Covariance estimation for associated random variables*

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Abstract

Considering an associated and strictly stationary sequence of random variables we introduce an histogram estimator for the covariances between indicator functions of those random variables. We find conditions on the covariance structure of the original random variables for the almost sure convergence of the estimator and for the convergence in distribution of the finite dimensional distributions. Finally we characterize the usual error criteria finding their convergence rates under assumptions on the convergence rate of the covariances.

1 Introduction

Let X_n, $n \geq 1$, be identically distributed random variables with common distribution function F. The study of the asymptotic behaviour of the empirical process

$$Z_n(t) = \frac{1}{\sqrt{n}} \sum_{i=1}^{n} \left(I_{[0,t]}(X_i) - F(t) \right),$$

where I_A represents the characteristic function of the set A, has attracted the interest of many statisticians as this function plays a central role in many statistical applications, whether directly or after some transformation. For example, Watson [19] proposed a "goodness of fit" test for distributions on the circle based on $W_n^2 = \frac{1}{2} \int_{[0,1]^2} (Z_n(t) - Z_n(s))^2 \, ds \, dt$. Another "goodness of fit" problem was considered by Anderson, Darling [1] based on the statistic $A_n^2 = \int_0^1 Z_n(t)\psi(t)\, dt$ for some suitably chosen weight function ψ. Other examples, where the asymptotic behaviour of Z_n is of interest include the Cramer-von Mises ω^2 test, some von Mises functionals or, more generally, functionals of the form $\int G(t, Z_n(t))\, dt$. Notice that, from a theoretical point of view it is enough to consider the empirical process based on random variables uniformly distributed on $[0,1]$. In fact, if this is not the case, define the quantile function $Q(y) = \inf\{x : F(x) \geq y\}$ on $[0,1]$. Then $Z_n(Q(t))$ has the same distribution as the empirical process constructed from uniform $[0,1]$ variables. So this is in fact the only theoretically relevant case, as general convergence conditions may be derived via this distribution characterization. So, throughout this article unless otherwise stated, the random variables X_n, $n \geq 1$, are supposed to be uniformly distributed on $[0,1]$, so the empirical process becomes

$$Z_n(t) = \frac{1}{\sqrt{n}} \sum_{i=1}^{n} \left(I_{[0,t]}(X_i) - t \right), \qquad t \in [0,1]. \tag{1}$$

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The first results characterizing the limit behaviour of the this sequence where obtained under the assumption of independence of the variables X_n, $n \geq 1$, dating back to Donsker [4], where the convergence in distribution of Z_n is proved with a gaussian limiting process Z such that $\mathrm{E}(Z) = 0$ and covariance function $\Gamma(s,t) = \mathrm{E}(Z(s)Z(t)) = s \wedge t - st$, the so called brownian bridge. The convergence is taken in the sense of convergence in distribution in the Skorohod space $\mathrm{D}[0,1]$, the natural space where the sequence Z_n lives. Extensions to nonindependent variables where eventually studied. Assuming the sequence X_n, $n \geq 1$, to be strictly stationary Billingsley [2] and later Sen [14] proved the convergence under some convergence rate on the ϕ-mixing coefficients, which where further replaced by the strong mixing coefficients α_n. For these coefficients Yoshihara [17] proved the convergence under the convergence rate $\alpha_n = O(n^{-r})$, with $r > 3$ (together with the strict stationarity of the sequence), later extended by Shao [15], who required only that $r > 2$. This rate on the strong mixing coefficients is, to the best knowledge of the authors, the slowest known convergence rate giving the convergence in distribution of Z_n to a limiting centered gaussian process Z which now has covariance function

$$\Gamma(s,t) = s \wedge t - st + \sum_{k=2}^{\infty} \left(\mathrm{P}\left(X_1 \leq s, X_k \leq t\right) - st\right) + \sum_{k=2}^{\infty} \left(\mathrm{P}\left(X_1 \leq t, X_k \leq s\right) - st\right). \qquad (2)$$

The use of other mixing coefficients lead to some other recent results, as in Shao, Yu [16] with ρ-mixing coefficients and in Doukhan, Massart, Rio [5] which considered the β-mixing coefficients.

Another way of controlling dependence is the so called association introduced by Esary, Proschan, Walkup [6], which we recall here: the random variables X_n, $n \geq 1$, are associated if

$$\mathrm{Cov}\left(f\left(X_1, ..., X_n\right), g\left(X_1, ..., X_n\right)\right) \geq 0$$

for any $n \in \mathbb{N}$ and real coordinatewise increasing functions f and g for which the covariance above exists. As follows from Newman's inequality (see Theorem 10 in Newman [10]) for associated variables, the covariances $\mathrm{Cov}(X_i, X_j)$ completely determine convergence in distribution, so it is natural to impose conditions in the rate of decrease of $\mathrm{Cov}(X_1, X_n)$ (supposing the strict stationarity of the sequence). Convergence results where first obtained by Yu [18] under the assumption $\mathrm{Cov}(X_1, X_n) = O(n^{-r})$ with $r > 7.5$, later improved by Shao, Yu [16] requiring only that $r > (3 + \sqrt{33})/2 \approx 4.373$.

All the above mentioned results are with respect to convergence in distribution in the Skorohod space $\mathrm{D}[0,1]$. Looking back at the examples presented, we notice that the topology on this space is to strong, so it is possible to derive the same convergence results under weaker conditions stating the problem in a weaker space. A natural choice of the space is $L^2[0,1]$. This problem was considered in Oliveira, Suquet [11] where the convergence was proved under one of the following hypotheses (always supposing the strict stationarity of the sequence)

- $\sum_n \alpha_n < \infty$ in the strong mixing case;
- $\sum_n \mathrm{Cov}^{1/3}(X_1, X_n) < \infty$ in the associated case.

These results where later extended by Oliveira, Suquet [12] replacing the space by $L^p[0,1]$, with $p \geq 2$, which may be a more convenient space depending on the use of a weight function. Again under the stationarity of the sequence, the convergence follows from one of the hypotheses

- $\alpha_n = O(n^{-r})$ with $r > p/2$, in the strong mixing case;

- $\mathrm{Cov}(X_1, X_n) = O(n^{-r})$ with $r > 3p/2$, in the associated case.

The appearance of the extra factor 3 in the convergence rate for associated variables is explained by the inequality (see Lemma 4.5 in Yu [18])

$$\mathrm{Cov}\left(I_{[0,s]}(X_1), I_{[0,t]}(X_2)\right) \le \left(\frac{3}{2}\right)^{1/3} \mathrm{Cov}^{1/3}\left(X_1, X_2\right), \qquad s,t \in [0,1], \tag{3}$$

where X_1, X_2 are associated and uniformly distributed in $[0,1]$.

The limiting process is always centered gaussian with covariance function $\Gamma(s,t)$ given by (2). The characterizations just described are of theoretical nature. In view of some statistical application one does not know the covariances that are summed in the expression (2). Our aim is to study the properties of an estimator for the terms

$$\varphi_k(s,t) = \mathrm{P}\left(X_1 \le s, X_{k+1} \le t\right) - st = \mathrm{Cov}\left(I_{[0,s]}(X_1), I_{[0,t]}(X_{k+1})\right),$$

with $k \in \mathbb{N}$ fixed, under the assumption of association. As the underlying variables X_n, $n \ge 1$, are supposed stationary a natural estimator of this covariance is the following histogram type estimator

$$\widehat{\varphi}_{k,n}(s,t) = \frac{1}{n-k} \sum_{i=1}^{n-k} \left(I_{[0,s]}(X_i), I_{[0,t]}(X_{i+k}) - st\right). \tag{4}$$

It is easily checked that, for each $s,t \in [0,1]$, $\mathrm{E}\left(\widehat{\varphi}_{k,n}(s,t)\right) = \varphi_k(s,t)$.

Estimation under association, or more generally, under positive dependence has been studied by Roussas [13] and Cai, Roussas [3], although the problem under consideration in these references was stated differently. In fact, the interest in [13] and [3] is on estimating the distribution function of the variables X_n, $n \ge 1$, (obviously not supposed to be uniform $[0,1]$). It is worth noticing that, in what regards convergence in distribution of the finite dimensional distributions of the estimator we are lead to the same condition on the covariance structure as the one found by Roussas (see Theorem 1.1 in [13]), namely that

$$\sum_{j=1}^{\infty} \mathrm{Cov}^{1/3}\left(X_1, X_j\right) < \infty.$$

In the sequel, unless otherwise stated, the variables X_n, $n \ge 1$, are strictly stationary associated and uniformly distributed on $[0,1]$.

2 Convergence of the estimator

In this section we will look at the convergence of the estimator (4). The first result gives a condition for the pointwise almost sure convergence of the estimator.

Theorem 1 *If the sequence X_n, $n \ge 1$, is such that*

$$\lim_{n\to\infty} \frac{1}{n} \sum_{j=1}^{n} \mathrm{Cov}^{1/3}\left(X_1, X_j\right) = 0, \tag{5}$$

then, for each fixed $k \in \mathbb{N}$,

$$\lim_{n\to\infty} \widehat{\varphi}_{k,n}(s,t) = \varphi_k(s,t) \quad a.s.$$

Proof: For each $k, n \in \mathbb{N}$ and $s, t \in [0,1]$ define the random variables $Y_{k,n} = I_{[0,s]}(X_n)I_{[0,t]}(X_{n+k})$. Since the sequence X_n, $n \geq 1$, is associated and strictly stationary and the $Y_{k,n}$ are decreasing functions of the X_n, the sequence $Y_{k,n}$, $n \geq 1$, is also associated and strictly stationary.

Our statement follows from the strong law of large numbers for the sequence $Y_{k,n}$, $n \geq 1$. In fact, letting $S_{k,n} = \sum_{i=1}^{n} Y_{k,i}$ then

$$
\begin{aligned}
&\frac{S_{k,n-k} - \mathrm{E}(S_{k,n-k})}{n-k} = \\
&= \frac{1}{n-k} \sum_{i=1}^{n-k} \left[\left(I_{[0,s]}(X_i) I_{[0,t]} X_{i+k} \right) \right) - \mathrm{P}\left(X_1 \leq s, X_{k+1} \leq t \right) \right] = \\
&= \widehat{\varphi}_{k,n}(s,t) - \varphi_k(s,t).
\end{aligned}
$$

According to Theorem 7 in Newman [10] the almost sure convergence to zero of this last expression follows from

$$
\lim_{n \to \infty} \frac{1}{n} \sum_{j=1}^{n} \mathrm{Cov}\left(Y_{k,1}, Y_{k,j} \right) = 0. \tag{6}
$$

To prove (6) we will find an upper bound to $\mathrm{Cov}\left(Y_{k,1}, Y_{k,j} \right)$. Applying a classical inequality by Lebowitz [8] and (3) we find

$$
\begin{aligned}
&0 \leq \mathrm{Cov}\left(Y_{k,1}, Y_{k,j} \right) \leq \\
&\leq \left(\frac{3}{2} \right)^{1/3} \left[\mathrm{Cov}^{1/3}(X_1, X_j) + \mathrm{Cov}^{1/3}(X_1, X_{k+j}) + \right. \\
&\qquad \left. + \mathrm{Cov}^{1/3}(X_{k+1}, X_j) + \mathrm{Cov}^{1/3}(X_{k+1}, X_{k+j}) \right].
\end{aligned} \tag{7}
$$

So (6) follows from (5) which completes the proof of the theorem. ■

It is worth noticing that the exponent $1/3$ in (5) is not relevant. In fact, this condition is equivalent to

$$
\lim_{n \to \infty} \frac{1}{n} \sum_{j=1}^{n} \mathrm{Cov}\left(X_1, X_j \right) = 0. \tag{8}
$$

This follows from an application of Hölder inequality and the fact that the covariances between each pair of variables of the sequence X_n, $n \geq 1$, are always between zero and one, since the variables are associated and uniformly distributed on $[0,1]$.

We note also that condition (5) is weaker than any of the conditions for the convergence of the empirical process mentioned in the Introduction. That (5) is weaker than Shao, Yu's condition follows immediately from the equivalence between (5) and (8). On the other side (5) is obviously weaker than Oliveira, Suquet's condition.

The next theorem establishes that (5) is still a sufficient condition for the almost sure convergence of the estimator even if the variables of the sequence X_n, $n \geq 1$, are not uniformly distributed. Proving this result reduces to applying the following generalization of Hoeffding's equality, which is contained in Theorem 2.3 of Yu [18], and is valid for any random variables X_1 and X_2,

$$
\mathrm{Cov}\left(f_1(X_1), f_2(X_2) \right) = \int_{\mathbb{R}^2} f_1'(x_1) f_2'(x_2) \mathrm{Cov}\left(I_{[0,x_1]}(X_1), I_{[0,x_2]}(X_2) \right) \, dx_1 \, dx_2 \tag{9}
$$

where f_1 and f_2 are absolutely continuous functions in any finite interval of $\mathbb{R}$.

Theorem 2 *Let X_n, $n \geq 1$, be a strictly stationary associated sequence of random variables having a bounded density function f and with distribution function F. If (5) holds then, for each fixed $k \in \mathbb{N}$,*

$$\lim_{n\to\infty} \frac{1}{n-k} \sum_{i=1}^{n-k} \left(I_{[0,s]}(X_i) I_{[0,t]}(X_{i+k}) - F(s)F(t) \right) = \mathrm{Cov}\left(I_{[0,s]}(X_1), I_{[0,t]}(X_{k+1}) \right) \qquad a.s.$$

Proof: Note first that due to the fact that F is a continuous and non decreasing function, $F(X_n)$, $n \geq 1$, is an associated strictly stationary sequence of random variables with uniform distribution on [0,1]. By means of (9) and since f is a bounded function we obtain,

$$\begin{aligned} \mathrm{Cov}\left(F(X_1), F(X_j)\right) &= \int_{\mathbb{R}^2} f(x_1) f(x_j) \mathrm{Cov}\left(I_{[0,x_1]}(X_1), I_{[0,x_j]}(X_j) \right) \, dx_1 \, dx_j \leq \\ &\leq \left(\sup_x \mid f(x) \mid \right)^2 \int_{\mathbb{R}^2} \mathrm{Cov}\left(I_{[0,x_1]}(X_1), I_{[0,x_j]}(X_j) \right) \, dx_1 \, dx_j = \\ &= \left(\sup_x \mid f(x) \mid \right)^2 \mathrm{Cov}(X_1, X_j). \end{aligned}$$

As mentioned before the sequence $F(X_n)$, $n \geq 1$, is associated, so from the last inequality and (5) it follows that

$$\lim_{n\to\infty} \frac{1}{n} \sum_{i=1}^{n} \mathrm{Cov}^{1/3}\left(F(X_1), F(X_i)\right) = 0.$$

At this point we have seen that the sequence $F(X_n)$, $n \geq 1$, satisfies all conditions of Theorem 1 from which so, almost surely

$$\begin{aligned} &\lim_{n\to\infty} \frac{1}{n-k} \sum_{i=1}^{n-k} \left(I_{[0,F(s)]}(F(X_i)), I_{[0,F(t)]}(F(X_{i+k})) - F(s)F(t) \right) = \\ &\qquad = \mathrm{Cov}\left(I_{[0,F(s)]}(F(X_1)), I_{[0,F(t)]}(F(X_{1+k})) \right) \end{aligned}$$

which is equivalent to

$$\lim_{n\to\infty} \frac{1}{n-k} \sum_{i=1}^{n-k} \left(I_{[0,F(s)]}(F(X_i), I_{[0,F(t)]}(F(X_{i+k}) \right) = \mathrm{P}\left(F(X_1) \leq F(s), F(X_{k+1}) \leq F(t)\right) \qquad a.s.$$

Now, the sets $\{\omega : F(X_i(\omega)) \leq F(s) \wedge F(X_{i+k}(\omega)) \leq F(t)\}$ and $\{\omega : X_i(\omega) \leq s \wedge X_{i+k}(\omega) \leq t\}$ are equal almost everywhere and consequently, $I_{[0,F(s)]}(F(X_i)) I_{[0,F(t)]}(F(X_{i+k})) = I_{[0,s]}(X_i) I_{[0,t]}(X_{i+k})$ almost everywhere, so the last convergence leads to

$$\lim_{n\to\infty} \frac{1}{n-k} \sum_{i=1}^{n-k} \left(I_{[0,s]}(X_i) I_{[0,t]}(X_{i+k}) \right) = \mathrm{P}\left(X_1 \leq s, X_{k+1} \leq t\right) \qquad a.s.$$

from which the result follows. ■

The uniform strong consistency of the estimator is established in the next theorem whose proof follows the same steps as the proof of the classical Glivenko-Cantelli Theorem.

Theorem 3 *If the sequence X_n, $n \geq 1$, satisfies (5) then, for each fixed $k \in \mathbb{N}$,*

$$\lim_{n\to\infty} \sup_{s,t\in[0,1]} |\widehat{\varphi}_{k,n}(s,t) - \varphi_k(s,t)| = 0 \qquad a.s.$$

Proof: In order to simplify the expressions in the course of the proof we define

$$\gamma_k(s,t) = \mathrm{P}\left(X_1 \leq s, X_{k+1} \leq t\right) = \varphi_k(s,t) + st$$

and

$$\widehat{\gamma}_{k,n}(s,t) = \frac{1}{n-k} \sum_{i=1}^{n-k} \left(I_{[0,s]}(X_i) I_{[0,t]}(X_{i+k}) \right) = \widehat{\varphi}_{k,n}(s,t) + st.$$

Hence we may write

$$D_n = \sup_{s,t\in[0,1]} |\widehat{\varphi}_{k,n}(s,t) - \varphi_k(s,t)| = \sup_{s,t\in[0,1]} |\widehat{\gamma}_{k,n}(s,t) - \gamma_k(s,t)|.$$

Let $M \geq 1$ be fixed. For any $0 \leq i,j \leq M$, we have from Theorem 1 that,

$$\lim_{n\to\infty} \widehat{\gamma}_{k,n}\left(\frac{i}{M}, \frac{j}{M}\right) = \gamma_k\left(\frac{i}{M}, \frac{j}{M}\right) \qquad a.s.$$

and consequently,

$$\lim_{n\to\infty} D_{M,n} = \lim_{n\to\infty} \max_{0\leq i,j\leq M} \left| \widehat{\gamma}_{k,n}\left(\frac{i}{M}, \frac{j}{M}\right) - \gamma_k\left(\frac{i}{M}, \frac{j}{M}\right) \right| = 0 \qquad a.s.$$

Also, since the variables X_i are uniformly distributed on $[0,1]$, it is easy to check that

$$\gamma_k\left(\frac{i}{M}, \frac{j}{M}\right) - \gamma_k\left(\frac{i-1}{M}, \frac{j-1}{M}\right) \leq \frac{2}{M}. \tag{10}$$

For $\frac{i-1}{M} \leq s \leq \frac{i}{M}$ and $\frac{j-1}{M} \leq t \leq \frac{j}{M}$ we have, by monotonicity, that,

$$\widehat{\gamma}_{k,n}(s,t) \leq \widehat{\gamma}_{k,n}\left(\frac{i}{M}, \frac{j}{M}\right) \leq \left| \widehat{\gamma}_{k,n}\left(\frac{i}{M}, \frac{j}{M}\right) - \gamma_k\left(\frac{i}{M}, \frac{j}{M}\right) \right| + \gamma_k\left(\frac{i}{M}, \frac{j}{M}\right) \leq$$

$$\leq D_{M,n} + \gamma_k\left(\frac{i}{M}, \frac{j}{M}\right) \leq D_{M,n} + \frac{2}{M} + \gamma_k\left(\frac{i-1}{M}, \frac{j-1}{M}\right) \leq D_{M,n} + \frac{2}{M} + \gamma_k(s,t),$$

taking account of (10). Proceeding in a similar way we obtain

$$\widehat{\gamma}_{k,n}(s,t) \geq -D_{M,n} - \frac{2}{M} + \gamma_k(s,t).$$

Thus, combining the last two inequalities, we have

$$\sup_{s,t\in[0,1]} |\widehat{\gamma}_{k,n}(s,t) - \gamma_k(s,t)| \leq D_{M,n} + \frac{2}{M}.$$

Since M is arbitrary and $\lim_{n\to\infty} D_{M,n} = 0$ almost surely, it follows that $\lim_{n\to\infty} D_n = 0$ almost surely. ∎

The following two theorems are concerned with the convergence in distribution of the finite dimensional distributions of the estimator.

Theorem 4 *If the sequence X_n, $n \geq 1$, is such that*

$$\sum_{j=1}^{\infty} \mathrm{Cov}^{1/3}\left(X_1, X_j\right) < \infty, \tag{11}$$

then, for each fixed $k \in \mathbb{N}$ and $s,t \in [0,1]$, $\sqrt{n-k}\,\left(\widehat{\varphi}_{k,n}(s,t) - \varphi_k(s,t)\right)$ converges in distribution to a gaussian random variable Z with $\mathrm{E}(Z) = 0$ and $\mathrm{Var}(Z) = \sigma^2$ where

$$\sigma^2 = \mathrm{Var}\left(I_{[0,s]}(X_1)I_{[0,t]}(X_{k+1})\right) + 2\sum_{j=2}^{\infty} \mathrm{Cov}\left(I_{[0,s]}(X_1)I_{[0,t]}(X_{k+1}), I_{[0,s]}(X_j)I_{[0,t]}(X_{k+j})\right) < \infty.$$

Proof: Considering the variables $Y_{k,n}$, $n \geq 1$, defined in the proof of Theorem 1 we may rewrite

$$\sigma^2 = \mathrm{Var}\left(Y_{k,1}\right) + 2\sum_{j=2}^{\infty} \mathrm{Cov}\left(Y_{k,1}, Y_{k,j}\right).$$

Letting $S_{k,n} = \sum_{i=1}^{n} Y_{k,i}$, we have

$$\frac{S_{k,n-k} - \mathrm{E}(S_{k,n-k})}{\sqrt{n-k}} = \sqrt{n-k}\,\left(\widehat{\varphi}_{k,n}(s,t) - \varphi_k(s,t)\right)$$

and consequently the result follows from a Central Limit Theorem for associated variables. According to Theorem 10 in Newman [9] it suffices to verify that the variables $Y_{k,i}$ have finite variance and that $\sigma^2 < \infty$. The first of these two conditions is easily verified

$$\mathrm{Var}\left(Y_{k,i}\right) = \mathrm{Var}\left(Y_{k,1}\right) = \mathrm{P}\left(X_1 \leq s, X_{k+1} \leq t\right) - \left(\mathrm{P}\left(X_1 \leq s, X_{k+1} \leq t\right)\right)^2 < \infty.$$

According to inequality (7), it follows from (11) that

$$\sum_{j=1}^{\infty} \mathrm{Cov}\left(Y_{k,1}, Y_{k,j}\right) < \infty$$

and consequently the second condition is also satisfied, that is $\sigma^2 < \infty$. ∎

Theorem 5 *If the random variables X_n, $n \geq 1$, satisfy (11) then, for each choice of $r \in \mathbb{N}$, $s_1, \ldots, s_r, t_1, \ldots, t_r \in [0,1]$ and $k \in \mathbb{N}$, the random vector*

$$\sqrt{n-k}\left(\widehat{\varphi}_{k,n}(s_1,t_1) - \varphi_k(s_1,t_1), \ldots, \widehat{\varphi}_{k,n}(s_r,t_r) - \varphi_k(s_r,t_r)\right) \tag{12}$$

converges in distribution to a centered gaussian random vector $(Z_1, \ldots, Z_r)$ with covariance matrix with entries σ_{ij} defined by

$$\sigma_{ij} = \mathrm{Cov}\left(Z_i, Z_j\right) = \mathrm{Cov}\left(I_{[0,s_i]}(X_1)I_{[0,t_i]}(X_{k+1}), I_{[0,s_j]}(X_1)I_{[0,t_j]}(X_{k+1})\right) +$$

$$+2\sum_{p=2}^{\infty} \mathrm{Cov}\left(I_{[0,s_i]}(X_1)I_{[0,t_i]}(X_{k+1}), I_{[0,s_j]}(X_p)I_{[0,t_j]}(X_{k+p})\right) < \infty.$$

Proof: Let $r \in \mathbb{N}, s_1, \ldots, s_r, t_1, \ldots, t_r \in [0,1]$ and $k \in \mathbb{N}$ be fixed. In order to simplify the notation we will drop the k and so, without loss of generality, we will always use n instead of $n-k$.

For each $j \in \mathbb{N}$ and $i \in 1, \ldots, r$, define the random variables $Y_j^i = I_{[0,s_i]\times[0,t_i]}(X_j, X_{k+j})$ and let $e_i = \mathrm{E}(Y_1^i) = \mathrm{E}(Y_j^i) = \mathrm{P}\left(X_1 \leq s_i, X_{k+1} \leq t_i\right)$. Note that the variables Y_j^i with $j \in \mathbb{N}$ and $i = 1, \ldots, r$, being decreasing functions of the X_n, are associated and also,

$$\mathrm{Cov}\left(Y_j^i, Y_{j'}^{i'}\right) = \mathrm{Cov}\left(Y_{j+h}^i, Y_{j'+h}^{i'}\right) \tag{13}$$

for each $j, j', h \in \mathbb{N}$ and $i, i' = 1, ..., r$ due to the stationarity of the sequence X_n, $n \geq 1$.

The i^{th} element of the random vector (12) can be written as

$$\sqrt{n}\left(\frac{1}{n}\sum_{j=1}^{n}\left(I_{[0,s_i]}(X_j)I_{[0,t_i]}(X_{k+j}) - s_it_i\right) - \left[\mathrm{P}(X_1 \leq s_i, X_{k+1} \leq t_i) - s_it_i\right]\right) =$$

$$= \sqrt{n}\left(\frac{1}{n}\sum_{j=1}^{n} Y_j^i - \mathrm{E}(Y_1^i)\right) = \frac{1}{\sqrt{n}}\sum_{j=1}^{n}(Y_j^i - e_i).$$

Let $a_1, \ldots, a_r \in \mathbb{R}$ be fixed and define, for each $n \in \mathbb{N}$, the random variables

$$W_n = \sum_{i=1}^{r} a_i\left(Y_n^i - e_i\right), \qquad \overline{W}_n = \sum_{i=1}^{r} |a_i|\left(Y_n^i - e_i\right),$$

$$S_n = \frac{1}{\sqrt{n}}\sum_{j=1}^{n} W_j, \qquad \overline{S}_n = \frac{1}{\sqrt{n}}\sum_{j=1}^{n} \overline{W}_j.$$

Obviously, the sequences W_n, $n \geq 1$, and $\overline{W}_n$, $n \geq 1$, are strictly stationary and the second one is also associated since the coefficients $|a_i|$ are non negative. Note also that S_n, $n \geq 1$, is the linear combination of coordinates of (12), needed to use the Cramer-Wold Theorem.

Proceeding as in the proof of Theorem 4 when verifying that the limit covariance is finite, condition (11) yields, by means of (7), that for each $i, j = 1, ..., r$,

$$\sum_{p=2}^{\infty} \mathrm{Cov}\left(Y_1^i, Y_p^j\right) < \infty, \tag{14}$$

and consequently,

$$\sigma_{ij} = \mathrm{Cov}\left(Y_1^i, Y_1^j\right) + 2\sum_{p=2}^{\infty} \mathrm{Cov}\left(Y_1^i, Y_p^j\right) < \infty.$$

By the Cramer-Wold Theorem the result follows from, the convergence in distribution of S_n to some gaussian centered random variable with finite variance given by

$$\sigma^2 = \mathrm{Var}\left(\sum_{i=1}^{r} a_iZ_i\right) = \sum_{i,j=1}^{r} a_ia_j\sigma_{ij} =$$

$$= \sum_{i,j=1}^{r} a_ia_j\mathrm{Cov}\left(Y_1^i - e_i, Y_1^j - e_j\right) + 2\sum_{p=2}^{\infty}\sum_{i,j=1}^{r} a_ia_j\mathrm{Cov}\left(Y_1^i - e_i, Y_p^j - e_j\right) =$$

$$= \mathrm{Var}\left(\sum_{i=1}^{r} a_i \left(Y_1^i - e_i\right)\right) + 2\sum_{p=2}^{\infty} \mathrm{Cov}\left(\sum_{i=1}^{r} a_i \left(Y_1^i - e_i\right), \sum_{i=1}^{r} a_i \left(Y_p^i - e_i\right)\right) =$$

$$= \mathrm{Var}(W_1) + 2\sum_{p=2}^{\infty} \mathrm{Cov}(W_1, W_p) < \infty.$$

The rest of the proof, consisting in establishing this convergence in distribution, will be accomplished in 4 steps. The main idea is to decompose the sum S_n in blocks with the same fixed width l and show that we may treat them as if they were independent.

For a fixed width $l \in \mathbb{N}$, denote by m the number of blocks (m is the largest integer less or equal than n/l).

Step 1: Let $l \in \mathbb{N}$ be fixed. We begin by approximating the characteristic function of S_n by the characteristic function of S_{ml}.

$$0 \le \left|\mathrm{E}\left(e^{itS_n}\right) - \mathrm{E}\left(e^{itS_{ml}}\right)\right| \le \mathrm{E}\left|e^{itS_n} - e^{itS_{ml}}\right| \le |t|\mathrm{E}\left|S_n - S_{ml}\right| \le$$

$$\le |t|\mathrm{E}^{1/2}\left((S_n - S_{ml})^2\right) = |t|\, ||S_n - S_{ml}||_2$$

$$= |t| \left|\left|\left(\frac{1}{\sqrt{n}} - \frac{1}{\sqrt{ml}}\right)\sum_{j=1}^{ml} W_j + \frac{1}{\sqrt{n}}\sum_{j=ml+1}^{n} W_j\right|\right|_2 \le$$

$$\le |t| \left(\frac{\sqrt{ml}}{\sqrt{ml}} - \frac{\sqrt{ml}}{\sqrt{n}}\right) \left|\left|\frac{1}{\sqrt{ml}}\sum_{j=1}^{ml} W_j\right|\right|_2 + \frac{|t|}{\sqrt{n}}\sum_{j=ml+1}^{n} ||W_j||_2$$

Since W_n, $n \ge 1$, is strictly stationary we have

$$0 \le \left|\mathrm{E}\left(e^{itS_n}\right) - \mathrm{E}\left(e^{itS_{ml}}\right)\right| \le$$

$$\le |t| \left(1 - \sqrt{\frac{ml}{n}}\right) \left|\left|\frac{1}{\sqrt{ml}}\sum_{j=1}^{ml} W_j\right|\right|_2 + \frac{|t|}{\sqrt{n}}(n - ml)\mathrm{Var}^{1/2}(W_1)$$

$$\le |t| \left(1 - \sqrt{\frac{ml}{n}}\right) \left|\left|\frac{1}{\sqrt{ml}}\sum_{j=1}^{ml} W_j\right|\right|_2 + \frac{|t|\, l}{\sqrt{n}}\mathrm{Var}^{1/2}(W_1).$$

We now prove that this sum converges to zero. The second term of the sum obviously converges to zero so we will concentrate only on the first term. For this we have,

$$\left|\left|\frac{1}{\sqrt{ml}}\sum_{j=1}^{ml} W_j\right|\right|_2 = \mathrm{Var}^{1/2}\left(\frac{1}{\sqrt{ml}}\sum_{j=1}^{ml} W_j\right) = \left(\frac{1}{ml}\sum_{i,j=1}^{ml} \mathrm{Cov}(W_i, W_j)\right)^{1/2} =$$

$$= \left(\frac{1}{ml}\sum_{i,j=1}^{ml}\sum_{k,k'=1}^{r} a_k a_{k'} \mathrm{Cov}\left(Y_i^k, Y_j^{k'}\right)\right)^{1/2} =$$

$$= \left(\sum_{k,k'=1}^{r} a_k a_{k'} \frac{1}{ml}\sum_{i,j=1}^{ml} \mathrm{Cov}\left(Y_i^k, Y_j^{k'}\right)\right)^{1/2}.$$

But, taking account of (13) and (14) we find that

$$\lim_{n\to\infty} \frac{1}{n} \sum_{p,p'=1}^{n} \operatorname{Cov}\left(Y_p^i, Y_{p'}^j\right) = \sigma_{ij}, \tag{15}$$

and consequently,

$$\lim_{m\to\infty} \left\| \frac{1}{\sqrt{ml}} \sum_{j=1}^{ml} W_j \right\|_2 = \left(\sum_{k,k'=1}^{r} a_k a_{k'} \sigma_{kk'} \right)^{1/2} = \sigma,$$

so the first term also converges to zero.

We have thus established that,

$$\lim_{n\to\infty} \left| \mathrm{E}\left(e^{itS_n}\right) - \mathrm{E}\left(e^{itS_{ml}}\right) \right| = 0.$$

Step 2: We now deal with the blocks of width l into which the sum S_{ml} is decomposed:

$$V_i^l = \frac{1}{\sqrt{l}} \sum_{j=(i-1)l+1}^{il} W_j.$$

Accordingly, for $\overline{S}_{ml}$ we have,

$$\overline{V}_i^l = \frac{1}{\sqrt{l}} \sum_{j=(i-1)l+1}^{il} \overline{W}_j.$$

Thus, we can rewrite S_{ml} and $\overline{S}_{ml}$, respectively as

$$S_{ml} = \frac{1}{\sqrt{m}} \sum_{i=1}^{m} V_i^l \qquad \text{and} \qquad \overline{S}_{ml} = \frac{1}{\sqrt{m}} \sum_{i=1}^{m} \overline{V}_i^l.$$

We note that, since W_n, $n \geq 1$, are strictly stationary, the variables V_i^l have the same distribution for $i \in \mathbb{N}$.

In order to approximate the characteristic function of S_{ml} by what we would obtain if the variables V_i^l were independent we will apply Theorem 16 of Newman [10]. For this purpose define, for each $i \in \mathbb{N}$, the functions

$$f_i\left(y_1^1, y_1^2, \ldots, y_1^r, y_2^1, y_2^2, \ldots, y_2^r, \ldots\right) = \frac{1}{\sqrt{l}} \sum_{j=(i-1)l+1}^{il} \sum_{k=1}^{r} a_k \left(y_j^k - e_k\right)$$

and

$$\overline{f}_i\left(y_1^1, y_1^2, \ldots, y_1^r, y_2^1, y_2^2, \ldots, y_2^r, \ldots\right) = \frac{1}{\sqrt{l}} \sum_{j=(i-1)l+1}^{il} \sum_{k=1}^{r} |a_k| \left(y_j^k - e_k\right).$$

Then, the functions $(\overline{f}_i + f_i)$ and $(\overline{f}_i - f_i)$ are both coordinatewise non decreasing since their coefficients, $|a_k| + a_k$ and $|a_k| - a_k$, are non-negative. As

$$V_i^l = f_i\left(Y_1^1, Y_1^2, \ldots, Y_1^r, Y_2^1, Y_2^2, \ldots, Y_2^r, \ldots\right)$$

and

$$\overline{V}_i^l = \overline{f}_i\left(Y_1^1, Y_1^2, \ldots, Y_1^r, Y_2^1, Y_2^2, \ldots, Y_2^r, \ldots\right),$$

and since the random variables $Y_1^1, Y_1^2, \ldots, Y_1^r, Y_2^1, Y_2^2, \ldots, Y_2^r, \ldots$ are associated it follows by Theorem 16 of Newman [10] that

$$\left|\mathrm{E}\left(e^{itS_{ml}}\right) - \mathrm{E}^m\left(e^{\frac{it}{\sqrt{m}}V_1^l}\right)\right| = \left|\mathrm{E}\left(e^{\frac{it}{\sqrt{m}}\sum_{i=1}^m V_i^l}\right) - \prod_{i=1}^m \mathrm{E}\left(e^{\frac{it}{\sqrt{m}}V_i^l}\right)\right| \leq$$

$$\leq 2\frac{t^2}{m}\sum_{\substack{k,k'=1\\k\neq k'}}^m \mathrm{Cov}\left(\overline{V}_k^l, \overline{V}_{k'}^l\right).$$

Expanding the last sum we find,

$$\sum_{\substack{k,k'=1\\k\neq k'}}^m \mathrm{Cov}\left(\overline{V}_k^l, \overline{V}_{k'}^l\right) = \sum_{\substack{k,k'=1\\k\neq k'}}^m \mathrm{Cov}\left(\frac{1}{\sqrt{l}}\sum_{j=(k-1)l+1}^{kl}\overline{W}_j, \frac{1}{\sqrt{l}}\sum_{i=(k'-1)l}^{k'l}\overline{W}_i\right) =$$

$$= \frac{1}{l}\sum_{\substack{k,k'=1\\k\neq k'}}^m \sum_{j=(k-1)l+1}^{kl}\sum_{i=(k'-1)l+1}^{k'l}\mathrm{Cov}\left(\overline{W}_i, \overline{W}_j\right)$$

which is the product of $1/l$ by the sum of all elements in the covariance matrix of $\left(\overline{W}_1, \ldots, \overline{W}_{ml}\right)$ except those elements in the m diagonal squares of side l, that is,

$$\sum_{\substack{k,k'=1\\k\neq k'}}^m \mathrm{Cov}\left(\overline{V}_k^l, \overline{V}_{k'}^l\right) = \frac{1}{l}\left(\sum_{i,j=1}^{ml}\mathrm{Cov}\left(\overline{W}_i, \overline{W}_j\right) - m\sum_{i,j=1}^{l}\mathrm{Cov}\left(\overline{W}_i, \overline{W}_j\right)\right) =$$

$$= \frac{1}{l}\left[\mathrm{Var}\left(\sum_{i=1}^{ml}\overline{W}_i\right) - m\mathrm{Var}\left(\sum_{i=1}^{l}\overline{W}_i\right)\right] =$$

$$= \mathrm{Var}\left(\frac{1}{\sqrt{l}}\sum_{i=1}^{ml}\overline{W}_i\right) - m\mathrm{Var}\left(\frac{1}{\sqrt{l}}\sum_{i=1}^{l}\overline{W}_i\right) =$$

$$= m\mathrm{Var}\left(\overline{S}_{ml}\right) - m\mathrm{Var}\left(\overline{S}_l\right)$$

Writing $\overline{\sigma}_n^2 = \mathrm{Var}\left(\overline{S}_n\right)$, we finally get,

$$\left|\mathrm{E}\left(e^{itS_{ml}}\right) - \mathrm{E}^m\left(e^{\frac{it}{\sqrt{m}}V_1^l}\right)\right| \leq 2\,\frac{t^2}{m}\,m\left(\overline{\sigma}_{ml}^2 - \overline{\sigma}_l^2\right) = 2t^2\left(\overline{\sigma}_{ml}^2 - \overline{\sigma}_l^2\right).$$

Step 3: From the classical Central Limit Theorem for independent and identically distributed variables we obtain,

$$\lim_{m\to\infty}\left|\mathrm{E}^m\left(e^{\frac{it}{\sqrt{m}}V_1^l}\right) - e^{-\frac{\sigma_l^2 t^2}{2}}\right| = \lim_{m\to\infty}\left|\prod_{i=1}^m \mathrm{E}\left(e^{\frac{it}{\sqrt{m}}V_i^l}\right) - e^{-\frac{\sigma_l^2 t^2}{2}}\right| = 0$$

where $\sigma_l^2 = \mathrm{Var}\left(V_1^l\right)$.

Step 4: To finish the proof we write,

$$\left|\mathrm{E}\left(e^{itS_n}\right)-e^{-\frac{t^2\sigma^2}{2}}\right|=\left|\mathrm{E}\left(e^{itS_n}\right)-\mathrm{E}\left(e^{itS_{ml}}\right)+\mathrm{E}\left(e^{itS_{ml}}\right)-\mathrm{E}^m\left(e^{\frac{it}{\sqrt{m}}V_1^l}\right)+\right.$$

$$\left.+\mathrm{E}^m\left(e^{\frac{it}{\sqrt{m}}V_1^l}\right)-e^{-\frac{\sigma_l^2t^2}{2}}+e^{-\frac{\sigma_l^2t^2}{2}}-e^{-\frac{t^2\sigma^2}{2}}\right|\leq$$

$$\leq\left|\mathrm{E}\left(e^{itS_n}\right)-\mathrm{E}\left(e^{itS_{ml}}\right)\right|+\left|\mathrm{E}\left(e^{itS_{ml}}\right)-\mathrm{E}^m\left(e^{\frac{it}{\sqrt{m}}V_1^l}\right)\right|+$$

$$+\left|\mathrm{E}^m\left(e^{\frac{it}{\sqrt{m}}V_1^l}\right)-e^{-\frac{\sigma_l^2t^2}{2}}\right|+\left|e^{-\frac{\sigma_l^2t^2}{2}}-e^{-\frac{t^2\sigma^2}{2}}\right|.$$

From *Step 2* the last inequality becomes,

$$\left|\mathrm{E}\left(e^{itS_n}\right)-e^{-\frac{t^2\sigma^2}{2}}\right|\leq\left|E\left(e^{itS_n}\right)-E\left(e^{itS_{ml}}\right)\right|+2t^2\left(\overline{\sigma}_{ml}^2-\overline{\sigma}_l^2\right)+$$

$$+\left|\mathrm{E}^m\left(e^{\frac{it}{\sqrt{m}}V_1^l}\right)-e^{-\frac{\sigma_l^2t^2}{2}}\right|+\left|e^{-\frac{\sigma_l^2t^2}{2}}-e^{-\frac{t^2\sigma^2}{2}}\right| \tag{16}$$

For a fixed $l\in\mathbb{N}$, the first and the third term in this upper bound converge to zero when $n\longrightarrow\infty$ according to *Step 1* and *Step 3*, respectively.

On account of (14) we get

$$\sum_{j=2}^{\infty}\mathrm{Cov}\left(\overline{W}_1,\overline{W}_j\right)=\sum_{j=2}^{\infty}\sum_{k,k'=1}^{r}|a_k|\,|a_{k'}|\,\mathrm{Cov}\left(Y_1^k,Y_j^{k'}\right)<\infty$$

and, since the sequence $\overline{W}_n$, $n\geq1$, is strictly stationary, it follows that

$$\lim_{n\to\infty}\overline{\sigma}_n^2=\lim_{n\to\infty}\mathrm{Var}\left(\overline{S}_n\right)=\lim_{n\to\infty}\frac{1}{n}\sum_{i,j=1}^{n}\mathrm{Cov}\left(\overline{W}_i,\overline{W}_j\right)=\overline{\sigma}^2$$

where

$$\overline{\sigma}^2=\mathrm{Var}\left(\overline{W}_1\right)+2\sum_{j=2}^{\infty}\mathrm{Cov}\left(\overline{W}_1,\overline{W}_j\right)<\infty.$$

Thus, when $n\longrightarrow\infty$, the second term in the last member of (16) converges to $2t^2\left(\overline{\sigma}^2-\overline{\sigma}_l^2\right)$.

So, for each $l\in\mathbb{N}$, we have,

$$\limsup_{n\to\infty}\left|\mathrm{E}\left(e^{itS_n}\right)-e^{-\frac{t^2\sigma^2}{2}}\right|\leq2t^2\left(\overline{\sigma}^2-\overline{\sigma}_l^2\right)+\left|e^{-\frac{t^2\sigma_l^2}{2}}-e^{-\frac{t^2\sigma^2}{2}}\right|. \tag{17}$$

Note that,

$$\sigma_l^2=\mathrm{Var}\left(V_1^l\right)=\mathrm{Var}\left(\frac{1}{\sqrt{l}}\sum_{j=1}^{l}W_j\right)=\sum_{k,k'=1}^{r}a_ka_{k'}\,\frac{1}{l}\sum_{i,j=1}^{l}\mathrm{Cov}\left(Y_i^k,Y_j^{k'}\right)$$

which, from (15), when $l \longrightarrow \infty$, converges to,

$$\sum_{k,k'=1}^{r} a_k a_{k'} \sigma_{kk'} = \sigma^2.$$

So, letting $l \longrightarrow \infty$, we get from (17)

$$\lim_{n\to\infty} \left| \mathrm{E}\left(e^{itS_n}\right) - e^{-\frac{t^2\sigma^2}{2}} \right| = 0,$$

thus proving the convergence in distribution of (12). ■

3 Error criteria

The discrepancy of the estimator $\widehat{\varphi}_{k,n}$ from the true φ_k, at a single point, is naturally measured by the mean square error (MSE)

$$\mathrm{MSE}\left\{\widehat{\varphi}_{k,n}(s,t)\right\} = \mathrm{E}\left(\widehat{\varphi}_{k,n}(s,t) - \varphi_k(s,t)\right)^2 =$$

$$= \frac{\mathrm{Var}\left(Y_{k,1}\right)}{n-k} + \frac{2}{(n-k)^2} \sum_{i=2}^{n-k} (n-k-i+1)\ \mathrm{Cov}\left(Y_{k,1}, Y_{k,i}\right)$$

where the variables $Y_{k,i}$, $k, i \in \mathbb{N}$, are defined as in the proof of Theorem 1.

For the evaluation of the global accuracy of $\widehat{\varphi}_{k,n}$ as an estimator of φ_k we consider the mean integrated square error (MISE)

$$\mathrm{MISE}\left\{\widehat{\varphi}_{k,n}\right\} = \mathrm{E}\left(\int_{[0,1]^2} \left(\widehat{\varphi}_{k,n}(s,t) - \varphi_k(s,t)\right)^2\ ds\ dt\right) =$$

$$= \int_{[0,1]^2} \mathrm{MSE}\left\{\widehat{\varphi}_{k,n}(s,t)\right\}\ ds\ dt.$$

We will see that, if the sequence X_n, $n \geq 1$, satisfies (5) then, for each fixed $k \in \mathbb{N}$,

$$\lim_{n\to\infty} \mathrm{MSE}\left\{\widehat{\varphi}_{k,n}(s,t)\right\} = 0 \qquad \text{and} \qquad \lim_{n\to\infty} \mathrm{MISE}\left\{\widehat{\varphi}_{k,n}\right\} = 0.$$

In fact, condition (5) implies condition (6), as seen in the proof of Theorem 1 and since,

$$0 \leq \frac{1}{(n-k)^2} \sum_{i=2}^{n-k} (n-k-i+1)\mathrm{Cov}\left(Y_{k,1}, Y_{k,i}\right) \leq \frac{1}{n-k} \sum_{i=1}^{n-k} \mathrm{Cov}\left(Y_{k,1}, Y_{k,i}\right),$$

it follows

$$\lim_{n\to\infty} \frac{2}{(n-k)^2} \sum_{i=2}^{n-k} (n-k-i+1)\mathrm{Cov}\left(Y_{k,1}, Y_{k,i}\right) = 0,$$

which yields

$$\lim_{n\to\infty} \mathrm{MSE}\left\{\widehat{\varphi}_{k,n}(s,t)\right\} = 0.$$

On the other side, by means of (7) and remembering that $0 \leq \mathrm{Cov}\,(X_i, X_j) \leq 1$, for each $i, j \in \mathbb{N}$, since the variables involved are associated and uniformly distributed on $[0,1]$, it follows that, for every $j \in \mathbb{N}$ and $s, t \in [0,1]$,

$$0 \leq \mathrm{Cov}\,(Y_{k,1}, Y_{k,j}) \leq \left(\frac{3}{2}\right)^{1/3} \Big[\mathrm{Cov}^{1/3}(X_1, X_j) + \mathrm{Cov}^{1/3}(X_1, X_{k+j}) +$$

$$+\mathrm{Cov}^{1/3}(X_{k+1}, X_j) + \mathrm{Cov}^{1/3}(X_{k+1}, X_{k+j})\Big] \leq$$

$$\leq \left(\frac{3}{2}\right)^{1/3} \left[4 \times 1^{1/3}\right] = 4\left(\frac{3}{2}\right)^{1/3} = C,$$

and consequently,

$$\mathrm{MSE}\,\{\widehat{\varphi}_{k,n}(s,t)\} = \frac{\mathrm{Var}\,(Y_{k,1})}{n-k} + \frac{2}{(n-k)^2} \sum_{i=2}^{n-k} (n-k-i+1)\mathrm{Cov}\,(Y_{k,1}, Y_{k,i}) \leq$$

$$\leq C + \frac{2}{n-k} \sum_{i=2}^{n-k} \mathrm{Cov}\,(Y_{k,1}, Y_{k,i}) \leq C + \frac{2}{n-k}(n-k)C = 3C$$

Since the $\mathrm{MSE}\,\{\widehat{\varphi}_{k,n}(s,t)\}$ is uniformly bounded, the Lebesgue convergence theorem applies and gives

$$\lim_{n\to\infty} \mathrm{MISE}\,\{\widehat{\varphi}_{k,n}\} = 0.$$

For the study of the convergence rate suppose that $\mathrm{Cov}^{1/3}(X_1, X_n) = O\left(n^{-(1+\varepsilon)}\right)$ for some $\varepsilon > 0$. Then from (7) it follows that $\mathrm{Cov}\,(Y_{k,1}, Y_{k,n}) = O\left(n^{-(1+\varepsilon)}\right)$, which implies $\sum_{j=1}^{\infty} \mathrm{Cov}\,(Y_{k,1}, Y_{k,j}) < \infty$, and consequently

$$\frac{1}{n} \sum_{j=1}^{n} \mathrm{Cov}\,(Y_{k,1}, Y_{k,j}) = O\left(n^{-1}\right).$$

Thus, we obtain the best convergence rate for the $\mathrm{MSE}\,\{\widehat{\varphi}_{k,n}(s,t)\}$,

$$\mathrm{MSE}\,\{\widehat{\varphi}_{k,n}(s,t)\} = O\left(n^{-1}\right),$$

and with similar arguments we find the same convergence rate for the $\mathrm{MISE}\,\{\widehat{\varphi}_{k,n}\}$,

$$\mathrm{MISE}\,\{\widehat{\varphi}_{k,n}\} = O\left(n^{-1}\right).$$

In a similar way it can be shown that the weaker condition $\mathrm{Cov}^{1/3}(X_1, X_n) = O\left(n^{-1}\right)$ leads to $\mathrm{MSE}\,\{\widehat{\varphi}_{k,n}(s,t)\} = O\left(n^{-(1-\varepsilon)}\right)$ and $\mathrm{MISE}\,\{\widehat{\varphi}_{k,n}\} = O\left(n^{-(1-\varepsilon)}\right)$ for every $\varepsilon > 0$, thus we do not lose much when compared with the previous case.

References

[1] Anderson, T. W., Darling, D. A. (1952). Asymptotic theory of certain "goodness of fit" criteria based on stochastic processes, Ann. Math. Statist. 23, 193-212.

[2] Billingsley, P. (1968). Convergence of Probability Measures. Wiley, New York.

[3] Cai, Z., Roussas, G. G. (1998). Efficient Estimation of a Distribution Function under Quadrant Dependence. Scand. J. Statis. 25, 211-224.

[4] Donsker, M. D. (1951). An invariance principle for certain probability limit theorems. Mem. Amer. Math. Soc. 6.

[5] Doukhan, P., Massart, P., Rio, E. (1995). Invariance principles for the empirical measure of a weakly dependent process. Ann. Inst. H. Poincaré 31, 393-427.

[6] Esary, J. D., Proschan, F., Walkup, D. W. (1967). Association of random variables, with applications. Ann. Math. Statist. 38, 1466-1474.

[7] Henriques, C. M., Oliveira, P. E.(1997). Estimação de covariências para variáveis associadas. Actas do V Congresso Anual da SPE, 483-488.

[8] Lebowitz, J.(1972). Bounds on the correlations and analycity properties of ferroma gnetic Ising spin systems. Comm. Math. Phys. 28, 313-321.

[9] Newman, C. M. (1980). Normal fluctuations and the FKG inequalities. Comm. Math Phys. 75, 119-128.

[10] Newman, C. M. (1984). Asymptotic independence and limit theorems for positively and negatively dependent random variables. Inequalities in Statistics and Probability, IMS Lecture Notes - Monograph Series 5, 127-140.

[11] Oliveira, P. E., Suquet, C. (1995). $L^2[0,1]$ weak convergence of the empirical process for dependent variables. Actes des XV Rencontres Franco-Belges de Statisticiens (Ondelettes et Statistique), Lecture Notes in Statistics 103, Wavelets and Statistics, Ed. A. Antoniadis, G. Oppenheim, 331-344.

[12] Oliveira, P. E., Suquet, C. (1998). Weak convergence in $L^p[0,1]$ of the uniform empirical process under dependence. Statist. Probab. Letters 39, 363-370.

[13] Roussas, G. G. (1995). Asymptotic normality of a smooth estimate of a random field distribution function under association. Statist. Probab. Letters 24, 77-90.

[14] Sen, P. K. (1971). A note on weak convergence of empirical processes for sequences of ϕ-mixing random variables. Ann. Math. Statist. 42, 2131-2133.

[15] Shao, Q. M. (1986). Weak convergence of multidimensional weighted empirical processes for strong mixing sequences. Chinese Ann. Math. Ser. A 7, 547-552.

[16] Shao, Q. M. e Yu, H. (1996). Weak convergence for weighted empirical process of dependent sequences. Ann. Probab. 24, 2098-2127.

[17] Yoshihara, K. (1975). Billingsley's theorems on empirical processes of strong mixing sequences. Yokohama Math. J. 23, 1-7.

[18] Yu, H. (1993). A Glivenko-Cantelli lemma and weak convergence for empirical processes of associated sequences. Probability Theory and Related Fields 95, 357-370.

[19] Watson, G. S. (1961). Goodness-of-fit tests on a circle. Biometrika 48, 109-114.