Natural Dualities

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Abstract

In 1936 M. Stone proved the duality between the category of zero dimensional compact Hausdorff spaces - we use the name **Stone** for this category - and the category of Boolean algebras. Several authors generalized these ideas and proved dualities between categories of "structured" sets and "structured" zero dimensional compact Hausdorffspaces. In this paper we use the language of limit sketches and study natural dual adjunctions between model categories of limit sketches in **Set** and **Stone** induced by totally finite objects. We give conditions to establish duality from the totally finite case.

Key words: natural duality, locally presentable category, limit sketch

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Introduction

In several papers Clark, Davey, Krauss and Werner (see [3] and references there) studied natural dual adjunctions between categories of varieties on one side and categories of structured boolean spaces on the other side induced by finite objects. Especially they were interested in dualities arising from these adjunctions between the full subcategories initialy cogenerated by these finite objects. In this paper we present a more general account to dualities of this kind employing the theory of limit sketches and locally presentable categories.

We call a model F of a limit sketch $\mathcal{S} = (\mathcal{A}, \mathcal{L}, \sigma)$ in **Set** or **Stone** totally finite if, for every $A \in \mathsf{Ob}\,\mathcal{A}$, $F(A)$ is finite. Our basic situation is now a natural dual adjunction between model categories $\mathbf{A} = \mathsf{Mod}\,(\mathcal{S}_1, \mathbf{Stone})$ and $\mathbf{B} = \mathsf{Mod}\,(\mathcal{S}_2, \mathbf{Set})$ of single sorted finitary limit sketches $\mathcal{S}_1$ and $\mathcal{S}_2$ induced by totally finite objects $\tilde{A}$ in $\mathbf{A}$ and $\tilde{B}$ in $\mathbf{B}$. This natural dual adjunction restricts to the full subcategories $\mathsf{InitCogen}(\tilde{A})$ and $\mathsf{InitCogen}(\tilde{B})$ initialy cogenerated by $\tilde{A}$ and $\tilde{B}$ respectively. The purpose of this paper is to give conditions to establish actually a duality between these subcategories whenever the restriction to the full subcategories of totally finite objects is allready a duality.

To do this, we shall use the fact that in model categories of limit sketches in **Set** every object is a directed colimit of finitely presentable objects. This implies that in $\mathsf{InitCogen}(\tilde{B})$ every object is a directed colimit of totally finite objects. It is well known, that in **Stone** the finitely copresentable objects are precisely the finite spaces. This implies, that for a finitary limit sketch $\mathcal{S} = (\mathcal{A}, \mathcal{L}, \sigma)$, with $\mathsf{Mor}\,\mathcal{A}$ finite, all totally finite objects in $\mathsf{Mod}\,(\mathcal{S}, \mathbf{Stone})$ are finitely

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copresentable. Putting all together, we can conclude that for every $B \in \mathsf{Ob}\,\mathbf{B}$ the unit ε_B of the adjunction is an isomorphism provided that $\varepsilon_{\bar{B}}$ is an isomorphism for every totally finite object $\bar{B}$.
Finally, compactness and (surjective, initial point separating source)-factorizations give the dual structure for $\mathbf{A}$: every object is a codirected limit of totally finite objects. This is also the last piece of information we need to prove our promised reduction to the totally finite case.
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1 Preliminaries

In this section we will introduce the basic definitions and constructions concerning natural dual adjunctions, limit sketches and locally presentable categories. First we will show very briefly how "schizophrenic objects" induce a duality. A detailed description can be found in [6].
Assume that we have categories $\mathbf{A}$ and $\mathbf{B}$ together with faithful functors $\mathbf{A} \xrightarrow{U} \mathbf{Set}$ and $\mathbf{B} \xrightarrow{V} \mathbf{Set}$ and objects $\tilde{A} \in \mathsf{Ob}\,\mathbf{A}$ and $\tilde{B} \in \mathsf{Ob}\,\mathbf{B}$ satisfying the equation $U(\tilde{A}) = V(\tilde{B})$. We may think of $\tilde{A}$ and $\tilde{B}$ as two "personalities" of the set $U(\tilde{A}) = V(\tilde{B})$. Under this conditions we have for every object $A \in \mathsf{Ob}\,\mathbf{A}$ and every element $a \in U(A)$ an evaluation map $\mathrm{ev}_{A,a} : \mathsf{hom}(A, \tilde{A}) \to U(\tilde{A}) = V(\tilde{B})$ and similiarly for every object $B \in \mathsf{Ob}\,\mathbf{B}$ and every element $b \in V(B)$ an evaluation map $\mathrm{ev}_{B,b} : \mathsf{hom}(B, \tilde{B}) \to V(\tilde{B}) = U(\tilde{A})$. We wish to construct contravariant functors $G : \mathbf{A} \to \mathbf{B}$ and $F : \mathbf{B} \to \mathbf{A}$ by lifting the contravariant hom functors $\mathsf{hom}(\_, \tilde{A})$ and $\mathsf{hom}(\_, \tilde{B})$.

$$\begin{tikzcd} \mathbf{A}^{\mathrm{op}} \arrow[r, "G"] \arrow[dr, "\mathsf{hom}(\_,\tilde{A})"'] & \mathbf{B} \arrow[d, "V"] \\ & \mathbf{Set} \end{tikzcd} \qquad \begin{tikzcd} \mathbf{B}^{\mathrm{op}} \arrow[r, "F"] \arrow[dr, "\mathsf{hom}(\_,\tilde{B})"'] & \mathbf{A} \arrow[d, "U"] \\ & \mathbf{Set} \end{tikzcd}$$

To do this we use the following assumptions:

1. For every object $A \in \mathsf{Ob}\,\mathbf{A}$ the V-structured source $(\mathsf{hom}(A, \tilde{A}) \xrightarrow{\mathrm{ev}_{A,a}} V(\tilde{B}))_{a \in U(A)}$ has a unique V-initial lift.

2. For every object $B \in \mathsf{Ob}\,\mathbf{B}$ the U-structured source $(\mathsf{hom}(B, \tilde{B}) \xrightarrow{\mathrm{ev}_{B,b}} U(\tilde{A}))_{b \in V(B)}$ has a unique U-initial lift.

These lifts define the functors G and F on objects and, because we chose the initial structure, for every morphism $f : A \to B$ in $\mathbf{A}$ ($\mathbf{B}$) $\mathsf{hom}(f, \tilde{A})$ ($\mathsf{hom}(f, \tilde{B})$) actually underlies an $\mathbf{B}$-morphism ($\mathbf{A}$-morphism).
We will often use the following simplified notation. If we have a concrete category $(\mathbf{C}, U)$ over

Set, we use the same notation for a **C**-morphism and its underlying map. If we have **C**-objects A and B and a map $g : U(A) \to U(B)$, we say that g is a **C**-morphism if it is the underlying map of a (necessary unique) **C**-morphism $\bar{g} : A \to B$.

Next we construct the units of the adjunction. For each object $A \in \mathsf{Ob}\,\mathbf{A}$ we have a map $\eta_A : U(A) \to UFG(A)$, given by $a \mapsto \mathrm{ev}_{A,a}$. Again, because of the initial structure, this map is actually an **A**-morphism and defines a natural transformation $\eta : 1_{\mathbf{A}} \to FG$. Symmetrically, we have a natural transformation $\varepsilon : 1_{\mathbf{B}} \to GF$. In fact, these natural transformations fulfil the equations of a dual adjunction. In the following a dual adjunction obtained in this way will be called *natural dual adjunction* induced by $(\tilde{A}, \tilde{B})$. It is given by the data $(\mathbf{A}, U, \mathbf{B}, V, G, F, \eta, \varepsilon, \tilde{A}, \tilde{B})$. If this dual adjunction is a dual equivalence, we will say that we have a *natural duality* induced by $(\tilde{A}, \tilde{B})$.

This dual adjunction gives an equivalence on the "fixed" full subcategories $\mathsf{Fix}\,(\eta) = \{\, A \in \mathsf{Ob}\,\mathbf{A} \mid \eta_A \text{ is an isomorphism} \,\}$ und $\mathsf{Fix}\,(\varepsilon) = \{\, B \in \mathsf{Ob}\,\mathbf{B} \mid \varepsilon_B \text{ is an isomorphism} \,\}$. If the objects $\tilde{A}$ and $\tilde{B}$ belong to $\mathsf{Fix}\,(\eta)$ and $\mathsf{Fix}\,(\varepsilon)$ respectively, then we have a natural duality in the sense above.

A natural question is now how to identify these subcategories. Of course we have, that say η is an isomorphism iff η is an embedding (i.e. it is initial and injective) and η is surjective. The first condition is equivalent to the fact that $\tilde{A}$ is an initial cogenerator in **A**.

Definition 1.1 *Let $\tilde{A}$ be an object of a concrete category $(\mathbf{A}, U)$ over* **Set**. *We will write* $\mathsf{InitCogen}(\tilde{A})$ *for the full subcategory of all objects of* **A** *for which* $\mathsf{hom}(A, \tilde{A})$ *is a point separating U-initial source. We call $\tilde{A}$ an initial cogenerator for $(\mathbf{A}, U)$ if* $\mathsf{InitCogen}(\tilde{A}) = \mathbf{A}$.

By construction, our dual adjunction restricts to the full subcategories $\mathsf{InitCogen}(\tilde{A})$ and $\mathsf{InitCogen}(\tilde{B})$. Let us denote these restrictions by $\tilde{U}$, $\tilde{V}$, $\tilde{G}$, $\tilde{F}$, $\tilde{\eta}$ and $\tilde{\varepsilon}$. Quite often this gives allready a duality. In this paper we are mainly interested in such situations. To establish a duality, in many cases we can prove a "Stone-Weierstrass"-like theorem.

Definition 1.2 *Assume that we have concrete categories $(\mathbf{A}, U)$ and $(\mathbf{B}, V)$ over* **Set** *together with an initial cogenerator $\tilde{A}$ of $(\mathbf{A}, U)$. Let $G : \mathbf{A} \to \mathbf{B}$ be a functor such that the diagram*

$$\begin{tikzcd} \mathbf{A}^{\mathrm{op}} \arrow[r, "G"] \arrow[dr, "\mathsf{hom}(\_, \tilde{A})"'] & \mathbf{B} \arrow[d, "V"] \\ & \mathbf{Set} \end{tikzcd}$$

commutes. We say that G satisfies the Stone-Weierstrass-condition *if for every object A of* **A** *and every $M \overset{m}{\to} G(A)$ with m an embedding and $m[V(M)]$ an initial point separating source m is an isomorphism.*

Note that, for every object A in $\mathsf{InitCogen}(\tilde{A})$, $\tilde{\eta}_A$ is an embedding and $\tilde{\eta}_A[U(A)]$ is an initial point separating source. The Stone-Weierstrass-condition for the restriction $\tilde{F}$ of F to $\mathsf{InitCogen}(\tilde{B})$ then implies that $\tilde{\eta}$ is an isomorphism.

We have the following facts:

Lemma 1.3 *Suppose we have a natural duality given by* $(\mathbf{A}, U, \mathbf{B}, V, G, F, \eta, \varepsilon, \tilde{A}, \tilde{B})$. *Assume that in addition* $\tilde{B}$ *is injective w.r.t. embeddings. Then* G *satisfies the Stone-Weierstrass-condition.*

Proof. Let A be an $\mathbf{A}$-object and $m : M \to G(A)$ an embedding in $\mathbf{B}$ with $m[V(M)]$ is an initial point separating source. Without loss of generality we may assume that $M = G(B)$ for an $B \in \mathsf{Ob}\,\mathbf{A}$ and $m = G(f)$ for $f : A \to B$ in $\mathbf{A}$. Since $\tilde{B}$ is injective f is surjective. On the other hand, the image of $G(f)$ is the set of all morphisms $h : A \to \tilde{A}$ which factorize through f. This implies that f is an embedding. Therefore f and also $G(f)$ are isomorphisms. □

Lemma 1.4 *Suppose we have a natural dual adjunction* $(\mathbf{A}, U, \mathbf{B}, V, G, F, \eta, \varepsilon, \tilde{A}, \tilde{B})$, $\tilde{A}$ *and* $\tilde{B}$ *are initial cogenerators for* $\mathbf{A}$ *and* $\mathbf{B}$ *respectively,* $\mathbf{B}$ *is (surjective, embedding)-factorizable for morphisms and* G *satisfies the Stone-Weierstrass-condition. Then* $\tilde{A}$ *is injective w.r.t. embeddings in* $\mathbf{A}$.

Proof. Let $m : A_0 \to A$ be an embedding in $\mathbf{A}$ and $G(A) \xrightarrow{e} B \xrightarrow{\bar{m}} G(A_0)$ is a (surjective, embedding)-factorization of $G(m)$. Then $\bar{m}[V(B)]$ contains all morphisms $h : A_0 \to \tilde{A}$ factorizing through m. Since m is an embedding and $\tilde{A}$ is an initial cogenerator we can conclude that $\bar{m}[V(B)]$ is an initial point separating source. Therefore $\bar{m}$ is isomorphic, hence $G(m)$ surjective and so $\tilde{A}$ is injective. □

In this paper we will use the theory of limit sketches and locally presentable categories. Therefore we will give the basic facts we will need. A detailed account can be found in [2] and [4].

Definition 1.5 *A* limit sketch *is a tripel* $\mathcal{S} = (\mathcal{A}, \mathcal{L}, \sigma)$ *consisting of a small category* $\mathcal{A}$, *a set of diagrams* $\mathcal{L}$ *in* $\mathcal{A}$ *and a function* σ *assigning to every diagram in* $\mathcal{L}$ *a compatible cone. Let* λ *be a regular cardinal. We call a limit sketch* $\mathcal{S}$ λ*-ary if, for all* $(D : I \to \mathcal{A}) \in \mathcal{L}$, $\#\mathsf{Mor}\,(I) < \lambda$ *holds.*
A model *of a limit sketch* $\mathcal{S}$ *in a category* $\mathbf{K}$ *is a functor* $F : \mathcal{A} \to \mathbf{K}$ *such that* $F(\sigma(\mathcal{D}))$ *is a limit of* FD *in* $\mathbf{K}$ *for every diagram* $\mathcal{D} = (D : I \to \mathcal{A})$ *in* $\mathcal{L}$.
The full subcategory of $\mathbf{K}^{\mathcal{A}}$ *consisting of all models of* $\mathcal{S}$ *in* $\mathbf{K}$ *is denoted by* $\mathsf{Mod}\,(\mathcal{S}, \mathbf{K})$. *If* $\mathbf{K} = \mathbf{Set}$ *we also write* $\mathsf{Mod}\,(\mathcal{S})$.

An example of a λ-ary limit sketch is the sketch consisting of a small λ-complete category $\mathcal{A}$, a set $\mathcal{L}$ of representatives of all λ-small diagrams and an assignment σ of a limit cone to each chosen λ-small diagram. Therefore every λ-complete small category is, in a canonical way, a λ-ary limit sketch. Such a limit sketch will be called λ-complete. A model of such a sketch in a category $\mathbf{K}$ is just a functor $F : \mathcal{A} \to \mathbf{K}$ which preserves all λ-small limits.
We also have the following

Theorem 1.6 *Let* $\mathcal{S} = (\mathcal{A}, \mathcal{L}, \sigma)$ *be a* λ*-ary limit sketch.* $\mathsf{Mod}\,(\mathcal{S})$ *is a full reflective subcategory of* $\mathbf{Set}^{\mathcal{A}}$ *closed under* λ*-directed colimits.*

Hence we have a reflector $R : \mathbf{Set}^{\mathcal{A}} \to \mathsf{Mod}\,(\mathcal{S})$ and in what follows we denote the reflection of a hom functor $\mathsf{hom}(A, \_)$ by $u(A)$ for each object A of $\mathcal{A}$. We have, for each model F, that

$$\mathsf{Nat}(u(A), F) \cong \mathsf{Nat}(\mathsf{hom}(A, \_), F) \cong F(A).$$

Definition 1.7 *Let λ be a regular cardinal. An object K of a category $\mathbf{K}$ is called λ-presentable, if the covariant hom functor* $\mathsf{hom}(K, \_)$ *preserves λ-directed colimits.*
A category $\mathbf{K}$ is called locally λ-presentable if

1. *$\mathbf{K}$ is cocomplete,*
2. *$\mathbf{K}$ has, up to isomorphism, only a set of λ-presentable objects and*
3. *every object of $\mathbf{K}$ is a λ-directed colimit of λ-presentable objects.*

Lemma 1.8 *Let $(\eta, \varepsilon) : F \dashv G : \mathbf{A} \to \mathbf{B}$ be an adjoint situation and $\mathbf{B}$ be locally λ-presentable. Then G preserves λ-directed colimits iff F preserves λ-presentable objects.*

The next theorem shows the connection between locally λ-presentable categories and λ-ary limit sketches.

Theorem 1.9 1. *Let $\mathbf{K}$ be a locally λ-presentable category and denote by $\mathsf{Pres}_\lambda(\mathbf{K})$ any set of representatives of all λ-presentable objects of $\mathbf{K}$. Then $\mathsf{Pres}_\lambda(\mathbf{K})$ is λ-cocomplete and $\mathbf{K}$ is equivalent to $\mathsf{Mod}\,(\mathsf{Pres}_\lambda(\mathbf{K})^{\mathrm{op}})$.*

2. *Let $\mathcal{S} = (\mathcal{A}, \mathcal{L}, \sigma)$ be a λ-ary limit sketch. Then $\mathsf{Mod}\,(\mathcal{S})$ is locally λ-presentable. If $\mathcal{S}$ is λ-complete, $\mathcal{A}$ is equivalent to $\mathsf{Pres}_\lambda(\mathsf{Mod}\,(\mathcal{S}))^{\mathrm{op}}$.*

3. *Two locally λ-presentable categories $\mathbf{K_1}$ and $\mathbf{K_2}$ are equivalent iff $\mathsf{Pres}_\lambda(\mathbf{K_1})$ and $\mathsf{Pres}_\lambda(\mathbf{K_2})$ are equivalent.*

4. *A λ-directed colimit preserving right adjoint functor $G : \mathbf{K_1} \to \mathbf{K_2}$ between locally λ-presentable categories $\mathbf{K_1}$ and $\mathbf{K_2}$ is an equivalence iff the restriction of its leftadjoint to the full subcategories of all λ-presentable objects is an equivalence.*

From now on all limit sketches will be finitary (i.e. $\aleph_0$-ary).
The model category of a limit sketch $\mathcal{S}$ is in general many sorted, there is no reason to have in general a faithful functor $U : \mathsf{Mod}\,(\mathcal{S}, \mathbf{K}) \to \mathbf{K}$. Therefore we introduce the concept of a single sorted limit sketch.
Let $\mathcal{S} = (\mathcal{A}, \mathcal{L}, \sigma)$ be a limit sketch. A $\mathcal{S}$-monomorphism is an $\mathcal{A}$-morphism $m : A \to B$ such that the diagram

$$\begin{tikzcd}
\mathbf{A} \arrow[dr, "m"] & \\
& B \\
\mathbf{A} \arrow[ur, "m"'] &
\end{tikzcd}$$

belongs to $\mathcal{L}$ and σ assigns the cone

$$\begin{tikzcd}
& A \\
A \arrow[ur, "\mathrm{id}"] \arrow[dr, "\mathrm{id}"'] & \\
& A
\end{tikzcd}$$

to this diagram. Without loss of generality we assume that every section is a $\mathcal{S}$-monomorphism and that the class of all $\mathcal{S}$-monomorphisms is closed under composition. Given a full subcategory $\mathcal{A}_0$ of $\mathcal{A}$, we define the full subcategory

$$\mathsf{Sub}_{\mathcal{S}}(\mathcal{A}_0) = \{\, C \in \mathsf{Ob}\,\mathcal{A} \mid \exists C \xrightarrow{m} A \text{ in } \mathcal{A} \text{ with } m \text{ is a } \mathcal{S}\text{-monomorphism and } A \in \mathsf{Ob}\,\mathcal{A}_0 \,\}$$

of $\mathcal{A}$. Note that $\mathcal{A}_0 \subset \mathsf{Sub}_{\mathcal{S}}(\mathcal{A}_0)$ holds. Let $A \in \mathsf{Ob}\,\mathcal{A}$. We define inductively a chain of full subcategories U_n^A of $\mathcal{A}$ by

1. $U_0^A = \mathsf{Sub}_{\mathcal{S}}(\{A\})$,
2. $U_{n+1}^A = \mathsf{Sub}_{\mathcal{S}}(\{C \in \mathsf{Ob}\,\mathcal{A} \mid$ there exist a diagram $\mathcal{D} = (D : I \to \mathcal{A})$ in $\mathcal{L}$ with $D(i) \in \mathsf{Ob}\,U_n^A$ for all $i \in \mathsf{Ob}\,I$ and $C = \mathsf{dom}\,(\sigma(\mathcal{D}))\,\})$.

Definition 1.10 *A limit sketch $\mathcal{S} = (\mathcal{A}, \mathcal{L}, \sigma)$ is called single sorted if there exists an element $A_0 \in \mathsf{Ob}\,\mathcal{A}$ such that $\mathcal{A} = \bigcup_{n \in \mathbb{N}} U_n^{A_0}$ holds. Such an element is called generator for $\mathcal{S}$.*

The evaluation functor at A_0 is then a faithful functor from $\mathsf{Mod}\,(\mathcal{S}, \mathbf{K})$ to $\mathbf{K}$, since we have

Lemma 1.11 *Let $\mathcal{S} = (\mathcal{A}, \mathcal{L}, \sigma)$ be a single sorted limit sketch with generator A_0. For every object A of $\mathcal{A}$ exists a finite subset $M \subset \mathsf{hom}(A, A_0)$ such that, for any model F of $\mathcal{S}$ in any category $\mathbf{K}$, $(F(A) \overset{F(f)}{\to} F(A_0))_{f \in M}$ is a monosource.*

We will need also the following

Definition 1.12 *Let $(\mathbf{K}, U)$ be a concrete category over* **Set** *and $\mathcal{S} = (\mathcal{A}, \mathcal{L}, \sigma)$ be a limit sketch. A model F of $\mathcal{S}$ in $\mathbf{K}$ is called totally finite if, for every $A \in \mathsf{Ob}\,\mathcal{A}$, $UF(A)$ is finite.*

If U preserves monosources F is totally finite if and only if $F(A_0)$ is finite for any generator A_0 for $\mathcal{S}$.

2 Stone-type Dualities

In this section we will study natural dual adjunctions between model categories of finitary limit sketches in **Set** and **Stone** induced by totally finite objects $\tilde{A}$ and $\tilde{B}$. The following will be our basic situation until the end of this section. We have single sorted, finitary limit sketches $\mathcal{S}_1 = (\mathcal{A}_1, \mathcal{L}_1, \sigma_1)$ with generator A_1 and $\mathcal{S}_2 = (\mathcal{A}_2, \mathcal{L}_2, \sigma_2)$ with generator A_2 and consider $\mathbf{A} = \mathsf{Mod}\,(\mathcal{S}_1, \mathbf{Stone})$ and $\mathbf{B} = \mathsf{Mod}\,(\mathcal{S}_2, \mathbf{Set})$. The evaluation functor at A_1 is a faithful functor from $\mathbf{A}$ to **Stone**, the composition of this functor with the usual forgetful functor from **Stone** to **Set** will be denoted by U. The evaluation functor at A_2 from $\mathbf{B}$ to **Set** is denoted by V. Therefore we have concrete categories $(\mathbf{A}, U)$ and $(\mathbf{B}, V)$ over **Set**. Furthermore we have totally finite objects $\tilde{A} \in \mathsf{Ob}\,\mathbf{A}$ and $\tilde{B} \in \mathsf{Ob}\,\mathbf{B}$. We assume that these objects induce a natural dual adjunction between $\mathbf{A}$ and $\mathbf{B}$ and use for this adjunction the notation we introduced in the previous section.

Lemma 2.1 $\mathsf{InitCogen}(\tilde{B})$ *is a reflective subcategory of* $\mathbf{B}$.

Proof. $\mathsf{InitCogen}(\tilde{B})$ is closed under limits in $\mathbf{B} = \mathsf{Mod}\,(\mathcal{S}, \mathbf{Set})$, hence it is complete, wellpowered and the inclusionfunctor preserves limits. Since $\mathsf{InitCogen}(\tilde{B})$ has also a cogenerator it is a reflective subcategory of $\mathbf{B}$. □

Let us denote the reflector by $R_{\tilde{B}}$. We have, for every object A of $\mathcal{A}$,

$$\mathsf{Nat}(R_{\tilde{B}}(u(A)), \tilde{B}) \cong \mathsf{Nat}(u(A), \tilde{B}) \cong \tilde{B}(A).$$

Since $\tilde{B}$ is totally finite we can conclude that $R_{\tilde{B}}(u(A))$ is totally finite for every object A of $\mathcal{A}$. Therefore the reflection of any finitely presentable object is totally finite. We have proved

Lemma 2.2 *Every object of* $\mathsf{InitCogen}(\tilde{B})$ *is a directed colimit of totally finite objects.*

Next we wish to show that $G : \mathbf{A}^{\mathrm{op}} \to \mathbf{B}$ preserves directed colimits. For that we look for the finitely copresentable objects of $\mathbf{A}$.

Proposition 2.3 *Let* $\mathcal{S} = (\mathcal{A}, \mathcal{L}, \sigma)$ *be a limit sketch with* $\mathsf{Mor}\,\mathcal{A}$ *finite. Every totally finite object* $\tilde{M}$ *of* $\mathsf{Mod}\,(\mathcal{S}, \mathbf{Stone})$ *is finitely copresentable.*

Proof. Let $D : I \to \mathsf{Mod}\,(\mathcal{S}, \mathbf{Stone})$ be a codirected diagram and $F \overset{\pi_i}{\to} D(i)$ a limit cone. For $i \leq j$ in I we denote by $\alpha_{i,j}$ the corresponding map from $D(i)$ to $D(j)$. Suppose we have $F \overset{\eta}{\to} \tilde{M}$.
For every $A \in \mathsf{Ob}\,\mathcal{A}$ there exists an $i_A \in I$ and a continuous map $f_A^{\#} : D(i_A)(A) \to \tilde{M}(A)$ such that the diagram

$$\begin{tikzcd} F(A) \arrow[r, "\eta_A"] \arrow[d, "(\pi_{i_A})_A"'] & \tilde{M}(A) \\ D(i_A)(A) \arrow[ur, "f_A^{\#}"'] & \end{tikzcd}$$

commutes, since $\tilde{M}(A)$ is a finite discrete space and so it is finitely copresentable in **Stone**. Let i_0 be a lower bound of $\{\, i_A \mid A \in \mathsf{Ob}\,\mathcal{A}\,\}$. We define

$$\bar{f}_A = f_A^{\#} \circ (\alpha_{i_0, i_A})_A.$$

For every $A \in \mathsf{Ob}\,\mathcal{A}$ we have $\eta_A = \bar{f}_A \circ (\pi_{i_0})_A$.
But the family $(\bar{f}_A)_A$ needs not to define a natural transformation. Let $h : A \to B$ be a morphism in $\mathcal{A}$. Consider the diagram

$$\begin{tikzcd} F(A) \arrow[r, "(\pi_{i_0})_A"] \arrow[d, "F(h)"'] & D(i_0)(A) \arrow[r, "\bar{f}_A"] \arrow[d, "D(i_0)(h)"] & \tilde{M}(A) \arrow[d, "\tilde{M}(h)"] \\ F(B) \arrow[r, "(\pi_{i_0})_B"'] & D(i_0)(B) \arrow[r, "\bar{f}_B"'] & \tilde{M}(B) \end{tikzcd}$$

The lefthand side and the outer square commute. Therefore we have

$$(\bar{f}_B \circ D(i_0)(h)) \circ (\pi_{i_0})_A = (\tilde{M}(h) \circ \bar{f}_A) \circ (\pi_{i_0})_A.$$

Since $\tilde{M}(B)$ is finitely copresentable in **Stone** we can find a $i_f \in I$ with

$$(\bar{f}_B \circ D(i_0)(h)) \circ (\alpha_{i_f, i_0})_A = (\tilde{M}(h) \circ \bar{f}_A \circ (\alpha_{i_f, i_0})_A.$$

Let i_1 be a lower bound of $\{\, i_f \mid f \in \mathsf{Mor}\,\mathcal{A} \,\}$. We define

$$f_A = \bar{f}_A \circ (\alpha_{i_1,i_0})_A = \bar{f}_A \circ (\alpha_{i_f,i_0})_A \circ (\alpha_{i_1,i_f})_A.$$

We have

$$\begin{aligned} \tilde{M}(h) \circ f_A &= \tilde{M}(h) \circ \bar{f}_A \circ (\alpha_{i_1,i_0})_A \\ &= \bar{f}_B \circ D(i_0)(h) \circ (\alpha_{i_1,i_0})_A \\ &= f_A \circ D(i_1)(h). \end{aligned}$$

Therefore $f = (f_A)_A$ is a natural transformation and $\eta = f \circ \pi_{i_1}$ holds, η factors through π_{i_1}. Assume now that we have $\eta = f \circ \pi_{i_0} = g \circ \pi_{i_0}$ for some $i_0 \in I$ and natural transformations $f : D(i_0) \to \tilde{M}$ and $g : D(i_0) \to \tilde{M}$.
For every $A \in \mathsf{Ob}\,\mathcal{A}$ there exists an $i_A \in I$ such that

$$f_A \circ (\alpha_{i_A,i_0})_A = g_A \circ (\alpha_{i_A,i_0})_A.$$

Let i_1 be a lower bound of $\{\, i_A \mid A \in \mathsf{Ob}\,\mathcal{A} \,\}$, we have for every $A \in \mathsf{Ob}\,\mathcal{A}$

$$f_A \circ (\alpha_{i_1,i_0})_A = f_B \circ (\alpha_{i_1,i_0})_A.$$

Therefore the factorization is essentially unique. $\square$

Corollary 2.4 *Let $\mathbf{K}$ be a subcategory of $\mathsf{Mod}\,(\mathcal{S}, \mathbf{Stone})$ closed under codirected limits where $\mathcal{S}$ is given like in the proposition above. Then every totally finite object of $\mathbf{K}$ is finitely copresentable.*

Theorem 2.5 *In addition to our general assumptions suppose that $\mathsf{Mor}\,\mathcal{A}_1$ is finite. Then $\tilde{\varepsilon}$ is an isomorphism iff $\tilde{\varepsilon}_B$ is an isomorphism for every totally finite object $B \in \mathsf{Ob}\,\mathsf{InitCogen}(\tilde{B})$.*

Proof. Since F preserves finitely presentable objects, $G : \mathbf{A}^{\mathrm{op}} \to \mathbf{B}$ preserves directed colimits. Since $\mathsf{InitCogen}(\tilde{A}) \hookrightarrow \mathbf{A}$ preserves (codirected) limits and $\mathsf{InitCogen}(\tilde{B}) \hookrightarrow \mathbf{B}$ reflects (directed) colimits, $\tilde{G} : \mathsf{InitCogen}(\tilde{A})^{\mathrm{op}} \to \mathsf{InitCogen}(\tilde{B})$ preserves directed colimits as well.
Let be $B \in \mathsf{Ob}\,\mathsf{InitCogen}(\tilde{B})$. We can express it as a directed colimit $B_i \xrightarrow{m_i} B$ of totally finite objects. We have the following commutative diagram.

$$\begin{tikzcd} B \arrow[r, "\tilde{\varepsilon}_B"] & \tilde{G}\tilde{F}(B) \\ B_i \arrow[u, "m_i"] \arrow[r, "\tilde{\varepsilon}_{B_i}"] & \tilde{G}\tilde{F}(B_i) \arrow[u, "\tilde{G}\tilde{F}(m_i)"'] \end{tikzcd}$$

On the left and the right hand side we have colimits, all $\tilde{\varepsilon}_{B_i}$ are isomorphisms, we can conclude that $\tilde{\varepsilon}_B$ is an isomorphism. $\square$

Remark 2.6 Suppose we have in our setting a duality between $\mathsf{InitCogen}(\tilde{B})$ and $\mathsf{Fix}\,\eta$ where $\mathsf{Mor}\,\mathcal{A}_1$ is finite. The totally finite objects of $\mathsf{Fix}\,\eta$ are finitely copresentable in $\mathsf{Fix}\,\eta$, and so, by duality, the totally finite objects in $\mathsf{InitCogen}(\tilde{B})$ are finitely presentable. But this implies that the inclusionfunctor $\mathsf{InitCogen}(\tilde{B}) \hookrightarrow \mathbf{B}$ preserves directed colimits. Turning things around, if we can find a single sorted limit sketch $\mathcal{S}_2$ and a totally finite object $\tilde{B}$ in $\mathsf{Mod}\,(\mathcal{S}_2, \mathbf{Set})$ such that $\mathsf{InitCogen}(\tilde{B})$ is not closed under directed colimits in $\mathbf{B}$, then we can't find a single sorted limit sketch $\mathcal{S}_1$ with $\mathsf{Mor}\,\mathcal{A}_1$ is finite and a totally finite object $\tilde{A}$ such that $\tilde{\varepsilon}$ is an isomorphism.

Next we will show that in presence of (surjective, initial point separating source)-factorizations every object of $\mathsf{InitCogen}(\tilde{A})$ is a codirected limit of totally finite objects.
Let $\mathbf{A}$ be a fibre small category with forgetful functor $U : \mathbf{A} \to \mathbf{Stone}$. We denote by $\mathcal{M}$ a set of representatives of all objects M of $\mathbf{A}$ with $U(M)$ is finite. Let $A \in \mathsf{Ob}\,\mathbf{A}$. We define a diagram scheme I_A in the following way. The objects of I_A are all $\mathbf{A}$-morphism $p : A \to M$ with $M \in \mathcal{M}$ and p is surjective. A morphism from $A \xrightarrow{p_1} M_1$ to $A \xrightarrow{p_2} M_2$ is an $\mathbf{A}$-morphism $h : M_1 \to M_2$ with $h \circ p_1 = p_2$. We have a canonical functor $D_A : I_A \to \mathbf{A}$ sending $A \xrightarrow{p} M$ to M. The cone $\mathsf{S} = (A \xrightarrow{p} D_A(A \xrightarrow{p} M))_{(A \xrightarrow{p} M) \in I_A}$ is a compatible cone for this diagram. We have

Lemma 2.7 *Suppose that* $(\mathbf{A}, U)$ *has (surjective,* U*-initial point separating source)-factorizations. For every object* $A \in \mathsf{Ob}\,\mathbf{A}$ *the diagram scheme* I_A *is codirected. If furthermore* S *is* U*-initial and point separating we have that* S *is a limit cone.*

Proof. The factorizations give the codirectedness of the diagramm. Since the source is initial and point separating we only have to show that for every compatible cone $C \xrightarrow{f_p} M$ there exists a map $f : C \to A$ with $p \circ f = f_p$ for every surjection $p : A \to M$ with $M \in \mathcal{M}$. But this follows from the fact that $U(A)$ is compact and that S is codirected. $\square$

Theorem 2.8 *We consider our basic situation where, in addition,* $\mathsf{Mor}\,\mathcal{A}_1$ *is finite. Furthermore suppose that* $(\mathsf{InitCogen}(\tilde{A}), \tilde{U})$ *has (surjective,*$\tilde{U}$*-initial point separating source)-factorizations. Then* $(\tilde{G}, \tilde{F}, \tilde{\eta}, \tilde{\varepsilon}) : \mathsf{InitCogen}(\tilde{A}) \to \mathsf{InitCogen}(\tilde{B})$ *is a dual equivalence iff its restriction to the full subcategories of totally finite objects is a dual equivalence.*

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