regular epimorphisms are descent morphisms in all categories with pullbacks ([6], 2.2).

Conversely, the pullback of a descent morphism $p : E \to B$, along any morphism, is the coequalizer of its kernel pair in $\mathcal{X}/B$ and so it is a regular epimorphisms in this category. Indeed, the descent condition for a morphism $p : E \to B$ implies the existence of coequalizers of some kernel pairs in $\mathcal{X}/B$. These are the coequalizers of the corresponding pairs in $\mathcal{X}$, provided they exist. In this case, the classes of universal regular epimorphisms and of descent morphisms coincide.

If, furthermore, $\mathcal{X}$ has a (Reg Epi, Mono)-factorization of morphisms then p is a descent morphism if and only if p^* reflects isomorphisms as it follows from the proof of Theorem 1.1 of [6].

The universal regular epimorphisms in $\mathcal{T}op$ where characterized by Day and Kelly in [5]. They are the morphisms $p : E \to B$ such that, for each $b \in B$ and directed open cover $\mathcal{D}$ of $p^{-1}(b)$, $p(V)$ is a neighbourhood of b, for some $V \in \mathcal{D}$.

For a morphism $p : E \to B$ in $\mathcal{T}op$ and $(C, \gamma; \xi) \in Des(p)$, let $q = coeq(\pi_2, \xi)$ and δ be the unique morphism such that $\delta \cdot q = p \cdot \gamma$. Then $(Q, \delta) = \Psi^p(C, \gamma; \xi)$, for the left adjoint Ψ^p to $\Phi^p : \mathcal{X}/B \to Des(p)$.

The diagram

$$\begin{tikzcd} E \times_B C \arrow[r, shift left, "\pi_2"] \arrow[r, shift right, "\xi"'] & C \arrow[r, "q"] \arrow[d, "\gamma"] & Q \arrow[d, "\delta"] \\ & E \arrow[r, "p"'] & B \end{tikzcd}$$

will be called *a descent situation defining* Q.

We recall that, for the comparison adjunction $\Psi^p \dashv \Phi^p(\alpha, \beta)$, the unit and counit are defined by

$$\alpha_{(C,\gamma;\xi)} = <\gamma, q> \text{ and } \beta_{(A,f)} \cdot g = \pi_2,$$

for π_2 the pullback of p along f and g the coequalizer of its kernel pair. Furthermore, they are pointwise bijective maps if p is surjective.

The following criterion will be our main tool in the sequel.

Theorem 3.1 *([10], 2.8) In $\mathcal{T}op$, p is an effective descent morphism if and only if it is a universal regular epimorphism and, for every descent situation defining Q, the square is a pullback.*

Let $\mathbb{T}$ be the monad defined in the introduction. For a $\mathbb{T}$-algebra $(C, \gamma; \xi)$, we have that

$$\gamma \cdot \xi = \pi_1, \text{ because } \xi \text{ is a morphism of } \mathcal{T}op/E,$$

$$\xi \cdot \eta = 1 \text{ and } \xi \cdot 1 \times_B \xi = \xi \cdot 1 \times_B \pi_2,$$

because ξ is a $\mathbb{T}$-structure map.

From the equality $\gamma \cdot \xi = \pi_1$ and the fact that (π_2, ξ) is an effective equivalence relation (see e.g. [10], 2.2 and 2.4), it is easy to prove the following:

Proposition 3.2 *For a morphism $p : E \to B$ in $\mathcal{T}op$ and a descent situation as above, the following holds:*

(i) If $p^{-1}(b) \cap \gamma(C) \neq \emptyset$ then $p^{-1}(b) \subseteq \gamma(C)$;

(ii) For $c, c' \in C$, $q(c) = q(c')$ if and only if $\xi(\gamma(c), c') = c$ or, equivalently, $\xi(\gamma(c'), c) = c'$.

From (i), we conclude that, for $(C, \gamma; \xi) \in Des(p)$, the subspace $\gamma(C)$ of E is the pullback along p of a subspace of B.

The second item tells us how to define the coequalizer of the pair (π_2, ξ).

We present now a very simple example of a non-effective descent map.

Example 3.3 Let E be the set $\{e_{11}, e_{12}, e_{21}, e_{22}, e_{31}, e_{32}\}$ with the topology generated by the subsets $U_1 = \{e_{11}, e_{21}\}$ and $U_2 = \{e_{22}, e_{31}\}$ and B be the set $\{b_1, b_2, b_3\}$ with the indiscrete topology.

The function $p : E \to B$ defined by $p(e_{ir}) = b_i$ is a universal regular epimorphism but it is not effective for global descent. To prove the latter, consider $(C, \gamma; \xi) \in Des(p)$, where C has the same underlying set as E and the topology generated by the topology of E and the open set $\{e_{21}\}$. Then we obtain a bundle (C, γ), with $\gamma(x) = x$, equipped with descent data in the only possible way: the $\mathbb{T}$-structure map is the function $\xi : E \times_B C \to C$ defined by $\xi(x, y) = x$. Indeed, the function ξ satisfies the equalities

$$\gamma \cdot \xi = \pi_1, \ \xi \cdot \eta = 1 \text{ and } \xi \cdot 1 \times_B \xi = \xi \cdot 1 \times_B \pi_2,$$

and is continuous because

$$\xi^{-1}(e_{21}) = U \times_B (V \cup \{e_{21}\}).$$

Hence we have a descent situation defining B

$$\begin{tikzcd} E \times_B C \arrow[r, shift left, "\pi_2"] \arrow[r, shift right, "\xi"'] & C \arrow[r, "q"] \arrow[d, "\gamma"] & B \arrow[d, "id"] \\ & E \arrow[r, "p"'] & B \end{tikzcd}$$

in which the square is not a pullback.

More general examples can be defined using a similar method as we describe next.

Let B be a three element space $\{b_1, b_2, b_3\}$ which has a non-open indiscrete subspace with two elements, say $\{b_2, b_3\}$. Consider a universal quotient $p : E \to B$ satisfying the following conditions: there exist non-empty open sets U_i in E, for $i = 1, 2$, such that

- $p^{-1}(b_2) \subset U_1 \cup U_2$;
- $p^{-1}(b_2) \cap U_i$ is not open in E, for $i = 1, 2$;
- $p^{-1}(x) \cap U_i = \emptyset$ whenever $p^{-1}(x) \cap U_j \neq \emptyset$, for $x \neq b_2$ and $i \neq j$.

It is obvious that the open sets U_1 and U_2 are not contained in $p^{-1}(b_2)$ and, without loss of generality, we can assume that $p^{-1}(b_1) \cap U_1 \neq \emptyset$ and $p^{-1}(b_3) \cap U_2 \neq \emptyset$. Moreover, $p^{-1}(b_2)$ has at least two elements, otherwise it would be open in E and so $\{b_2\}$ would be open in B.

If we add to the topology of E the set $W = p^{-1}(b_2) \cap U_1$ we obtain a bundle (C, γ), where the underlying function of γ is the identity, equipped with descent data $\xi : E \times_B C \to C$ defined by $\xi(x, y) = x$, as in the above example.

Also here $p \cdot \gamma$ is the coequalizer of (π_2, ξ), i.e. C has the same p-saturated open sets as E. To prove that, we recall that $\mathcal{O}(C)$ is a singly generated frame extension of $\mathcal{O}(E)$, as introduced by B. Banaschewski in [2], and the open sets of C are of the form

$$W' = L_1 \cup (W \cap L_2),$$

where L_i, for $i = 1, 2$, are open sets in E. Furthermore, we may take $L_1 \subseteq L_2$.

Let $W' = p^{-1}(S)$ be an open set in C which is not open in E. Since $W \cap L_2 \neq \emptyset$, we have that $p^{-1}(b_2) \subseteq L_2$, by the definition of W' and because $L_1 \subseteq L_2$. Hence $W' = L_1 \cup W$. Suppose that $S = \{b_2\}$ or $S = \{b_1, b_2\}$. Then $p^{-1}(b_2) \subseteq L_1 \cup U_1$ and, by the universality of the quotient p, this implies that b_2 belongs to some open subset of $p(L_1), p(U_1)$ or $p(L_1 \cup U_1)$, which is false. Also $S \neq \{b_2, b_3\}$, otherwise $p^{-1}(b_3) \subset L_1$ and so b_3 would belong to some open set contained in $p(L_1) = \{b_2, b_3\}$. Therefore, $S = B$ and so that $W' = E$, which contradicts our assumption.

Under the above conditions, we can define a topology on the underlying set of E generated by $\mathcal{O}(E) \cup W$, where W has no descent data with respect to p, to obtain a bundle (C, γ) equipped with descent data ξ for which $q=$ coeq(π_2, ξ) is $p \cdot \gamma$. Thus, like in the example above, we have a descent situation defining B where the square is not a pullback.

Since effective descent morphisms are stable under pullback ([10], 3.1) all universal quotients whose pullback along the subspace embedding of some three-element subspace is a map p satisfying the prescribed conditions, are non-effective descent morphisms.

4 Characterizations of effective descent maps

Let $\mathbb{E} \subseteq \mathbb{E}'$ be classes of continuous functions stable under pullback along a morphism p and closed under composition with isomorphisms.

Proposition 4.1 *([7], 2.6) The map p is $\mathbb{E}$-descent if it is $\mathbb{E}'$-descent. An effective $\mathbb{E}'$-descent map p is effective for $\mathbb{E}$-descent if and only if for each pullback diagram*

$$\begin{tikzcd} E \times_B D \arrow[r, "\pi_2"] \arrow[d, "\pi_1"'] & D \arrow[d, "\delta"] \\ E \arrow[r, "p"'] & B \end{tikzcd}$$

if $\pi_1 \in \mathbb{E}$ *and* $\delta \in \mathbb{E}'$ *then* $\delta \in \mathbb{E}$.

When this transferability condition holds for $\mathbb{E}'$ the class of all morphisms in $\mathcal{T}op$, we say that $\mathbb{E}$ *descends along* p.

In this case,

- the $\mathbb{E}$-descent maps are exactly the $\mathbb{E}$-*universal regular epimorphisms* (as it follows from Proposition 1.6 in [7]), that is the morphisms whose pullbacks along $\mathbb{E}$-morphisms are regular epimorphisms.
- effective descent morphisms are effective $\mathbb{E}$-descent maps.

Theorem 4.2 *A map in* $\mathcal{T}op$ *is global-descent if and only if it is a bijective-descent map.*

Proof. It remains to prove each bijective-descent morphism $p : E \to B$ is a descent map. For a directed open cover $\mathcal{D}$ of $p^{-1}(b)$, for some $b \in B$, consider a space B' with the same underlying set as B and the coarsest topology containing the open sets of the space B and the sets of the form

$$\{b\} \cup B \setminus p(V)$$

for all $V \in \mathcal{D}$, such that $V \cap p^{-1}(b) \neq \emptyset$.

The pullback $p' : E' \to B'$ of p along the map $i : B' \to B$, with $i(x) = x$, is a quotient. Indeed, bijective-descent maps are bijective-regular epimorphisms because bijective maps descend along surjections and p is surjective. Consequently, they are quotients because identities are bijective maps.

The set $p^{-1}(b)$ is open in E' because each $x \in p^{-1}(b)$ belongs to

$$V \cap p^{-1}(\{b\} \cup B \setminus p(V)) \subseteq p^{-1}(b),$$

for some $V \in \mathcal{D}$, which is an open subset of E'. Hence $\{b\}$ is open in B' and so

$$\{b\} = U \cap (\{b\} \cup B \setminus p(V))$$

for some U open in E and $V \in \mathcal{D}$. Therefore, $b \in U \subseteq p(V)$, i.e. $b \in int(p(V))$. □

We remark that, since only the neighbourhoods of b in B' are relevant in the proof 4.2, the space defined in the proof of Theorem 1 in [5] can also be used to prove our claim.

From 4.1 and 4.2 one immediatly obtains the result below.

Corollary 4.3 *For classes* $\mathbb{E}$ *stable under pullback and containing the continuous bijections,* $\mathbb{E}$-*descent maps are exactly the descent maps.*

For $\mathbb{E}$ the class of bijective maps, the subfibration given by $\mathbb{E}(X)$ of the basic fibration given by $\mathcal{T}op/X$ is a small category, because $\mathcal{T}op$ is well-powered. Since there is at most one morphism between any two objects, $\mathbb{E}(X)$ is, up to equivalence, the complete lattice of all topologies on spaces X' for which $1 : X' \to X$ is continuous.

If p is surjective, the functor $p^* : \mathbb{E}(B) \to \mathbb{E}(E)$ has a left adjoint L defined on objects by

$$L(C, \gamma) = (D, \delta),$$

where $\delta \cdot q$ is the (Reg Epi, Mono)-factorization of $p \cdot \gamma$. The counit $\overline{\epsilon}$ of the adjunction has as components the maps induced by the diagonal property of the factorization: for each $(A, f) \in \mathbb{E}(B)$ the counit $\overline{\epsilon}_{(A,f)}$ is the unique map such that $\overline{\epsilon}_{(A,f)} \cdot q = \pi_2$ and $f \cdot \overline{\epsilon}_{(A,f)} = \delta$, for $(D, \delta) = L(E \times_B A, \pi_1)$. Then, since p^* restricted to $\mathbb{E}(B)$ is obviously faithful, we have that

$$p \text{ is a descent morphism } \Leftrightarrow p^* : \mathbb{E}(B) \to \mathbb{E}(E) \text{ is full}$$

and this occurs if and only if $\mathbb{E}(B)$ is a sublattice of $\mathbb{E}(E)$.

From Theorem 1.1 in [6], already referred to at the beginning of Section 3, and the above equivalence we conclude that

$$p^* : \mathcal{T}op/B \to \mathcal{T}op/E \text{ reflects isomorphisms} \Leftrightarrow p^* : \mathbb{E}(B) \to \mathbb{E}(E) \text{ is a full functor.}$$

Also the effective descent maps can be characterized in terms of maps which are effective for bijective-descent, as we show next.

Proposition 4.4 *A map is effective for global-descent if and only if its pullback along an arbitrary morphism is effective for bijective-descent.*

Proof. Let $\mathbb{E}$ denote the class of bijective maps. Then $\mathbb{E}$ descends along surjections.

Since effective descent morphisms are effective for $\mathbb{E}$-descent, the necessity of the condition follows from the fact that pullbacks of effective morphisms are effective descent morphisms.

Conversely, if the pullback of p along an arbitrary morphism is an $\mathbb{E}$-descent morphism, then p itself is a descent map.

Let α denote the component of the unit of the $\Psi^p \dashv \Phi^p$ at $(C, \gamma; \xi) \in Des(p)$. It is easy to check that $(C, \alpha; \zeta) \in Des_{\mathbb{E}}(\pi_2)$, for $\zeta = \xi f$, where $f : (E \times_B Q) \times_Q C \to E \times_B C$ is the canonical isomorphism as shown in the diagram

$$\begin{tikzcd}
(E \times_B Q) \times_Q C \arrow[d, "f"] \arrow[dr, shift left] \arrow[dr, shift right] & & \\
E \times_B C \arrow[r, shift left, "\pi_2"] \arrow[r, shift right, "\xi"'] & C \arrow[r, "q"] \arrow[d, "\alpha"] & Q \arrow[d, "1"] \\
& E \times_B Q \arrow[r, "\pi_2"] \arrow[d, "\pi_1"] & Q \arrow[d, "\delta"] \\
& E \arrow[r, "p"] & B
\end{tikzcd}$$

Since α is a bijective map and $\pi_2 : E \times_B Q \to Q$ is effective for $\mathbb{E}$-descent, there exists some $(D, \delta) \in \mathbb{E}(Q)$ such that $\Phi^{\pi_2}_{\mathbb{E}}(D, \delta)$ is, up to isomorphism, $(C, \alpha; \zeta)$. But the pullback of π_2 along δ is the coequalizer of (π_2, ξ), so δ is an isomorphism. Consequently, α is an isomorphism and so p is effective for global-descent. $\square$

With very little change, the same proof works if we consider surjective maps instead of bijective maps. In this case we can consider just pullbacks along subspace embeddings.

Proposition 4.5 *A map is effective for global-descent morphism if and only if its pullbacks along subspace embeddings are effective for surjective-descent.*

Proof. Let now $\mathbb{E}$ denote the class of surjective maps. Also in this case $\mathbb{E}$ descends along surjective maps.

If p is an effective descent map, its pullback along any morphism, being an effective descent map, is effective for surjective-descent.

Now, for $(C, \gamma; \xi) \in Des(p)$ we consider the diagram

$$\begin{tikzcd}
\gamma(C) \times_A C \arrow[d, "f"] \arrow[dr, shift left] \arrow[dr, shift right] & & \\
E \times_B C \arrow[r, shift left, "\pi_2"] \arrow[r, shift right, "\xi"'] & C \arrow[r, "q"] \arrow[d, "g"] & Q \arrow[d, "h"] \\
& \gamma(C) \arrow[r, "p'"] \arrow[d, "m"] & A \arrow[d, "n"] \\
& E \arrow[r, "p"] & B
\end{tikzcd}$$

where $m \cdot g$ is the (RegEpi, Mono)-factorization of γ and A is the subspace of B such that $p^*(A, n) = (\gamma(C), m)$. If p' is effective for surjective-descent, as in the proof of 4.4, we conclude that the upper square is a pullback. Therefore, also the outer rectangle is a pullback and so p is effective for descent. $\square$

For arbitrary categories with pullbacks and classes $\mathbb{E}$ satisfying suitable conditions, effective $\mathbb{E}$-descent morphisms are stable under pullback along $\mathbb{E}$-morphisms. Indeed, if $\mathbb{E}$ contains $\mathrm{Iso}(\mathcal{X})$, is closed under composition and weakly left cancellable (i.e. $gf, g \in \mathbb{E} \Rightarrow f \in \mathbb{E}$), then the class of effective $\mathbb{E}$-descent morphisms is stable under pullback along $\mathbb{E}$-morphisms.

This is Theorem 2.4 in [11], where, though clear in the proof that preceeds it, the restriction to pullbacks along morphisms in $\mathbb{E}$, instead of along arbitrary morphisms, is not stated.

We are going to show that effective surjective-descent maps are stable under pullback along arbitrary maps and so, by 4.5, that effective global-descent maps are exactly the maps that are effective for $\mathbb{E}$-descent, for the class $\mathbb{E}$ of surjective maps.

First we prove an auxiliary result.

Lemma 4.6 *For the pullback $(E \times_B A, \pi_1, \pi_2)$ of the pair (p, f), let $(C, \gamma; \xi)$ be an object of $Des(\pi_2)$ and D be the complement of $\pi_1 \cdot \gamma(C)$ in E. Then the bundle $(C \coprod D, \sigma)$, where $\sigma : C \coprod D \to E$ is the map induced by $\pi_1 \cdot \gamma$ and the subspace embedding of D in E, is equipped with descent data with respect to p.*

Proof. We define a function

$$\zeta : E \times_B (C \coprod D) \to C \coprod D$$

by

$$\zeta(e, x) = \begin{cases} \xi(e, x) & \text{if } x \in C \\ e & \text{otherwise} \end{cases}$$

and prove that it is continuous.

Identifying $(E \times_B A) \times_A C$ with $E \times_B C$ and denoting by τ_C the coproduct injection, the following

$$\begin{tikzcd}
E \times_B C \arrow[r, shift left, "\pi_2"] \arrow[r, shift right, "\xi"'] \arrow[d, "1 \times_B \tau_C"'] & C \arrow[d, "\tau_C"] & \\
E \times_B (C \coprod D) \arrow[r, shift left, "\pi_2"] \arrow[r, shift right, "\zeta"'] & C \coprod D \arrow[d, "\sigma"] & \\
 & E \arrow[r, "p"] & B
\end{tikzcd}$$

is a commutative diagram in $\mathcal{T}op$ as we show next.

The morphism $1 \times_B \tau_C$ is an open embedding, because it is the pullback of the open embedding τ_C along π_2. Also $\zeta \cdot 1 \times_B \tau_C = \tau \cdot \xi$.

For open subsets U of C,

$$\zeta^{-1}(U) = \zeta^{-1}(\tau_C(U)) = 1 \times_B \tau_C(\xi^{-1}(U))$$

which is open in $E \times_B (C \coprod D)$.

By 3.2(i),

$$\zeta^{-1}(D) = D \times_B (C \coprod D) = E \times_B D$$

and for $U = V \cap D$, with V open in E,

$$\zeta^{-1}(U) = U \times_B (C \coprod D) = V \times_B D,$$

which are open sets. Hence ζ is a continuous function.

It is easy to check that ζ is a $\mathbb{T}$-structure map and this completes the proof of the lemma. $\square$

Proposition 4.7 *Effective maps for surjective-descent are pullback stable.*

Proof. With the notation of the previous lemma, let $(C, \gamma; \xi) \in Des_{\mathbb{E}}(\pi_2)$, where $\mathbb{E}$ denotes the class of surjective maps.

Identifying again $(E \times_B A) \times_A C$ with $E \times_B C$, we have that $(C, \gamma'; \xi) \in Des(p)$ for $\gamma' = \pi_1 \gamma$. Let $q = coeq(\pi_2, \xi)$ and consider the diagram

$$\begin{tikzcd}
E \times_B C \arrow[r, shift left, "\pi_2"] \arrow[r, shift right, "\xi"'] & C \arrow[r, "q"] \arrow[d, "\gamma"] & Q \arrow[d, "h"] \\
 & E \times_B A \arrow[r, "\pi_2"] \arrow[d, "\pi_1"] & A \arrow[d, "f"] \\
 & E \arrow[r, "p"] & B
\end{tikzcd}$$

For $D = E \setminus \pi_1 \gamma(C)$, let $\sigma : C \coprod D \to E$ be the morphism induced by γ' and by the subspace embedding of D in E. By 4.6, $(C \coprod D, \sigma; \zeta) \in Des(p)$, for the map ζ defined there. Furthermore, since σ is surjective, $(C \coprod D, \sigma; \zeta)$ belongs to $Des_{\mathbb{E}}(p)$.

In the diagram

$$\begin{tikzcd}
E \times_B C \arrow[r, shift left, "\pi_2"] \arrow[r, shift right, "\xi"'] \arrow[d, "1 \times_B \tau_C"'] & C \arrow[r, "q"] \arrow[d, "\tau_C"] & Q \arrow[d, "g"] \\
E \times_B (C \coprod D) \arrow[r, shift left, "\pi_2"] \arrow[r, shift right, "\zeta"'] & C \coprod D \arrow[r, "q'"] \arrow[d, "\sigma"] & Q' \arrow[d, "\delta'"] \\
& E \arrow[r, "p"] & B
\end{tikzcd}$$

where $q' = coeq(\pi_2, \zeta)$, the bottom square is a pullback, because p is effective for surjective-descent.

For each subset U of Q

$$\tau_C(q^{-1}(U)) = q'^{-1}(g(U))$$

and so g is an open embedding because τ_C is an open embedding and q' is a quotient. Since open embeddings are stable under pullback and weakly left cancellable, also the right-upper square is a pullback.

Now, since $\sigma \cdot \tau_C = \pi_1 \cdot \gamma$ and $\delta' \cdot g = f \cdot h$, the outer rectangle in

$$\begin{tikzcd}
C \arrow[r, "q"] \arrow[d, "\gamma"'] & Q \arrow[d, "h"] \\
E \times_B A \arrow[r, "\pi_2"] \arrow[d, "\pi_1"'] & A \arrow[d, "f"] \\
E \arrow[r, "p"] & B
\end{tikzcd}$$

is a pullback. Since the bottom square is a pullback, the same holds for the upper square. Thus, π_2 is effective for surjective-descent as claimed. $\square$

Combining 4.7 and 4.5 we conclude the following:

Theorem 4.8 *Effective descent morphism in $\mathcal{T}op$ are exactly the maps which are effective for surjection-descent.*

Corollary 4.9 *A map is effective for descent if and only if it is effective for $\mathbb{E}$-descent, for each class $\mathbb{E}$ stable under pullback which contains the surjective maps and descends along universal quotients.*

Proof. Under each one of the conditions, the morphism is a universal quotient.

Effective descent morphisms are effective $\mathbb{E}$-descent morphisms, because $\mathbb{E}$ descends along universal quotients, and, by the same reason, effective $\mathbb{E}$-descent morphisms are effective for surjective-descent. Now the conclusion follows from the previous result. $\square$

References

[1] J. Adámek, H. Herrlich, G. E. Strecker, *Abstract and Concrete Categories* (Wiley, New York 1990.

[2] B. Banaschewski, *Singly generated frame extensions*, J. Pure Appl. Alg. 83 (1990) 1-21.

[3] J. Bénabou and J. Roubaud, *Monades et descente*, C.R. Acad. Sc. (1970) 96-98.

[4] C. Cassidy, M. Hébert, G. M. Kelly, *Reflective subcategories, localizations and factorization systems*, J. Austral. Math. Soc. (Series A) 38 (1985) 287-329.

[5] B. J. Day and G. M. Kelly, *On topological quotient maps preserved by pullbacks or products*, Proc. Cambridge Phil. Soc. 67 (1970) 553-558.

[6] G. Janelidze and W. Tholen, *How algebraic is the change-of-base functor?*, Lect. Notes in Math. 1448 (Springer, Berlin 1991) pp. 157-173.

[7] G. Janelidze and W. Tholen, *Facets of descent I*, Applied Cat. Struct. 2 (1994) 245-281.

[8] G. Janelidze and W. Tholen, *Facets of descent II*, Applied Cat. Struct. 5 (1997) 229-248.

[9] J. Reiterman and W. Tholen, *Effective descent maps of topological spaces*, Top. and its Appl. 57 (1994) 53-69.

[10] M. Sobral and W. Tholen, *Effective descent morphisms and effective equivalence relations*, CMS Conference Proceedings Vol. 13 (AMS, Providence, R. I., 1992) pp. 421-433.

[11] M. Sobral, *Some Aspects of Topological Descent*, Applied Cat. Struct. 4 (1996) 97-106.

Departamento de Matemática
Universidade de Coimbra
3000 Coimbra, Portugal