Separable Kripke structures are algebraically universal

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Abstract

For every poset $(I, \leq)$ and every family $(G_i)_{i \in I}$ of groups there exists a family of separable Kripke structures $(K_i)_{i \in I}$ on the same set, such that $G_i \cong Aut(K_i)$ and K_i is subalgebra of K_j *iff* $i \leq j$, for $i, j \in I$. More generally, this work is concerned with representations of algebraic categories by means of the category of separable Kripke structures. Consequences thereof are mentioned. Thus, in contrast to the algebraic non-universality of the category of Boolean algebras we conclude the algebraic universality of the category of separable dynamic algebras. Perfect classes of Kripke structures are introduced.

1. Introduction

1.1. On Kripke structures and dynamic algebras

Dynamic algebras were introduced by Kozen [19] and Pratt [29] to provide models of Propositional Dynamic Logic (PDL). For relation to Computer Science and examples see [29].

Following [29] a *dynamic algebra* is a two-sorted algebra $(\mathcal{B}, \mathcal{A}, <>)$ where

$$\mathcal{B} = (B, \vee, \neg, 0) \ \text{ and } \ \mathcal{A} = (A, ;, \cup, *)$$

are one-sorted algebras with $\quad 0:$ 0-ary operation (constant),
$\neg, *:$ 1-ary operations,
$\vee, ;, \cup:$ 2-ary operations,
and an operation named *diamond* $\quad <\cdot,\cdot>: A \times B \longrightarrow B:$ mixed operation,
satisfying the following conditions (where ";" is omitted, for brevity):

(i) $\mathcal{B}$ is a Boolean algebra;

(ii) $<a, 0> = 0$ and $<a, p \vee q> = <a, p> \vee <a, q>$;

(iii) $<a \cup b, p> = <a, p> \vee <b, p>$;

(iv) $<ab, p> = <a, <b, p>>$;

(v) $p \vee <aa^*, p> \leq <a^*, p> \leq p \vee <a^*, \neg p \wedge <a, p>>$

for $a, b \in A$ and $p, q \in B$.

Obs.: $(p \leq q)$ is abbreviation of $(p \wedge q = p)$ and $(p \wedge q)$ is abbreviation of $(\neg(\neg p \vee \neg q))$.

Homomorphisms of dynamic algebras $h : (\mathcal{B}, \mathcal{A}, <>) \longrightarrow (\mathcal{B}', \mathcal{A}', <>)$ are defined in the usual way, as homomorphisms of two-sorted algebras, *i.e.*, $h = (h_1, h_2)$ with $h_1 : \mathcal{B} \longrightarrow \mathcal{B}'$,

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$h_2 : \mathcal{A} \longrightarrow \mathcal{A}'$ such that h_1 and h_2 preserve the mentioned one-sorted operations and, moreover, h preserves the mixed operation, that is, $h_1(<a,p>) = <h_2(a), h_1(p)>$ for every $a \in \mathcal{A}$ and every $p \in \mathcal{B}$. Let *DA* denote the category of all dynamic algebras (whenever we consider a category mentioning only its objects, we assume that its morphisms are all the homomorphisms among those objects).

A dynamic algebra $(\mathcal{B}, \mathcal{A}, <>)$ is said to be *separable* if for all $a, b \in \mathcal{A}$,

$$<a,p> = <b,p> \quad \text{for every} \quad p \in \mathcal{B} \qquad \Longrightarrow \qquad a = b.$$

We denote by *SDA* the full subcategory of *DA* whose objects are the separable dynamic algebras.

Kripke structures, the traditional models of PDL, were presented in [29] as examples of dynamic algebras. They are defined as follows. The *full Kripke structure* on a given non-empty set S is the pair $(exp(S), exp(S\times S))$ where $exp(S)$ is the Boolean algebra of all subsets of S (with the usual set theoretical operations) and $exp(S\times S)$ is the set of all binary relations on S. The operations ;, $\cup$ and $*$ on $exp(S \times S)$ are the composition, the union and the reflective-transitive closure of binary relations, respectively. The reflective-transitive closure of $a \in exp(S\times S)$ is defined,as usually, by

$$a^* = \bigcup_{n=0}^{\infty} a^n,$$

where a^0 is the identity on S, and a^n $(n > 0)$ is the composition $a; a; \ldots ; a$, n times. The diamond operation $<a,p>$ (for $a \in exp(S\times S)$ and $p \in exp(S)$) is defined to be the pre-image of p under a,

$$\{s \in S : (s, s') \in a, \text{ for some } s' \in p\}.$$

Informally, the set S can be interpreted as the set of states of a computer, the subsets of S can be interpreted as propositions, the binary relations as computer programs. Then, "$s \in p$" can be interpreted as "*state s satisfies proposition p*", $(s, s') \in a$ as "*program a may run from initial state s to final state s'*", $a; b$ as "*execute program a, then program b*", $a \cup b$ as "*execute program a or program b non-deterministically*", a^* as "*execute program a zero or more times*", $<a,p>$ as "*the proposition satisfied when a is executed and stops in a state satisfying p*". *Kripke structures* are defined as the dynamic subalgebras of full Kripke structures. They intend to reflect the input-output behaviour of computer programs. Whenever we want to emphasize the fact that a certain Kripke structure $(\mathcal{B}, \mathcal{A})$ is a Kripke structure on a state set S we will indicate it by $(\mathcal{B}, \mathcal{A}, S)$.

Full Kripke structures are separable dynamic algebras, though Kripke structures are not necessarily separable. Let us denote by *SKri* the full subcategory of *DA* whose objects are the separable Kripke structures.

The simplest examples of dynamic algebras are the so called Boolean-trivial dynamic algebras $(\mathcal{B}, \mathcal{A}, <>)$ with $\mathcal{B} = \{0 = 1\}$, $\mathcal{A} = (A, ;, \cup, *)$ of the required type and $<a, 0>= 0$. They are, obviously, non-separable.

Denote by *FKri* the class of all finite full Kripke structures and by T the class of all Boolean-trivial dynamic algebras. Allow us to reuse the symbol *DA* to denote the variety of dynamic algebras.

The importance of the above two types of examples was shown by Pratt [29] in the establishment of the Theorem,

$$(*)\ DA = \mathrm{HSP}(FKri \cup T),$$

where H, S and P stand for the closure under homomorphic images, subalgebras and products, respectively. Thus, DA is the smallest variety containing *FKri* and determined by a set of Boolean equations only (*i.e.*, equations on Boolean sort), namely, the set of Boolean equations satisfied in *FKri* . The equality $(*)$ is the algebraic counterpart of the Segerberg-Parikh-Gabbay completeness Theorem for PDL, which was established by means of the Segerberg's axioms (*cf.* [8], [28], [35]). Thus a Boolean equation is satisfied in every finite full Kripke structure *iff* it can be "deduced" from the equations (i)-(v) above. See, *e.g.* [33] and [13], to define a formal way to deduce new equations from old equations in such a way that Completeness Theorem can be established.

More about dynamic algebras and Kripke structures can be found in [17], [18], [20], [21], [27], [31].

1.2. Formulation of the Main Theorem

We introduce *perfect classes* of Kripke structures. More details on those classes of Kripke structures will be given in section 2. Thus, a class $\mathcal{C}$ of Kripke structures is called *perfect* if, for every $(\mathcal{B},\mathcal{A},S),(\mathcal{B}',\mathcal{A}',S')\in\mathcal{C}$ and every homomorphism $(h_1,h_2):(\mathcal{B},\mathcal{A},S)\longrightarrow(\mathcal{B}',\mathcal{A}',S')$ there is a unique mapping $f:S'\longrightarrow S$ such that $h_1(p)=f^{-1}(p)$ for every $p\in\mathcal{B}$ and there is no other $h_2':\mathcal{A}\longrightarrow\mathcal{A}'$ such that $(f^{-1}/_{\mathcal{B}},h_2')$ is a homomorphism. The aim of this paper is to prove the following,

MAIN THEOREM. *There exists a perfect class of separable Kripke structures such that the corresponding full subcategory of SKri is algebraically universal, which means that it contains any category of universal algebras as a full subcategory.*

In section 2 of this paper we introduce state mappings, its relations with Kripke structure homomorphisms and define perfect class of Kripke structures. In section 3 we recall the representation problems concerning universality. In section 4 we refer an almost full embedding from which the main result will be proved and in sections 5 and 6 we prove it. In section 7 we list some of the consequences of the algebraic universality of *SKri*. In section 8 we refer test algebras.

2. State mappings

Since Kripke structures are dynamic algebras, their homomorphisms are defined as dynamic algebra homomorphisms. However, homomorphisms of Kripke structures can carry more information. We define a *state mapping* $f:(\mathcal{B},\mathcal{A},S)\longleftarrow(\mathcal{B}',\mathcal{A}',S')$ as an arbitrary mapping $f:S'\longrightarrow S$.

There is no relation between homomorphisms and state mappings in general, as we show below.

PROPOSITION 2.1. *Let $f:S'\longrightarrow S$ be a mapping. Then*

$$(f^{-1},(f\times f)^{-1}):(exp(S),exp(S\times S),S)\longrightarrow(exp(S'),exp(S'\times S'),S')$$

is a homomorphism iff f is a bijection.

Obs.:Given a mapping $f:S'\longrightarrow S$ and a subset $p\subseteq S$, we denote by $f^{-1}(p)$ the pre-image of p under f.

Proof. If f is a bijection, then $(f^{-1},(f\times f)^{-1})$ is an isomorphism, evidently.

If f is not a bijection, then either it is not one-to-one or it is not surjective.

In $(exp(S), exp(S\times S), S)$ it happens $\emptyset^* = \Delta_S$. If f is not one-to-one, then

$$((f\times f)^{-1}(\emptyset))^* = \emptyset^* = \Delta_{S'} \neq (f\times f)^{-1}(\Delta_S).$$

If f is not surjective, then the proposition just below can be used. $\square$

PROPOSITION 2.2. *Let $f : S' \to S$ be a mapping not surjective. Then there is no homomorphism*

$$(h_1, h_2) : (exp(S), exp(S\times S), S) \to (exp(S'), exp(S'\times S'), S')$$

for which $h_1 = f^{-1}$.

Proof. Let there exists $h_2 : exp(S\times S) \longrightarrow exp(S'\times S')$ such that $h = (f^{-1}, h_2)$ is a homomorphism. Since f is not surjective, we can choose $s \in S\backslash f(S')$. Choose $a = \{s\}\times S$. Then $<a, S>= \{s\}$, hence $<h_2(a), S'>= \emptyset$. So that $h_2(a) = \emptyset$. Choose $b = S\times\{s\}$. Then $<ba, S>= S$, hence $<h_2(ba), S'>= S'$. But then $h_2(ab) \neq h_2(a)h_2(b)$. $\square$

State mappings can induce Kripke structure homomorphisms. Thus, we consider

DEFINITION 2.3. A state mapping

$$f : (\mathcal{B}, \mathcal{A}, S) \longleftarrow (\mathcal{B}', \mathcal{A}', S')$$

is *compatible* if there exists $h_2 : \mathcal{A} \longrightarrow \mathcal{A}'$ such that $h = (f^{-1}/_{\mathcal{B}}, h_2)$ is a homomorphism. We say that f is *uniquely compatible* if there exists precisely one such h_2.

EXAMPLE 2.4. Consider a state set S with more than one element. Choose $s \in S$. Define the Kripke structure $(\mathcal{B}, \mathcal{A}, S)$ where $\mathcal{B} = \{\emptyset, S\}$ and $\mathcal{A} = \{\Delta_S, a, \Delta_S\cup a\}$ with $a = S\times\{s\}$. Then the identity on S is a compatible, but not uniquely compatible state mapping $1_S : (\mathcal{B}, \mathcal{A}, S) \longleftarrow (\mathcal{B}, \mathcal{A}, S)$, since $(1_S^{-1}/_{\mathcal{B}}, h_2')$ and $(1_S^{-1}/_{\mathcal{B}}, h_2'')$ are homomorphisms where h_2' is the identity mapping on $\mathcal{A}$ and h_2'' is the constant mapping constantly equal to Δ_S.

On the other hand Kripke structure homomrphisms can be induced by state mappings. So then, we consider

DEFINITION 2.5. A homomorphism

$$(h_1, h_2) : (\mathcal{B}, \mathcal{A}, S) \longrightarrow (\mathcal{B}', \mathcal{A}', S')$$

is *grounded* if there exists a state mapping $f : S' \longrightarrow S$ such that $h_1 = f^{-1}/_{\mathcal{B}}$. We say that (h_1, h_2) is *uniquely grounded* if there exists precisely one state mapping f with $h_1 = f^{-1}/_{\mathcal{B}}$.

EXAMPLE 2.6. Let S be an infinite set. Consider the Kripke structure $(\mathcal{B}, \mathcal{A}, S) = (exp(S), \{\Delta_S\}, S)$. Let $\mathcal{U}$ be an ultrafilter on $exp(S)$ satisfying $\{s\} \notin \mathcal{U}$ for every $s \in S$ (such an ultrafilter exists by Zorn's Lemma). Let $(\mathcal{B}', \mathcal{A}', S')$ be an arbitrary Kripke structure such that $\mathcal{A}'$ contains $\Delta_{S'}$. Then the homomorphism

$$(h_1, h_2) : (\mathcal{B}, \mathcal{A}, S) \longrightarrow (\mathcal{B}', \mathcal{A}', S')$$

defined by

$$h_1(p) = \begin{cases} S' & \text{if } p \in \mathcal{U} \\ \emptyset & \text{otherwise} \end{cases}$$

and $h_2(\Delta_S) = \Delta_{S'}$ is not grounded because h_1 doesn't preserve infinite unions.

EXAMPLE 2.7. Consider the Kripke structure $(\mathcal{B}, \mathcal{A}, S)$ on $S = \{s_1, s_2\}$ with $\mathcal{B} = \{\emptyset, S\}$, $\mathcal{A} = \{\Delta_S\}$. Then, for every Kripke structure $(\mathcal{B}', \mathcal{A}', S')$ such that $\mathcal{A}'$ contains $\Delta_{S'}$, the homomorphism

$$(h_1, h_2) : (\mathcal{B}, \mathcal{A}, S) \longrightarrow (\mathcal{B}', \mathcal{A}', S')$$

defined by $h_1(\emptyset) = \emptyset$, $h_1(S) = S'$ and $h_2(\Delta_S) = \Delta_{S'}$ is grounded but not uniquely grounded, since $h_1 = f^{-1}/_{\mathcal{B}} = g^{-1}/_{\mathcal{B}}$ for $f, g : S' \longrightarrow S$ state mappings defined by $f(s) = s_1$, $g(s) = s_2$ for $s \in S'$.

Now, we can rewrite the definition of perfect class of Kripke structures.

DEFINITION 2.8. A class $\mathscr{C}$ of Kripke structures is called *perfect* if, for every $(\mathcal{B}, \mathcal{A}, S)$, $(\mathcal{B}', \mathcal{A}', S') \in \mathcal{C}$, every homomorphism $h : (\mathcal{B}, \mathcal{A}, S) \longrightarrow (\mathcal{B}', \mathcal{A}', S')$ is uniquely grounded and the corresponding state mapping is uniquely compatible.

3. On algebraic universality

3.1. Representation problems

The representation problem for groups mentioned in the abstract has its simplest form when we choose $(I, \leq)$ to be a singular set trivially ordered, that is, "*Given a category $\mathcal{U}$, is every group isomorphic to the group of all automorphisms of some object of $\mathcal{U}$?*". This is the case with the category of symmetric graphs, the category of distributive lattices, the category of topological spaces, (*cf.* [7], [34], [3], [9]).

A similar question can be formulated for monoids instead of groups as "*Given a category $\mathcal{U}$, is every monoid M isomorphic to the monoid of all endomorphisms of some object of $\mathcal{U}$?*". That is, "*Is there some object $A \in Obj\,\mathcal{U}$ such that $M \cong \mathcal{U}(A, A)$?*". The category of binary relations, the category of semigroups, the category of topological spaces and open continuous mappings satisfy that condition.

Often, the answer to those questions requires a category-theoretical reasoning not only in the part where one represents but in the part which we represent, also. Since a monoid can be seen as a (only one object) category, we are led to a generalization from a problem of representation of monoids to a problem of representation of categories.

We are interested in the representation of the *concretizable categories.* Recalling the definition, we say that a category $\mathcal{C}$ is *concretizable* if there exists a faithful functor

$$\mathcal{F} : \mathcal{C} \longrightarrow Set$$

where *Set* is the category of all sets and all mappings. Now, our representation problem can be reformulated as follows "*Given a category $\mathcal{U}$, can every concretizable category $\mathcal{C}$ be represented in $\mathcal{U}$ in such a way that objects of $\mathcal{C}$ are substituted by objects of $\mathcal{U}$ and the morphisms between every two objects of $\mathcal{C}$ get represented exactly by the morphisms prescribed in $\mathcal{U}$ between the corresponding image objects*". In other words, "*Is $\mathcal{C}$ isomorphic to a full subcategory of $\mathcal{U}$?*". Whenever a category $\mathcal{U}$ has the above property it is called an *universal category.* An example of an universal category is the category $S(P^+)$ whose objects are pairs $(X, \mathscr{S})$ where X is a set and $\mathscr{S}$ is a family of subsets of X and morphisms from $(X, \mathscr{S})$ to $(X', \mathscr{S}')$ are mappings $f : X \to X'$, such that each S belonging to $\mathscr{S}$ has image under f, $f(S)$, lying in $\mathscr{S}'$.

A category is said to be *algebraic* if it is isomorphic to a category of some algebras of a given type and all their homomorphisms. A category is *algebraically universal* (briefly, *alg-universal*) if every algebraic category is fully embeddable into it. We recall that a category

$\mathcal{A}$ is fully embeddable into a category $\mathcal{B}$ if there is a full one-to-one functor $F : \mathcal{A} \to \mathcal{B}$. The category $Alg(\Delta)$ of all algebras of a given type Δ is alg-universal *iff* the sum of the arities of the operations of type Δ is not less than 2. The category *Graph* of all graphs is also alg-universal. Moreover, this category provides a criterion to decide about the alg-universality of any category, since

A category is alg-universal iff the category Graph fully embeds into it.

The above results are contributions of Kučera, Hedrlín, Pultr and Trnková. A full account of universality and alg-universality can be found in the Pultr and Trnková's monograph [30].

Under the set-theoretical assumption that there are not too many measurable cardinals, more precisely, that

(M) *There exists a cardinal α such that every α-additive two-valued measure is trivial,*

every concretizable category is algebraic (*cf.* [10], [22]). Then, Kučera and Pultr concluded in [23] that concretizability coincides with algebraicity *iff* (M) (*e.g.* the category of compact Hausdorff spaces and continuous mappings is non-algebraic under non (M)).

Clearly, under condition (M) both the concepts of universality and of algebraic universality coincide. Therefore, in this paper we restrict ourselves to the problem of the alg-universality of the category of separable Kripke structures. The category of dynamic algebras is, trivially, alg-universal since its full subcategory whose objects are the Boolean-trivial dynamic algebras is isomorphic to the category $Alg(2, 2, 1)$. Moreover, the category of Boolean algebras is not alg-universal because $\mathbb{Z}_3$, the cyclic group of order 3, is an automorphism group of no Boolean algebra [26]. This is of relevance to our problem in that the investigation of the alg-universality of separable Kripke structures may now become one of choosing a class of Boolean algebras with a parallel interfering structure, the action part of the Kripke structure, which will determine the compatible Boolean homomorphisms.

3.2. Remarks on the definition of algebraic category

The concept of algebraic category was introduced by Isbell in [14], where homogeneous algebras, that is, algebras with one underlying set only, were understood. Actually, the algebraicity doesn't need to be described by (full) operations, since each $Rel(\Delta)$, the category of all relational systems of type Δ and all their homomorphisms, is algebraic as shown by Hedrlín and Pultr in [11].

Here, we deal with dynamic algebras, which are heterogeneous algebras, that is, algebras with possibly many underlying sets (*cf.* [4] and [2] for the formal definition). However, categories of heterogeneous algebras and all their homomorphisms are algebraic, too. It is enough to see each heterogeneous algebra $\mathcal{A}$ as relational system defined on the set-theoretical disjoint union of the underlying sets of $\mathcal{A}$ with a relation corresponding to each operation of A (the relation which is the operation seen as a relation) plus a unary relation per each, and equal to each, one of the underlying sets of $\mathcal{A}$. The unary relations are added to choose the right relation homomorphisms.

4. Preliminary construction

Let $UndGraph_0$ denote the category of connected undirected graphs without loops with more than one vertex and all their homomorphisms. It is known (*cf.* [30]) that $UndGraph_0$ is alg-universal. Therefore, to prove the Main Theorem it is enough to construct a full embedding of $UndGraph_0$ into *SKri* in such a way that the image subcategory defines a perfect class of Kripke structures. The desired full embedding will be a composition of two contravariant embeddings, say $\phi\circ\psi$. We will restrict the domain of ϕ to the image $\psi(UndGraph_0^{op})$. So, we need to describe this category. This is will be the aim of this section (we follow the monograph [30]). The embedding ϕ is constructed in section 5. We preserve the notation used in this section in the forthcoming sections.

By *Comp* we denote the category of compact Hausdorff spaces and continuous mappings. *Comp* is not alg-universal, although its dual is almost alg-universal, that is the required embedding is full up to the constant morphisms, more exactly

THEOREM 4.1.([37],[30]). *There exists an almost full embedding*

$$\psi : UndGraph_0^{op} \longrightarrow Comp.$$

Thus, by definition, we have

(i) ψ is one-to-one on objects and on morphisms;

(ii) For any morphism f of $UndGraph_0$, $\psi(f)$ is a non-constant continuous mapping;

(iii) For every pair of graphs (X,R), (X',R') belonging to $UndGraph_0$ and every non-constant morphism $g : \psi(X',R') \to \psi(X,R)$ in *Comp* there exists a morphism $f : (X,R) \to (X',R')$ in $UndGraph_0$, such that $\psi(f) = g$.

In this case, *Comp* is said to be dual to an almost alg-universal category. In order to describe the category $\psi(UndGraph^{op})$, we recall some definitions.

We define *continua* as connected compact Hausdorff spaces with more than one point. Such spaces are, consequently, infinite sets. A *Cook* continuum is a continuum D such that for any subcontinuum S of D, each continuous mapping $f : S \to D$ is either a constant mapping or the inclusion. Continua with this property were given the name Cook continuum, since it was H. Cook [6] who firstly constructed such a continuum. Each continuum has a countable pairwise disjoint system of its subcontinua. See the Appendix A of [30] for details.

We recall that a topological space D is said *rigid* if each continuous mapping $f : D \to D$ is either a constant mapping or the identity. The following property is satisfied:

(1)*Let X, X' be sets and D a rigid space. Denote by D^X the product space (i.e., with the topology of the pointwise convergence). Let a and b be distinct elements of D. Denote by c_a (resp. c'_a) the element of D^X (resp. $D^{X'}$) constantly equal to a. Consider, similarly, c_b and c'_b. Let $g : D^{X'} \to D^X$ be a continuous mapping such that $g(c'_a) = c_a$ and $g(c'_b) = g(c_b)$. Then, there exists a mapping $f : X \to X'$ such that $g(\alpha) = \alpha\circ f$ for all $\alpha \in D^{X'}$.*

Given a graph (X,R) we say that $Y \subseteq X$ is an *independent set* of (X,R) if no two vertices of Y are joined by an edge, *i.e.*, if $\{x,y\} \subseteq Y \Longrightarrow \{x,y\} \notin R$. A *characteristic* mapping $h_Y : X \longrightarrow \{0,1\}$ of a set $Y \subseteq X$ is defined to be $h_Y(y) = 1$ for $y \in Y$ and $h_Y(y) = 0$ otherwise.

Now, we proceed with the construction of ψ, which is done after several steps, as follows:

- Choose A, B, C, H pairwise disjoint subcontinua of a *Cook continuum*. Thus, between two distinct continua of the system A, B, C, H no other continuous mappings are allowed than the constant ones. Moreover, each one of those spaces is rigid.

- Choose distinct elements $a_0, a_1 \in A$, $b_0, b_1 \in B$, $c_0, c_1 \in C$, $0, 1, a, b \in H$.

- Given a graph $(X,R) \in UndGraph_0$ a compact Hausdorff space $\psi(X,R)$ is constructed considering the following elements:

– the product space H^X;

– c_a and c_b the elements of H^X constantly equal, respectively, to a and to b;

– the subspace $\chi_R \subseteq \{0,1\}^X \subseteq H^X$ consisting of the characteristic mappings of all independent sets of (X,R). The set χ_R is a closed subset of H^X ([30, VI.16.9]);

– the following identifications in the topological sum

$$S = A \vee B \vee H^X \vee (\chi_R \times C)$$

$$\begin{array}{lcl} a_0 & \sim & c_a, \\ b_0 & \sim & c_b, \\ h_Y & \sim & (h_Y, c_0) \text{ for all } h_Y \in \chi_R, \\ a_1 & \sim & b_1 \sim (h_Y, c_1) \text{ for all } h_Y \in \chi_R. \end{array}$$

- Define $\psi(X,R)$ as the quotient space $S/_\sim$. Since S is a finite sum of compact spaces, S is still compact. Then, $\psi(X,R)$ is compact and it is easily seen that it is Hausdorff, too.

To simplify the notation, allow us to suppose that A, B, H^X, $\chi_R \times C$ are subsets of $\psi(X,R)$ and $a_0 = c_a$, …, $a_1 = b_1 = (h_Y, c_1)$.

Let $f : (X,R) \longrightarrow (X',R')$ be a morphism in *UndGraph*$_0$,

$$\psi(f) = g : \psi(X',R') \longrightarrow \psi(X,R)$$

is defined as follows,

$$\begin{array}{l} g(z) = z \text{ for } z \in A \cup B, \\ g(\alpha) = \alpha \circ f \text{ for } \alpha \in H^{X'}, \\ g(h_Y, z) = (h_Y \circ f, z) \text{ for } (h_Y, z) \in \chi_{R'} \times C. \end{array}$$

The mapping g is correctly defined, since it preserves the identifications made in the definition of the space $\psi(X,R)$. It is straightforward that g is continuous. Clearly, ψ is a one-to-one functor. A detailed proof of the fullness can be found in [30, VI.16.8-VI.16.14]. For the sake of clarity, we present a sketch of that proof.

Let

$$g : \psi(X',R') \longrightarrow \psi(X,R)$$

be a non-constant continuous mapping. According to the following properties:

(2) *Let L be one of the spaces H, A, B, C. Let $d : L \longrightarrow \psi(X,R)$ be a non-constant continuous mapping. Then, either $L \in \{A, B\}$ and $d(z) = z$ for all $z \in L$, or $L = C$ and there exists precisely one independent set Y such that $d(z) = (h_Y, z)$ for all $z \in L$, or $L = H$ and $g(L) \subseteq H^X$;*

(3) *Let $d : H^{X'} \to \psi(X,R)$ be a non-constant continuous mapping. Then $d(H^{X'}) \subseteq H^X$;*

the mapping g cannot be constant neither on $A, B, H^{X'}$ nor $\{h_Y\} \times C$ for no $h_Y \in \chi_{R'}$. Thus, by (2) and (3) again $g(H^{X'}) \subseteq H^X$ and $g(z) = z$ for all $z \in A \cup B$. Therefore $g(c'_a) = c_a$ and $g(c'_b) = c_b$. By property (1) there exists a mapping $f : X \to X'$ such that $g(\alpha) = \alpha \circ f$ for all $\alpha \in H^{X'}$. Then, $g(h_Y, c_1) = (h_Y \circ f, c_1)$ for every $h_Y \in \chi_{R'}$, which means that f is a homomorphism of undirected graphs. Again, by property (2) we have $g(h_Y, z) = (h_Y \circ f, z)$ for all $h_Y \in \chi_{R'}$, $z \in C$. We conclude $\psi(f) = g$. □

The next Lemma is for later use. It will help us to calculate the interiors and the closures of subsets of the spaces of the form $\psi(X,R)$ for some graph (X,R).

Before proceeding we will fix some notation. By use of "cl", "int", "ext" and "b" we denote, respectively, the closure, the interior, the exterior and the boundary taken in the space which will index those keywords. Whenever the index space is Q, we won't use the index if there is no ambiguity. We also we use the bar $\overline{}$ to denote the closure of subsets of Q. Given a set S we denote by S^c the complement of S. We denote the canonical projection by $p : S \longrightarrow S/_{\sim} = \psi(X, R)$.

LEMMA 4.2. *Let G be a subset of $\psi(X, R)$. Then*

(i) $\overline{G} = p(cl_S(p^{-1}(G)))$;

(ii) $p(int_S(p^{-1}(G)))$ *is open* $\Longrightarrow int(G) = p(int_S(p^{-1}(G)))$.

Proof. (i) Firstly, consider F closed in S. Then F is compact. Therefore, $p(F)$ is compact in the Hausdorff space $\psi(X, R)$. That is, $p(F)$ is closed. The immediate consequence is that $p(cl_S(p^{-1}(G))$ is closed. Now, the question is whether $p(cl_S(p^{-1}(G))$ is the least closed set in $\psi(X, R)$ containing G, which is obvious.

(ii) *Mutatis mutandis.* Note that $p(int_S(p^{-1}(G)))$ is not open in general. Choose, for instance, $G = A$. □

5. Construction of the embedding

$$\phi : \psi(UndGraph_0^{op}) \longrightarrow SKri^{op}$$

The construction of ϕ is made after two steps. Several lemmas are established to insure the correct definition of ϕ. Let (X, R) be a graph of $UndGraph_0$. Let's denote $Q = \psi(X, R)$. In order to define $\phi(Q)$ we are interested in a family of subsets of Q stable under pre-images under non-constant continuous mappings. For that we will choose a particular family $\mathcal{T}$ of regularly open subsets of Q and prove that it is a basis of Q. We recall that an open subset is said regularly open (r.o. for brevity) if it is equal to the interior of its closure. The basis $\mathcal{T}$ will determine the Boolean part and also the action part of $\phi(Q)$ and was refined in a way that, concerning the required fullness for ϕ, the Boolean part will contribute to the definition of a convenient mapping and the action part will force the continuity of that mapping. We choose y_1, y_2 distinct points of $A\backslash\{a_0, a_1\}$ to be added to the structure of $\phi(Q)$ in a manner that the continuous mapping defined won't be constant.

We divide the construction of $\mathcal{T}$ into 8 parts to describe the specific situation around the points $a_0, a_1, \ldots, h_Y$ (for $h_Y \in \mathcal{X}_R$) in Q. That is, $\mathcal{T}$ is defined to be the union

$$\mathcal{T} = \mathcal{T}_1 \cup \mathcal{T}_2 \cup \cdots \cup \mathcal{T}_8,$$

where

$\mathcal{T}_1 = \Big\{U_A \subseteq Q : U_A \text{ r.o. of } A,\ a_0, a_1 \notin cl_A(U_A),\ y_1, y_2 \notin b_A(U_A)\Big\}$,
$\mathcal{T}_2 = \Big\{U_B \subseteq Q : U_B \text{ r.o. of } B,\ b_0, b_1 \notin cl_B(U_B)\Big\}$,
$\mathcal{T}_3 = \Big\{U_{H^X} \subseteq Q : U_{H^X} \in \mathcal{T}_0,\ c_a, c_b \notin cl_{H^X}(U_{H^X}) \text{ and } cl_{H^X}(U_{H^X}) \bigcap \mathcal{X}_R = \emptyset\Big\}$,

here, $\mathcal{T}_0$ denotes the set
$\Big\{\prod_{x\in X} U_x : U_x \text{ r.o. of } H, x \in X \text{ and only a finite number of } U_x\text{'s is different of } H\Big\}$

$$\mathcal{T}_4 = \left\{(U_{H^X} \bigcap \chi_R)\times U_C \subseteq Q : U_{H^X} \in \mathcal{T}_0, U_C \text{ is a r.o. of } C, c_0, c_1 \notin cl_C(U_C)\right\},$$
$$\mathcal{T}_5 = \Big\{U_A \bigcup U_{H^X} \subseteq Q : U_A \text{ r.o. of } A,\ a_0 \in U_A, a_1 \notin cl_A(U_A), y_1, y_2 \notin b_A(U_A)$$
$$U_{H^X} \in \mathcal{T}_0,\ c_a \in U_{H^X},\ c_b \notin cl_{H^X}(U_{H^X}), cl_{H^X}(U_{H^X}) \bigcap \chi_R = \emptyset\Big\},$$
$$\mathcal{T}_6 = \Big\{U_B \bigcup U_{H^X} \subseteq Q : U_B \text{ r.o. of } B,\ b_0 \in U_B, b_1 \notin cl_B(U_B),$$
$$U_{H^X} \in \mathcal{T}_0,\ c_b \in U_{H^X}, c_a \notin cl_{H^X}(U_{H^X}), cl_{H^X}(U_{H^X}) \bigcap \chi_R = \emptyset\Big\},$$
$$\mathcal{T}_7 = \Big\{U_{H^X} \bigcup (U_{H^X} \bigcap \chi_R)\times U_C \subseteq Q : U_{H^X} \in \mathcal{T}_0,\ c_a, c_b \notin cl_{H^X}(U_{H^X}),$$
$$cl_{H^X}(U_{H^X}) \bigcap \chi_R = U_{H^X} \bigcap \chi_R \neq \emptyset, U_C \text{ r.o. of } C,\ c_0 \in U_C, c_1 \notin cl_C(U_C)\Big\},$$
$$\mathcal{T}_8 = \Big\{U_A \bigcup U_B \bigcup (\chi_R \times U_C) \subseteq Q : U_A \text{ r.o. of } A,\ a_1 \in U_A, a_0 \notin cl_A(U_A),$$
$$y_1, y_2 \notin b_A(U_A), U_B \text{ r.o. of } B,\ b_1 \in U_B, b_0 \notin cl_B(U_B), U_C \text{ r.o. of } C,\ c_1 \in U_C,$$
$$c_0 \notin cl_C(U_C)\Big\}.$$

LEMMA 5.1. *The family $\mathcal{T}$ is a basis for the topological space Q.*

Proof. Let G be an open in Q, that is, $p^{-1}(G)$ is an open in S, where p is the canonical projection. Consider $z \in G$. We are going to show that there exists $U \in \mathcal{T}$ such that $z \in U \subseteq G$.

The spaces A, B, C, H are regular, since they are compact Hausdorff spaces. Thus, each one of them has a basis of regularly open sets.

We have several cases for z:

1. $z \in A \backslash \{a_0, a_1\}$.
   Therefore there exists a regularly open U_A in A such that
   $$z \in U_A \subseteq p^{-1}(G) \bigcap A.$$
   A is Hausdorff and regular, so it is possible to choose U_A such that $a_0, a_1 \notin cl_A(U_A)$ and $y_1, y_2 \notin b_A(U_A)$. Thus, $U_A = p^{-1}(U_A)$ is an open in Q belonging to $\mathcal{T}_1$.

2. $z \in B \backslash \{b_0, b_1\}$.
   Mutatis mutandis for 1.

3. $z \in H^X \backslash \left(\{c_a, c_b\} \bigcup \chi_R\right)$.
   Therefore there exists $\prod_{x \in X} U_x \in \mathcal{T}_0$, such that
   $$z \in \prod_{x \in X} U_x \subseteq p^{-1}(G) \bigcap H^X.$$
   Since χ_R is closed in H^X (*cf.* 4) and since H^X is Hausdorff and regular, we can choose $\prod_{x \in X} U_x$ such that
   $$cl_{H^X}\left(\prod_{x \in X} U_x\right) \bigcap \left(\chi_R \bigcup \{c_a, c_b\}\right) = \emptyset.$$
   Moreover $\prod_{x \in X} U_x$ is open in Q since $p^{-1}\left(\prod_{x \in X} U_x\right) = \prod_{x \in X} U_x$. Thus, $\prod_{x \in X} U_x \in \mathcal{T}_3$.

4. $z \in \chi_R \times (C \backslash \{c_0, c_1\})$.
   Mutatis mutandis for 1.

5. $z = a_0 = c_a$.

   On one hand, there exists U_A regularly open in A such that
$$a_0 \in U_A \subseteq p^{-1}(G) \bigcap A.$$
   Again, since A is by Hausdorff and regular it is possible to choose U_A such that $a_1 \notin cl_A(U_A)$ and $y_1, y_2 \notin b_A(U_A)$. On the other hand, there exists $U_{H^X} \in \mathcal{T}_0$ such that
$$c_a \in U_{H^X} \subseteq p^{-1}(G) \bigcap H^X.$$
   And since H^X is Hausdorff and regular it is possible to choose U_{H^X} such that
$$cl_{H^X}(U_{H^X}) \bigcap (\chi_R \bigcup \{c_b\}) = \emptyset.$$
   Therefore $U = U_A \bigcup U_{H^X}$ is an open in Q belonging to $\mathcal{T}_5$.

6. $z = b_0 = c_b$.

   Mutatis mutandis for 5.

7. $z = h_Y = (h_Y, c_0) \in \chi_R$.

   In this case there exists $U_1 \in \mathcal{T}_0$ such that
$$h_Y \in U_1 \subseteq p^{-1}(G) \bigcap H^X.$$
   We can choose U_1 in such a way that
$$c_a, c_b \notin cl_{H^X}(U_1).$$
   Concurrently, there are $U_2 \in \mathcal{T}_0$ and U_C regularly open in C, such that
$$(h_Y, c_0) \in \left(U_2 \bigcap \chi_R\right) \times U_C \subseteq p^{-1}(G) \cap (\chi_R \times C).$$
   U_C can be chosen such that
$$c_1 \notin cl_C(U_C).$$
   Let's put $U' = U_1 \cap U_2$. We can choose $\prod_{x \in X} U_x \in \mathcal{T}_0$, such that $\prod_{x \in X} U_x \subseteq U'$, $h_Y(x) \in U_x$ and $cl_H(U_x) \bigcap \{0,1\} = U_x \bigcap \{0,1\}$ for each $x \in X$. Thus, $U = \left(\prod_{x \in X} U_x\right) \bigcup \left(\left(\prod_{x \in X} U_x\right) \bigcap \chi_R\right) \times U_C$ is an open in Q belonging to $\mathcal{T}_7$.

8. $z = a_1 = b_1 = (h_Y, c_1), \forall h_Y \in \chi_R$.

   Therefore, there are regularly open sets U_A, U_B, U_C in each space such that
$$z \in U_A \bigcup U_B \bigcup (\chi_R \times U_C) \subseteq p^{-1}(G) \bigcap \left(A \bigcup B \bigcup (\chi_R \times C)\right).$$
   Moreover, U_A, U_B, U_C can be chosen such that $a_0 \notin cl_A(U_A)$, $y_1, y_2 \notin b_A(U_A)$, $b_0 \notin cl_B(U_B)$, $c_0 \notin cl_C(U_C)$. The open set $U_A \bigcup U_B \bigcup (\chi_R \times U_C)$ belongs to $\mathcal{T}_8$. $\square$

STEP 1. We define $\phi(Q) = (\mathcal{B}_Q, \mathcal{A}_Q)$, where

- $\mathcal{B}_Q$ is the Boolean subalgebra of $exp(Q)$ generated by $\mathcal{T} \bigcup \Big\{\{y_1\}, \{y_2\}\Big\}$;
- $\mathcal{A}_Q$ is the subalgebra of $exp(Q \times Q)$ relative to the operations $\cup, \circ$ and $*$ generated by $\Big\{\Delta_U : U \in \mathcal{T}\Big\} \bigcup \Big\{Q \times \{y_1\}, Q \times \{y_2\}\Big\}$ for $\Delta_U = \Big\{(x, x) : x \in U\Big\}$.

LEMMA 5.2. *$\mathcal{A}_Q$ is the set of binary relations on Q of the following type*

$$\Delta_G \cup G_1 \times \{y_1\} \cup G_2 \times \{y_2\}$$

for every G, G_1, G_2 belonging to the subalgebra $\mathcal{B}_\mathcal{T}$ of $(exp(Q), \cup, \cap)$ generated by $\mathcal{T}$.

The proof is an easy exercise in the Kripke structure operations. Notice that the elements of $\mathcal{B}_\mathcal{T}$ are open sets in Q. $\square$

LEMMA 5.3. *$(\mathcal{B}_Q, \mathcal{A}_Q)$ is a separable Kripke structure.*

Proof. We have to show the closure under $<>$. Let $p \in \mathcal{B}_Q$. It is enough to prove the closure for $\Delta_G, G \times \{y\} \in \mathcal{A}_Q$:

- $<\Delta_G, p> = G \cap p \in \mathcal{B}_Q$,
- $<G \times \{y\}, p> = \begin{cases} G & \text{if } y \in p \\ \emptyset & \text{otherwise} \end{cases} \in \mathcal{B}_Q$.

It remains to prove that $(\mathcal{B}_Q, \mathcal{A}_Q)$ is separable. Let R_1, R_2 be binary relations belonging to $\mathcal{A}_Q$, such that $R_1 \neq R_2$. Suppose $(x, y) \in R_1 \backslash R_2$.

- If $y \in \{y_1, y_2\}$ then $\{y\} \in \mathcal{B}_Q$. We have $x \in <R_1, \{y\}>$, but $x \notin <R_2, \{y\}>$.
- If $y \notin \{y_1, y_2\}$ then there exists $\Delta_G \subseteq R_1$, such that $y \in G$ and $y = x$. Hence, $x \in < R_1, G \backslash \{y_1, y_2\} >$. Suppose that $x \in < R_2, G \backslash \{y_1, y_2\} >$. Then, there exists $y' \in G \backslash \{y_1, y_2\}$, such that, $(x, y') \in R_2$. Furthermore $y' \neq y$, *i.e.*, $y' \neq x$. Therefore $y' \in \{y_1, y_2\}$, which is a contradiction. $\square$

STEP 2. Let (X', R') be another graph in $UndGraph_0$. Let $Q', \mathcal{T}_0', \mathcal{T}_1', \ldots, \mathcal{T}_8', \mathcal{T}', c_a', c_b'$ stand for (X', R') as $Q, \mathcal{T}_0, \mathcal{T}_1, \ldots, \mathcal{T}_8, \mathcal{T}, c_a, c_b$ stand for (X, R). If $g : Q' \longrightarrow Q$ is a morphism of $\psi(UndGraph_0^{op})$, we define

$$\phi(g) = (\phi_g^1, \phi_g^2) : (\mathcal{B}_Q, \mathcal{A}_Q) \longrightarrow (\mathcal{B}_{Q'}, \mathcal{A}_{Q'})$$

by putting

(i) $\phi_g^1(p) = g^{-1}(p)$ for $p \in \mathcal{B}_Q$;

(ii) $\phi_g^2(\Delta_G) = \Delta_{g^{-1}(G)}$ for $\Delta_G \in \mathcal{A}_Q$;

(iii) $\phi_g^2(G \times \{y\}) = g^{-1}(G) \times \{y\}$ for $G \times \{y\} \in \mathcal{A}_Q$;

(iv) For the remaining elements of $\mathcal{A}_Q$ use (ii), (iii) and the fact that ϕ_g^2 preserves finite unions.

REMARK 5.4. Suppose that ϕ is a well defined full embedding. Then all the homomorphisms $(h_1, h_2) : (\mathcal{B}_Q, \mathcal{A}_Q) \longrightarrow (\mathcal{B}_{Q'}, \mathcal{A}_{Q'})$ among the Kripke structures of the image subcategory of ϕ are of the described form, that is $(h_1, h_2) = \phi(g)$ for some continuous state mapping $g : Q' \longrightarrow Q$. In particular, $h_1(p) = g^{-1}(p)$ for each p belonging to the basis $\mathcal{T}$. Besides, for every $x \in Q$ the singular set $\{x\}$ is the intersection of all its open neighbourhoods belonging to $\mathcal{T}$ and pre-images under g preserve those intersections. Therefore (h_1, h_2) is uniquely grounded and g is uniquely compatible. Thus, the mentioned class of Kripke structures will be perfect.

Since ψ is a full embedding, we have $g = \psi(f)$ for one and only one morphism of graphs $f : (X, R) \longrightarrow (X', R')$.

The rest of this section is devoted to the proof that ϕ is a well defined embedding.

LEMMA 5.5. *We have*

(i) $g^{-1}(\{y_1\}) = \{y_1\} \in \mathcal{B}_{Q'}$ *and* $g^{-1}(\{y_2\}) = \{y_2\} \in \mathcal{B}_{Q'}$*;*

(ii) *For* $U \in \mathcal{T}$, $g^{-1}(U) \in \mathcal{T}'$*;*

(iii) *For* $p \in \mathcal{B}_Q$, $g^{-1}(p) \in \mathcal{B}_{Q'}$*.*

Proof. (i) Follows from last remark.
(iii) Follows from (i) and (ii).
(ii) Beginning with some remarks we will reach the aimed conclusions for each set $U \in \mathcal{T}$.

(a) For $\prod\limits_{x \in X} U_x \in \mathcal{T}_0$, $g^{-1}\left(\prod\limits_{x \in X} U_x\right) = \prod\limits_{x' \in X'} U'_{x'} \in \mathcal{T}'_0$ where $U'_{x'} = \bigcap\limits_{x \in f^{-1}(\{x'\})} U_x$, as we are going to verify.

It happens that $g^{-1}\left(\prod\limits_{x \in X} U_x\right) \subseteq g^{-1}(H^X) \subseteq H^{X'}$. Then,

$$\alpha \in g^{-1}\left(\prod_{x \in X} U_x\right) \iff \alpha \circ f \in \prod_{x \in X} U_x \iff \forall x \in X, \alpha(f(x)) \in U_x \iff$$

$$\iff \forall x' \in X',\ \alpha(x') \in \bigcap_{x \in \{x : f(x) = x'\}} U_x.$$

Notice that it may happen $\{x : x' = f(x)\} = \emptyset$. In that case, we make the usual convention $\bigcap\limits_{x \in \emptyset} U_x = H$. Moreover, each U_x appears in the expression of only one of the $U'_{x'}$ otherwise the mapping f wouldn't be a well defined mapping. Consequently, only finitely many U_x's are distinct from H in the expression of $U'_{x'}$, then $U'_{x'}$ is a regularly open set. Therefore, we can conclude that the product $\prod\limits_{x' \in X'} U'_{x'}$ is in $\mathcal{T}'_0$.

(b) For $U \in \mathcal{T}_0$, we have

$$cl_{H^{X'}}(g^{-1}(U)) \subseteq g^{-1}(cl_{H^X}(U)).$$

Now, we consider the different types of elements of $\mathcal{T}$.

1. If $U \in \mathcal{T}_1$, trivially we have $g^{-1}(U) = U \in \mathcal{T}'_1$.

2. If $U \in \mathcal{T}_2$, *mutatis mutandis* for 1.

3. If $U \in \mathcal{T}_3$, it follows from the above remarks that $g^{-1}(U) \in \mathcal{T}'_3$.

4. If $U = \left(\left(\prod_{x\in X} U_x\right) \cap \chi_R\right) \times U_C \in \mathcal{T}_4$, we have

$$g^{-1}\left(\left(\left(\prod_{x\in X} U_x\right) \cap \chi_R\right) \times U_C\right) \subseteq g^{-1}(\chi_R \times U_C) \subseteq \chi_{R'} \times C.$$

As a consequence, we can write

$$g^{-1}\left(\left(\left(\prod_{x\in X} U_x\right) \cap \chi_R\right) \times U_C\right) = \left(\left(\prod_{x'\in X'} U'_{x'}\right) \cap \chi_R\right) \times U_C \in \mathcal{T}'_4.$$

5. If $U = U_A \bigcup \prod_{x\in X} U_x \in \mathcal{T}_5$, it happens that

$$g^{-1}\left(U_A \bigcup \left(\prod_{x\in X} U_x\right)\right) = U_A \bigcup \left(\prod_{x'\in X'} U'_{x'}\right).$$

Moreover, if $c_a \in \left(\prod_{x\in X} U_x\right)$ we have $c'_a \in \prod_{x'\in X'} U'_{x'}$. Therefore, the conclusion that $g^{-1}(U) \in \mathcal{T}'_5$ follows easily from the above remarks.

6. If $U \in \mathcal{T}_6$, *mutatis mutandis* for 5.

7. If $U = \left(\prod_{x\in X} U_x\right) \bigcup \left(\left(\prod_{x\in X} U_x\right) \cap \chi_R\right) \times U_C \in \mathcal{T}_7$, we will have $g^{-1}(U) \in \mathcal{T}'_7 \bigcup \mathcal{T}'_3$. In fact,

$$g^{-1}\left(\left(\prod_{x\in X} U_x\right) \bigcup \left(\left(\prod_{x\in X} U_x\right) \cap \chi_R\right) \times U_C\right) =$$
$$= \left(\prod_{x'\in X'} U'_{x'}\right) \bigcup \left(\left(\prod_{x'\in X'} U'_{x'}\right) \cap \chi_{R'}\right) \times U_C.$$

By remark (b), if a characteristic function $h_Y \in cl_{H^{X'}}\left(\prod_{x'\in X'} U'_{x'}\right)$ then

$g(h_Y) = h_{f^{-1}(Y)} \in cl_{H^X}\left(\prod_{x\in X} U_x\right)$. Since $U \in \mathcal{T}_7$, we have $g(h_Y) \in \prod_{x\in X} U_x$. Consequently, $h_Y \in \prod_{x'\in X'} U'_{x'}$. Therefore,

$$cl_{H^{X'}}\left(\prod_{x'\in X'} U'_{x'}\right) \cap \chi_{R'} = \left(\prod_{x'\in X'} U'_{x'}\right) \cap \chi_{R'}.$$

The conditions on c'_a and c'_b are also verified. Finally, if

$$\left(\prod_{x'\in X'} U'_{x'}\right) \cap \chi_{R'} \neq \emptyset$$

then $g^{-1}(U) \in \mathcal{T}'_7$. Otherwise, $g^{-1}(U) \in \mathcal{T}'_3$.

8. If $U = U_A \bigcup U_B \bigcup (\chi_R \times U_C) \in \mathcal{T}_8$ then $g^{-1}(U) = U_A \bigcup U_B \bigcup (\chi_{R'} \times U_C) \in \mathcal{T}'_8$. □

LEMMA 5.6. *$\phi(g)$ is a well defined homomorphism of Kripke structures.*

Proof. It follows from Lemma 5.5 that ϕ_g^1 is a well defined map. Besides, ϕ_g^2 is a well defined mapping too, since each $R \in \mathcal{A}_Q$ admits only one decomposition $R = \Delta_G \cup G_1 \times \{y_1\} \cup G_2 \times \{y_2\}$ (is a consequence of the choice of the elements of $\mathcal{T}_1, \mathcal{T}_5$ and $\mathcal{T}_8$). ϕ_g^1 is, trivially, a homomorphism of Boolean algebras. ϕ_g^2 is additive by construction. The preservation of the composition is a direct consequence of the following fact

$$(*) \text{ for } p \in \mathcal{B}_Q \text{ and } y \in \{y_1, y_2\}, \quad y \in p \textit{ iff } y \in g^{-1}(p).$$

We will verify that ϕ_g^2 preserves $*$. For $\Delta_G \cup G_1 \times \{y_1\} \cup G_2 \times \{y_2\} \in \mathcal{A}_Q$ we have

$$\begin{aligned}
&\phi_g^2\left((\Delta_G \cup G_1 \times \{y_1\} \cup G_2 \times \{y_2\})^*\right) = \\
&= \Delta_{Q'} \cup g^{-1}(G_1) \times \{y_1\} \cup g^{-1}(G_2) \times \{y_2\} \cup \left(Q' \times \delta_{y_1}^{g^{-1}(G_2)} \cap g^{-1}(G_1) \times \{y_2\}\right) \cup \\
&\qquad\qquad \cup \left(Q' \times \delta_{y_2}^{g^{-1}(G_1)} \cap g^{-1}(G_2) \times \{y_1\}\right) = \\
&= \left(\Delta_{Q'} \cup g^{-1}(G_1) \times \{y_1\} \cup g^{-1}(G_2) \times \{y_2\}\right)^* = \\
&= \left(\phi_g^2(\Delta_G \cup G_1 \times \{y_1\} \cup G_2 \times \{y_2\})\right)^*,
\end{aligned}$$

where the Kronecker symbol δ is used with the following meaning, for $y \in Q'$ and $F \subseteq Q'$:

$$\delta_y^F = \begin{cases} Q' \text{ if } y \in F \\ \emptyset \text{ otherwise.} \end{cases}$$

It remains to prove that ϕ_g preserves $<>$. It is enough to analyze the cases $\Delta_G, G \times \{y\} \in \mathcal{A}_Q$, for $p \in \mathcal{B}_Q$:

- $\phi_g^1(<\Delta_G, p>) = \phi_g^1(G \cap p) = g^{-1}(G) \cap g^{-1}(p) = <\Delta_{g^{-1}(G)}, g^{-1}(p)> = \\ = <\phi_g^2(\Delta_G), \phi_g^1(p)>$.
- $\phi_g^1(<G \times \{y\}, p>) = \begin{cases} g^{-1}(G) & \text{if } y \in p \\ \emptyset & \text{otherwise,} \end{cases}$

$$<\phi_g^2(G \times \{y\}), \phi_g^1(p)> = <g^{-1}(G) \times \{y\}, g^{-1}(p)> = \begin{cases} g^{-1}(G) & \text{if } y \in g^{-1}(p) \\ \emptyset & \text{otherwise,} \end{cases}$$

and then the conclusion follows from (*). $\square$

PROPOSITION 5.7. *ϕ is an embedding.*

Proof. It is trivial to show that ϕ is a one-to-one functor on objects. In order to show that ϕ is also faithful consider

$$g, g' : Q' \longrightarrow Q,$$

non-constant continuous mappings such that $g \neq g'$. So, there exists $z \in Q'$ such that $g(z) \neq g'(z)$. Since Q is Hausdorff there are opens U, U' in $\mathcal{T}$ verifying,

$$g(z) \in U,\ g'(z) \in U' \text{ and } U \cap U' = \emptyset,$$

that is,

$$z \in \phi_g^1(U),\ z \notin \phi_{g'}^1(U).$$

Consequently, $\phi(g) \neq \phi(g')$. $\square$

6. The embedding is full

This section is devoted to the proof of the fullness of the functor ϕ constructed in the previous section. The proof is still rather involving. We maintain the notation of the previous section.

Let

$$(h_1, h_2) : (\mathcal{B}_Q, \mathcal{A}_Q) \longrightarrow (\mathcal{B}_{Q'}, \mathcal{A}_{Q'})$$

be a homomorphism of Kripke structures. We are going to construct a non-constant continuous mapping $g : Q' \longrightarrow Q$ and show that g satisfies $\phi(g) = (h_1, h_2)$.

Let $x \in Q'$. Consider

$$\mathcal{F}_x = \{F \in \mathcal{B}_Q : F \text{ is closed in } Q \text{ and } x \in h_1(F)\}\,.$$

We have

(i) $F \in \mathcal{F}_x \Longrightarrow F \neq \emptyset$;

(ii) $F_1, F_2 \in \mathcal{F}_x \Longrightarrow F_1 \cap F_2 \in \mathcal{F}_x$;

(iii) $I_x = \bigcap \{F : F \in \mathcal{F}_x\} \neq \emptyset$ (it follows from the compactness of Q, (ii) and (i));

(iv) I_x contains one element only.

We prove (iv). If there are $z_1, z_2 \in I_x$ with $z_1 \neq z_2$, then there are $U_1, U_2 \in \mathcal{T}$ such that

$$z_1 \in U_1,\ z_2 \in U_2 \ \text{ and } \ U_1 \cap U_2 = \emptyset.$$

Thus, $h_1(U_1) \cap h_1(U_2) = \emptyset$, since h_1 is a Boolean homomorphism.
Therefore, if $x \in h_1(U_1)$ we would have $x \notin h_1(U_2)$, *i.e.*, $x \in h_1(U_2^c)$. It would then follow $U_2^c \in \mathcal{F}_x$ and $z_2 \in U_2^c$, which is a contradiction.
If $x \notin h_1(U_1)$ we would reach a contradiction in the same way.

Thus, we define $g(x)$ to be the unique element of the set I_x. Let us write

$$g(x) = \bigcap_{F \in \mathcal{F}_x} F.$$

LEMMA 6.1. *g is a non-constant mapping.*

Proof. We have,

$$<Q \times \{y_1\}, \{y_1\}>= Q \ \text{ and } \ <Q \times \{y_2\}, \{y_2\}>= Q.$$

Applying the homomorphism h to these equalities, necessarily we conclude

$$h_1(\{y_1\}) \neq \emptyset \ \text{ and } \ h_1(\{y_2\}) \neq \emptyset.$$

Taking one element x_1 in the first set and one element x_2 in the second set, we get $g(x_1) = y_1 \neq y_2 = g(x_2)$. □

LEMMA 6.2. *The mapping g is continuous.*

Proof. The conclusion comes after the following sub-lemmas 6.3-6.7. □

LEMMA 6.3. *If $U \in \mathcal{T}$ then $h_1(U)$ is open in Q'.*

Proof. If $U \in \mathcal{T}$ we have $\Delta_U \in \mathcal{A}_Q$. Thus,

$$<\Delta_U, U>= U \text{ and } <\Delta_U, U^c>= \emptyset.$$

Applying the homomorphism h we conclude

$$(*)\ <h_2(\Delta_U), h_1(U)>= h_1(U) \text{ and } (**)\ <h_2(\Delta_U), (h_1(U))^c>= \emptyset.$$

Besides,

$$h_2(\Delta_U) = \Delta_G \cup G_1 \times \{y_1\} \cup G_2 \times \{y_2\}$$

for some open sets $G, G_1, G_2 \in \mathcal{B}_{\mathcal{T}'}$.

Then, (**) yields

$$<\Delta_G \cup G_1 \times \{y_1\} \cup G_2 \times \{y_2\}, (h_1(U))^c>= \emptyset.$$

Thus,

$$G \cap (h_1(U))^c = \emptyset,$$
$$G \subseteq h_1(U).$$

On the other hand, (*) gives, successively,

$$<\Delta_G \cup G_1 \times \{y_1\} \cup G_2 \times \{y_2\}, h_1(U)>= h_1(U),$$

$$\underbrace{\left(G \cap h_1(U)\right)}_{G} \cup \left(G_1 \cap \delta_{y_1}^{h_1(U)}\right) \cup \left(G_2 \cap \delta_{y_2}^{h_1(U)}\right) = h_1(U).$$

Consequently, $h_1(U)$ is open. □

LEMMA 6.4. *For $U \in \mathcal{T}$ it holds:*

(i) *$ext_Q(U)$ is a finite union of elements of $\mathcal{T}$;*

(ii) *$\overline{U} \in \mathcal{B}_Q$;*

(iii) *$h_1(\overline{U})$ is closed in Q'.*

Proof. (ii) Follows immediately from (i).

(iii) By (i) there are $U_1, \ldots, U_n \in \mathcal{T}$ such that

$$ext(U) = \bigcup_{i=1}^{n} U_i.$$

Therefore

$$h_1(\overline{U}) = h_1\left(ext(U)^c\right) = \left(\bigcup_{i=1}^{n} h_1(U_i)\right)^c.$$

The conclusion follows from Lemma 6.3.

(i) Firstly, we decompose each space A, B, C and H^X in some finite union of regularly open sets in the respective space. Then, Q will be a finite union of elements of the basis by means of those decompositions.

Since A is Hausdorff and regular there are regularly open sets U_A^0 and U_A^1 in A such that

$$a_0 \in U_A^0,\ a_1 \in U_A^1 \text{ and } U_A^0 \bigcap U_A^1 = \emptyset.$$

Thus, $U_A^0 \subseteq (U_A^1)^c$ and since U_A^1 is open, we get $cl_A(U_A^0) \subseteq (U_A^1)^c$. Therefore, $a_1 \ \not\in\ cl_A(U_A^0)$ and the same applying to a_0. Now, by regularity there are regularly open sets G^0 and G^1 in A such that

$$a_0 \in G^0 \subseteq cl_A(G^0) \subseteq U_A^0,$$

$$a_1 \in G^1 \subseteq cl_A(G^1) \subseteq U_A^1.$$

The exterior of regularly open sets is still regularly open. Denoting by U_A^2 the set $ext_A(G^0) \bigcap ext_A(G^1)$, which is still regularly open, we have

$$A = U_A^0 \bigcup U_A^1 \bigcup U_A^2.$$

The same applying to B and C.

Since $\{0,1\}^X$ is a closed set, the same reasoning allows us to choose in $\mathcal{T}_0$ regularly open neighborhoods $U_{H^X}^a$ and $U_{H^X}^b$ of c_a and c_b, respectively, in such a way that they have their closure disjoint of $\{0,1\}^X$ and moreover, $c_a \notin cl_{H^X}(U_{H^X}^b)$, $c_b \notin cl_{H^X}(U_{H^X}^a)$. Then, it is possible to choose $U_{H^X}^\chi = \prod_{x \in X} U_x \in \mathcal{T}_0$ such that $\chi_R \subseteq U_{H^X}^\chi$, (thus $cl_{H^X}(U_{H^X}^\chi) \cap \chi_R = U_{H^X}^\chi \cap \chi_R$), c_a, $c_b \notin cl_{H^X}(U_{H^X}^\chi)$ and

$$H^X = U_{H^X}^a \bigcup U_{H^X}^b \bigcup U_{H^X}^\chi.$$

Actually, as $U_{H^X}^a \bigcup U_{H^X}^b$ is strictly contained in H^X, there exists $x_0 \in X$ such that $U_{x_0}^a \cup U_{x_0}^b$ is strictly contained in H as well, where $U_{x_0}^a$ and $U_{x_0}^b$ are the x_0-factors of $U_{H^X}^a$ and $U_{H^X}^b$, respectively. Then, it is enough to consider in $\prod_{x \in X} U_x$ all the U_x's equal to H, but U_{x_0}. As it was made for the space A, U_{x_0} can be chosen such that

$$H = U_{x_0}^a \cup U_{x_0}^b \cup U_{x_0}$$

and $a, b \notin cl_H(U_{x_0})$.

We decompose $\chi_R \times C$ by means of the corresponding decomposition of $C = U_C^0 \bigcup U_C^1 \bigcup U_C^2$:

$$\chi_R \times C = (\chi_R \times U_C^0) \bigcup (\chi_R \times U_C^1) \bigcup (\chi_R \times U_C^2).$$

Given $U \in \mathcal{T}$, we find a finite decomposition of $ext_Q(U)$ in terms of the elements of $\mathcal{T}$, after replacing A, B, H^X and $\chi_R \times C$ in the following expression by their decompositions:

$$ext(U) = (ext(U) \bigcap A) \cup (ext(U) \bigcap B) \cup \left(ext(U) \bigcap H^X\right) \cup \left(ext(U) \bigcap (\chi_R \times C)\right)$$

$\square$

Denote by $\mathcal{R}_\mathcal{T}$ the family of open sets of Q which are finite unions of open sets of the basis $\mathcal{T}$.

LEMMA 6.5. *For $G_1 \in \mathcal{R}_\mathcal{T}$ and $G_2 \in \mathcal{B}_Q$, if $\overline{G_1} \subseteq G_2$ then $\overline{h_1(G_1)} \subseteq h_1(G_2)$.*

Proof. Let $G_1 \in \mathcal{T}$, since the conclusion about the more general case follows immediately from this one.

Thus, the monotony of h_1 yields

$$h_1(G_1) \subseteq h_1(\overline{G_1}) \text{ and } h_1(\overline{G_1}) \subseteq h_1(G_2).$$

The required inclusion holds since $h_1(\overline{G_1})$ is closed. $\square$

The following Lemma is valid in every regular compact space.

LEMMA 6.6. *For every closed F in Q we have*

$$F = \bigcap_{\{G \in \mathcal{R}_{\mathcal{T}} : G \supseteq F\}} \overline{G}.$$

Proof. The inclusion ($\subseteq$) is obvious. Suppose that it is a strict one. Let x be one element of the intersection which doesn't belong to F. By regularity, there are open sets V_1, V_2 such that

$$x \in V_1,\ F \subseteq V_2 \text{ and } V_1 \cap V_2 = \emptyset.$$

For each $y \in F$ there exists $U_y \in \mathcal{T}$ such that $y \in U_y \subseteq V_2$. F is compact, so there exists a finite union of those U_y, let's say G_2, satisfying $F \subseteq G_2 \subseteq V_2$. The hypothesis implies $x \in \overline{G_2}$. Now, we have successively,

$V_1 \cap V_2 = \emptyset \Longrightarrow V_2 \subseteq V_1^c \Longrightarrow \overline{G_2} \subseteq V_1^c \Longrightarrow x \in V_1^c$, which is a contradiction. □

LEMMA 6.7. *For every closed F in Q the following equalities hold*

$$g^{-1}(F) = \bigcap_{F \subseteq G \in \mathcal{R}_{\mathcal{T}}} h_1(G) = \bigcap_{F \subseteq G \in \mathcal{R}_{\mathcal{T}}} \overline{h_1(G)}.$$

Proof. We begin with the proof of the second equality. The inclusion ($\subseteq$) is obvious. For the proof of the other inclusion we consider $G \in \mathcal{R}_{\mathcal{T}}$ such that $F \subseteq G$. Q is a normal space, so there exists an open U' such that

$$F \subseteq U' \subseteq \overline{U'} \subseteq G.$$

By an argument similar to the one used to prove the last Lemma, there exists $U \in \mathcal{R}_{\mathcal{T}}$ such that

$$F \subseteq U \subseteq U'.$$

Consequently,

$$F \subseteq U \subseteq \overline{U} \subseteq G.$$

By Lemma 6.5 we have $\overline{h_1(U)} \subseteq h_1(G)$.

Next, we prove the first equality.

($\subseteq$) Let $G \in \mathcal{R}_{\mathcal{T}}$ such that $F \subseteq G$. Then, G^c is closed and still belongs to $\mathcal{B}_Q$. By definition of g we write $x \in h_1(G^c) \Longrightarrow g(x) \in G^c$. That is, $h_1(G^c) \subseteq g^{-1}(G^c)$, which is the same as $g^{-1}(G) \subseteq h_1(G)$, since h_1 is a Boolean homomorphism. From this together with $F \subseteq G$, we conclude

$$g^{-1}(F) \subseteq h_1(G)$$

as it was required.

($\supseteq$) Let $x \in \bigcap_{F \subseteq G \in \mathcal{R}_{\mathcal{T}}} h_1(G)$. Thus, for every $G \in \mathcal{R}_{\mathcal{T}}$, we have

$$F \subseteq G \Longrightarrow x \in h_1(G) \subseteq h_1(\overline{G}).$$

It follows $g(x) \in \overline{G}$, then by Lemma 6.6, $g(x) \in F$, that is, $x \in g^{-1}(F)$. □

REMARK 6.8. If $g : Q' \longrightarrow Q$ is a non-constant continuous mapping, we can guarantee the existence of a homomorphism of graphs

$$f : (X, R) \longrightarrow (X', R')$$

such that $\psi(f) = g$. Therefore, we know the form of g.

The following proposition will complete the establishment of the fullness.

PROPOSITION 6.9. *$\phi(g) = (h_1, h_2)$.*

Proof. To conclude that $\phi_g^1 = h_1$ is enough to prove that $\phi_g^1(U) = h_1(U)$, for every $U \in \mathcal{T} \bigcup \Big\{\{y_1\}, \{y_2\}\Big\}$, since this set generates $\mathcal{B}_Q$. The proof of this fact is finished after the Lemmas 6.10-6.14.

LEMMA 6.10. *It holds $h_1(\{y_1\}) = g^{-1}(\{y_1\})$ and $h_1(\{y_2\}) = g^{-1}(\{y_2\})$.*

Proof. We have $<Q \times \{y_1\}, \{y_1\}>= Q$. Applying h_1, it follows that

$$<h_2(Q \times \{y_1\}), h_1(\{y_1\})>= Q',$$

and so $h_1(\{y_1\}) \neq \emptyset$.

On the other hand, by definition of g, $y \in h_1(\{y_1\}) \Longrightarrow g(y) = y_1$. Remark 6.8 implies $y = y_1$. Therefore,

$$h_1(\{y_1\}) = g^{-1}(\{y_1\}).$$

Similarly for y_2. □

Now, we have to prove that $h_1(U) = g^{-1}(U)$ for $U \in \mathcal{T}$. The definition of g yields $h_1(F) \subseteq g^{-1}(F)$ for every closed $F \in \mathcal{B}_Q$.

So, we can write immediately

$$g^{-1}(U) \subseteq h_1(U) \subseteq h_1(\overline{U}) \subseteq g^{-1}(\overline{U}).$$

If we show that $int(g^{-1}(\overline{U})) \subseteq g^{-1}(U)$, then we will be able to conclude the required equality, since $h_1(U)$ is open (Lemma 6.3).

LEMMA 6.11. *For $U \in \mathcal{T}$, we have*

$$int\left(g^{-1}(\overline{U})\right) \subseteq g^{-1}(U).$$

Proof. We consider the proof for $U \in \mathcal{T}_7$, the other cases being similar or straightforward.

Note that $g = \psi(f)$ for some graph homomorphism f.

We begin with the calculation of $g^{-1}\left(\overline{\prod\limits_{x \in X} U_x}\right)$ for $\prod\limits_{x \in X} U_x \in \mathcal{T}_0$. As in the proof of Lemma 5.5(ii), we can write

$$g^{-1}\left(\overline{\prod_{x \in X} U_x}\right) = g^{-1}\left(\prod_{x \in X} cl_H(U_x)\right) = \prod_{x' \in X'} V_{x'}, \text{ where } V_{x'} = \bigcap_{x \in f^{-1}(\{x'\})} cl_H(U_x).$$

Now, if $U = \left(\prod\limits_{x \in X} U_x\right) \bigcup \left(\left(\prod\limits_{x \in X} U_x\right) \cap \chi_R\right) \times U_C \in \mathcal{T}_7$, we have

$$\overline{U} = \left(\prod_{x \in X} cl_H(U_x)\right) \bigcup cl_{\chi_R}\left(\left(\prod_{x \in X} U_x\right) \cap \chi_R\right) \times cl_C(U_C).$$

Since $U \in \mathcal{T}_7$, the set $\left(\prod\limits_{x \in X} U_x\right) \cap \chi_R$ is closed in χ_R.

Therefore,

$$g^{-1}(\overline{U}) = \Big(\prod_{x'\in X'} V_{x'}\Big) \cup \Big(\Big(\prod_{x'\in X'} U'_{x'}\Big) \cap \chi_{R'}\Big) \times cl_C(U_C),$$

where

$$U'_{x'} = \bigcap_{x\in f^{-1}(\{x'\})} U_x.$$

Since

$$cl_{H^X}(\Big(\prod_{x'\in X'} U'_{x'}\Big) \cap \chi_R) = \Big(\prod_{x'\in X'} V_{x'}\Big) \cap \chi_{R'} = \Big(\prod_{x'\in X'} U'_{x'}\Big) \cap \chi_{R'},$$

and each $V_{x'}$ is, effectively, a finite intersection, we have, successively,

$$int(g^{-1}(\overline{U})) =$$

$$= \Big(\prod_{x'\in X'} int_H(V_{x'})\Big) \cup int_{\chi_{R'}}\Big(\Big(\prod_{x'\in X'} U'_{x'}\Big) \cap \chi_{R'}\Big) \times int_n\left(cl_C(U_C)\right) =$$

$$= \Big(\prod_{x'\in X'} U'_{x'}\Big) \cup \Big(\Big(\prod_{x'\in X'} U'_{x'}\Big) \cap \chi_{R'}\Big) \times U_C = g^{-1}(U) \qquad \square$$

To finalize the proof of the fullness it remains to prove

$$\phi_g^2 = h_2.$$

It is sufficient to show that

$$h_2(\Delta_U) = \Delta_{g^{-1}(U)}, \quad h_2(Q\times\{y_1\}) = Q'\times\{y_1\} \quad \text{and} \quad h_2(Q\times\{y_2\}) = Q'\times\{y_2\}$$

since those binary relations generate $\mathcal{A}_Q$.

We begin with the proof of a special case, which will be useful to solve the other cases.

LEMMA 6.12. $h_2(\Delta_Q) = \Delta_{Q'}$.

Proof. Let's suppose that $h_2(\Delta_Q) \nsubseteq \Delta_{Q'}$. Then, there exists $x, y \in Q'$, with $x \neq y$, such that $(x, y) \in h_2(\Delta_Q)$. Necessarily, $y \in \{y_1, y_2\}$.

Applying the homomorphism h to $<\Delta_Q, \{y\}>= \{y\}$, we have

$$<h_2(\Delta_Q), h_1(\{y\})>= h_1(\{y\}).$$

Since $h_1(\{y\}) = \{y\}$, it follows that

$$x \in <h_2(\Delta_Q), \{y\}>= \{y\}$$

which is a contradiction. Consequently, $h_2(\Delta_Q) \subseteq \Delta_{Q'}$, *i.e.*,

$$h_2(\Delta_Q) = \Delta_G \text{ for some subset } G \subseteq Q'.$$

But, from $<\Delta_Q, Q>= Q$, we have, successively,

$$<\underbrace{h_2(\Delta_Q)}_{\Delta_G}, \underbrace{h_1(Q)}_{Q'}>= \underbrace{h_1(Q)}_{Q'} \Longrightarrow G \cap Q' = Q' \Longrightarrow G = Q'. \qquad \square$$

LEMMA 6.13. *Let* $\Delta_G \in \mathcal{A}_Q$. *Then* $h_2(\Delta_G) = \Delta_{g^{-1}(G)}$.

Proof. We have $\Delta_Q = (\Delta_G)^*$. Therefore $h_2(\Delta_Q) = (h_2(\Delta_G))^*$, that is,

$$h_2(\Delta_Q) = \Delta_{Q'} \cup h_2(\Delta_G) \cup (h_2(\Delta_G); h_2(\Delta_G)) \cup \ldots$$

Then $h_2(\Delta_G) \subseteq h_2(\Delta_Q) = \Delta_{Q'}$. Thus, $h_2(\Delta_G) = \Delta_{G'}$, for some $G' \subseteq Q'$. Applying h to $<\Delta_G, Q>= G$ we get

$$<\Delta_{G'}, Q'>= h_1(G).$$

Consequently, $G' = h_1(G) = g^{-1}(G)$. □

LEMMA 6.14. *It holds $h_2(Q\times\{y_1\}) = Q'\times\{y_1\}$ and $h_2(Q\times\{y_2\}) = Q'\times\{y_2\}$.*

Proof. Applying h to $<Q\times\{y_1\}, \{y_1\}>= Q$ we can write

$$<h_2(Q\times\{y_1\}), \{y_1\}>= Q'$$

due to Lemma 6.10. Then $Q'\times\{y_1\} \subseteq h_2(Q\times\{y_1\})$.

Similarly,

$$<Q\times\{y_1\}, \{y_1\}^c>= \emptyset \Longrightarrow <h_2(Q\times\{y_1\}), (\{y_1\})^c>= \emptyset.$$

Consequently,

$$(Q'\times(\{y_1\}^c)) \bigcap h_2(Q\times\{y_1\}) = \emptyset.$$

Thus, we can finally conclude that

$$h_2(Q\times\{y_1\}) = Q'\times\{y_1\}.$$

Mutatis mutandis for y_2. □

7. Applications

Now, we list some of the consequences of the alg-universality of *SKri* .

COROLLARY 7.1. *The category SDA is alg-universal.*

COROLLARY 7.2. *Any monoid is isomorphic to the endomorphism monoid of some separable Kripke structure.*

Proof. This result is due to the fact that every small category is algebraic. Thus, it is enough to fully embed in *SKri* the small category with one object and having the given monoid as its endomorphism monoid. □

COROLLARY 7.3. *For every group G there exists a separable Kripke structure K such that $G \cong Aut(K) = End(K)$.*

Proof. Consider the small category with one object and having G as its set of endomorphisms. Notice that every endomorphism in this category is an isomorphism. □

The following property is established in every algebraic and algebraically universal category as it was proved by Hedrlín and Sichler [12].

COROLLARY 7.4. *SKri contains a proper class of mutually disjoint copies of itself.*

It was independently proved by Magill [24], Maxson [25] and Schein [36] that a Boolean algebra is uniquely determined by its endomorphism monoid (that is, for Boolean algebras B, B', if $End(B) \cong End(B')$ then $B \cong B'$). This property is not shared by the objects of alg-universal categories as we show bellow.

COROLLARY 7.5. *For any monoid M and for every cardinal n, there exists a family of separable Kripke structures $(K_i)_{i<n}$ with $End(K_i) \cong M$ and $Hom(K_i, K_j) = \emptyset$ for distinct $i, j < n$.*

Proof. It is enough to fully embed in $SKri$ a small category $\mathcal{S}$ with n objects and with morphisms

$$\mathcal{S}(i,j) = \begin{cases} M & \text{if } i = j \\ \emptyset & \text{otherwise} \end{cases}$$

for objects i and j of $\mathcal{S}$ (composition in $\mathcal{S}$ is defined according to the product in the monoid M). □

COROLLARY 7.6. *Let $(I, \leq)$ be a poset and let $(G_i)_{i \in I}$ be a family of groups. Then, there exists a set S and a family $(K_i)_{i \in I}$ of separable Kripke structures on S, such that $G_i \cong Aut(K_i)$, for $i \in I$. Moreover, a Kripke structure K_i is a subalgebra of K_j iff $i \leq j$.*

Proof. The statement above is, once more, due to the possibility of fully embedding a convenient small category $\mathcal{S}$ in $SKri$.

The construction of $\mathcal{S}$ is as follows. We suppose that there exists the meet $i \wedge i'$ of each pair $i, i' \in (I, \leq)$ and that there exists the maximum 1 of $(I, \leq)$ as well. Otherwise, the set I is enlarged, and the corresponding new groups G_i's are taken arbitrarily. Thus, we define

$$Obj\mathcal{S} = I$$

and for every $i, i' \in I$ the set $\mathcal{S}(i, i')$ of all morphisms of $\mathcal{S}$ from i to i' is

$$\mathcal{S}(i, i') = \{(\rho_{ij}, g, \gamma_{ji'}) : j \in I, j \leq i \wedge i', g \in G_j\},$$

where ρ_{ij} and $\gamma_{ji'}$ are formal symbols making the set of morphisms disjoint for different pairs of objects in order to have the domain and the codomain of an object in $\mathcal{S}$ correctly defined. The composition of morphisms in $\mathcal{S}$ (which is written, for convenience, from the left to the right) is defined by

$$(\rho_{ij}, g, \gamma_{ji'}) \circ (\rho_{i'j'}, g', \gamma_{j'i''}) = \begin{cases} (\rho_{ij}, g.g', \gamma_{ji''}) & \text{if } j=(j \wedge j')=j' \\ (\rho_{ij}, g, \gamma_{ji''}) & \text{if } j=(j \wedge j') \neq j' \\ (\rho_{ij'}, g', \gamma_{j'i''}) & \text{if } j \neq (j \wedge j')=j' \\ (\rho_{ij_0}, 1, \gamma_{j_0 i'}) & \text{if } j_0=(j \wedge j'), j_0 \neq j, j_0 \neq j', \end{cases}$$

where $g \in G_j$, $g' \in G_{j'}$ and 1 is the unit of G_{j_0}. It is routine to check the associativity of the composition, hence $\mathcal{S}$ is really a category. This category was defined in [16].

For every pair $i, j \in I$, with $i \leq j$, let us denote $c_{ij} = (\rho_{ii}, 1, \gamma_{ij})$, (where 1 is the unit of the group G_i) and $r_{ji} = (\rho_{ji}, 1, \gamma_{ii})$. Then, for every i, $r_{ii} = c_{ii}$ is the identity 1_i. For the sake of simplicity, we denote by g ($\in G_i$) the morphism $(\rho_{ii}, g, \gamma_{ii})$. Thus, each morphism $(\rho_{ij}, g, \gamma_{ji'})$ is the result of the composition $r_{ij} \circ g \circ c_{ji'}$.

(*) For each $i \leq j$, c_{ij} is a monomorphism (moreover, since $c_{ij} \circ r_{ji} = 1_i$, each c_{ij} is a coretraction and r_{ji} is the correspondent retraction). Besides, whenever $i \not\leq j$, there are no monomorphisms in $\mathcal{S}(i,j)$ since every morphism in $\mathcal{S}(i,j)$ is the composition of $r_{i,i \wedge j} \circ c_{i \wedge j, i}$ with itself.

(**) Every $g \in G_i$, seen as a morphism in $\mathcal{S}(i,i)$ is an isomorphism, and there is no other isomorphism in $\mathcal{S}(i,i)$.

Let $F : \mathcal{S} \longrightarrow SKri$ be a full embedding. Then, $K_1 = F(1)$ is a separable Kripke structure on a set, let's say, S. Consider, for each $i \in I$, the separable Kripke structure $K_i = (F(c_{i1}))\,(F(i))$ on S, which is the image of $F(i)$ under the monomorphism $F(c_{i1})$.

The conclusion follows from remarks (*) and (**). □

Recalling the condition (M) referred in the section 3, since *SKri* is algebraic we can establish,

COROLLARY 7.7. *SKri it is universal iff* (M) *happens.*

Every class of finitary algebras closed under the formation of homomorphic images, subalgebras and products is a variety, that is, can be presented by a set of equations. This is the well known Birkhoff theorem. Therefore, a countable set of variables is enough to describe the equations satisfied by all the algebras of such classes of algebras.

Now, the question to be addressed is about the number of variables necessary to describe classes of finitary algebras presented by implications. Regarding the existing flexibility in the definition of the concepts used bellow, we follow [2].

To define implication we fix a type Ω of finitary algebras and a set V of variables. So, a V-*implication* for Ω-algebras is a formal expression

$$\{t_i = t'_i\}_{i\in I} \Longrightarrow s = s'$$

where I is an index set and all t_i, t'_i, s and s' are terms of the absolutely free Ω-algebra $\mathrm{V}^{\#}$ generated by V.

We say that an Ω-algebra A *satisfies* that implication if for each substitution of variables

$$f : V \longrightarrow A$$

we have $f^{\#}(s) = f^{\#}(s')$, whenever

$$f^{\#}(t_i) = f^{\#}(t'_i) \text{ for all } i \in I,$$

where $f^{\#} : V^{\#} \longrightarrow A$ is the homomorphic extension of f.

Classes of algebras closed under subalgebras and products, which we will designate by SP-classes, are presented by implications. This is a Theorem of Mal'cev (*cf.* [2]). The set of premises of an implication can be infinite, which gives rise to the question whether the number of variables to describe the class of implications satisfied by a given SP-class of algebras is always bounded or not. If we define a *quasivariety* to be a class of similar algebras presented by a set of implications then the question is whether SP-classes are always quasivarieties. (Although, in [1] a *quasivariety* is defined to be an SP-class of similar algebras.). The answer turns out to be dependent of the set theory we live in.

The problem was solved by Adámek in [1] where he proved that each SP-class of algebras is a quasivariety *iff* the following condition is stated:

Vopěnka's Principle*: every class of models of a first order theory in which no model has an embedding to another one is a set.*

Vopěnka's Principle can be reformulated in terms of the category of graphs yielding:

The category of graphs does not have a discrete full subcategory whose objects form a proper class.

Notice that, by Theorem 7.5, in every algebraically universal category there exists a discrete full subcategory whose objects form a set of prescribed cardinality.

In Bernays-Gödel set theory with the axiom of choice for classes (BGC) measurable cardinals do exist if Vopěnka's Principle holds. Therefore, the negation of Vopěnka's Principle is consistent with BGC set theory.

Simultaneously, Vopěnka's Principle is also consistent with BGC set theory if huge cardinals exist. For details see [2] and [15]. From the alg-universality of *SKri* another formulation of Vopěnka's Principle is possible:

COROLLARY 7.8. *Vopěnka's Principle holds iff there is no discrete full subcategory of S Kri whose objects form a proper class.*

Moreover, as in [1] one can deduce,

COROLLARY 7.9. *Under the negation of Vopěnka's Principle there exists an* SP*-class of Kripke structures which is not a quasivariety.*

8. Open problem on test algebras

A *test algebra* is a dynamic algebra endowed with one more operation ([29]), the test:

$$? : \mathcal{B} \longrightarrow \mathcal{A}$$

and one more equation, namely

$$<p?, q>= p \wedge q.$$

A Kripke *test* structure $(\mathcal{B}, \mathcal{A}, S)$ is a Kripke structure equipped with a binary relation $p? = \{(s,s) : s \in p\}$ for every $p \in \mathcal{B}$. In the presence of test, Kripke structures gain expressiveness, since the meaning of the programs *if p then a else b* and *while p do a* can be captured, respectively, by $(p?)a \cup (\neg p?)b$ and $((p?)a)^*(\neg p?)$. Some results on test algebras can be obtained with slight modifications on results on dynamic algebras. See, *e.g.* [32].

The open problem is " *Is the category of test algebras and their homomorphisms alg-universal*"?

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