A noncoercive Volterra variational inequality arising in a Hele-Shaw problem

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Abstract

We study a free boundary problem arising from the injection-compression moulding process of a fluid into a finite Hele-Shaw closed cell. We show that there exists a unique solution with some additional properties of the corresponding noncoercive evolutionary variational inequality with a Volterra term.

1 Introduction

We consider the analysis of a noncoercive evolution problem arising in a mathematical model of the injection and compression moulding of molten polymers. Recently Steinbach [6],[7] has extended the variational inequality approach to the following variant of a non-homogeneous Hele-Shaw flow, corresponding to find a pressure field $p = p(x,t)$, $x \in \mathbb{R}^2$ and a free boundary domain $\Lambda(t)$ (the liquid region) satisfying at each instant $t \in]0,T]$:

$$-\nabla \cdot (a\nabla p) = j \quad \text{in} \quad \Lambda(t), \tag{1}$$

$$p = 0 \quad \text{and} \quad -a\nabla p \cdot \vec{n} = dV_n \quad \text{on} \quad \partial\Lambda(t), \tag{2}$$

where V_n is the normal velocity of $\partial\Lambda(t)$ in the outward direction $\vec{n}$, $\nabla = (\frac{\partial}{\partial x_1}, \frac{\partial}{\partial x_2})$, $j = j(x,t)$ is a given function accounting for the flow rate and $a = a(x,t) > 0$ is the known flow conductivity which may depend on the fluid shear viscosity and on the possible variable gap thickness $d = d(x,t) > 0$ of the cavity, described by a given two-dimensional bounded domain $\Omega \supset \Lambda(t)$, $t \in]0,T]$.

The initial condition $\Lambda(0) = \Lambda_0$ is given and on the known boundary $\partial\Omega$ of $\Omega \subset \mathbb{R}^2$, a Neumann type condition is prescribed

$$a\nabla p \cdot \vec{n} + \lambda p = 0 \quad \text{on} \quad \partial\Omega \cap \partial\Lambda(t), \tag{3}$$

where the constant $\lambda = 0$, if the flow is tangential (no leakage), or $\lambda > 0$, if the wall of the lateral container is not completely impermeable.

In the classical Hele-Shaw problem ($a = d = 1, \lambda = 0$), Elliott - Janovský [2] have used the Baiocchi type transformation for the nonnegative pressure p extended by zero in $\Omega \setminus \Lambda(t)$:

$$u(x,t) = \int_0^t p(x,\tau)d\tau, \tag{4}$$

and considered the problem as a one parameter (the time) family, of elliptic noncoercive variational inequalities, for which there exists a unique solution, monotone in time, up to the instant at which the mould is filled. In fact this case can be regarded as a one phase Stefan problem with zero specific heat for which the variational approach is also well developped (see, for instance, Rodrigues [5] and its references).

Assuming the natural condition $p \geq 0$ and the front advances with time, considering the integral-differential operator

$$Au(x,t) = -\nabla \cdot (a(x,t)\nabla u(x,t)) + \int_0^t \nabla \cdot (\frac{\partial a}{\partial t}(x,\tau)\nabla u(x,\tau))d\tau,$$

the time integration (4) transforms the boundary problem (1) - (2) in the unilateral condition for $u = u(x,t)$ (see [6],[7] for details)

$$u \geq 0, \quad Au \geq f \quad \text{and} \quad u(Au - f) = 0 \quad \text{in} \quad \Omega, \quad \text{for} \quad t > 0, \tag{5}$$

where $f = f(x,t) = \int_0^t j(x,\tau)d\tau - d(x,t) + d(x,0)\chi_{\Lambda_0}$, χ_{Λ_0} being the characteristic function of the initial set Λ_0. The associated variational inequality corresponding to the fixed boundary condition (3) with $\lambda > 0$ is coercive in the Sobolev space $H^1(\Omega)$ and was studied in detail in [7].

Here we consider the Neumann boundary condition (3) with $\lambda = 0$ which leads to a noncoercive problem of the type considered by Duvaut [1] in linear viscoelasticity. In section 2 we show that up to a compatible time there exists a unique solution, while in section 3 we give some regularity and monotonicity properties of the variational solution. In particular, we extend the results of Steinbach [7] to the noncoercive case.

2 The noncoercive Volterra variational inequality

Letting K be the closed convex subset of $H^1(\Omega)$ such that

$$K = \{v \in H^1(\Omega) : v \geq 0 \;\text{ a.e. in }\; \Omega\}, \tag{6}$$

the unilateral problem (5) with homogeneous Neumann condition can be written in variational form, for each t, and the problem for u may be stated as being:

Problem(V): Find $u = u(t) \in K$ for a.e. $t \in]0, T]$, such that

$$\begin{aligned}\int_\Omega a\nabla u(t)\cdot\nabla(v-u(t))dx \;&\geq\; \int_\Omega \Big(\int_0^t \frac{\partial a}{\partial t}\nabla u d\tau\Big)\cdot\nabla(v-u(t))dx \\ &+\; \int_\Omega f(t)(v-u(t))dx, \quad \forall v \in K\end{aligned} \tag{7}$$

We remark that time enters in Problem(V) basically as a parameter and with a memory effect. However the free boundary flow domain may be obtained at a.e. time $t > 0$ by setting $\Lambda(t) = \{x \in \Omega : u(x,t) > 0\}$. We also observe that from (4) we have $u(0) = 0$ and the initial condition has been incorporated in the definition of f. Since the gap thickness $d(x,t)$ should be decreasing in time, we expect $f(x,t) \leq 0$, but from (7) with $v = u(t)+1$ we see that $\int_\Omega f(x,t)dx \leq 0$ is a necessary condition for the existence of a solution. This should hold up to the time T^* at which the mould is filled. The instant $T^* > 0$ is the solution of the equation

$$0 = \int_\Omega f(x,T^*)dx = \int_\Omega \{\int_0^{T^*} j(x,\tau)d\tau - d(x,T^*) + d(x,0)\chi_{\Lambda_0}\}dx,$$

and we shall consider the problem only up to times $T < T^*$ for which the condition

$$\sup_{0<t<T} \int_\Omega f(x,t)dx \equiv -\beta_T < 0 \tag{8}$$

holds, for a given function $f \in L^\infty(0,T;L^2(\Omega))$.

With respect to the coefficient $a = a(x,t) \in W^{1,\infty}(0,T;L^\infty(\Omega))$ we assume

$$a(x,t) \geq \alpha > 0 \;\text{ and }\; \Big| \frac{\partial a}{\partial t}(x,t) \Big| \leq \alpha' \;\text{ for a.e. }\; x \in \Omega,\; t \in]0,T[, \tag{9}$$

for some positive constants α, α'.

Theorem 1: Under the assumptions (8) and (9) there exists a unique solution $u \in L^\infty(0,T;H^1(\Omega))$ to Problem(V).

Proof: We show first the uniqueness. Suppose $u = u(x,t)$ and $\hat{u} = \hat{u}(x,t)$ are two solutions to (7). Taking $v = u(t)$ in the problem (7) for $\hat{u}$, at time t, and vice-versa, we obtain for $w = u - \hat{u}$

$$\int_\Omega a \mid \nabla w(t) \mid^2 dx \leq \int_0^t \int_\Omega \frac{\partial a}{\partial t} \nabla w(\tau) \cdot \nabla w(t) dx d\tau.$$

Using (9), we get first with the Hölder inequality,

$$\alpha \parallel \nabla w(t) \parallel_{L^2(\Omega)} \leq \alpha' \int_0^t \parallel \nabla w(\tau) \parallel_{L^2(\Omega)} d\tau,$$

and, applying Gronwall inequality, we conclude

$$\parallel \nabla w(t) \parallel_{L^2(\Omega)} = 0 \;\text{ for }\; a.e. \;\; t > 0.$$

Then $w = w(t)$ is a function of t only. Let's suppose $w(t^*) \neq 0$ for some t^*. Since $w(t^*) = u(t^*) - \hat{u}(t^*) \in \mathbb{R}$, from (7) it follows

$$0 = w(t^*) \int_\Omega f(x, t^*) dx$$

which contradicts the assumption (8). It must then be $u = \hat{u}$ for a.e. t.

For the existence we follow the method suggested, without proofs, by Duvaut in [1]. Let $u_n = u_n(x,t)$ denote the unique solution of the

Problem(V_n): Find $u_n = u_n(t) \in K$ for a.e. $t \in]0,T]$, such that

$$\begin{aligned} \int_\Omega a \nabla u_n(t) \cdot \nabla (v - u_n(t)) dx \;\; &\geq \;\; \int_\Omega (\int_0^t \frac{\partial a}{\partial t} \nabla u_{n-1} d\tau) \cdot \nabla (v - u_n(t)) dx \\ &+ \int_\Omega f(t)(v - u_n(t)) dx, \quad \forall v \in K \end{aligned} \tag{10}$$

where $u_0 = 0$ and $n \in \mathbb{N}$. By the assumption (8), well-known results for noncoercive variational inequalities (see, for instance, [3], § III.2) yield a sequence of solutions to Problem (V_n)

$$u_n \in L^\infty(0,T; H^1(\Omega)), \quad n = 1, 2, 3, ... \tag{11}$$

Letting $v = u_{n+1}(t)$ in (10) for Problem V_n and $v = u_n(t)$ for Problem V_{n+1}, using the assumption (9), we easily get for $z_n = u_{n+1} - u_n$:

$$\alpha \parallel \nabla z_n(t) \parallel_{L^2(\Omega)} \leq \alpha' \int_0^t \parallel \nabla z_{n-1}(\tau) \parallel_{L^2(\Omega)} d\tau, \quad n = 1, 2, 3, ... \tag{12}$$

Denoting $\varphi_n(t) = \parallel \nabla z_n(t) \parallel_{L^2(\Omega)}$, $n = 0, 1, 2, ...$, we remark that φ_0 is bounded in $L^\infty(0,T)$ by a constant c_0 (depending only on β_T and on $L^\infty(0,T;L^2(\Omega))$ norm of f). From (12), first we get

$$\varphi_1(t) \leq \frac{\alpha'}{\alpha} c_0 t,$$

and, by sucessive iterations, for any $n \in \mathbb{N}$

$$\varphi_n(t) \leq c_0 \frac{(\frac{\alpha'}{\alpha})^n t^n}{n!}, \quad \forall t \in]0, T].$$

This implies that

$$\sup_{0<t<T} \varphi_n(t) = \| \nabla z_n \|_{L^\infty(0,T;L^2(\Omega))} \to 0, \quad \text{as} \quad n \to \infty$$

and, consequently, ∇u_n is a Cauchy sequence in $L^\infty(0, T; (L^2(\Omega))^2)$. In particular, it is bounded independently of n and we have

$$\nabla u_n \to \xi \quad \text{in} \;\; L^\infty(0, T; (L^2(\Omega))^2). \tag{13}$$

Introducing the orthogonal projection in $L^2(\Omega)$ of $H^1(\Omega)$ into $\mathbb{R}$ by $Pv = \frac{1}{|\Omega|}\int_\Omega v dx$, the remainder $Rv = v - Pv$, for all $v \in H^1(\Omega)$ satisfies the Poincaré -Wirtinger inequality. Hence, for a.e. t, we have

$$\| u_n - Pu_n \|_{L^2(\Omega)} \leq c \| \nabla u_n \|_{L^2(\Omega)}, \quad n \in \mathbb{N}. \tag{14}$$

Since $u_n(t) \in K$ implies $Pu_n(t) > 0$, it suffices to obtain an upper bound for $Pu_n(t)$ in order to show that the sequence (11) is bounded in $L^\infty(0, T; H^1(\Omega))$ independently of $n \in \mathbb{N}$.

Letting $v = 0$ in (10) and using the above decomposition, we get

$$\begin{aligned} -Pu_n(t) \int_\Omega f(t) dx + \int_\Omega a \mid \nabla u_n(t) \mid^2 dx \quad &\leq \quad \int_\Omega (\int_0^t \frac{\partial a}{\partial t} \nabla u_{n-1} d\tau) \cdot \nabla u_n(t) dx + \\ &+ \quad \int_\Omega f(t)(u_n(t) - Pu_n(t)) dx, \end{aligned}$$

and using the properties (13) - (14) and the assumptions (8) - (9) we conclude

$$0 < Pu_n(t) \leq \frac{c}{\beta_T} \quad \text{for a.e.} \;\; t \in]0, T], \tag{15}$$

where the constant $c > 0$ is independent of $n \in \mathbb{N}$ and of t.

Then we may assume that there exists a subsequence, still denoted by $\{u_n\}$, such that

$$u_n \rightharpoonup u \;\; \text{in} \;\; L^\infty(0, T; H^1(\Omega)) - \text{weakly}^*$$

for some function $u \in L^\infty(0,T;H^1(\Omega))$. Clearly $u(t) \in K$, for a.e. t , and from (13) we get $\xi = \nabla u$ a.e.. By letting $n \to \infty$ in (10) it is clear that u is the unique solution of Problem(V), and Theorem 1 is proved.

Remark 1: The analogous problem corresponding to $\lambda > 0$ in the Neumann condition (3) corresponds to the variational inequality for $u_\lambda = u_\lambda(t) \in K$ such that, for a.e. $t \in]0,T]$

$$\int_\Omega a\nabla u_\lambda(t)\cdot\nabla(v-u_\lambda(t))dx + \lambda\int_{\partial\Omega} u_\lambda(t)(v-u_\lambda(t))ds \geq$$
$$\int_\Omega(\int_0^t \frac{\partial a}{\partial t}\nabla u_\lambda d\tau)\cdot\nabla(v-u_\lambda(t))dx + \int_\Omega f(t)(v-u_\lambda(t))dx, \quad \forall v \in K \tag{16}$$

Due to the Poincaré type inequality for a Lipschitz boundary $\partial\Omega$, (see, for instance [5],pg.75)

$$\| v \|_{L^2(\Omega)} \leq C_\Omega(\| \nabla v \|_{L^2(\Omega)} + \| v \|_{L^2(\partial\Omega)})$$

this problem is now coercive in $H^1(\Omega)$ and using the argument of the proof of Theorem 1 we find, in the corresponding Problem(V_n) with the additional boundary term, that u_n is also a Cauchy sequence in $L^\infty(0,T;H^1(\Omega))$. This implies that we may solve (16) for any $f \in L^\infty(0,T;L^2(\Omega))$ satisfying (8), which extends the results of Steinbach [6],[7], that require $f \in C([0,T],L^2(\Omega))$, that is, we may allow discontinuous compressions in this model.

Remark 2: We may also consider a non-homogeneous Neumann condition with a lateral injection rate $\zeta = \zeta(x,t)$ on the right-hand side of (3). In the noncoercive case ($\lambda = 0$) we need to add the additional boundary integral $\int_{\partial\Omega} g(t)(v-u(t))ds$ in the right-hand side of (7), with $g(x,t) = \int_0^t \zeta(x,\tau)d\tau \in L^\infty(0,T;L^2(\partial\Omega))$ and to replace the assumption (8) by

$$\sup_{0<t<T}\{\int_\Omega f(x,t)dx + \int_{\partial\Omega} g(x,t)ds\} < 0$$

This assumption is also sufficient for the approximation of the solution to Problem(V) by the solutions of the regularized-penalized Neumann problem in $H^1(\Omega)$, for a.e. $t \in]0,T]$:

$$-\nabla\cdot(a\nabla u_\epsilon(t) - \int_0^t \frac{\partial a}{\partial t}\nabla u_\epsilon d\tau) - \frac{1}{\epsilon}u_\epsilon^-(t) = f(t) \quad \text{in } \Omega$$

$$(a\nabla u_\epsilon(t) - \int_0^t \frac{\partial a}{\partial t}\nabla u_\epsilon d\tau)\cdot\vec{n} + \epsilon u_\epsilon(t) = g(t) \quad \text{on } \partial\Omega$$

where $\epsilon > 0$ is a small parameter tending to zero, and $w^- = \sup(0,-w)$ (see, for instance, [2] or [4], pg. 280).

3 Properties of the variational solutions

We start by showing a Lipschitz continuity in the time variable by assuming also $f \in W^{1,\infty}(0,T;L^2(\Omega))$ i. e. there is a positive constant F' such that

$$\|f(t_1) - f(t_2)\|_{L^2(\Omega)} \leq F'|t_1 - t_2| \text{ for all } t_1, t_2 \in]0,T] \tag{17}$$

Theorem 2 Under the assumptions (8)-(9) and (17) the solution u of Problem (V) is such that

$$u \in W^{1,\infty}(0,T;H^1(\Omega)). \tag{18}$$

Proof: We observe that, by Poincaré-Wirtinger inequality, the $H^1(\Omega)$ norm is equivalent to the norm $|||v||| = |Pv| + \|\nabla v\|_{L^2(\Omega)}$ where $Pv = \frac{1}{|\Omega|}\int_\Omega v dx$, with the orthogonal decomposition $v = Pv + Rv$.

We estimate first, for all $t_1, t_2 \in]0,T]$

$$|Pu(t_1) - Pu(t_2)| \leq \frac{C}{\beta_T}\|\nabla u(t_1) - \nabla u(t_2)\|_{L^2(\Omega)} \tag{19}$$

for some positive constant C depending only on the $L^\infty(0,T;H^1(\Omega))$ norm of u, and other known constants. If $Pu(t_1) > Pu(t_2)$, we take $v = u(t_2)$ in (7) at time t_1 and we get, due to (8),

$$\begin{aligned}
\beta_T[Pu(t_1) - Pu(t_2)] &\leq -\int_\Omega f(t_1)[Pu(t_1) - Pu(t_2)]dx \\
&\leq \int_\Omega f(t_1)[Ru(t_1) - Ru(t_2)]dx + \\
&+ \int_\Omega \left[a(t_1)\nabla u(t_1) - \int_0^{t_1} \frac{\partial a}{\partial \tau}\nabla u d\tau\right] \cdot \nabla(u(t_2) - u(t_1))dx \\
&\leq C\|\nabla u(t_1) - \nabla u(t_2)\|_{L^2(\Omega)}
\end{aligned} \tag{20}$$

by using the Hölder and Poincaré-Wirtinger inequalities. If $Pu(t_2) > Pu(t_1)$ we change t_1 with t_2 in the argument (20) and we may conclude (19).

In order to estimate the gradients, we take $v = u(t_2)$ in (7) at $t = t_1$ and $v = u(t_1)$ at $t = t_2$ and we get for their difference $w = u(t_1) - u(t_2)$.

$$\begin{aligned}
&\alpha\|\nabla w\|^2_{L^2(\Omega)} \leq \int_\Omega a(t_1)|\nabla(u(t_1) - u(t_2))|^2 dx \\
&\leq \int_\Omega \left[(a(t_1) - a(t_2))\nabla u(t_2) + \int_{t_1}^{t_2} \frac{\partial a}{\partial \tau}\nabla u d\tau\right] \cdot \nabla w + \int_\Omega |f(t_1) - f(t_2)||Pw + Rw|dx \\
&\leq 2\alpha' C_1|t_2 - t_1|\,\|\nabla w\|_{L^2(\Omega)} + F'|t_2 - t_1|\Big(\|Pw\|_{L^2(\Omega)} + \|Rw\|_{L^2(\Omega)}\Big) \\
&\leq C'|t_2 - t_1|\,\|\nabla w\|_{L^2(\Omega)}
\end{aligned} \tag{21}$$

where we have used the assumption (17), the estimate (19) and the Poincaré-Wirtinger inequality to control the L^2-norms of Pw and Rw, respectively.

From (21) we find first that $t \longmapsto \|\nabla u(t)\|_{L^2(\Omega)}$ is Lipschitz continuous and, using again (19), the conclusion (18) follows by the equivalence of the norms in $H^1(\Omega)$.

Remark 3 Using the same argument of the proof of Theorem 2, if instead of (17) we assume only $f \in C^{0,\gamma}([0,T];L^2(\Omega))$, $0<\gamma<1$, or only $f \in C^0([0,T];L^2(\Omega))$, we easily conclude from (21), instead of (18) that $u \in C^{0,\gamma}([0,T];H^1(\Omega))$ or $u \in C^0([0,T];H^1(\Omega))$, respectively.

Assuming further assumptions on the data it is possible to improve the regularity of the variational solution, namely its monotonicity in time,

$$t_2 > t_1 \text{ implies } f(x,t_2) \geq f(x,t_1) \text{ for a.e. } x \in \Omega \tag{22}$$

and the $W^{2,p}$-regularity in the space variables, provided the associated linear problem has also this regularity, for instance, under the additional assumptions

$$f \in L^\infty(0,T;L^\infty(\Omega)),\ a \in W^{1,\infty}(0,T;C^1(\Omega)),\ \partial\Omega \in C^{1,1} \tag{23}$$

In fact, arguing as in Steinbach [7], we may also approximate the solution of Problem (V) by letting $\epsilon \to 0^+$ in the semilinear regularized Problem (V$_\epsilon$): find $u_\epsilon(t) \in H^1(\Omega)$, such that for $t \in]0,T]$

$$\begin{aligned}&\int_\Omega \Big[\Big(a\nabla(u_\epsilon(t)) - \int_0^t \frac{\partial a}{\partial t}\nabla u_\epsilon\, d\tau\Big)\cdot\nabla v + \Big(F\theta_\epsilon(u_\epsilon(t)) + \epsilon\, u_\epsilon(t)\Big)v\Big]dx = \\ &= \int_\Omega f(t)\, v\, dx\end{aligned} \tag{24}$$

where $\theta_\epsilon \in C^{0,1}(\mathrm{I\!R})$ is a nondecreasing function satisfying $\theta_\epsilon(r) = -1$ for $r \leq 0$ and $\theta_\epsilon(r) = 0$ for $r \geq \epsilon > 0$ and $F = \|f(0)\|_{L^\infty(\Omega)}$. Using the same arguments as in Theorems 1 and 2, under the assumption (8), we have an *a priori* estimate of u_ϵ in $W^{1,\infty}(0,T;H^1(\Omega))$ independently of ϵ. Hence, arguing as in Theorem 3.5 of [7], we may prove that $u_\epsilon(t) \in K$, $\partial u_\epsilon/\partial t \geq 0$ a.e. in $\Omega\times]0,T]$, $u_\epsilon \to u$ as $\epsilon \to 0$ (see also [4], § 5.3). Therefore, as in Theorem 4.5 of [7], we also have u_ϵ bounded in $L^\infty(0,T;W^{2,q}(\Omega))$ for all finite q, and we may state the following result.

Theorem 3 Under the previous assumptions, namely (8), (9), (17), (22) and (23) the unique solution u of Problem (V) is such that

$$u \in W^{1,\infty}(0,T;H^1(\Omega)) \cap L^\infty(0,T;W^{2,q}(\Omega)),\ \ \forall q < \infty$$

and

$$\frac{\partial u}{\partial t} \geq 0 \text{ a.e. in } \Omega \times]0, T]$$

Remark 4 The assumption (22) is satisfied, for instance, if $j \geq 0$ (injection of polymer) and if $d = d(x, t)$ is non-increasing in time (which holds in moulding compressing). The conclusion $p = \partial u / \partial t \geq 0$ not only is physically natural but also implies that the subsets $\Lambda(t) = \{x : u(x, t) > 0\}$ are increasing in time.

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