ON DECOMPOSITIONS IN GENERALISED LORENTZ-ZYGMUND SPACES

J. S. NEVES

ABSTRACT. Various characterisations are given of the generalised Lorentz-Zygmund (GLZ) spaces $L_{p,q;\boldsymbol{\alpha}}(R)$, with $p,q \in (0,+\infty]$, $m \in \mathbb{N}$, $\boldsymbol{\alpha} \in \mathbb{R}^m$ and (R,μ) a finite measure space. Given a measure space (R,μ) and $\boldsymbol{\alpha} \in \mathcal{R}^m_-$, we obtain equivalent representations for the (quasi-) norm of the GLZ space $L_{\infty,\infty;\boldsymbol{\alpha}}(R)$. Moreover, when (R,μ) is a finite measure space and $\boldsymbol{\alpha} \in \mathcal{R}^m_+$, we present an equivalent norm for the space $L_{1,1;\boldsymbol{\alpha}}(R)$ in terms of decompositions. We show how the equivalent norms considered for $L_{\infty,\infty;\boldsymbol{\alpha}}(R)$, with (R,μ) a finite measure space, and the decomposition norm in $L_{1,1;\boldsymbol{\alpha}}(R)$ can be employed to get simple proofs of some extrapolation results involving these spaces.

1. INTRODUCTION

In [EK], Edmunds and Krbec obtained some decompositions for the exponential Orlicz space $L_{\Phi_1}(\Omega)$, usually denoted by $E_\alpha(\Omega)$, with Young function Φ_1 given by $\Phi_1(t) = \exp t^\alpha$ for large t, where $\alpha > 0$ and Ω is a measurable subset of $\mathbb{R}^n$ with finite n-dimensional Lebesgue measure $|\Omega|_n$. Without loss of generality, it was assumed that $|\Omega|_n = 1$. They showed that considering a suitable decomposition of $(0,1)$ into a union of disjoint intervals $\{(t_k, t_{k-1})\}_{k\in\mathbb{N}}$ it is enough to control only the blow up of the norms $\|f^*\|_{L_k(t_k,t_{k-1})}$, where f^* is the non-increasing rearrangement of f, by the same power $k^{-1/\alpha}$ to have $L_{\Phi_1}(\Omega)$. The proof was based on the fact that $L_{\Phi_1}(\Omega)$ coincides with the Zygmund space $L^\infty(\log L)^{-1/\alpha}(\Omega)$ (see [BR80, Theorem D] or [BS88, Lemma IV.6.2]). In Section 3, we extend this result to the generalised Lorentz-Zygmund (GLZ) spaces $L_{p,q;\boldsymbol{\alpha}}(R)$, with $p,q \in (0,+\infty]$, $m \in \mathbb{N}$, $\boldsymbol{\alpha} \in \mathbb{R}^m$, and (R,μ) a finite measure space, cf. Theorem 3.2. The method of the proof is different from, and in our opinion easier than, that used in [EK].

In [Tri93], Triebel gave an equivalent norm for the exponential Orlicz space $L_{\Phi_1}(\Omega)$, where Ω is a measurable subset of $\mathbb{R}^n$ with finite volume; see also [EGO98]. With this equivalent norm, he proved that the embeddings $id : B^{n/p}_{pp}(\Omega) \to E_\alpha(\Omega)$ and $id : H^{n/p}_p(\Omega) \to E_\alpha(\Omega)$, with $1 < p < +\infty$, $0 < \alpha < p'$ and Ω a bounded C^∞-domain in $\mathbb{R}^n$, are compact and obtained estimates for the approximation and entropy numbers of those embeddings. Let us just mention that $B^{n/p}_{pp}(\Omega)$ and $H^{n/p}_p(\Omega)$ are classical Besov spaces and fractional Sobolev spaces, respectively. We refer to [Tri93] for more details. Equivalent norms for the double exponential Orlicz space $L_{\Phi_2}(\Omega)$, usually denoted by $EE_\alpha(\Omega)$, with Young function Φ_2 given by $\Phi_2(t) = \exp\exp t^\alpha$ for large t, where $\alpha > 0$ and Ω is a measurable subset of $\mathbb{R}^n$ with finite volume, were obtained by Edmunds, Gurka and Opic in [EGO98]. The proof was also based on the fact that $L_{\Phi_2}(\Omega)$ coincides with the GLZ space $L_{\infty,\infty;0,-\frac{1}{\alpha}}(\Omega)$, see [EGO95, Lemma 3.9]. Following the same technique as in [EGO98], we obtain in Section 4 equivalent representations for the (quasi-) norms of the GLZ spaces $L_{\infty,\infty;\boldsymbol{\alpha}}(R)$, with (R,μ) a measure space and $\boldsymbol{\alpha} \in \mathcal{R}^m_-$, *i.e.*

Date: May 28, 1999.

1991 *Mathematics Subject Classification.* 46E30.

Key words and phrases. Generalised Lorentz-Zygmund spaces, equivalent (quasi-) norms, extrapolation.

$\boldsymbol{\alpha} = (\alpha_1, \ldots, \alpha_m) \in \mathbb{R}^m$, $\alpha_1, \ldots, \alpha_{m-1} \leq 0$ and $\alpha_m < 0$, cf. Theorem 4.1 and its Corollaries. In particular, when (R, μ) has finite measure we obtain equivalent norms for the GLZ spaces $L_{\infty,\infty;\boldsymbol{\alpha}}(R)$, with $\boldsymbol{\alpha} \in \mathcal{R}_-^m$, extending in this way the results in [Tri93] and [EGO98]. Still in Section 4, we give an equivalent norm for the spaces $L_{1,1;\boldsymbol{\alpha}}(R)$, with (R, μ) a non-atomic finite measure space and $\boldsymbol{\alpha} \in \mathcal{R}_+^m$, *i.e.* $\boldsymbol{\alpha} = (\alpha_1, \ldots, \alpha_m) \in \mathbb{R}^m$, $\alpha_1, \ldots, \alpha_{m-1} \geq 0$ and $\alpha_m > 0$, in terms of decompositions. This result extends a result obtained by Edmunds and Triebel, cf. [ET96, Theorem 2, p. 72], for the spaces $L^1(\log L)^\alpha(\Omega)$, with $\alpha > 0$ and Ω a measurable subset of $\mathbb{R}^n$ with finite volume. We refer to [FK98, Theorem 3.4] for a different proof of this result.

In Section 5, we show how the equivalent norms obtained in Section 4 for $L_{\infty,\infty;\boldsymbol{\alpha}}(R)$, with $\boldsymbol{\alpha} \in \mathcal{R}_-^m$, and the decomposition norm in $L_{1,1;\boldsymbol{\alpha}}(R)$, with $\boldsymbol{\alpha} \in \mathcal{R}_+^m$, can be employed to get simple proofs of some extrapolation results involving these spaces. Let us remark that we do not follow a general setting in terms of abstract extrapolation methods considered by Jawerth and Milman, cf. [JM91] (see also [Mil94]). We mention that the starting point of the extrapolation theory was the Theorem of Yano [Yan51] which can be described as follows. Suppose that T is a bounded linear operator on $L_p(0,1)$ for $p > 1$ with $\|T\|_{L_p \to L_p} = \mathcal{O}((p-1)^{-\alpha})$ as $p \downarrow 1$, for some $\alpha > 0$; then these estimates can be extrapolated to $L^1(\log L)^\alpha(0,1) \to L_1(0,1)$; see [Zyg59, Theorem XII.4.11 (ii), p. 119] for a more general formulation. We refer to [Tor86, Theorem IV.5.3, p.92] where T was supposed to be sublinear. We also refer to [FK98, Theorem 4.2] where T was supposed to be subadditive. In [Ste70, p. 23] and [ET96, p. 74] the case was considered when T is the Hardy-Littlewood maximal operator. It should be emphasised that the decomposition approach, used in [ET96] and [FK98], skips completely the machinery of weak type inequalities and the Marcinkiewicz interpolation Theorem, since it follows at once from the expression of the norm in $L^1(\log L)^\alpha(\Omega)$, with $\alpha > 0$. There is also a dual statement for operators acting from $L_p(R_0)$ into $L_p(R_1)$, with (R_0, μ_0) and (R_1, μ_1) finite measure spaces, for p close to $+\infty$, such that $\|T\|_{L_p \to L_p} = \mathcal{O}(p^{1/\alpha})$ as $p \to +\infty$, for some $\alpha > 0$; then there exist positive constants λ, K such that $\int_{R_1} \exp(\lambda |Tf|^\alpha)\, d\mu_1 \leq K$ for each f with $|f| \leq 1$; see [Zyg59, Theorem XII.4.11 (i), p. 119]. There is also a version of this result for sublinear operators. We refer to Section 5 for more details.

2. Notation and Preliminaries

As usual, $\mathbb{R}^n$ denotes Euclidean n-dimensional space. Let (R, Σ, μ), usually denoted by (R, μ), be a totally σ-finite measure space and referred in the sequel only as a measure space. A set $E \in \Sigma$ is called an atom of (R, Σ, μ) if $\mu(E) > 0$ and $F \subset E$, $F \in \Sigma$ implies either $\mu(F) = 0$ or $\mu(E \backslash F) = 0$. If there are no atoms, then (R, Σ, μ) is called non-atomic. A measure space (R, μ) is called resonant if it is one of the following two types: (i) non-atomic; (ii) completely atomic, with all atoms having equal measure. We refer to [BS88, pp.45–51] for more details and for a different, but equivalent, definition. When $R = \mathbb{R}^n$ we shall always take μ to be Lebesgue measure μ_n, and shall write $|\Omega|_n = \mu_n(\Omega)$ for any measurable subset Ω of $\mathbb{R}^n$. The family of all extended scalar-valued (real or complex) μ-measurable functions on R will be denoted by $\mathcal{M}(R, \mu)$; $\mathcal{M}_0(R, \mu)$ will stand for the subset of $\mathcal{M}(R, \mu)$ consisting of all those functions which are finite μ-a.e.

Definition 2.1. *Let $f \in \mathcal{M}_0(R, \mu)$. The distribution function μ_f of f is defined by*

$$\mu_f(\lambda) = \mu\{x \in R : |f(x)| > \lambda\}, \quad \text{for all } \lambda \geq 0, \tag{2.1}$$

and the non-increasing rearrangement of f is the function f^* defined on $[0, +\infty)$ by

$$f^*(t) = \inf\{\lambda \geq 0 : \mu_f(\lambda) \leq t\}, \quad \text{for all } t \geq 0. \tag{2.2}$$

If (R, μ) is a finite measure space, then the distribution function μ_f is bounded by $\mu(R)$ and so $f^*(t) = 0$ for all $t \geq \mu(R)$. In this case we may regard f^* as a function defined on the interval $[0, \mu(R))$; for more details we refer to [BS88].

Definition 2.2. *Two functions $f \in \mathcal{M}_0(R,\mu)$ and $g \in \mathcal{M}_0(S,\nu)$ are said to be equimeasurable if they have the same distribution function, i.e., if $\mu_f(\lambda) = \nu_g(\lambda)$ for all $\lambda \geq 0$.*

Now let $m \in \mathbb{N}$ and $\boldsymbol{\alpha} = (\alpha_1, \ldots, \alpha_m) \in \mathbb{R}^m$. Let us denote by ϑ_α^m and ω_α^m the real functions defined by

$$\vartheta_\alpha^m(t) = \prod_{i=1}^m l_i^{\alpha_i}(t), \quad \text{for all } t \in (0,+\infty), \tag{2.3}$$

and

$$\omega_\alpha^m(t) = \prod_{i=1}^m l_{i-1}^{\alpha_i}(t), \quad \text{for all } t \in [1,+\infty), \tag{2.4}$$

where $l_0, l_1, \ldots, l_m$ are non-negative functions defined on $(0,+\infty)$ by

$$l_0(t) = t,\ \ l_1(t) = 1 + |\log t|,\ \ l_i(t) = 1 + \log l_{i-1}(t), \quad i \in \{2, \ldots, m\}. \tag{2.5}$$

Definition 2.3 (cf. [EGO97])**.** *Let $p, q \in (0,+\infty]$, $m \in \mathbb{N}$ and $\boldsymbol{\alpha} = (\alpha_1, \ldots, \alpha_m) \in \mathbb{R}^m$. The generalised Lorentz-Zygmund (GLZ) space $L_{p,q;\boldsymbol{\alpha}}(R)$ is defined to be the set of all functions $f \in \mathcal{M}_0(R,\mu)$ such that*

$$\|f\|_{p,q;\boldsymbol{\alpha};R} := \|t^{\frac{1}{p}-\frac{1}{q}} \vartheta_\alpha^m(t)\ f^*(t)\|_{q,(0,+\infty)} \tag{2.6}$$

is finite. Here $\|.\|_{q,(0,+\infty)}$ stands for the usual L_q (quasi-) norm over the interval $(0,+\infty)$.

We remark that in [EGO97], the space $L_{p,q;\boldsymbol{\alpha}}(R)$ and the quasi-norm $\|.\|_{p,q;\boldsymbol{\alpha};R}$ defined above are denoted by $L_{p,q;\alpha_1,\ldots,\alpha_m}(R)$ and $\|.\|_{p,q;\alpha_1,\ldots,\alpha_m;R}$, respectively. We use the notation in [EGO97] only when we are considering particular cases.

Let us observe that when we consider $\boldsymbol{\alpha} = (0, \ldots, 0)$ in the previous Definition, we get the Lorentz space $L_{p,q}(R)$ endowed with the (quasi-) norm $\|.\|_{p,q;R}$, which is just the Lebesgue space $L_p(R)$ endowed with the (quasi-) norm $\|.\|_{p;R}$ when $p = q$; if $p = q$, $m = 1$ and $(R,\mu) = (\Omega, \mu_n)$, we get the Zygmund space $L^p(\log L)^{\alpha_1}(\Omega)$ endowed with the (quasi-) norm $\|.\|_{p;\alpha_1;\Omega}$.

Let us introduce some more notation, that will be needed in Section 4. Let $m \in \mathbb{N}$ with $m \geq 2$. We define the numbers $exp_0, \ldots, exp_m$ by

$$exp_0 = 1, \ \ exp_i = e^{exp_{i-1}},\ i \in \{1, \ldots, m\}.$$

Let $\boldsymbol{\alpha} = (\alpha_1, \ldots, \alpha_m) \in \mathbb{R}^m$. Let us denote by γ_α^m the non-negative function defined by

$$\gamma_\alpha^m(t) = \prod_{i=1}^m \ell_{i-1}^{\alpha_i}(t), \quad \text{for all } t \in [exp_{m-2}, +\infty), \tag{2.7}$$

where $\ell_0, \ldots, \ell_m$ are the non-negative functions defined by

$$\ell_0(t) = t,\ t \geq 1;\ \ell_i(t) = \log \ell_{i-1}(t),\ t \geq exp_{i-1},\ i \in \{1, \ldots, m\}.$$

We are going to need in Section 3 the following Lemma, which is very easy to prove.

Lemma 2.1. *(i) Let $m, k \in \mathbb{N}$. Then*

$$l_m(e^{-k+1}) = l_{m-1}(k).$$

(ii) Let $m \in \mathbb{N}_0$ and $k \in \mathbb{N}$. Then

$$l_m(k) \leq l_m(k+1) \leq e\ l_m(k).$$

(iii) Let $\alpha \in \mathbb{R}$ and $m, k \in \mathbb{N}$. Then for each $t \in (e^{-k}, e^{-k+1})$, we have the inequalities

$$\min\{1, e^\alpha\}\ l_{m-1}^\alpha(k) \leq l_m^\alpha(t) \leq \max\{1, e^\alpha\}\ l_{m-1}^\alpha(k).$$

(iv) *Let* $\alpha \in \mathbb{R}$, $m \in \mathbb{N}$ *and* $k \geq 2$. *Then the inequalities*

$$\min\{1, e^{-\alpha}\}\, l_{m-1}^{\alpha}(k) \leq l_m^{\alpha}(t) \leq \max\{1, e^{-\alpha}\}\, l_{m-1}^{\alpha}(k)$$

hold for each $t \in (e^{-k+1}, e^{-k+2})$.

The following Lemma, with an obvious proof, will be used later on.

Lemma 2.2. *Let* $k \in \mathbb{N}$ *and* $q_0 > exp_{k-1}$. *Then*

(i) $\ell_k(q) \leq l_k(q)$, *for each* $q \in [exp_{k-1}, +\infty)$;

(ii) $l_k(q) \leq e^k \ell_k(q)$, *for each* $q \in [exp_k, +\infty)$;

(iii) $l_k(q) \leq \left(\dfrac{k}{\ell_k(q_0)} + 1\right) \ell_k(q)$, *for each* $q \in [q_0, +\infty)$.

By a Young function Φ we mean a continuous non-negative, strictly increasing and convex function on $[0, +\infty)$ satisfying

$$\lim_{t \to 0+} \frac{\Phi(t)}{t} = \lim_{t \to +\infty} \frac{t}{\Phi(t)} = 0.$$

Given a Young function Φ and any measurable subset Ω of $\mathbb{R}^n$, $L_\Phi(\Omega)$ will denote the corresponding Orlicz space, *i.e.* the collection of functions $f \in \mathcal{M}_0(\Omega, \mu_n)$ for which there is a $\lambda > 0$ such that $\int_\Omega \Phi\left(\frac{|f(x)|}{\lambda}\right) dx < +\infty$, equipped with the Luxemburg norm $\|.\|_{\Phi,\Omega}$ given by

$$\|f\|_{\Phi,\Omega} = \inf\left\{\lambda > 0 : \int_\Omega \Phi\left(\frac{|f(x)|}{\lambda}\right) dx \leq 1\right\}.$$

We refer to [Ada75, Chapter VIII] and [KJF77, Chapter III] for more details.

Let Φ_1 and Φ_2 be Young functions. Recall that Φ_2 *dominates* Φ_1 *globally* if there is a positive constant κ such that

$$\Phi_1(t) \leq \Phi_2(\kappa t) \tag{2.8}$$

for all $t \geq 0$. Similarly, Φ_2 *dominates* Φ_1 *near infinity* if there are positive constants κ and t_0 such that (2.8) holds for all $t \in [t_0, +\infty)$. Two Young functions are said to be *equivalent globally* (*near infinity*) if each dominates the other globally (near infinity). We have from [Ada75, Theorem 8.12, pp. 234-235] the following result: If Φ_1 and Φ_2 are equivalent globally (or near infinity and $|\Omega|_n < +\infty$), then $L_{\Phi_1}(\Omega) = L_{\Phi_2}(\Omega)$ and the corresponding norms are equivalent.

Lemma 2.3 (cf. [EGO98]). *Let* Ω *be a measurable subset of* $\mathbb{R}^n$ *with finite volume and let* $\alpha > 0$. *Then*

(i) *the space* $L^\infty(\log L)^{-1/\alpha}(\Omega) = L_{\infty,\infty;-\frac{1}{\alpha}}(\Omega)$ *coincides with the Orlicz space* $L_{\Phi_1}(\Omega)$, *where* $\Phi_1(t) = \exp t^\alpha$ *for all* $t \geq t_0$ *with some* $t_0 \in (0, +\infty)$, *and the corresponding (quasi-) norms are equivalent;*

(ii) *the space* $L^\infty(\log\log L)^{-1/\alpha}(\Omega) = L_{\infty,\infty;0,-\frac{1}{\alpha}}(\Omega)$ *coincides with the Orlicz space* $L_{\Phi_2}(\Omega)$, *where* $\Phi_2(t) = \exp\exp t^\alpha$ *for all* $t \geq t_0$ *with some* $t_0 \in (0, +\infty)$, *and the corresponding (quasi-) norms are equivalent.*

We will denote the Orlicz spaces $L_{\Phi_1}(\Omega)$ and $L_{\Phi_2}(\Omega)$, considered in Lemma 2.3, by $E_\alpha(\Omega)$ and $EE_\alpha(\Omega)$, respectively. In view of the same Lemma, we may endow these spaces with the quasi-norms $\|.\|_{E_\alpha(\Omega)} := \|.\|_{\infty,\infty;-\frac{1}{\alpha};\Omega}$ and $\|.\|_{EE_\alpha(\Omega)} := \|.\|_{\infty,\infty;0,-\frac{1}{\alpha};\Omega}$. For more details we refer to [EGO98].

Let $m \in \mathbb{N}$. We denote by $\mathcal{R}_+^m$ and $\mathcal{R}_-^m$ the following subsets of $\mathbb{R}^m$:

$$\mathcal{R}_+^m = \{(\alpha_1, \ldots, \alpha_m) \in \mathbb{R}^m : \alpha_1, \ldots, \alpha_{m-1} \geq 0 \ \text{ and } \ \alpha_m > 0\}$$

$$\mathcal{R}_-^m = \{(\alpha_1, \ldots, \alpha_m) \in \mathbb{R}^m : \alpha_1, \ldots, \alpha_{m-1} \leq 0 \ \text{ and } \ \alpha_m < 0\}.$$

Given a Banach space X let us denote by X^* its dual space.

Let $j_0 \in \mathbb{N}$ and let $\{A_j\}_{j\geq j_0}$ be a sequence of Banach spaces. We denote by $l_1(A_j)$ the space of all sequences $a = \{a_j\}_{j\geq j_0}$ with $a_j \in A_j$, $j \geq j_0$, such that

$$\|a\|_{l_1(A_j)} = \sum_{j=j_0}^{+\infty} \|a_j\|_{A_j} < +\infty.$$

By $l_\infty(A_j)$ we denote the space of all sequences $a = \{a_j\}_{j\geq j_0}$ with $a_j \in A_j$, $j \geq j_0$, for which $\|a\|_{l_\infty(A_j)} = \sup_{j\geq j_0} \|a_j\|_{A_j}$ is finite. The space $c_0(A_j)$ is the subspace of $l_\infty(A_j)$ consisting of all sequences $a = \{a_j\}_{j\geq j_0}$ such that

$$\lim_{j\to+\infty} \|a_j\|_{A_j} = 0.$$

By Lemma 1.11.1 in [Tri78, pp.68-69], generalised in an obvious way,

$$[c_0(A_j)]^* = l_1(A_j^*), \tag{2.9}$$

with the usual interpretation (not only isomorphic but also isometric); see [Tri78] for more details.

For two non-negative expressions (*i.e.* functions or functionals) $\mathcal{A}$, $\mathcal{B}$ we use the symbol $\mathcal{A} \precsim \mathcal{B}$ to mean that $\mathcal{A} \leq c\,\mathcal{B}$, for some positive constant c independent of the variables in the expressions $\mathcal{A}$ and $\mathcal{B}$. If $\mathcal{A} \precsim \mathcal{B}$ and $\mathcal{B} \precsim \mathcal{A}$, we write $\mathcal{A} \approx \mathcal{B}$.

We adopt the convention that $\frac{a}{+\infty} = 0$ and $\frac{a}{0} = +\infty$ for all $a > 0$. If $p \in [1, +\infty]$, the conjugate number p' is given by $\frac{1}{p} + \frac{1}{p'} = 1$.

3. Decompositions

As was said in the Introduction, the following results extend the decompositions considered in [EK] for the exponential Orlicz spaces $E_\alpha(\Omega)$.

Let us assume, in this Section, that (R, μ) is a finite measure space. Without loss of generality we suppose that $\mu(R) = 1$; see Remark 3.1. In the sequel, we shall consider the decomposition of $(0, 1)$ into $\{(e^{-k}, e^{-k+1})\}_{k\geq 1}$.

Theorem 3.1. *Let $p, q \in (0, +\infty]$, $m \in \mathbb{N}$ and $\boldsymbol{\alpha} = (\alpha_1, \ldots, \alpha_m) \in \mathbb{R}^n$. Then for each $f \in L_{p,q;\boldsymbol{\alpha}}(R)$ we have*

(i) if $0 < q < +\infty$,

$$\|f\|_{p,q;\boldsymbol{\alpha};R} \approx \left[\sum_{k=1}^{+\infty} \left(e^{-k/p}\, \omega_\alpha^m(k)\, f^*(e^{-k})\right)^q\right]^{1/q} \tag{3.1}$$

$$\approx \left[\sum_{k=2}^{+\infty} \left(e^{-k/p}\, \omega_\alpha^m(k)\, f^*(e^{-k+1})\right)^q\right]^{1/q}; \tag{3.2}$$

(ii) if $q = +\infty$,

$$\|f\|_{p,q;\boldsymbol{\alpha};R} \approx \sup_{k\geq 1} \left\{e^{-k/p}\, \omega_\alpha^m(k)\, f^*(e^{-k})\right\} \tag{3.3}$$

$$\approx \sup_{k\geq 2} \left\{e^{-k/p}\, \omega_\alpha^m(k)\, f^*(e^{-k+1})\right\}. \tag{3.4}$$

Proof. (i) Let $0 < q < +\infty$ and suppose $f \in L_{p,q;\boldsymbol{\alpha}}(R)$. Then by Lemma 2.1 it follows that

$$\begin{aligned}\|f\|^q_{p,q;\boldsymbol{\alpha};R} &\geq c_1 \sum_{k=2}^{+\infty} \left(e^{-k(\frac{1}{p}-\frac{1}{q})}\vartheta_\alpha^m(e^{-k+1})\; f^*(e^{-k+1})\right)^q\; e^{-k}\\ &\geq c_2 \sum_{k=1}^{+\infty} \left(e^{-\frac{k}{p}}\, \omega_\alpha^m(k)\; f^*(e^{-k})\right)^q.\end{aligned}$$

Conversely, for $f \in L_{p,q;\boldsymbol{\alpha}}(R)$, we have again by Lemma 2.1

$$\|f\|^q_{p,q;\boldsymbol{\alpha};R} \;\le\; c_3 \sum_{k=2}^{+\infty} \left(e^{-\frac{k}{p}}\, \omega^m_\alpha(k)\; f^*(e^{-k+1})\right)^q \le c_4 \sum_{k=1}^{+\infty} \left(e^{-\frac{k}{p}}\, \omega^m_\alpha(k)\; f^*(e^{-k})\right)^q,$$

which gives the desired inequalities.

(ii) The proof of the case $q = +\infty$ is similar to the previous one. □

Let Ω be a measurable subset of $\mathbb{R}^n$ such that $|\Omega|_n = 1$. By Theorem 3.1 we conclude that

$$\|f\|_{E_\alpha(\Omega)} \approx \sup_{k\ge 1} \frac{f^*(e^{-k})}{k^{1/\alpha}} \approx \sup_{k\ge 2} \frac{f^*(e^{-k+1})}{k^{1/\alpha}}, \quad \text{for each} \quad f \in E_\alpha(\Omega),$$

and

$$\|f\|_{EE_\alpha(\Omega)} \approx \sup_{k\ge 1} \frac{f^*(e^{-k})}{(1+\log k)^{1/\alpha}} \approx \sup_{k\ge 2} \frac{f^*(e^{-k+1})}{\log^{1/\alpha} k}, \quad \text{for each} \quad f \in EE_\alpha(\Omega).$$

The next Lemma, with an easy proof, will be used to prove the last result of this Section.

Lemma 3.1. *Let $f \in \mathcal{M}_0(R,\mu)$, $J_k = (e^{-k}, e^{-k+1})$, $k \ge 1$. Then*

(i) for each $k \in \mathbb{N}$ we have

$$c_1 f^*(e^{-k+1}) \le \|f^*\|_{k,J_k} \le c_2 f^*(e^{-k}), \tag{3.5}$$

where c_1 and c_2 are positive constants independent of f and k;

(ii) for each $k \ge 2$ we have

$$c_1 f^*(e^{-k+2}) \le \|f^*\|_{k,J_{k-1}} \le c_2 f^*(e^{-k+1}), \tag{3.6}$$

where c_1 and c_2 are positive constants independent of f and k.

Theorem 3.2. *Let $p, q \in (0,+\infty]$, $m \in \mathbb{N}$ and $\boldsymbol{\alpha} = (\alpha_1, \dots, \alpha_m) \in \mathbb{R}^n$. Let $J_k = (e^{-k}, e^{-k+1})$, $k \ge 1$, and $I_k = J_{k-1}$, $k \ge 2$. Then for each $f \in L_{p,q;\boldsymbol{\alpha}}(R)$ we have*

(i) if $0 < q < +\infty$,

$$\|f\|_{p,q;\boldsymbol{\alpha};R} \;\approx\; \left[\sum_{k=1}^{+\infty} \left(e^{-k/p}\, \omega^m_\alpha(k)\, \|f^*\|_{k,J_k}\right)^q\right]^{1/q} \tag{3.7}$$

$$\approx\; \left[\sum_{k=2}^{+\infty} \left(e^{-k/p}\, \omega^m_\alpha(k)\, \|f^*\|_{k,I_k}\right)^q\right]^{1/q}; \tag{3.8}$$

(ii) if $q = +\infty$,

$$\|f\|_{p,q;\boldsymbol{\alpha};R} \;\approx\; \sup_{k\ge 1} \left\{e^{-k/p}\, \omega^m_\alpha(k)\, \|f^*\|_{k,J_k}\right\} \tag{3.9}$$

$$\approx\; \sup_{k\ge 2} \left\{e^{-k/p}\, \omega^m_\alpha(k)\, \|f^*\|_{k,I_k}\right\}. \tag{3.10}$$

Proof. (i) Suppose $0 < q < +\infty$ and let $f \in L_{p,q;\boldsymbol{\alpha}}(R)$. Then by (3.1) and by (3.5), we have

$$\|f\|^q_{p,q;\boldsymbol{\alpha};R} \;\ge\; c_1 \sum_{k=1}^{+\infty} \left(e^{-\frac{k}{p}}\, \omega^m_\alpha\; \|f^*\|_{k,J_k}\right)^q.$$

By (3.2) and by (3.6), we also have

$$\|f\|^q_{p,q;\boldsymbol{\alpha};R} \;\ge\; c_2 \sum_{k=2}^{+\infty} \left(e^{-\frac{k}{p}}\, \omega^m_\alpha(k)\; \|f^*\|_{k,I_k}\right)^q.$$

Conversely, for $f \in L_{p,q;\boldsymbol{\alpha}}(R)$, by (3.2) and by (3.5), we have

$$\|f\|^q_{p,q;\boldsymbol{\alpha};R} \quad \leq \quad c_3 \sum_{k=1}^{+\infty} \left(e^{-\frac{k}{p}} \, \omega^m_\alpha(k) \, \|f^*\|_{k,J_k} \right)^q .$$

By (3.2), by Lemma 2.1 and by (3.6), we have

$$\|f\|^q_{p,q;\boldsymbol{\alpha};R} \quad \leq \quad c_4 \sum_{k=3}^{+\infty} \left(e^{-\frac{k}{p}} \, \omega^m_\alpha(k) \, f^*(e^{-k+2}) \right)^q \leq c_5 \sum_{k=2}^{+\infty} \left(e^{-\frac{k}{p}} \, \omega^m_\alpha(k) \, \|f^*\|_{k,I_k} \right)^q ,$$

which gives the desired inequalities.

(ii) The proof of the case $q = +\infty$ is similar to the previous one. □

Let Ω be a measurable subset of $\mathbb{R}^n$ such that $|\Omega|_n = 1$. By Theorem 3.2 we conclude that for each $f \in E_\alpha(\Omega)$

$$\|f\|_{E_\alpha(\Omega)} \quad \approx \quad \sup_{k \geq 1} \frac{\|f^*\|_{k,J_k}}{k^{1/\alpha}} \approx \sup_{k \geq 2} \frac{\|f^*\|_{k,I_k}}{k^{1/\alpha}}. \tag{3.11}$$

The first estimate in (3.11) is given in [EK] by Corollary 2.3. The counterpart for the spaces $EE_\alpha(\Omega)$ is given by

$$\|f\|_{EE_\alpha(\Omega)} \approx \sup_{k \geq 1} \frac{\|f^*\|_{k,J_k}}{(1+\log k)^{1/\alpha}} \approx \sup_{k \geq 2} \frac{\|f^*\|_{k,I_k}}{\log^{1/\alpha} k}, \quad \text{for all} \quad f \in EE_\alpha(\Omega).$$

Remark 3.1. If (R,μ) is a finite measure space with measure $\mu(R)$, $m \in \mathbb{N}$ and $\boldsymbol{\alpha} \in \mathbb{R}^m$, we have $\vartheta^m_\alpha(s) \approx \vartheta^m_\alpha(s\,\mu(R))$, for all $s \in (0,1)$. This follows from the estimates $e^{-j}\, l_i(s) \leq l_i(s\mu(R)) \leq e^j\, l_i(s)$, for all $s \in (0,1)$ and $i = 1,\dots,m$ where j is a positive integer such that $e^{j-1} \leq l_1(\mu(R)) \leq e^j$.

With the previous considerations, it is easy to see that the estimates in Theorem 3.1 and Theorem 3.2 still hold, up to constants, if we replace $f^*(e^{-k})$ by $f^*(e^{-k}\,\mu(R))$, for each $k \in \mathbb{N}$, and $J_k = (e^{-k}, e^{-k+1})$ by $J_k = (e^{-k}\,\mu(R), e^{-k+1}\,\mu(R))$, for each $k \in \mathbb{N}$, respectively.

4. Equivalent (Quasi-) Norms for some Generalised Lorentz-Zygmund Spaces

In this Section, we are going to consider in the first part the GLZ spaces $L_{\infty,\infty;\boldsymbol{\alpha}}(R)$, with $\boldsymbol{\alpha} \in \mathcal{R}^m_-$, and in the second part the GLZ spaces $L_{1,1;\boldsymbol{\alpha}}(R)$, with $\boldsymbol{\alpha} \in \mathcal{R}^m_+$.

4.1. The GLZ spaces $L_{\infty,\infty;\boldsymbol{\alpha}}(R)$.

First we are going to recall a Lemma.

Lemma 4.1 (cf. [GO98], Lemma 5.1)**.** *Let $m \in \mathbb{N}$ and $\nu > 0$. Then there is a constant $c \in (0,+\infty)$ such that for all $s \in (0,1)$,*

$$\sup_{q \in [1,+\infty)} l^{-\nu}_{m-1}(q) s^{1/q} \leq c\, l^{-\nu}_m(s).$$

With the help of the previous result, it is not difficult to prove the next Lemma.

Lemma 4.2. *Let $m \in \mathbb{N}$ and $\boldsymbol{\alpha} \in \mathcal{R}^m_-$. Let $t_0 \in (0,+\infty)$. Then there is a positive constant c such that $\omega^m_\alpha(q) s^{1/q} \leq c\, \vartheta^m_\alpha(s)$, for all $s \in (0,t_0)$ and all $q \in [1,+\infty)$.*

The following result generalises Theorem 3.1 in [EGO98].

Theorem 4.1. *Let $m \in \mathbb{N}$ and $\boldsymbol{\alpha} \in \mathcal{R}^m_-$. Let $t_0 \in (0,+\infty)$.*

(i) Let $p \in (0,+\infty]$. Then for each $f \in L_{p,\infty;\boldsymbol{\alpha}}(R)$,

$$\|f\|_{p,\infty;\boldsymbol{\alpha};R} \approx \sup_{q \in [1,+\infty)} \omega^m_\alpha(q) \, \|f^*\|_{\frac{q}{q/p+1},\infty;(0,t_0)} + \sup_{t_0 \leq t < +\infty} \{t^{1/p} \vartheta^m_\alpha(t) f^*(t)\}. \tag{4.1}$$

(ii) Then for each $f \in L_{\infty,\infty;\boldsymbol{\alpha}}(R)$,

$$\|f\|_{\infty,\infty;\boldsymbol{\alpha};R} \approx f^*(t_0) + \sup_{q\in[1,+\infty)} \omega_\alpha^m(q)\,\|f^*\|_{q;(0,t_0)}. \tag{4.2}$$

Proof. We follow the proof of Theorem 3.1 in [EGO98], where the case $p = +\infty$, $m = 2$, $\alpha_1 = 0$, $\alpha_2 < 0$ and $\mu(R) < +\infty$ with $t_0 = \mu(R)$ was considered.

(i) Let $t_0 \in (0, +\infty)$ and $\mathcal{A} := \mathcal{B} + \mathcal{C}$ where

$$\mathcal{B} := \sup_{q\in[1,+\infty)} \omega_\alpha^m(q)\,\|f^*\|_{\frac{q}{q/p+1},\infty;(0,t_0)} \quad \text{and} \quad \mathcal{C} := \sup_{t_0\le t<+\infty} \{t^{1/p}\vartheta_\alpha^m(t)f^*(t)\}.$$

Suppose $f \in L_{p,\infty;\boldsymbol{\alpha}}(R)$. By Lemma 4.2 there is a constant $c_1 > 0$ such that for all $q \in [1, +\infty)$,

$$\omega_\alpha^m(q)\,\|f^*\|_{\frac{q}{q/p+1},\infty;(0,t_0)} \quad\le\quad c_1 \sup_{0<s<t_0} \{\vartheta_\alpha^m(s)\; s^{\frac{1}{p}} f^*(s)\}.$$

Passing to the supremum over all $q \in [1, +\infty)$, we get the inequality $\mathcal{B} \le c_1\,\|f\|_{p,\infty;\boldsymbol{\alpha};R}$. Hence

$$\mathcal{A} \quad \le 2\max\{1, c_1\}\|f\|_{p,\infty;\boldsymbol{\alpha};R}. \tag{4.3}$$

Conversely, suppose the right hand-side of (4.1) is finite. Fix $s \in (0, t_0)$ and set $q = 1 + |\log s|$. Then $\mathcal{B} \ge \omega_\alpha^m(q)\; s^{\frac{1}{q}+\frac{1}{p}} f^*(s) \ge e^{-1}\vartheta_\alpha^m(s)s^{\frac{1}{p}} f^*(s)$. Taking the supremum over all $s \in (0, t_0)$, we obtain the inequality

$$\mathcal{B} \ge e^{-1} \sup_{0<t<t_0} \{t^{1/p}\vartheta_\alpha^m(t)f^*(t)\}.$$

So $\mathcal{A} \ge e^{-1}\|f\|_{p,\infty;\boldsymbol{\alpha};R}$, which together with (4.3) gives the estimate (4.1).

(ii) Let $t_0 \in (0, +\infty)$. First we prove the following estimate

$$\mathcal{A} + \mathcal{B} \approx f^*(t_0) + \sup_{q\in[1,+\infty)} \omega_\alpha^m(q)\,\|f^*\|_{q;(0,t_0)}, \tag{4.4}$$

where

$$\mathcal{A} := \sup_{q\in[1,+\infty)} \omega_\alpha^m(q)\,\|f^*\|_{q,\infty;(0,t_0)} \quad \text{and} \quad \mathcal{B} := \sup_{t_0\le t<+\infty} \{\vartheta_\alpha^m(t)f^*(t)\}.$$

Suppose the right hand-side of (4.4) is finite. Since $\|f^*\|_{q,\infty;(0,t_0)} \le \|f^*\|_{q;(0,t_0)}$, cf. Proposition IV.4.2 of [BS88], and $\mathcal{B} \le f^*(t_0)$, we immediately obtain the inequality

$$\mathcal{A} + \mathcal{B} \le f^*(t_0) + \sup_{q\in[1,+\infty)} \omega_\alpha^m(q)\,\|f^*\|_{q;(0,t_0)}.$$

Now we prove the converse inequality. Suppose that $\mathcal{A} + \mathcal{B} < +\infty$. If $1 \le q < q_1$ then

$$\|f^*\|_{q;(0,t_0)} \quad\le\quad \|f^*\|_{q_1,\infty;(0,t_0)} t_0^{\frac{1}{q}-\frac{1}{q_1}} (1 - \frac{q}{q_1})^{-1/q}. \tag{4.5}$$

Let $q \in [1, +\infty)$. Since $l_j(q) \le l_j(2q) \le e l_j(q)$, for all $j \in \mathbb{N}_0$ we have by (4.5), with $q_1 = 2q$, the following inequalities

$$\omega_\alpha^m(q)\,\|f^*\|_{q;(0,t_0)} \quad\le\quad c_1\omega_\alpha^m(2q)\,\|f^*\|_{2q,\infty;(0,t_0)} \le c_1 \sup_{r\in[2,+\infty)} \omega_\alpha^m(r)\,\|f^*\|_{r,\infty;(0,t_0)}.$$

Therefore, passing to the supremum over all $q \in [1, +\infty)$, we get the inequality

$$\sup_{q\in[1,+\infty)} \omega_\alpha^m(q)\,\|f^*\|_{q;(0,t_0)} \le c_1\,\mathcal{A}. \tag{4.6}$$

Now it easily follows from (4.6) that

$$f^*(t_0) + \sup_{q\in[1,+\infty)} \omega_\alpha^m(q)\,\|f^*\|_{q;(0,t_0)} \le \max\{c_1, \vartheta_{-\alpha}^m(t_0)\}\,(\mathcal{A} + \mathcal{B})$$

and (4.4) is proved. The estimate (4.2) follows from (4.1), with $p = +\infty$, and from (4.4). $\square$

When (R,μ) is a finite measure space the previous estimates are much nicer.

Corollary 4.1. *Suppose (R,μ) is a measure space such that $\mu(R)<+\infty$. Let $m\in\mathbb{N}$ and $\boldsymbol{\alpha}\in\mathcal{R}^m_-$.*

(i) Let $p\in(0,+\infty]$. Then for each $f\in L_{p,\infty;\boldsymbol{\alpha}}(R)$,

$$\|f\|_{p,\infty;\boldsymbol{\alpha};R}\approx\sup_{q\in[1,+\infty)}\omega^m_\alpha(q)\,\|f\|_{\frac{q}{q/p+1},\infty;R}. \tag{4.7}$$

(ii) Then for each $f\in L_{\infty,\infty;\boldsymbol{\alpha}}(R)$,

$$\|f\|_{\infty,\infty;\boldsymbol{\alpha};R}\approx\sup_{q\in[1,+\infty)}\omega^m_\alpha(q)\,\|f\|_{q;R}. \tag{4.8}$$

Proof. The results follow from the theorem with $t_0=\mu(R)$ and from the fact that $f^*(t)=0$, $t\ge\mu(R)$. For the part (ii) we use also Proposition II.1.8 in [BS88]; see also Theorem 1.8.5 in [Zie89]. □

From (ii) of Corollary 4.1 we recover the results of Theorem 3.1 in [EGO98] for the spaces $E_\alpha(\Omega)$ and $EE_\alpha(\Omega)$, where Ω is a measurable subset of $\mathbb{R}^n$ with $|\Omega|_n<+\infty$.

Corollary 4.2. *Let $m\in\mathbb{N}$ and $\boldsymbol{\alpha}\in\mathcal{R}^m_-$. Let $t_0\in(0,+\infty)$. If $j_0\in\mathbb{N}$ and $q_0\ge1$ then for all $f\in L_{\infty,\infty;\boldsymbol{\alpha}}(R)$,*

$$\begin{aligned}\|f\|_{\infty,\infty;\boldsymbol{\alpha};R}&\approx f^*(t_0)+\sup_{j\in\mathbb{N},j\ge j_0}\omega^m_\alpha(j)\,\|f^*\|_{j;(0,t_0)} &(4.9)\\ &\approx f^*(t_0)+\sup_{q\in[q_0,+\infty)}\omega^m_\alpha(q)\,\|f^*\|_{q;(0,t_0)}. &(4.10)\end{aligned}$$

Proof. We follow the proof of Corollary 3.2 in [EGO98], where the case $m=2$, $\alpha_1=0$, $\alpha_2<0$ and $\mu(R)<+\infty$ with $t_0=\mu(R)$ was proved. For $f\in L_{\infty,\infty;\boldsymbol{\alpha}}(R)$, $j_0\in\mathbb{N}$ and $q_0\ge1$ we denote

$$\begin{aligned}S_1(f)&=f^*(t_0)+\sup_{q\in[1,+\infty)}\omega^m_\alpha(q)\,\|f^*\|_{q;(0,t_0)},\\ S_2(f)&=f^*(t_0)+\sup_{q\in[1,+\infty)}\omega^m_\alpha(q)\,t_0^{-1/q}\,\|f^*\|_{q;(0,t_0)},\\ S_3(f)&=f^*(t_0)+\sup_{q\in[q_0,+\infty)}\omega^m_\alpha(q)\,\|f^*\|_{q;(0,t_0)},\\ \sigma_1(f)&=f^*(t_0)+\sup_{j\in\mathbb{N},j\ge j_0}\omega^m_\alpha(j)\,\|f^*\|_{j;(0,t_0)},\\ \sigma_2(f)&=f^*(t_0)+\sup_{j\in\mathbb{N},j\ge j_0}\omega^m_\alpha(j)\,t_0^{-1/j}\,\|f^*\|_{j;(0,t_0)},\\ \sigma_3(f)&=f^*(t_0)+\sup_{j\in\mathbb{N},j\ge[q_0]+1}\omega^m_\alpha(j)\,\|f^*\|_{j;(0,t_0)},\end{aligned}$$

where $[q_0]$ denotes the integer part of q_0.

(i) Let $t_0\in(0,+\infty)$, $j_0\in\mathbb{N}$ and $f\in L_{\infty,\infty;\boldsymbol{\alpha}}(R)$. First we prove that

$$\|f\|_{\infty,\infty;\boldsymbol{\alpha};R}\approx f^*(t_0)+\sup_{j\in\mathbb{N},j\ge j_0}\omega^m_\alpha(j)\,\|f^*\|_{j;(0,t_0)}.$$

If $q\in[1,+\infty)$, we put $j=\max\{j_0,[q]+1\}$ and choose $n\in\mathbb{N}$ such that $e^{n-1}\ge j_0$. Then

$$j\le j_0([q]+1)<j_0q\le e^{n-1}(q+1)\le e^{n-1}2q\le e^nq$$

and hence

$$l_{k-1}(j)\le e^n l_{k-1}(q),\quad k=2,\dots,m.$$

Therefore

$$e^{n(\alpha_1+\dots+\alpha_m)}\omega^m_\alpha(q)\le\omega^m_\alpha(j). \tag{4.11}$$

Since $j \geq [q]+1 > q$, we get by Hölder's inequality together with (4.11) the inequality

$$\omega_\alpha^m(q) t_0^{-1/q} \|f^*\|_{q;(0,t_0)} \leq c\, \omega_\alpha^m(j) t_0^{-1/j} \|f^*\|_{j;(0,t_0)},$$

where $c = e^{-n(\alpha_1+\cdots+\alpha_m)} > 1$, and hence

$$S_2(f) \leq c\, \sigma_2(f). \tag{4.12}$$

It is easy to see that $S_1(f) \approx S_2(f)$, $\sigma_1(f) \approx \sigma_2(f)$, and since $\sigma_1(f) \leq S_1(f)$ we have, together with (4.12), the estimates

$$\sigma_1(f) \leq S_1(f) \approx S_2(f) \leq c\, \sigma_2(f) \approx \sigma_1(f). \tag{4.13}$$

So (4.9) it follows from (4.2) and (4.13).

(ii) Let $t_0 \in (0,+\infty)$, $q_0 \geq 1$ and $f \in L_{\infty,\infty;\boldsymbol{\alpha}}(R)$. From (4.2) it follows that

$$S_3(f) \leq S_1(f) \approx \|f\|_{\infty,\infty;\boldsymbol{\alpha};R}. \tag{4.14}$$

Since $\sigma_3(f) = \sigma_1(f)$ if $j_0 = [q_0]+1$, we have by (4.9)

$$\|f\|_{\infty,\infty;\boldsymbol{\alpha};R} \approx \sigma_3(f) \leq S_3(f). \tag{4.15}$$

Therefore, by (4.14) and (4.15) we get (4.10) and the proof is finished. □

When (R,μ) is a measure space of finite measure we obtain simple equivalent norms.

Corollary 4.3. *Suppose (R,μ) is a measure space such that $\mu(R) < +\infty$. Let $m \in \mathbb{N}$ and $\boldsymbol{\alpha} \in \mathcal{R}_-^m$. If $j_0 \in \mathbb{N}$ and $q_0 \geq 1$ then for all $f \in L_{\infty,\infty;\boldsymbol{\alpha}}(R)$,*

$$\|f\|_{\infty,\infty;\boldsymbol{\alpha};R} \approx \sup_{j\in\mathbb{N}, j\geq j_0} \omega_\alpha^m(j)\, \|f\|_{j;R} \tag{4.16}$$

$$\approx \sup_{q\in[q_0,+\infty)} \omega_\alpha^m(q)\, \|f\|_{q;R}. \tag{4.17}$$

Proof. The results follow from Corollary 4.2 with $t_0 = \mu(R)$ and Proposition II.1.8 in [BS88]; see also Theorem 1.8.5 in [Zie89]. □

If we consider $m = 1$, $\alpha_1 < 0$ and Ω a measurable subset of $\mathbb{R}^n$ with $|\Omega|_n < +\infty$ in the above Corollary we recover part (i) of Corollary 3.2 in [EGO98].

Corollary 4.4. *Let $m \in \mathbb{N}$, $m \geq 2$ and $\boldsymbol{\alpha} \in \mathcal{R}_-^m$. Let $t_0 \in (0,+\infty)$. If $j_0 \in \mathbb{N}$, $j_0 \geq [exp_{m-2}]+1$ and $q_0 > exp_{m-2}$ then for all $f \in L_{\infty,\infty;\boldsymbol{\alpha}}(R)$,*

$$\|f\|_{\infty,\infty;\boldsymbol{\alpha};R} \approx f^*(t_0) + \sup_{j\in\mathbb{N}, j\geq j_0} \gamma_\alpha^m(j)\, \|f^*\|_{j;(0,t_0)} \tag{4.18}$$

$$\approx f^*(t_0) + \sup_{q\in[q_0,+\infty)} \gamma_\alpha^m(q)\, \|f^*\|_{q;(0,t_0)}. \tag{4.19}$$

Proof. (i) Let $j_0 \in \mathbb{N}$, $j_0 \geq [exp_{m-2}]+1$. Since $j_0 > exp_{m-2}$, it follows from (i) and (iii) of Lemma 2.2 that, for each $k \in \{1,\ldots,m-1\}$, $\ell_k(j) \approx l_k(j)$, for all $j \geq j_0$. Therefore, the estimate (4.18) follows from (4.9).

(ii) Let $q_0 > exp_{m-2}$. Then for $k = 1,\ldots,m-1$, the estimate $\ell_k(q) \approx l_k(q)$, for all $q \geq q_0$, follows from (i) and (iii) of Lemma 2.2. Therefore, the estimate (4.19) follows from (4.10). □

Corollary 4.5. *Suppose (R,μ) is a measure space such that $\mu(R) < +\infty$. Let $m \in \mathbb{N}$, $m \geq 2$ and $\boldsymbol{\alpha} \in \mathcal{R}_-^m$. If $j_0 \in \mathbb{N}$, $j_0 \geq [exp_{m-2}]+1$ and $q_0 > exp_{m-2}$ then for all $f \in L_{\infty,\infty;\boldsymbol{\alpha}}(R)$,*

$$\|f\|_{\infty,\infty;\boldsymbol{\alpha};R} \approx \sup_{j\in\mathbb{N}, j\geq j_0} \gamma_\alpha^m(j)\, \|f\|_{j;R} \tag{4.20}$$

$$\approx \sup_{q\in[q_0,+\infty)} \gamma_\alpha^m(q)\, \|f\|_{q;R}. \tag{4.21}$$

Proof. The results follow from Corollary 4.4 with $t_0 = \mu(R)$ and Proposition II.1.8 in [BS88]; see also Theorem 1.8.5 in [Zie89]. □

If we consider $m = 2$, $\alpha_1 = 0$, $\alpha_2 < 0$, and Ω a measurable subset of $\mathbb{R}^n$ with $|\Omega|_n < +\infty$ in the above Corollary we recover part (ii) of Corollary 3.2 in [EGO98].

4.2. **The GLZ spaces $L_{1,1;\boldsymbol{\alpha}}(R)$.**

Let us assume, in this Subsection, that (R, μ) is a finite measure. Again, without loss of generality we suppose that $\mu(R) = 1$. Let $m \in \mathbb{N}$ and $\boldsymbol{\alpha} \in \mathcal{R}^m_+$. Let us consider the spaces $L_{1,1;\boldsymbol{\alpha}}(R)$ and $L_{\infty,\infty;-\boldsymbol{\alpha}}(R)$ endowed with $\|.\|_{1,1;\boldsymbol{\alpha};R}$ and $\|.\|_{\infty,\infty;-\boldsymbol{\alpha};R}$, respectively.

The triangle inequality for $\|.\|_{1,1;\boldsymbol{\alpha};R}$ follows immediately by the property,

$$\int_0^t \varphi(s)(f+g)^*(s)ds \le \int_0^t \varphi(s)f^*(s)ds + \int_0^t \varphi(s)g^*(s)ds, \quad 0 < t < 1,$$

whenever φ is a non-negative decreasing function on $(0,1)$, cf. [Lor51, p. 38] or [BR80, p.23].

Let us introduce the functional $\|f\|_{(\infty,\infty;-\boldsymbol{\alpha};R)} = \sup_{0<t<1} \vartheta^m_{-\alpha}(t) f^{**}(t)$. Then by Lemma 3.2 in [EGO97], we have

$$\|f\|_{\infty,\infty;-\boldsymbol{\alpha};R} \le \|f\|_{(\infty,\infty;-\boldsymbol{\alpha};R)} \precsim \|f\|_{\infty,\infty;-\boldsymbol{\alpha};R},$$

for all $f \in L_{\infty,\infty;-\boldsymbol{\alpha}}(R)$. The triangle inequality for $\|.\|_{(\infty,\infty;-\boldsymbol{\alpha};R)}$ it follows from the sub-additivity of $f \mapsto f^{**}$, cf. Theorem II.3.4 in [BS88].

Before we give the next result, we refer to [BS88] for the definitions of mutually associate and rearrangement-invariant Banach function spaces. Given a Banach function space X let us denote by X' its associate space.

Lemma 4.3. *Let $m \in \mathbb{N}$ and $\boldsymbol{\alpha} \in \mathcal{R}^m_+$. If (R, μ) is a resonant measure space, then*

$$X = (L_{1,1;\boldsymbol{\alpha}}(R), \|.\|_{1,1;\boldsymbol{\alpha};R})$$

and

$$Y = (L_{\infty,\infty;-\boldsymbol{\alpha}}(R), \|.\|_{(\infty,\infty;-\boldsymbol{\alpha};R)})$$

are rearrangement-invariant Banach function spaces and they are mutually associate (up to equivalence of norms).

Proof. There is no difficulty in verifying that X and Y are Banach function spaces and the rearrangement invariance is obvious, since two equimeasurable functions have the same non-increasing rearrangement.

Now we are going to prove that X and Y are mutually associate. We follow the proof of Theorem IV.6.5 in [BS88] and the proof of Lemma 3.4 in [EGO97].

Suppose $g \in Y$. Then for any $f \in X$ with $\|f\|_X \le 1$, we have by the Hardy-Littlewood inequality, cf. Theorem II.2.2 in [BS88],

$$\int_R |fg|\, d\mu \quad \le \quad \int_0^1 f^*(t)g^*(t)\, dt \le \sup_{0<t<1} \{g^{**}(t)\vartheta^m_{-\alpha}(t)\} \|f\|_X = \|g\|_Y \, \|f\|_X.$$

Hence taking the supremum over all $f \in X$ with $\|f\|_X \le 1$, we get

$$\|g\|_{X'} = \sup\{\int_R |fg|\, d\mu : f \in X, \|f\|_X \le 1\} \le \|g\|_Y. \tag{4.22}$$

To establish an inequality reverse to (4.22), it is sufficient by the Luxemburg representation Theorem (see [BS88, Theorem II.4.10, p. 62]) to do so for the measure space $(\mathbb{R}^+, \mu_1)$ and functions g in $\mathbb{R}^+$ for which $g = g^*$. Suppose g belongs to the associate space X' of X, and also under the previous conditions, then by Hölder's inequality, cf. Corollary II.4.5 in [BS88], for $0 < t < 1$,

$$tg^{**}(t) = \int_0^1 \chi_{[0,t]}(s) g^*(s)\, ds \le \|\chi_{[0,t]}\|_X \|g\|_{X'}.$$

Since

$$\|\chi_{[0,t]}\|_X = \int_0^1 \chi_{[0,t]}(s)\vartheta_\alpha^m(s)\, ds = \int_0^t \vartheta_\alpha^m(s)\, ds \quad \approx \quad t\, \vartheta_\alpha^m(t),$$

we get

$$\|g\|_Y \lesssim \|g\|_{X'}. \tag{4.23}$$

The estimates (4.22) and (4.23) together show that Y is equivalent to the associate of X and hence, by the Lorentz-Luxemburg Theorem, cf. Theorem I.2.7 in [BS88], the spaces X and Y are mutually associate. □

Proposition 4.1. *Suppose (R,μ) is a non-atomic measure space. Let $m \in \mathbb{N}$ and $\boldsymbol{\alpha} \in \mathcal{R}_-^m$. Then, up to equivalence of norms,*

$$(L^0_{\infty,\infty;\boldsymbol{\alpha}}(R))^* = L_{1,1;-\boldsymbol{\alpha}}(R), \tag{4.24}$$

where $L^0_{\infty,\infty;\boldsymbol{\alpha}}(R)$ is the completion of $L_\infty(R)$ in $L_{\infty,\infty;\boldsymbol{\alpha}}(R)$.

Proof. We apply Theorem II.5.5 in [BS88, pp. 67-68] to the space $X = L_{\infty,\infty;\boldsymbol{\alpha}}(R)$. It is easy to see that $\lim_{t\to 0+} \varphi_X(t) = 0$, where φ_X is the fundamental function of X, see [BS88, p. 65]. Therefore, by Theorem II.5.5 in [BS88], $(X_b)^* = X'$. But by Lemma 4.3, X' coincides with $L_{1,1;-\boldsymbol{\alpha}}(R)$ up to equivalence of norms and, by Proposition I.3.10 in [BS88, p. 17], X_b coincides with the space $L^0_{\infty,\infty;\boldsymbol{\alpha}}(R)$. □

Let $j_0, m \in \mathbb{N}$ and $\boldsymbol{\alpha} \in \mathcal{R}_-^m$. We denote by $c_0^s(L_j(R))$ the subspace of $c_0(L_j(R))$ which consists of all elements $\{F_j\}_{j\ge j_0}$ of $c_0(L_j(R))$ with $F_j = \omega_\alpha^m(j) f$, for all $j \ge j_0$, where $f \in L_{\infty,\infty;\boldsymbol{\alpha}}(R)$. In what follows, and according to Corollary 4.3, we consider the space $L_{\infty,\infty;\boldsymbol{\alpha}}(R)$ endowed with the norm

$$\|.\|^d_{\infty,\infty;\boldsymbol{\alpha};R} = \sup_{j\in\mathbb{N}, j\ge j_0} \omega_\alpha^m(j)\, \|.\|_{j;R}.$$

Proposition 4.2. *Let $j_0, m \in \mathbb{N}$ and $\boldsymbol{\alpha} \in \mathcal{R}_-^m$. Then*

$$L^0_{\infty,\infty;\boldsymbol{\alpha}}(R) = \{f \in L_{\infty,\infty;\boldsymbol{\alpha}}(R) : \lim_{j\to+\infty} \omega_\alpha^m(j)\, \|f\|_{j;R} = 0\}$$

and $(L^0_{\infty,\infty;\boldsymbol{\alpha}}(R), \|.\|^d)$ is isometric to $(c_0^s(L_j(R)), \|.\|_{l_\infty(L_j(R))})$.

Proof. If $f \in L^0_{\infty,\infty;\boldsymbol{\alpha}}(R)$, the results follow easily.

Conversely, suppose $f \in L_{\infty,\infty;\boldsymbol{\alpha}}(R)$ with $\lim_{j\to+\infty} \omega_\alpha^m(j)\|f\|_{j;R} = 0$. Let $\epsilon > 0$. Then there is $j_1 \in \mathbb{N}$, with $j_1 \ge j_0$, such that for all $j \ge j_1$ we have the inequality

$$\omega_\alpha^m(j)\|f\|_{j;R} < \frac{\epsilon}{2}. \tag{4.25}$$

Since $f \in L_{\infty,\infty;\boldsymbol{\alpha}}(R)$, f is finite $\mu - a.e.$ For each $n \in \mathbb{N}$ let us consider the set $R_n = \{x \in R : |f(x)| > n\}$. Now we introduce a sequence $\{f_n\}_{n\in\mathbb{N}}$ in $L_\infty(R)$ by $f_n(x) = f(x)$ if $x \in R\backslash R_n$, and $f_n(x) = 0$ otherwise. Then, for each $n \in \mathbb{N}$, we have by (4.25)

$$\|f - f_n\|^d_{\infty,\infty;\boldsymbol{\alpha};R} \le \max_{j\in\mathbb{N}, j_0\le j\le j_1} \omega_\alpha^m(j)\|f\|_{j;R_n} + \frac{\epsilon}{2} = \omega_\alpha^m(k)\|f\|_{k;R_n} + \frac{\epsilon}{2}. \tag{4.26}$$

Now

$$\|\omega_\alpha^m(k) f\|^k_{k;R_n} = \|\, (\omega_\alpha^m(k) f)^k\, \chi_{R_n}\|_{1;R}.$$

Let us consider, for each $n \in \mathbb{N}$, a function defined $\mu - a.e.$ on R by

$$g_n = (\omega_\alpha^m(k)|f|)^k\, \chi_{R_n}.$$

We note that for all $n \in \mathbb{N}$, $|g_n| \leq h$, $\mu - a.e$ on R, where $h = (\omega_\alpha^m(k)|f|)^k$, $\mu - a.e$ on R, is a function in $L_1(R)$. Since $\lim_{n\to+\infty} \chi_{R_n} = 0$, $\mu - a.e.$ it follows from the Lebesgue dominated convergence Theorem that $\lim_{n\to+\infty} \|\omega_\alpha^m(k)f\|_{k;R_n}^k = 0$. Hence, there is $n_0 \in \mathbb{N}$ such that

$$\omega_\alpha^m(k)\|f\|_{k;R_n} < \frac{\epsilon}{2}, \quad \text{for each} \quad n \geq n_0. \tag{4.27}$$

Therefore, from (4.26) and (4.27), we get $\lim_{n\to+\infty} \|f - f_n\|_{\infty,\infty;\boldsymbol{\alpha};R}^d = 0$, which shows that $f \in L^0_{\infty,\infty;\boldsymbol{\alpha}}(R)$.

Now we can define a linear mapping H from $L^0_{\infty,\infty;\boldsymbol{\alpha}}(R)$ onto $c_0^s(L_j(R))$ by

$$H(f) = \{\omega_\alpha^m(j)f\}_{j\geq j_0}, \qquad \text{for all} \quad f \in L^0_{\infty,\infty;\boldsymbol{\alpha}}(R).$$

We also have $\|H(f)\|_{c_0^s(L_j(R))} = \|f\|^d_{\infty,\infty;\boldsymbol{\alpha};R}$, for all $f \in L^0_{\infty,\infty;\boldsymbol{\alpha}}(R)$, and the proof is finished. ☐

The next result gives an equivalent norm for the GLZ spaces $L_{1,1;\boldsymbol{\alpha}}(R)$, with $\boldsymbol{\alpha} \in \mathcal{R}_+^m$, in terms of decompositions.

Theorem 4.2. *Suppose (R,μ) is a non-atomic measure space. Let $m \in \mathbb{N}$ and $\boldsymbol{\alpha} \in \mathcal{R}_+^m$. Let $j_0 \in \mathbb{N}$ with $j_0 \geq 2$. Then $L_{1,1;\boldsymbol{\alpha}}(R)$ is the set of all measurable functions $g : R \to \mathbb{C}$ which can be represented as*

$$g = \sum_{j=j_0}^{+\infty} g_j, \tag{4.28}$$

with g_j a measurable function on R that belongs to $L_{j'}(R)$, for each $j \geq j_0$, such that

$$\sum_{j=j_0}^{+\infty} \omega_\alpha^m(j)\|g_j\|_{j';R} < +\infty. \tag{4.29}$$

The infimum of the expression (4.29) taken over all admissible representations (4.28) is an equivalent norm on $L_{1,1;\boldsymbol{\alpha}}(R)$

Proof. Let $j_0 \in \mathbb{N}$. Let us consider a measurable function $h : R \to \mathbb{C}$ that can be represented as

$$h = \sum_{j=j_0}^{+\infty} g_j, \tag{4.30}$$

with g_j a measurable function on R that belongs to $L_{j'}(R)$, for each $j \geq j_0$, such that

$$\sum_{j=j_0}^{+\infty} \omega_\alpha^m(j)\|g_j\|_{j';R} < +\infty$$

and let us define

$$\Phi_h(f) = \int_R hf \, d\mu, \quad \text{for all} \quad f \in L^0_{\infty,\infty;-\boldsymbol{\alpha}}(R). \tag{4.31}$$

Then $\Phi_h \in (L^0_{\infty,\infty;-\boldsymbol{\alpha}}(R))^*$ and

$$\|\Phi_h|(L^0_{\infty,\infty;-\boldsymbol{\alpha}}(R))^*\| \leq \inf \sum_{j=j_0}^{+\infty} \omega_\alpha^m(j)\|g_j\|_{j';R}, \tag{4.32}$$

where the infimum is taken over all admissible representations (4.30). In fact, for all $f \in L^0_{\infty,\infty;-\boldsymbol{\alpha}}(R)$, we have by Theorem 1.27 in [Rud86, p.22] and by Hölder's inequality, the following

$$|\Phi_h(f)| \quad \leq \quad \sum_{j=j_0}^{+\infty} \|g_j\|_{j';R}\|f\|_{j;R} \leq \|f\|^d_{\infty,\infty;-\boldsymbol{\alpha};R} \sum_{j=j_0}^{+\infty} \omega^m_\alpha(j)\|g_j\|_{j';R}.$$

Thus, Φ_h is a bounded linear functional on $L^0_{\infty,\infty;\boldsymbol{\alpha}}(R)$ (the linearity of Φ_h is obvious) such that

$$\|\Phi_h|(L^0_{\infty,\infty;-\boldsymbol{\alpha}}(R))^*\| \leq \sum_{j=j_0}^{+\infty} \omega^m_\alpha(j)\|g_j\|_{j';R}$$

and we get (4.32).

Now we follow the reasoning in the proof of Theorem 2.6.2/2 in [ET96, pp. 72-74]. Let $G \in (L^0_{\infty,\infty;-\boldsymbol{\alpha}}(R))^*$. Since $L^0_{\infty,\infty;-\boldsymbol{\alpha}}(R)$ is isometric to $c^s_0(L_j(R))$, cf. Proposition 4.2, $G \circ H^{-1} \in (c^s_0(L_j(R)))^*$, where H is the isometry considered in the referred proposition. By Hahn-Banach theorem, there exists a bounded linear functional $\widetilde{G \circ H^{-1}}$ on $c_0(L_j(R))$, which is an extension of $G \circ H^{-1}$ to $c_0(L_j(R))$ and has the same norm

$$\|\widetilde{G \circ H^{-1}}|(c_0(L_j(R)))^*\| = \|G \circ H^{-1}|(c^s_0(L_j(R)))^*\|.$$

But by (2.9), $\widetilde{G \circ H^{-1}}$ can be identified with an element $\{\tilde{G}_j\}_{j \geq j_0} \in l_1((L_j(R))^*)$ such that

$$\|G \circ H^{-1}|(c^s_0(L_j(R)))^*\| = \|\widetilde{G \circ H^{-1}}|(c_0(L_j(R)))^*\| = \sum_{j=j_0}^{+\infty} \|\tilde{G}_j|(L_j(R))^*\|. \tag{4.33}$$

Since each $\tilde{G}_j$ can be identified with a $\tilde{g}_j \in L_{j'}(R)$ by

$$\tilde{G}_j(f) = \int_R \tilde{g}_j\, f\, d\mu, \quad \text{for all } \ f \in L_j(R),$$

with $\|\tilde{G}_j|(L_j(R))^*\| = \|\tilde{g}_j\|_{j',R}$, it follows from (4.33) that

$$\|G|(L^0_{\infty,\infty;-\boldsymbol{\alpha}}(R))^*\| = \|G \circ H^{-1}|(c^s_0(L_j(R)))^*\| = \sum_{j=j_0}^{+\infty} \|\tilde{g}_j\|_{j',R}. \tag{4.34}$$

Using Theorem 1.38 in [Rud86, p. 29] we get

$$G(f) \quad = \quad \sum_{j=j_0}^{+\infty} \tilde{G}_j(\omega^m_{-\alpha}(j)f) = \int_R h\, f\, d\mu, \quad \text{for all } \ f \in L^0_{\infty,\infty;-\boldsymbol{\alpha}}(R),$$

with

$$h = \sum_{j=j_0}^{+\infty} g_j \quad \text{and} \quad g_j = \tilde{g}_j\, \omega^m_{-\alpha}(j),\ j \geq j_0,$$

because, for each $f \in L^0_{\infty,\infty;-\boldsymbol{\alpha}}(R)$,

$$\sum_{j=j_0}^{+\infty} \int_R |f\, \omega^m_{-\alpha}(j)\tilde{g}_j|\, d\mu \quad \leq \quad \|f\|^d_{\infty,\infty;-\boldsymbol{\alpha};R} \sum_{j=j_0}^{+\infty} \|\tilde{g}_j\|_{j';R} < +\infty.$$

From (4.34), we get

$$\|G|(L^0_{\infty,\infty;-\boldsymbol{\alpha}}(R))^*\| \quad \geq \quad \inf \sum_{j=j_0}^{+\infty} \omega^m_\alpha(j)\|g_j\|_{j',R}, \tag{4.35}$$

where the infimum is taken over all admissible representations of h that satisfy (4.29). But since $G = \Phi_h$, we have from (4.32) and (4.35) that

$$\|G|(L^0_{\infty,\infty;-\boldsymbol{\alpha}}(R))^*\| \quad = \quad \inf \sum_{j=j_0}^{+\infty} \omega^m_\alpha(j)\|g_j\|_{j',R},$$

where the infimum is taken over all admissible representations of h that satisfy (4.29).

Now given a function h represented as (4.28) and satisfying (4.29), we infer by (4.24) that there is a $g \in L_{1,1;\boldsymbol{\alpha}}(R)$ such that

$$\Phi_h(f) = \int_R f\, g\, d\mu, \quad \text{for all} \ \ f \in L^0_{\infty,\infty;-\boldsymbol{\alpha}}(R),$$

with

$$\|g\|_{1,1;\boldsymbol{\alpha};R} \approx \|\Phi_h|(L^0_{\infty,\infty;-\boldsymbol{\alpha}}(R))^*\|.$$

Then it follows, by Theorem 1.39 in [Rud86, p. 30], that $g = h\ \mu - a.e.$, because it is easy to see that $g, h \in L_1(R)$.

Conversely, let $g \in L_{1,1;\boldsymbol{\alpha}}(R)$. By (4.24), g defines a linear functional Λ_g on $L^0_{\infty,\infty;-\boldsymbol{\alpha}}(R)$ such that

$$\Lambda_g(f) = \int_R f\, g\, d\mu, \quad \text{for all} \ \ f \in L^0_{\infty,\infty;-\boldsymbol{\alpha}}(R),$$

with

$$\|g\|_{1,1;\boldsymbol{\alpha};R} \approx \|\Lambda_g|(L^0_{\infty,\infty;-\boldsymbol{\alpha}}(R))^*\|.$$

Since there is a function h that can be represented as (4.28) and satisfying (4.29) for which $\Lambda_g = \Phi_h$, it follows as above that $g = h\ \mu - a.e.$ □

5. Applications

As was referred in the Introduction, there is a version of the extrapolation result in [Zyg59, Theorem XII.4.11 (i), p. 119] for sublinear operators. Therefore we start this section by defining sublinear operator and by recalling that extrapolation result; see [Tor86, Theorem V.3.3, p.124] or [FK98, Theorem 4.1] for instance.

Definition 5.1. *Let (R_0, μ_0) and (R_1, μ_1) be measure spaces. Let T be an operator whose domain is some linear subspace of $\mathcal{M}_0(R_0, \mu_0)$ and whose range is contained in $\mathcal{M}(R_1, \mu_1)$. Then T is said to be sublinear if the relations*

$$|T(f+g)| \le |Tf| + |Tg| \quad \textit{and} \quad |T(\lambda f)| = |\lambda|\,|Tf|$$

hold $\mu_1 - a.e$ on R_1 for all f and g in the domain of T and for all scalars λ.

Theorem 5.1. *Suppose Ω is a measurable subset of $\mathbb{R}^n$ with finite volume. Let $\alpha > 0$ and $q_0 \in [1, +\infty)$. If A is a bounded sublinear operator in $L_q(\Omega)$, $q_0 \le q < +\infty$, such that*

$$\|Af\|_q \le c\, q^{1/\alpha}\|f\|_q, \ \ q \ge q_0 \ge 1,$$

then

$$\|Af\|_{E_\alpha(\Omega)} \le c\|f\|_\infty, \ \ \textit{for all} \ \ f \in L_\infty(\Omega).$$

Now, by the results of Section 4, the following Theorem is an obvious generalisation of the previous one.

Theorem 5.2. *Let $m \in \mathbb{N}$ and $\boldsymbol{\alpha} \in \mathcal{R}^m_-$. Suppose (R_0, μ_0) and (R_1, μ_1) are finite measure spaces.*

(i) Suppose A is a bounded sublinear operator from $L_q(R_0)$ into $L_q(R_1)$ such that either

$$\|Af\|_{q;R_1} \leq c\, \omega^m_{-\alpha}(q)\|f\|_{q;R_0}, \quad \text{for all } \ f \in L_q(R_0),$$

for each $q \in [q_0, +\infty)$ with $q_0 \geq 1$, or

$$\|Af\|_{q;R_1} \leq c\, \gamma^m_{-\alpha}(q)\|f\|_{q;R_0}, \quad \text{for all } \ f \in L_q(R_0),$$

for each $q \in [q_0, +\infty)$ with $q_0 > exp_{m-2}$ and $m \geq 2$. Then

$$A : L_\infty(R_0) \longrightarrow L_{\infty,\infty;\boldsymbol{\alpha}}(R_1),$$

and

$$\|Af\|_{\infty,\infty;\boldsymbol{\alpha};R_1} \leq c\|f\|_{\infty;R_0}, \quad \text{for all } \ f \in L_\infty(R_0)$$

(ii) Suppose A is a bounded sublinear operator from $L_q(R_0)$ into $L_q(R_1)$ such that either

$$\|Af\|_{q;R_1} \leq c\, \omega^m_{\alpha}(q)\|f\|_{q;R_0}, \quad \text{for all } \ f \in L_q(R_0),$$

for each $q \in [q_0, +\infty)$ with $q_0 \geq 1$, or

$$\|Af\|_{q;R_1} \leq c\, \gamma^m_{\alpha}(q)\|f\|_{q;R_0}, \quad \text{for all } \ f \in L_q(R_0),$$

for each $q \in [q_0, +\infty)$ with $q_0 > exp_{m-2}$ and $m \geq 2$. Then

$$A : L_{\infty,\infty;\boldsymbol{\alpha}}(R_0) \longrightarrow L_\infty(R_1),$$

and

$$\|Af\|_{\infty;R_1} \leq c\|f\|_{\infty,\infty;\boldsymbol{\alpha};R_0}, \quad \text{for all } \ f \in L_{\infty,\infty;\boldsymbol{\alpha}}(R_0).$$

Proof. The proof is a consequence of Corollaries 4.3, 4.5 and [KJF77, Theorem 2.11.4, p. 84]. □

If we take $m = 1$, $\boldsymbol{\alpha} = -1/\alpha$, with $\alpha > 0$ in part (i) of the previous Theorem, we recover Theorem 5.1.

Now we present an extrapolation result involving the GLZ spaces $L_{1,1;\boldsymbol{\alpha}}(R)$, with $\boldsymbol{\alpha} \in \mathcal{R}^m_+$.

Theorem 5.3. *Let (R_0, μ_0) and (R_1, μ_1) be non-atomic finite measure spaces. Let $m \in \mathbb{N}$, $j_0 \geq 2$ and $\boldsymbol{\alpha}, \boldsymbol{\beta} \in \mathcal{R}^m_+$. Suppose A is an operator whose domain is $\mathcal{M}_0(R_0, \mu_0)$ and whose range is contained in $\mathcal{M}(R_1, \mu_1)$ such that:*

(i) for every possible representation of $f \in \mathcal{M}_0(R_0, \mu_0)$ by $f = \sum_{j=j_0}^{+\infty} f_j$ (convergent μ_0-a.e. on R_0), with $\{f_j\}_j \subset \mathcal{M}_0(R_0, \mu_0)$, we have $\sum_{j=j_0}^{+\infty} Af_j$ convergent $\mu_1 - a.e.$ on R_1 and the inequality

$$|Af| \leq |\sum_{j=j_0}^{+\infty} Af_j| \quad \mu_1 - a.e. \text{ on } R_1; \tag{5.1}$$

(ii) for all $p \in (1, +\infty)$ and all $f \in L_p(R_0)$,

$$\|Af\|_{p;R_1} \leq c\, \omega^m_\beta(\frac{1}{p-1})\, \|f\|_{p;R_0}, \tag{5.2}$$

where c is independent of f, p and $\boldsymbol{\beta}$.

Then

$$\|Af\|_{1,1;\boldsymbol{\alpha};R_1} \leq c'\|f\|_{1,1;\boldsymbol{\alpha}+\boldsymbol{\beta};R_0}, \tag{5.3}$$

for all $f \in L_{1,1;\boldsymbol{\alpha}+\boldsymbol{\beta}}(R_0)$, for some constant c' independent of f, $\boldsymbol{\alpha}$ and $\boldsymbol{\beta}$.

Proof. Let $j_0 \geq 2$. Fix $f \in L_{1,1;\boldsymbol{\alpha}+\boldsymbol{\beta}}(R_0)$ and $f = \sum_{j=j_0}^{+\infty} f_j$, with

$$\sum_{j=j_0}^{+\infty} \omega_{\alpha+\beta}^m(j) \|f_j\|_{j',R_0} < +\infty. \tag{5.4}$$

We remark that $\sum_{j=j_0}^{+\infty} Af_j$ converges $\mu_1 - a.e.$ on R_1, because by Hölder's inequality and (5.2) we get

$$\sum_{j=j_0}^{+\infty} \int_{R_1} |Af_j| \, d\mu_1 \quad \leq c \sum_{j=j_0}^{+\infty} \omega_{\alpha+\beta}^m(j) \|f_j\|_{j',R_0},$$

and the rest it follows from (5.4) and from Theorem 1.38 in [Rud86, p. 29].

Now, by (5.1), (5.2) and Theorem 4.2,

$$\begin{aligned}
\|Af\|_{1,1;\boldsymbol{\alpha};R_1} &\leq \|\sum_{j=j_0}^{+\infty} Af_j\|_{1,1;\boldsymbol{\alpha};R_1} \leq c_1 \sum_{j=j_0}^{+\infty} \omega_{\alpha}^m(j) \|Af_j\|_{j',R_1} \\
&\leq c_2 \sum_{j=j_0}^{+\infty} \omega_{\alpha}^m(j) \, \omega_{\beta}^m(\frac{1}{j'-1}) \|f_j\|_{j',R_0} \\
&\leq c_2 \sum_{j=j_0}^{+\infty} \omega_{\alpha+\beta}^m(j) \|f_j\|_{j',R_0}. \qquad (5.5)
\end{aligned}$$

Taking the infimum over all the decompositions of f we get (5.3). □

Remark 5.1. *In the Theorem above we only need the condition (5.2) be satisfied for all p such that $1 < p \leq p_0$, for some $p_0 \in (1, +\infty)$, because in that case we can consider j_0 large enough. We could also replace (5.2) by the condition*

$$\|Af\|_{p;R_1} \leq c \, \omega_{\beta}^m(\frac{p}{p-1}) \|f\|_{p;R_0},$$

for all $p \in (1, +\infty)$ (or $p \in (1, p_0]$) and for all $f \in L_p(R_0)$, where c is independent of f, p and $\boldsymbol{\beta}$.

Since the Hardy-Littlewood maximal operator satisfies part (i) of the previous Theorem trivially and condition (5.2) with $m = 1$ and $\boldsymbol{\beta} = 1$, we recover the result already known for the maximal operator, *i.e.*

$$M : L^1(\log L)^{a+1}(\Omega) \longrightarrow L^1(\log L)^a(\Omega),$$

and

$$\|Mf|L^1(\log L)^a(\Omega)\| \leq c_2 \|f|L^1(\log L)^{a+1}(\Omega)\|,$$

for all $f \in L^1(\log L)^{a+1}(\Omega)$, where $a > 0$; see the literature mentioned in the Introduction.

Acknowledgements: It is a pleasure to thank Prof. D. E. Edmunds for his helpful suggestions during the preparation of this paper. The author is also grateful to Calouste Gulbenkian Foundation and to Mathematics Department of the University of Coimbra for the financial support.

References

[Ada75] R. A. Adams. *Sobolev Spaces*. Academic Press, New York, 1975.

[BR80] C. Bennett and K. Rudnick. On Lorentz-Zygmund spaces. *Dissertationes Math. (Rozprawy Mat.)*, 175:1–72, 1980.

[BS88] C. Bennett and R. Sharpley. *Interpolation of Operators*, volume 129 of *Pure and Applied Mathematics*. Academic Press, New York, 1988.

[EGO95] D. E. Edmunds, P. Gurka, and B. Opic. Double exponential integrability of convolution operators in generalised Lorentz-Zygmund spaces. *Indiana Univ. Math. J.*, 44:19–43, 1995.

[EGO97] D. E. Edmunds, P. Gurka, and B. Opic. On embeddings of logarithmic Bessel potential spaces. *J. Funct. Anal.*, 146:116–150, 1997.

[EGO98] D. E. Edmunds, P. Gurka, and B. Opic. Norms of embeddings of logarithmic Bessel potential spaces. *Proc. Amer. Math. Soc.*, 126(8):2417–2425, 1998.

[EK] D. E. Edmunds and M. Krbec. On Decomposition in Exponential Orlicz Spaces. *Math. Nachr.*, To appear.

[ET96] D. E. Edmunds and H. Triebel. *Function Spaces, Entropy Numbers and Differential Operators*, volume 120 of *Cambridge Tracts in Mathematics*. Cambridge University Press, 1996.

[FK98] A. Fiorenza and M. Krbec. On decompositions in $L(\log L)^{\alpha}$. Preprint n. 129. Academy of Sciences of the Czech Republic, 1998.

[GO98] P. Gurka and B. Opic. Global limiting embeddings of logarithmic Bessel potential spaces. *Math. Inequal. Appl.*, 1:565–584, 1998.

[JM91] B. Jawerth and M. Milman. Extrapolation theory with applications. *Mem. Amer. Math. Soc.*, 89(440), 1991.

[KJF77] A. Kufner, O. John, and S. Fučík. *Function Spaces*. Noordhoff International Publishing, Leyden. Academia, Publishing House of the Czechoslovak Academy of Sciences, Prague, 1977.

[Lor51] G. G. Lorentz. Some new functional spaces. *Ann. Math.*, 1:411–429, 1951.

[Mil94] M. Milman. *Extrapolation and optimal decomposition*, volume 1580 of *Lecture Notes in Mathematics*. Springer Verlag, Berlin, Heidelber, 1994.

[Rud86] W. Rudin. *Real and Complex Analysis*. Mcgraw-Hill Book Co., Singapore, 3^{rd} edition, 1986.

[Ste70] E. M. Stein. *Singular Integrals and Differentiability Properties of Functions*. Princeton University Press, Princeton, New Jersey, 1970.

[Tor86] A. Torchinsky. *Real Variable Methods in Harmonic Analysis*, volume 123 of *Pure and Applied Mathematics*. Academic Press, San Diego, 1986.

[Tri78] H. Triebel. *Interpolation Theory, Function Spaces, Differential Operators*, volume 18 of *North-Holland Mathematical Library*. VEB Deutscher Verlag der Wissenschaften, Berlin 1978; North-Holland Publishing Co., 1978.

[Tri93] H. Triebel. Approximation numbers and entropy numbers of embeddings of fractional Besov-Sobolev spaces in Orlicz spaces. *Proc. London Math. Soc. (3)*, 66(3):589–618, 1993.

[Yan51] S. Yano. Notes on Fourier analysis (XXIX): An Extrapolation Theorem. *J. Math. Soc. Japan*, 3, 1951.

[Zie89] W. P. Ziemer. *Weakly Differentiable Functions*. Springer-Verlag, 1989.

[Zyg59] A. Zygmund. *Trigonometric Series*, volume II. Cambridge University Press, Cambridge, 2^{nd} edition, 1959.

Júlio S. Neves, Departamento de Matemática da FCT, Universidade de Coimbra, Apartado 3008, 3000 Coimbra, Portugal

Current address: Júlio S. Neves, P/G Pigeonholes, School of Mathematical Sciences, University of Sussex, Falmer, Brighton, BN1 9QH, UK

E-mail address: `jsn@mat.uc.pt` or `J.S.Neves@sussex.ac.uk`