SOME PROPERTIES OF THE SPACES $F_{pq}^{(s,\Psi)}(I\!R^n)$ AND $B_{pq}^{(s,\Psi)}(I\!R^n)$

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1. Introduction

Our aim is to present a systematic study of some properties for the so called spaces of generalized smoothness $B_{pq}^{(s,\Psi)}(I\!R^n)$ and $F_{pq}^{(s,\Psi)}(I\!R^n)$, specially concerning atomic and subatomic decompositions.
The spaces $B_{pq}^{(s,\Psi)}(I\!R^n)$ were introduced by Edmunds and Triebel, in [ET98], in the context of a spectral theory for isotropic fractal drums. For $\Psi(t) = (1 + |\log t|)^b$, these spaces were already used by Leopold, in [Leo98], to get compactness of some limiting embeddings between functions spaces on bounded domains.
In an analogous way as $B_{pq}^{(s,\Psi)}(I\!R^n)$ were introduced one can also define the spaces $F_{pq}^{(s,\Psi)}(I\!R^n)$. Both of these spaces coincide with the usual Besov and Triebel-Lizorkin spaces $B_{pq}^{s}(I\!R^n)$ and $F_{pq}^{s}(I\!R^n)$, respectively, when $\Psi = 1$. Now a new parameter Ψ is coming, but s remains to be the main smoothness whether Ψ stands for a finer tuning.
Spaces of generalized smoothness were introduced in a different approach by Kalyabin in 1977. A survey has been given in [Kal87]. More historical references to this subject can be found in [Leo98].

2. Definitions and Basic Properties

2.1. Notation. As usual, $I\!R^n$ denotes the n-dimensional real euclidean space, $I\!N$ the collection of all natural numbers and $I\!N_0 = I\!N \cup \{0\}$. If $\alpha = (\alpha_1, \ldots, \alpha_n) \in I\!N_0^n$ is a multi-index its length is $|\alpha| = \alpha_1 + \cdots + \alpha_n$, the derivatives D^α have the usual meaning and if $x = (x_1, \ldots, x_n) \in I\!R^n$ then $x^\alpha = x_1^{\alpha_1} \ldots x_n^{\alpha_n}$. Let $\mathcal{S}(I\!R^n)$ be the Schwartz space of all complex-valued rapidly decreasing infinitely differentiable functions on $I\!R^n$ equipped with the usual topology. By $\mathcal{S}'(I\!R^n)$ we denote its topological dual, the space of all tempered distributions on $I\!R^n$. If $\varphi \in \mathcal{S}(I\!R^n)$ then

$$\widehat{\varphi}(\xi) = (\mathcal{F}\varphi)(\xi) = (2\pi)^{-\frac{n}{2}} \int_{I\!R^n} e^{-ix\xi}\, \varphi(x)\, dx, \quad \xi \in I\!R^n, \tag{2.1}$$

denotes the Fourier transform of φ. As for $\mathcal{F}^{-1}\varphi$ or $\check{\varphi}$, it stands for the inverse Fourier transform, given by the right-hand side of (2.1) with i in place of $-i$. Of course, $x\xi$ denotes the scalar product on $I\!R^n$. Both $\mathcal{F}$ and $\mathcal{F}^{-1}$ are extended to $\mathcal{S}'(I\!R^n)$ in the standard way. Furthermore, $L_p(I\!R^n)$ with $0 < p \leq \infty$, denotes the usual quasi-Banach space with respect to the Lebesgue measure, quasi-normed by

$$\|f \mid L_p(I\!R^n)\| = \left(\int_{I\!R^n} |f(x)|^p \, dx \right)^{\frac{1}{p}}$$

with the usual modification if $p = \infty$. The equivalence $\sim$ in $a_k \sim b_k$ or $\varphi(x) \sim \psi(x)$ means that there are two positive constants c_1 and c_2 such that

$$c_1\, a_k \leq b_k \leq c_2\, a_k \qquad \text{or} \qquad c_1\, \varphi(x) \leq \psi(x) \leq c_2\, \varphi(x)$$

for all admitted values of the discrete variable k or the continuous variable x.

2.2. Definitions. Let φ_0 be a C^∞ function on $I\!R^n$ with

$$\text{supp}\,\varphi_0 \subset \{\xi \in I\!R^n : |\xi| \le 2\}, \quad \varphi_0(\xi) = 1 \text{ if } |\xi| \le 1. \tag{2.2}$$

Let $j \in I\!N$ and

$$\varphi_j(\xi) = \varphi_0(2^{-j}\xi) - \varphi_0(2^{-j+1}\xi), \quad \xi \in I\!R^n. \tag{2.3}$$

Then, since

$$\text{supp}\,\varphi_j \subset \{\xi \in I\!R^n : 2^{j-1} \le |\xi| \le 2^{j+1}\}, \quad j \in I\!N, \tag{2.4}$$

and

$$\sum_{j=0}^{\infty} \varphi_j(\xi) = 1 \text{ for all } \xi \in I\!R^n, \tag{2.5}$$

the $(\varphi_j)_{j\in I\!N_0}$ is a smooth a dyadic resolution of unity. For any $f \in \mathcal{S}'(I\!R^n)$, by the Paley-Wiener-Schwartz theorem, $(\varphi_k \widehat{f})^\vee$, $k \in I\!N_0$, is an entire analytic function on $I\!R^n$. In particular $(\varphi_k \widehat{f})^\vee(x)$ makes sense pointwise. Moreover

$$f = \sum_{j=0}^{\infty} (\varphi_j \widehat{f})^\vee \quad (\text{convergence in } \mathcal{S}'(I\!R^n)).$$

Definition 2.1. *A positive monotone function Ψ on the interval $(0,1]$ is called admissible if*

$$\Psi(2^{-j}) \sim \Psi(2^{-2j}); \quad j \in I\!N_0.$$

Remark 2.2. *Let $0 < c < 1$ and $b \in I\!R$. Then*

$$\Psi(x) = |log\, cx|^b, \quad 0 < x \le 1,$$

is an admissible function.

Remark 2.3. *Let Ψ be an admissible function according to Definition 2.1. If Ψ is increasing, there exists $\theta \in I\!R_0^+$ such that*

$$\Psi(2^{-2j}) \le \Psi(2^{-j}) \le 2^\theta\, \Psi(2^{-2j}) \le 2^{\theta k}\, \Psi(2^{-2^k j}),\ j \in I\!N_0,\ k \in I\!N. \tag{2.6}$$

If Ψ is decreasing, there exists $\theta' \in I\!R_0^+$ such that

$$\Psi(2^{-j}) \le \Psi(2^{-2^k j}) \le 2^{\theta' k}\, \Psi(2^{-j}),\ k, j \in I\!N_0. \tag{2.7}$$

Proposition 2.4. *Let Ψ be an admissible function according to Definition 2.1.*
(i) Let $\chi \in I\!R$. Then Ψ^χ is also admissible.
(ii) There are positive numbers c_1, c_2, b and c, with $c \in (0,1)$ such that

$$c_1\, |log\, cx|^{-b} \le \Psi(x) \le c_2\, |log\, cx|^b, \quad x \in (0,1]. \tag{2.8}$$

Proof. Part (i) is obvious. To prove (ii) it is sufficient to prove it for a decreasing admissible function Ψ (the other case then follows using (i) and $\chi = -1$). Recall that we have (2.7). Let

$$2^{-2^{k+1}} \le x \le 2^{-2^k}, \quad \text{for some} \quad k \in I\!N_0.$$

Then, with $b = \theta'$, we have in one hand

$$\Psi(x) \le \Psi(2^{-2^{k+1}}) \le 2^{bk}\, \Psi(2^{-2}) \le \Psi(2^{-2})\, |\log x|^b,$$

and in another hand

$$\Psi(x) \ge \Psi(2^{-2^k}) \ge 2^{-bk}\, \Psi(2^{-2^k 2^k}) \ge 2^{-bk}\, \Psi(1) \ge \Psi(1)\, |\log x|^{-b}.$$

∎

Definition 2.5. *(i) Let $0 < p \le \infty$, $0 < q \le \infty$, $s \in I\!R$ and Ψ an admissible function according to Definition 2.1. Then $B_{pq}^{(s,\Psi)}(I\!R^n)$ is the collection of all $f \in \mathcal{S}'(I\!R^n)$ such that*

$$\|f \,|\, B_{pq}^{(s,\Psi)}(I\!R^n)\| = \left(\sum_{j=0}^{\infty} 2^{sjq}\, \Psi(2^{-j})^q\, \|(\varphi_j \widehat{f})^\vee \,|\, L_p(I\!R^n)\|^q \right)^{\frac{1}{q}} \tag{2.9}$$

(with usual modification if $q = \infty$) is finite.
(ii) Let $0 < p < \infty$, $0 < q \le \infty$, $s \in I\!R$ and Ψ an admissible function according to Definition 2.1. Then $F_{pq}^{(s,\Psi)}(I\!R^n)$ is the collection of all $f \in \mathcal{S}'(I\!R^n)$ such that

$$\|f \,|\, F_{pq}^{(s,\Psi)}(I\!R^n)\| = \left\| \left(\sum_{j=0}^{\infty} 2^{sjq}\, \Psi(2^{-j})^q\, |(\varphi_j \widehat{f})^\vee|^q \right)^{\frac{1}{q}} \,|\, L_p(I\!R^n) \right\| \tag{2.10}$$

(with usual modification if $q = \infty$) is finite.

Remark 2.6. *If $\Psi = 1$ then the spaces $B_{pq}^{(s,\Psi)}(I\!R^n)$ and $F_{pq}^{(s,\Psi)}(I\!R^n)$ coincide with the usual Besov and Triebel-Lizorkin spaces, $B_{pq}^{s}(I\!R^n)$ and $F_{pq}^{s}(I\!R^n)$, respectively. The theory of these last spaces has been developed in its full extend in* [Tri83] *and* [Tri92]*. As far as more recent topics are concerned we refer to* [ET96], [RuS96] *and* [Tri97]*. If $\Psi(x) = (1 + |log x|)^b$, $b \in I\!R$, we obtain the spaces $B_{pq}^{s,b}(I\!R^n)$ used by Leopold in* [Leo98].
Both $B_{pq}^{(s,\Psi)}(I\!R^n)$ and $F_{pq}^{(s,\Psi)}(I\!R^n)$ are quasi-Banach spaces which are independent of the particular function $\varphi_0 \in \mathcal{S}(I\!R^n)$ chosen according to (2.2), in the sense of equivalent quasi-norms. This justifies our omission of subscript φ_0 in (2.9) and (2.10). If $p \ge 1$ and $q \ge 1$, then $B_{pq}^{(s,\Psi)}(I\!R^n)$ and $F_{pq}^{(s,\Psi)}(I\!R^n)$ are Banach spaces. These properties we get in the usual way, using the multiplier theorem 1.6.3 in [Tri83] *and the properties of the admissible function Ψ.*
As for $B_{pq}^{s}(I\!R^n)$ and $F_{pq}^{s}(I\!R^n)$, see Theorem 2.3.3 in [Tri83]*, the embeddings $\mathcal{S}(I\!R^n) \subset B_{pq}^{(s,\Psi)}(I\!R^n) \subset \mathcal{S}'(I\!R^n)$ and $\mathcal{S}(I\!R^n) \subset F_{pq}^{(s,\Psi)}(I\!R^n) \subset \mathcal{S}'(I\!R^n)$ hold true for all admissible values of p, q, s and Ψ. If in addition $0 < p < \infty$ and $0 < q < \infty$, then $\mathcal{S}(I\!R^n)$ is dense in $B_{pq}^{(s,\Psi)}(I\!R^n)$ and in $F_{pq}^{(s,\Psi)}(I\!R^n)$.*

2.3. Equivalent quasi-norms. Let $(\varphi_k)_{k \in I\!N_0}$ be the same system of functions as in 2.2 with the generating function φ_0. We introduce the maximal functions

$$(\varphi_k^* f)_a(x) = \sup_{z \in I\!R^n} \frac{|(\varphi_k \widehat{f})^\vee(x-z)|}{1 + |2^k z|^a}, \quad f \in \mathcal{S}'(I\!R^n), \quad a > 0, \tag{2.11}$$

where $x \in I\!R^n$ and $k \in I\!N_0$. The result below is the counterpart of Theorem 2.3.2 in [Tri92] for the spaces $B_{pq}^{(s,\Psi)}(I\!R^n)$ and $F_{pq}^{(s,\Psi)}(I\!R^n)$, and it is also obtained as a simple consequence of Theorem 1.6.2 in [Tri83].

Theorem 2.7. *(i) Let $0 < p \le \infty$, $0 < q \le \infty$, $s \in I\!R$, Ψ an admissible function as in Definition 2.1 and $a > \frac{n}{p}$. Then*

$$B_{pq}^{(s,\Psi)}(I\!R^n) = \left\{ f \in \mathcal{S}'(I\!R^n) : \left(\sum_{j=0}^{\infty} 2^{sjq}\, \Psi(2^{-j})^q\, \left\|(\varphi_j^* f) \,|\, L_p(I\!R^n)\right\|^q \right)^{\frac{1}{q}} < \infty \right\} \tag{2.12}$$

(with modification if $q = \infty$) in the sense of equivalent quasi-norms.
(ii) Let $0 < p < \infty$, $0 < q \le \infty$, $s \in I\!R$, Ψ an admissible function as in Definition 2.1 and

$a > \frac{n}{\min(p,q)}$. *Then*

$$(2.13)\ F^{(s,\Psi)}_{pq}(I\!R^n) = \left\{ f \in \mathcal{S}'(I\!R^n) : \left\| \left(\sum_{j=0}^{\infty} 2^{sjq}\, \Psi(2^{-j})^q \, |(\varphi_j^* f)_a(.)|^q \right)^{\frac{1}{q}} | \, L_p(I\!R^n) \right\| < \infty \right\}$$

(with modification if $q = \infty$) in the sense of equivalent quasi-norms.

2.4. Lifting Property and Embeddings. Let $\sigma \in I\!R$, then

$$I_\sigma : f \to \left((1+|\xi|^2)^{\frac{\sigma}{2}} \widehat{f} \right)^{\vee}, \tag{2.14}$$

is a one-to-one map of $\mathcal{S}(I\!R^n)$ onto itself and of $\mathcal{S}'(I\!R^n)$ onto itself. Obviously $I_\sigma I_\eta = I_{\sigma+\eta}$. As for the B and F spaces, I_σ acts as a lift.

Theorem 2.8. *Let $s \in I\!R$, $\sigma \in I\!R$, $0 < q \leq \infty$ and Ψ an admissible function according to Definition* 2.1.
(i) Let $0 < p \leq \infty$. Then I_σ maps $B^{(s,\Psi)}_{pq}(I\!R^n)$ isomorphically onto $B^{(s-\sigma,\Psi)}_{pq}(I\!R^n)$ and $\|I_\sigma f \,|\, B^{(s-\sigma,\Psi)}_{pq}(I\!R^n)\|$ is an equivalent quasi-norm on $B^{(s,\Psi)}_{pq}(I\!R^n)$.
(ii) Let $0 < p < \infty$. Then I_σ maps $F^{(s,\Psi)}_{pq}(I\!R^n)$ isomorphically onto $F^{(s-\sigma,\Psi)}_{pq}(I\!R^n)$ and $\|I_\sigma f \,|\, F^{(s-\sigma,\Psi)}_{pq}(I\!R^n)\|$ is an equivalent quasi-norm on $F^{(s,\Psi)}_{pq}(I\!R^n)$.

Proof. We prove (ii). Let $f \in F^{(s,\Psi)}_{pq}(I\!R^n)$. We have

$$\|I_\sigma f \,|\, F^{(s-\sigma,\Psi)}_{pq}(I\!R^n)\| = \left\| \left\{ 2^{(s-\sigma)j}\, \Psi(2^{-j})\, (\varphi_j\, (1+|\xi|^2)^{\frac{\sigma}{2}}\, \widehat{f})^{\vee} \right\}_{j \in I\!N_0} | \, L_p(l_q) \right\|. \tag{2.15}$$

Let $\phi \in \mathcal{S}(I\!R^n)$ with

$$\phi(x) = 1 \quad \text{if} \quad \frac{1}{2} \leq |x| \leq 2 \qquad \text{and} \quad \operatorname{supp} \phi \subset \{\xi : \frac{1}{4} \leq |\xi| \leq 4\}.$$

Then

$$(\varphi_j\, (1+|\xi|^2)^{\frac{\sigma}{2}}\, \widehat{f})^{\vee} = \left((1+|\xi|^2)^{\frac{\sigma}{2}}\, \phi(2^{-j}\xi)\, (\varphi_j\, \widehat{f}) \right)^{\vee}, \quad j \in I\!N.$$

Applying the multiplier theorem in [Tri83, 1.6.3, p.31], with $\eta \in I\!N$ such that $\eta > \frac{n}{2} + \frac{n}{\min(p,q)}$ and

$$M_j(\xi) = 2^{-\sigma j}\, (1+|\xi|^2)^{\frac{\sigma}{2}}\, \phi(2^{-j}\xi)$$

we get

$$\begin{aligned} &\left\| \left\{ 2^{(s-\sigma)j}\, \Psi(2^{-j})\, (\varphi_j\, (1+|\xi|^2)^{\frac{\sigma}{2}}\, \widehat{f})^{\vee} \right\}_{j \in I\!N} | \, L_p(l_q) \right\| \\ &\leq c_1 \sup_{l \in I\!N} \|M_l(2^{l+2}.) \,|\, H_2^\eta\| \, \left\| 2^{sj}\, \Psi(2^{-j})\, (\varphi_j\, \widehat{f})^{\vee} \,|\, L_p(l_q) \right\|. \end{aligned} \tag{2.16}$$

For a multi-index $\alpha \in I\!N_0^n$ with $|\alpha| \leq \eta$ we have

$$\begin{aligned} |D^\alpha[M_l(2^{l+2}.)](x)| &\leq 2^{2\sigma} \sum_{\beta \leq \alpha} \binom{\alpha}{\beta} |D^\beta[(2^{-2(l+2)} + |x|^2)^{\frac{\sigma}{2}}]|\, |(D^{\alpha-\beta}\phi)(4x)|\, 4^{|\alpha-\beta|} \\ &\leq c_2 \sum_{\beta \leq \alpha} \binom{\alpha}{\beta} |D^\beta[(2^{-2(l+2)} + |x|^2)^{\frac{\sigma}{2}}]|. \end{aligned}$$

But

$$|D^\beta[(2^{-2(l+2)} + |x|^2)^{\frac{\sigma}{2}}]| \leq c_\sigma^\beta\, |x|^{|\beta|}\, (2^{-2(l+2)} + |x|^2)^{\frac{\sigma}{2} - |\beta|},$$

and for $x \in \operatorname{supp} M_l(2^{l+2}.)$ we have $\frac{1}{16} \leq |x| \leq 1$. Hence using $|x| \leq 1$ for the values of β such that $\frac{\sigma}{2} - |\beta| \geq 0$ or $|x| \geq \frac{1}{16}$ for the other values of β one would come to

$$\sup_{l \in I\!N} \|M_l(2^{l+2}.) \,|\, H_2^{\eta}\| < \infty.$$

This in (2.16) together with the term corresponding to $j = 0$, that can be treated in a similar way, gives

$$\|I_\sigma\, f \,|\, F^{(s-\sigma,\Psi)}_{pq}(I\!R^n)\| \leq c_3\, \|f \,|\, F^{(s,\Psi)}_{pq}(I\!R^n)\|.$$

Observing that $I_\sigma\, I_{-\sigma}\, f = f$ the proof is complete. ■

We finish this section with some embedding assertions. This is the counterpart of Proposition 2.3.2/2 and Theorem 2.7.1 in [Tri83]. In the following $A_1 \subset A_2$ always means that the quasi-normed space A_1 is continuously embedded in the quasi-normed space A_2, i.e. there exists a constant c such that $\|a \,|\, A_2\| \leq c\,\|a \,|\, A_1\|$ holds for all $a \in A_1$.

Proposition 2.9. *(i) Let $0 < p \leq \infty$, $0 < q_0 \leq q_1 \leq \infty$, $s \in I\!R$ and Ψ an admissible function according to Definition 2.1. Then*

$$B^{(s,\Psi)}_{pq_0}(I\!R^n) \subset B^{(s,\Psi)}_{pq_1}(I\!R^n).$$

Analogous for F-spaces, with $0 < p < \infty$.
(ii) Let $0 < p \leq \infty$, $0 < q_0 \leq \infty$, $0 < q_1 \leq \infty$, $s \in I\!R$, $\varepsilon > 0$, Ψ and $\widetilde{\Psi}$ admissible functions according to Definition 2.1. Then

$$B^{(s+\varepsilon,\Psi)}_{pq_0}(I\!R^n) \subset B^{(s,\widetilde{\Psi})}_{pq_1}(I\!R^n).$$

Analogous for F-spaces, with $0 < p < \infty$.
(iii) Let $0 < q \leq \infty$, $0 < p < \infty$, $s \in I\!R$ and Ψ an admissible function according to Definition 2.1. Then

$$B^{(s,\Psi)}_{p\,\min(p,q)}(I\!R^n) \subset F^{(s,\Psi)}_{pq}(I\!R^n) \subset B^{(s,\Psi)}_{p\,\max(p,q)}(I\!R^n).$$

(iv) Let $0 < p_0 \leq p_1 < \infty$, $0 < q \leq \infty$, Ψ an admissible function according to Definition 2.1 and $s_0, s_1 \in I\!R$ with

$$s_0 - \frac{n}{p_0} = s_1 - \frac{n}{p_1}.$$

Then

$$B^{(s_0,\Psi)}_{p_0 q}(I\!R^n) \subset B^{(s_1,\Psi)}_{p_1 q}(I\!R^n).$$

(v) Let $0 < p_0 < p_1 < \infty$, $0 < q_0 \leq \infty$, $0 < q_1 \leq \infty$, Ψ an admissible function according to Definition 2.1 and $s_0, s_1 \in I\!R$ with

$$s_0 - \frac{n}{p_0} = s_1 - \frac{n}{p_1}.$$

Then

$$F^{(s_0,\Psi)}_{p_0 q_0}(I\!R^n) \subset F^{(s_1,\Psi)}_{p_1 q_1}(I\!R^n).$$

(vi) Let $0 < p \leq \infty$, $0 < q \leq \infty$, $s \in I\!R$ and Ψ an admissible function according to Definition 2.1. Then

$$B^{(s,\Psi)}_{pq}(I\!R^n) \subset B^{s}_{pq}(I\!R^n), \quad \text{if } \Psi \text{ is decreasing}$$

or

$$B^{(s,\Psi)}_{pq}(I\!R^n) \subset B^{s-\varepsilon}_{pq}(I\!R^n), \quad \text{if } \Psi \text{ is increasing},$$

for any $\varepsilon > 0$. Analogous for F-spaces, with $0 < p < \infty$.

Proof. As for (i), (iii) and (v) one can simply follow the proof in case $\Psi \equiv 1$ in [Tri83] and inserting the factor $\Psi(2^{-j})$. As for (iv) we can do the same as before with $s_1 = 0$ and then use the lift according to Proposition 2.8.
For the proof of (ii) for B-spaces (similar for F-spaces): from part (i) of this proposition, it is enough to take $q_0 = \infty$. Since Ψ and $\widetilde{\Psi}$ are admissible functions, by Proposition 2.4, there exist positive constants c_1, c_2, $\widetilde{c}_1$, $\widetilde{c}_2$, b and $\widetilde{b}$ such that

$$c_1\, j^{-b} \le \Psi(2^{-j}) \le c_2\, j^{b} \quad \text{and} \quad \widetilde{c}_1\, j^{-\widetilde{b}} \le \widetilde{\Psi}(2^{-j}) \le \widetilde{c}_2\, j^{\widetilde{b}}, \quad j \in I\!N.$$

Let ε_1 be such that $0 < \varepsilon_1 < \varepsilon$. Then,

$$\begin{aligned} 2^{sj}\, \widetilde{\Psi}(2^{-j}) &= 2^{(s+\varepsilon)j}\, 2^{(\varepsilon_1-\varepsilon)j}\, 2^{-\varepsilon_1 j}\, \widetilde{\Psi}(2^{-j}) \\ &\le c_\varepsilon\, 2^{(s+\varepsilon)j}\, j^{-(b+\widetilde{b})}\, 2^{-\varepsilon_1 j}\, \widetilde{c}_2\, j^{\widetilde{b}} \le c_\varepsilon\, \frac{\widetilde{c}_2}{c_1}\, 2^{(s+\varepsilon)j}\, 2^{-\varepsilon_1 j}\, \Psi(2^{-j}). \end{aligned}$$

Hence

$$\begin{aligned} \|f \mid B^{(s,\widetilde{\Psi})}_{pq_1}(I\!R^n)\| &= \left(\sum_{j=0}^{\infty} 2^{sjq_1}\, \widetilde{\Psi}(2^{-j})^{q_1}\, \|(\varphi_j \widehat{f})^\vee \mid L_p(I\!R^n)\|^{q_1} \right)^{\frac{1}{q_1}} \\ &\le c'_\varepsilon \left(\sum_{j=0}^{\infty} 2^{-\varepsilon_1 q_1 j}\, 2^{(s+\varepsilon)jq_1}\, \Psi(2^{-j})^{q_1}\, \|(\varphi_j \widehat{f})^\vee \mid L_p(I\!R^n)\|^{q_1} \right)^{\frac{1}{q_1}} \\ &\le c'_\varepsilon \left(\sum_{j=0}^{\infty} 2^{-\varepsilon_1 q_1 j} \right)^{\frac{1}{q_1}} \sup_{j \in I\!N_0} 2^{(s+\varepsilon)j}\, \Psi(2^{-j})\, \|(\varphi_j \widehat{f})^\vee \mid L_p(I\!R^n)\| \\ &\le c''_\varepsilon\, \|f \mid B^{(s+\varepsilon,\Psi)}_{p\infty}(I\!R^n)\|. \end{aligned}$$

In what concerns the justification for (vi), we have

$$2^{(s-\varepsilon)j} = 2^{sj}\, 2^{-\varepsilon j} \le c_\varepsilon\, 2^{sj}\, j^{-b} \le c'_\varepsilon\, 2^{sj}\, \Psi(2^{-j}), \quad j \in I\!N,$$

and if in addition Ψ is decreasing, we have $2^{sj} \le \Psi(1)^{-1}\, 2^{sj}\, \Psi(2^{-j}), \quad j \in I\!N_0$.

■

3. Characterization by local means

Let $B = \{y \in I\!R^n : |y| \le 1\}$ be the unit ball in $I\!R^n$, and let k be a C^∞ function in $I\!R^n$ with $\operatorname{supp} k \subset B$, then we introduce the local means

$$k(t,f)(x) = \int_{I\!R^n} k(y) f(x+ty)\, dy, \quad x \in I\!R^n,\ t > 0, \tag{3.1}$$

which makes sense for any $f \in \mathcal{S}'(I\!R^n)$ (appropriately interpreted). Let k_0 and k^0 be two C^∞ functions in $I\!R^n$ with

$$\operatorname{supp} k_0 \subset B, \quad \operatorname{supp} k^0 \subset B \tag{3.2}$$

and

$$\widehat{k_0}(0) \ne 0, \quad \widehat{k^0}(0) \ne 0. \tag{3.3}$$

For $N \in I\!N$, we define

$$k(y) = \Delta^N k^0(y) = \left(\sum_{j=1}^{n} \frac{\partial^2}{\partial x_j^2} \right)^N k^0(y), \quad y \in I\!R^n. \tag{3.4}$$

Remark that

$$\check{k}(x) = |x|^{2N}\check{k}^0(x),\ x \in I\!R^n. \tag{3.5}$$

Theorem 3.1. *Let $0 < p < \infty$, $0 < q \le \infty$, $s \in I\!R$ and Ψ an admissible function as in Definition 2.1. Let $N \in I\!N_0$ with*

$$\begin{cases} 2N > s, & \text{if } \Psi \text{ is increasing} \\ 2N > s+\theta', & \text{if } \Psi \text{ is decreasing.} \end{cases} \tag{3.6}$$

Then, there exists $h \in I\!N_0$ such that

$$\|k_0(2^{-h}, f)\,|\,L_p(I\!R^n)\| + \left\| \left(\sum_{j=1}^{\infty} 2^{jsq}\,\Psi(2^{-j})^q\,|k(2^{-j}, f)(.)|^q \right)^{\frac{1}{q}} \,|\, L_p(I\!R^n) \right\| \tag{3.7}$$

(modification if $q=\infty$) is an equivalent quasi-norm in $F_{pq}^{(s,\Psi)}(I\!R^n)$.

Proof. The proof is divided in two cases (a) and (b), whether the admissible function Ψ is increasing or decreasing. The idea of the proof goes back to Theorem 2.4.1 in [Tri92].
(a) Suppose that the admissible function Ψ is monotone increasing and so we have (2.6). Notice that we always have

$$k(2^{-j}, f)(x) = (2\pi)^{\frac{n}{2}}(\check{k}(2^{-j}.)\widehat{f})^{\vee}(x), \quad x \in I\!R^n,\ j \in I\!N, \tag{3.8}$$

and an analogous equality for k_0.
Step 1. Let $f \in F_{pq}^{(s,\Psi)}(I\!R^n)$. In the first two steps we prove that the quasi-norm in (3.7) can be estimated from above by $c\,\|f\,|\,F_{pq}^{(s,\Psi)}(I\!R^n)\|$. Let $(\varphi_k)_{k\in I\!N_0}$ the dyadic resolution of unity introduced in 2.2 and let $\varphi_l = 0$ if $-l \in I\!N$. We write

$$\begin{aligned} 2^{sj}\,\Psi(2^{-j})\,k(2^{-j}, f)(x) &= (2\pi)^{\frac{n}{2}}\,2^{sj}\,\Psi(2^{-j}) \sum_{l=-\infty}^{M} (\check{k}(2^{-j}.)\varphi_{j+l}\widehat{f})^{\vee}(x) + \\ &\quad + (2\pi)^{\frac{n}{2}}\,2^{sj}\,\Psi(2^{-j}) \sum_{l=M+1}^{\infty} (\check{k}(2^{-j}.)\varphi_{j+l}\widehat{f})^{\vee}(x), \quad j \in I\!N, \end{aligned} \tag{3.9}$$

where $M \in I\!N$ will be chosen later on. We take for granted that the convergence in (3.9) is not only in $\mathcal{S}'(I\!R^n)$ but also pointwise a.e. (to prove later on in Step 3). We estimate the first sum in (3.9), where there is no problem of convergence, because is finite ($\varphi_{j+l} = 0$ if $l < -j$). Let

$$\widetilde{\varphi}_j(x) = |2^{-j}x|^{2N}\varphi_j(x), \quad j \in I\!N_0. \tag{3.10}$$

For $j \in I\!N$ and $l \in Z\!\!Z$ with $l \le M$, we have $j + l \le j + M \le j + 2^M \le 2^{M+1}j$, which gives

$$\Psi(2^{-j}) \le 2^{\theta(M+1)}\Psi(2^{-2^{M+1}j}) \le 2^{\theta(M+1)}\Psi(2^{-(j+l)}),$$

because Ψ is increasing and by (2.6). Recalling also (3.5), for $j \in I\!N$ we have

$$\begin{aligned} &\left| \sum_{l=-\infty}^{M} 2^{sj}\,\Psi(2^{-j}) \left(\check{k}(2^{-j}.)\varphi_{j+l}\widehat{f}\right)^{\vee}(x) \right| \\ &\le 2^{\theta(M+1)} \sum_{l=-\infty}^{M} 2^{(2N-s)l} \left| \left(\check{k}^0(2^{-j}z)2^{(j+l)s}\Psi(2^{-(j+l)})\widetilde{\varphi}_{j+l}(z)\widehat{f}\right)^{\vee}(x) \right|. \end{aligned} \tag{3.11}$$

But

$$\left|\left(\check{k}^0(2^{-j}z)2^{(j+l)s}\Psi(2^{-(j+l)})\widetilde{\varphi}_{j+l}(z)\widehat{f}\right)^{\vee}(x)\right|$$

$$\leq (2\pi)^{-\frac{n}{2}}\int_{I\!R^n}\left|k^0(-\xi)\right|\left|\left(2^{(j+l)s}\Psi(2^{-(j+l)})\widetilde{\varphi}_{j+l}\widehat{f}\right)^{\vee}(x-2^{-j}\xi)\right|\,d\xi. \tag{3.12}$$

Let a be a real number such that $a > \frac{n}{\min(p,q)}$. Obviously

$$|(\widetilde{\varphi}_{j+l}\widehat{f})^{\vee}(x-2^{-j}\xi)| \leq \left(\widetilde{\varphi}^*_{j+l}f\right)_a(x)(1+|2^l\xi|^a).$$

Using this in (3.12), leads us to

$$\left|\left(\check{k}^0(2^{-j}z)2^{(j+l)s}\Psi(2^{-(j+l)})\widetilde{\varphi}_{j+l}(z)\widehat{f}\right)^{\vee}(x)\right|$$

$$\leq (2\pi)^{-\frac{n}{2}}\,2^{(j+l)s}\,\Psi(2^{-(j+l)})\left(\widetilde{\varphi}^*_{j+l}f\right)_a(x)\int_{I\!R^n}\left|k^0(-\xi)\right|(1+|2^M\xi|^a)\,d\xi$$

$$\leq c_1\,2^{(j+l)s}\,\Psi(2^{-(j+l)})\left(\widetilde{\varphi}^*_{j+l}f\right)_a(x), \quad \text{for } \ l\leq M, \tag{3.13}$$

since $k^0 \in \mathcal{D}(I\!R^n)$. Putting (3.13) in (3.11), gives

$$\left|\sum_{l=-\infty}^{M} 2^{sj}\,\Psi(2^{-j})\left(\check{k}(2^{-j}.)\varphi_{j+l}\widehat{f}\right)^{\vee}(x)\right|$$

$$\leq c_1\,2^{\theta(M+1)}\sum_{l=-\infty}^{M} 2^{(2N-s)l}\,2^{(j+l)s}\,\Psi(2^{-(j+l)})\,\left(\widetilde{\varphi}^*_{j+l}f\right)_a(x), \quad j\in I\!N. \tag{3.14}$$

We apply first in (3.14) the l_q-quasi-norm with respect to j and afterwards the L_p-quasi-norm with respect to x. Because $2N > s$ we obtain

$$\left\|\left(\sum_{j=1}^{\infty}\left|\sum_{l=-\infty}^{M} 2^{sj}\,\Psi(2^{-j})\left(\check{k}(2^{-j}.)\varphi_{j+l}\widehat{f}\right)^{\vee}(x)\right|^q\right)^{\frac{1}{q}} |\, L_p(I\!R^n)\right\|$$

$$\leq c_2\left\|\left(\sum_{m=0}^{\infty} 2^{smq}\,\Psi(2^{-m})^q\,(\widetilde{\varphi}^*_m f)^q_a(x)\right)^{\frac{1}{q}} |\, L_p(I\!R^n)\right\|. \tag{3.15}$$

Using Theorem 2.2.4(i) in [Tri92], the right hand side of (3.15) can be estimated from above by

$$c_3\left\|\left(\sum_{m=0}^{\infty} 2^{smq}\,\Psi(2^{-m})^q|(\widetilde{\varphi}_m\widehat{f})^{\vee}(.)|^q\right)^{\frac{1}{q}} |\, L_p(I\!R^n)\right\|. \tag{3.16}$$

Applying now Theorem 2.2.4(ii) in [Tri92], (3.16) can be estimated from above by

$$c_4\left\|\left(\sum_{m=0}^{\infty} 2^{smq}\,\Psi(2^{-m})^q|(\varphi_m\widehat{f})^{\vee}(.)|^q\right)^{\frac{1}{q}} |\, L_p(I\!R^n)\right\| = c_4\,\|f\,|\,F^{(s,\Psi)}_{pq}(I\!R^n)\|. \tag{3.17}$$

Finally in (3.15), we obtain

$$\left\|\left(\sum_{j=1}^{\infty}\left|\sum_{l=-\infty}^{M} 2^{sj}\,\Psi(2^{-j})(\check{k}(2^{-j}.)\varphi_{j+l}\widehat{f})^{\vee}(.)\right|^q\right)^{\frac{1}{q}} |\, L_p(I\!R^n)\right\| \leq c_5\,\|f\,|\,F^{(s,\Psi)}_{pq}(I\!R^n)\|. \tag{3.18}$$

Step 2. We estimate the second sum in (3.9) and we have to make sure now that (3.9) converges a.e. and in some space $L_r(I\!R^n)$ with $1 \le r \le \infty$. However the latter comes as a by product. Let $s_0 \in I\!R$ be such that

$$s_0 + \theta + 2\sigma_{pq} < s, \tag{3.19}$$

and introduce

$$\varphi'_j(x) = |2^{-j}x|^{s_0}\varphi_j(x), \quad x \in I\!R^n,\ j \in I\!N. \tag{3.20}$$

For $j \in I\!N$ and $l \ge M+1 \ge 1$, $j + l \le j + 2^l \le 2^{l+1}j$, and then, since Ψ is increasing and by (2.6), we have

$$\Psi(2^{-j}) \le 2^{\theta(l+1)}\Psi(2^{-2^{l+1}j}) \le 2^{\theta(l+1)}\Psi(2^{-(j+l)}).$$

Also by (3.5) we have

$$\begin{aligned}
&\left|\sum_{l=M+1}^{\infty} 2^{sj}\,\Psi(2^{-j})\left(\check{k}(2^{-j}.)\varphi_{j+l}\widehat{f}\right)^{\vee}(x)\right| \\
&\le 2^{\theta}\sum_{l=M+1}^{\infty} 2^{(s_0+\theta-s)l}\left|\left(\check{k}^0(2^{-j}z)|2^{-j}z|^{2N-s_0}2^{(j+l)s}\,\Psi(2^{-(j+l)})\varphi'_{j+l}\widehat{f}\right)^{\vee}(x)\right|.
\end{aligned}\tag{3.21}$$

Let χ be a function in $\mathcal{D}(I\!R^n)$ such that

$$\chi(x) = 1 \quad \text{if} \quad \frac{1}{2} \le |x| \le 2 \qquad \text{and} \qquad \operatorname{supp}\chi \subset \{\xi \in I\!R^n : \frac{1}{4} \le |\xi| \le 4\}. \tag{3.22}$$

Each term in (3.21) can be estimated from above as follows

$$\begin{aligned}
&\left|\left(\check{k}^0(2^{-j}z)|2^{-j}z|^{2N-s_0}2^{(j+l)s}\Psi(2^{-(j+l)})\varphi'_{j+l}(z)\widehat{f}\right)^{\vee}(x)\right| \\
&\le (2\pi)^{-\frac{n}{2}}\int_{I\!R^n}\left|\left(\check{k}^0(2^{-j}z)|2^{-j}z|^{2N-s_0}\chi(2^{-j-l}z)\right)^{\vee}(y)\right|\left|\left(2^{(j+l)s}\Psi(2^{-(j+l)})\varphi'_{j+l}\widehat{f}\right)^{\vee}(x-y)\right|dy \\
&\le 2(2\pi)^{-\frac{n}{2}}2^{(j+l)s}\Psi(2^{-(j+l)})(\varphi'^{*}_{j+l}f)_a(x)\int_{I\!R^n}\left|\left(\frac{\check{k}(2^l.)}{|2^l.|^{s_0}}\chi(.)\right)^{\vee}(\xi)\right|(1+|\xi|)^a\,d\xi,
\end{aligned}\tag{3.23}$$

with a as in Step 1. Using Theorem 4.1 in [Far98] the integral in (3.23) can be estimated from above as follows

$$\begin{aligned}
&\int_{I\!R^n}\left|\left(\frac{\check{k}(2^l z)}{|2^l z|^{s_0}}\chi(z)\right)^{\vee}(\xi)\right|(1+|\xi|)^a\,d\xi \\
&\le c_1\,2^{-ls_0}\left\|\check{k}(2^l z)\chi(z)\,|\,H^{\lambda}_2(I\!R^n)\right\|\left\|(1+|\xi|)^a\left(\frac{h(z)}{|z|^{s_0}}\right)^{\vee}(\xi)\,|\,L_1(I\!R^n)\right\|,
\end{aligned}\tag{3.24}$$

where $h \in \mathcal{D}(I\!R^n)$ is such that

$$h(x) = 1 \quad \text{if} \quad \frac{1}{4} \le |x| \le 4 \qquad \text{and} \qquad \operatorname{supp}h \subset \{\xi \in I\!R^n : \frac{1}{8} \le |\xi| \le 8\}, \tag{3.25}$$

and $\lambda > a + \frac{n}{2}$. The second factor in (3.24) is obviously constant since $\frac{h(z)}{|z|^{s_0}} \in \mathcal{D}(I\!R^n)$. In what concerns the other factor in (3.24) it can be proved that

$$\sup_{l\in I\!N} 2^{-ls_0}\left\|\check{k}(2^l z)\chi(z)\,|\,H^{\lambda}_2\right\| < \infty, \tag{3.26}$$

(see Remark 3.2). From what has been said and from (3.23),(3.24), there exists a constant c_2 such that

$$\begin{aligned}&\left|\left(\check{k}^0(2^{-j}z)|2^{-j}z|^{2N-s_0}2^{(j+l)s}\Psi(2^{-(j+l)})\varphi'_{j+l}(z)\widehat{f}\right)^\vee(x)\right|\\ &\le c_2\,2^{(j+l)s}\,\Psi(2^{-(j+l)})(\varphi'^*_{j+l}f)_a(x).\end{aligned} \tag{3.27}$$

Applying (3.27) in (3.21) we obtain

$$\begin{aligned}&\left|\sum_{l=M+1}^{\infty} 2^{sj}\,\Psi(2^{-j})\left(\check{k}(2^{-j}.)\varphi_{j+l}\widehat{f}\right)^\vee(x)\right|\\ &\le c_2\,2^\theta\sum_{l=M+1}^{\infty}2^{(s_0+\theta-s)l}\,2^{(j+l)s}\,\Psi(2^{-(j+l)})\,(\varphi'^*_{j+l}f)_a(x),\quad j\in I\!N.\end{aligned} \tag{3.28}$$

We apply first in (3.28) the l_q-quasi-norm with respect to j and afterwards the L_p-quasi-norm with respect to x. Since $s > s_0+\theta$ we get

$$\begin{aligned}&\left\|\left(\sum_{j=1}^{\infty}\left|\sum_{l=M+1}^{\infty}2^{sj}\,\Psi(2^{-j})\left(\check{k}(2^{-j}.)\varphi_{j+l}\widehat{f}\right)^\vee(x)\right|^q\right)^{\frac{1}{q}}|\,L_p(I\!R^n)\right\|\\ &\le c_3\,2^{\frac{s_0+\theta-s}{2}M}\left\|\left(\sum_{m=1}^{\infty}2^{smq}\,\Psi(2^{-m})^q\,(\varphi'^*_m f)^q_a(x)\right)^{\frac{1}{q}}|\,L_p(I\!R^n)\right\|.\end{aligned} \tag{3.29}$$

Acting in a similar way as in Step 1, from (3.29) we obtain

$$\begin{aligned}&\left\|\left(\sum_{j=1}^{\infty}\left|\sum_{l=M+1}^{\infty}2^{sj}\,\Psi(2^{-j})\left(\check{k}(2^{-j}.)\varphi_{j+l}\widehat{f}\right)^\vee(x)\right|^q\right)^{\frac{1}{q}}|\,L_p(I\!R^n)\right\|\\ &\le c_4\,2^{\frac{s_0+\theta-s}{2}M}\,\|f\,|\,F^{(s,\Psi)}_{pq}(I\!R^n)\|.\end{aligned} \tag{3.30}$$

Now by (3.9), (3.18) and (3.30), we have

$$\left\|\left(\sum_{j=1}^{\infty}2^{sjq}\,\Psi(2^{-j})^q\,|k(2^{-j},f)(x)|^q\right)^{\frac{1}{q}}|\,L_p(I\!R^n)\right\|\le c_5\,\|f\,|\,F^{(s,\Psi)}_{pq}(I\!R^n)\|. \tag{3.31}$$

In an analogous way one can prove that

$$\|k_0(2^{-h},f)\,|\,L_p(I\!R^n)\|\le c_6\,\|f\,|\,F^{(s,\Psi)}_{pq}(I\!R^n)\|, \tag{3.32}$$

With (3.31) and (3.32) we have proved one of the desired inequalities between quasi-norm (3.7) and $\|.\,|\,F^{(s,\Psi)}_{pq}(I\!R^n)\|$.

Step 3. We have to care about the convergence on the right hand side of (3.9) pointwise a.e. and in some space $L_r(I\!R^n)$, $1\le r\le\infty$. We can rewrite (3.28) as follows

$$\begin{aligned}&\left|\sum_{l=M+1}^{L} 2^{sj}\,\Psi(2^{-j})\left(\check{k}(2^{-j}.)\varphi_{j+l}\widehat{f}\right)^\vee(x)\right|\\ &\le c_2\,2^\theta\sum_{l=M+1}^{L}2^{(s_0+\theta-s)l}\,2^{(j+l)s}\,\Psi(2^{-(j+l)})\,(\varphi'^*_{j+l}f)_a(x),\quad j\in I\!N,\end{aligned} \tag{3.33}$$

with $L > M$. Using $s_0 + \theta - s < 0$ and $l_q \hookrightarrow l_1$, if $0 < q \leq 1$, or Hölder inequality, if $1 < q \leq \infty$, we conclude that if M is large enough then the right hand side of (3.33) can be estimated from above by

$$\varepsilon \left(\sum_{l=M+1}^{\infty} 2^{lsq}\, \Psi(2^{-l})^q\, (\varphi_l'^* f)_a^q(x) \right)^{\frac{1}{q}}, \tag{3.34}$$

for given $\varepsilon > 0$. Because $f \in F_{pq}^{(s,\Psi)}(I\!R^n)$, Theorem 2.7 and considerations as those done in Step 2 give

$$\left\| \left(\sum_{m=1}^{\infty} 2^{smq}\, \Psi(2^{-m})^q (\varphi_m'^* f)_a(x) \right)^{\frac{1}{q}} \,|\, L_p(I\!R^n) \right\| < \infty.$$

Therefore, the expression in (3.34) is finite a.e. and this proves the desired pointwise convergence. Next we prove the $\mathcal{S}'$ convergence and assume $0 < p < 1$. Let $\sigma = s - \sigma_p$. Putting this in (3.33) we obtain

$$\begin{aligned} &\left| \sum_{l=M+1}^{L} 2^{\sigma j}\, \Psi(2^{-j})\, \left(\check{k}(2^{-j}.)\varphi_{j+l}\widehat{f} \right)^{\vee}(x) \right| \\ &\leq c_2\, 2^{\theta} \sum_{l=M+1}^{L} 2^{(s_0+\theta-\sigma)l}\, 2^{(j+l)\sigma}\, \Psi(2^{-(j+l)})\, (\varphi_{j+l}'^* f)_a(x). \end{aligned} \tag{3.35}$$

From (3.19), we have

$$\sigma = s - \sigma_p > s_0 + \theta.$$

Using this last inequality, instead of $s > s_0 + \theta$, and proceeding as to obtain (3.34) we conclude that there exists M sufficiently large such that the right hand side of (3.35) can be estimated from above by

$$\varepsilon \left(\sum_{l=M+1}^{\infty} 2^{l\sigma q}\, \Psi(2^{-l})^q\, (\varphi_l'^* f)_a^q(x) \right)^{\frac{1}{q}}, \tag{3.36}$$

for given $\varepsilon > 0$. From the embedding $F_{pq}^{(s,\Psi)}(I\!R^n) \subset F_{1q}^{(\sigma,\Psi)}(I\!R^n)$, consequence of Proposition 2.9(iv), we have

$$\left\| \left(\sum_{m=1}^{\infty} 2^{m\sigma q}\, \Psi(2^{-m})^q\, (\varphi_m'^* f)_a^q(.) \right)^{\frac{1}{q}} \,|\, L_1(I\!R^n) \right\| < \infty.$$

And then, from (3.35) and (3.36)

$$\begin{aligned} &\left\| \sum_{l=M+1}^{L} 2^{\sigma j}\, \Psi(2^{-j})\, \left(\check{k}(2^{-j}.)\varphi_{j+l}\widehat{f} \right)^{\vee}(.) \,|\, L_1(I\!R^n) \right\| \\ &\leq \varepsilon \left\| \left(\sum_{l=M+1}^{\infty} 2^{l\sigma q}\, \Psi(2^{-l})^q\, (\varphi_l'^* f)_a^q(.) \right)^{\frac{1}{q}} \,|\, L_1(I\!R^n) \right\|, \end{aligned}$$

for given $\varepsilon > 0$. It follows that (3.35) and hence (3.9) converges in $L_1(I\!R^n)$. If $1 \le p < \infty$, then by (3.33) and (3.34)

$$\left\| \sum_{l=M+1}^{L} 2^{sj}\, \Psi(2^{-j}) \left(\check{k}(2^{-j}.)\varphi_{j+l}\widehat{f}\right)^{\vee}(.) \,|\, L_p(I\!R^n) \right\|$$
$$\le \varepsilon \left\| \left(\sum_{l=M+1}^{\infty} 2^{lsq}\, \Psi(2^{-l})^q\, (\varphi_l'^{*} f)_a^q(.) \right)^{\frac{1}{q}} \,|\, L_p(I\!R^n) \right\|,$$

where $\varepsilon > 0$ is given. So (3.33) and hence (3.9) converges in $L_p(I\!R^n)$, therefore in $\mathcal{S}'(I\!R^n)$.

Step 4. Let $f \in F_{pq}^{(s,\Psi)}(I\!R^n)$. We want now to prove that the quasi-norm $\|f \,|\, F_{pq}^{(s,\Psi)}(I\!R^n)\|$ can be estimated from above by the quasi-norm in (3.7). By hypothesis $\widehat{k_0}(0) \neq 0$ and $\widehat{k^0}(0) \neq 0$. Then also $\check{k_0}(0) \neq 0$ and $\check{k^0}(0) \neq 0$. Since $k_0, k^0 \in \mathcal{S}(I\!R^n)$, $\check{k_0}, \check{k^0} \in \mathcal{S}(I\!R^n)$ are C^∞ functions. So, we can say that there exists a neighbourhood of the origin where both $\check{k_0}$ and $\check{k^0}$ are non zero. Recall (3.5). Therefore, there exists $\varepsilon > 0$ such that

$$\check{k^0}(x) \neq 0 \text{ for } |x| \le 2\varepsilon, \quad \check{k_0}(x) \neq 0 \text{ for } |x| \le 2\varepsilon \quad \text{and} \quad \check{k}(x) \neq 0 \text{ for } \frac{\varepsilon}{2} \le |x| \le 2\varepsilon. \tag{3.37}$$

If useful one can choose ε to be of the form $\varepsilon = 2^{-h}$ for some fixed $h \in I\!N_0$. Let $\phi \in \mathcal{S}(I\!R^n)$ be a function with

$$\operatorname{supp}\phi \subset \{\xi \in I\!R^n : |\xi| \le 2^{M+1}\} \qquad \text{and} \qquad \phi(x) = 1 \quad \text{if } |x| \le 2^M,$$

where we choose the natural number M later on. By (2.4) and (3.37), we have

$$\begin{aligned} &|(\varphi_j \widehat{f})^{\vee}(x)| = |(\varphi_j \phi(2^{-j}.)\widehat{f})^{\vee}(x)| \le \\ &\le (2\pi)^{-\frac{n}{2}} \int_{I\!R^n} \left| \left(\frac{\varphi_j}{\check{k}(\varepsilon 2^{-j}.)}\right)^{\vee}(y) \left(\check{k}(\varepsilon 2^{-j}.)\, \phi(2^{-j}.)\, \widehat{f}\right)^{\vee}(x-y) \right| dy, \quad j \in I\!N. \end{aligned} \tag{3.38}$$

A corresponding estimate holds for $j = 0$, in this case with k_0 instead of k. We assume this latter interpretation throughout this Step. Let $0 < r < \min(1, p, q)$. Using an inequality of Plancherel-Polya-Nikol'skij type as in [Tri83, 1.3.2(5)], by (3.38) we get

$$\begin{aligned} &|(\varphi_j \widehat{f})^{\vee}(x)|^r \le c_1'\, 2^{(M+j)n(1-r)} \times \\ &\int_{I\!R^n} \left| \left(\frac{\varphi_j}{\check{k}(\varepsilon 2^{-j}.)}\right)^{\vee}(y) \left(\check{k}(\varepsilon 2^{-j}.)\, \phi(2^{-j}.)\, \widehat{f}\right)^{\vee}(x-y) \right|^r dy,\ j \in I\!N_0. \end{aligned} \tag{3.39}$$

If $j \in I\!N$, then we have $\varphi_j(x) = \overline{\varphi}(2^{-j}x)$, with $\overline{\varphi}(x) = \varphi_0(x) - \varphi_0(2x)$, see (2.3), hence

$$\left| \left(\frac{\varphi_j}{\check{k}(\varepsilon 2^{-j}.)}\right)^{\vee}(y) \right|^r = 2^{jnr} \left| \left(\frac{\overline{\varphi}}{\check{k}(\varepsilon .)}\right)^{\vee}(2^j y) \right|^r \le c_b^r\, 2^{jnr}\, (1 + |2^j y|)^{-br}, \tag{3.40}$$

where $b \in I\!N$ is at our disposal, since $\left(\frac{\overline{\varphi}}{\check{k}(\varepsilon .)}\right)^{\vee} \in \mathcal{S}(I\!R^n)$. Putting (3.40) in (3.39) leads us to

$$\begin{aligned} &|(\varphi_j \widehat{f})^{\vee}(x)|^r \le c_2'\, 2^{(M+j)n(1-r)+jnr} \times \\ &\sum_{l=0}^{\infty} 2^{-brl} \int_{\{\xi \in I\!R^n : |\xi| \le 2^{-j+l}\}} \left| \left(\check{k}(\varepsilon 2^{-j}.)\phi(2^{-j}.)\widehat{f}\right)^{\vee}(x-y) \right|^r dy. \end{aligned} \tag{3.41}$$

Recall that in (3.41) $b \in I\!N$ is at our disposal. Now we estimate from above each integral in (3.41)

$$\int_{\{\xi\in I\!R^n:|\xi|\le 2^{-j+l}\}} \left|\left(\check{k}(\varepsilon 2^{-j}.)\phi(2^{-j}.)\widehat{f}\right)^{\vee}(x-y)\right|^r dy$$
$$\le 2^{(-j+l)n}\mathcal{M}\left[\left|\left(\check{k}(\varepsilon 2^{-j}.)\phi(2^{-j}.)\widehat{f}\right)^{\vee}\right|^r\right](x) \tag{3.42}$$

where $\mathcal{M}$ stands for the Hardy-Littlewood maximal function. We put the estimative obtained in (3.42), in (3.41), and choosing $b \in I\!N$ such that $br > n$ we arrive at

$$|(\varphi_j\widehat{f})^{\vee}(x)|^r \le c_3'\, 2^{Mn(1-r)}\,\mathcal{M}\left[\left|\left(\check{k}(\varepsilon 2^{-j}.)\phi(2^{-j}.)\widehat{f}\right)^{\vee}\right|^r\right](x). \tag{3.43}$$

Since $0 < r < \min(1,p,q)$, we have $1 < \frac{p}{r} < \infty$ and $1 < \frac{q}{r} \le \infty$. We multiply (3.43) with $2^{sjr}\,\Psi(2^{-j})^r$, apply the $l_{\frac{q}{r}}$-norm with respect to j and afterwards the $L_{\frac{p}{r}}$-norm with respect to x, then by Theorem 2.2.2 in [Tri92] we obtain

$$\left\|\left(\sum_{j=0}^{\infty} 2^{sjq}\,\Psi(2^{-j})^q|(\varphi_j\widehat{f})^{\vee}(x)|^q\right)^{\frac{1}{q}} \,|\, L_p(I\!R^n)\right\|^r \le$$
$$\le c_4'\, 2^{Mn(1-r)}\left\|2^{sjr}\,\Psi(2^{-j})^r\left|\left(\check{k}(\varepsilon 2^{-j}.)\phi(2^{-j}.)\widehat{f}\right)^{\vee}\right|^r \,|\, L_{\frac{p}{r}}(l_{\frac{q}{r}})\right\|$$
$$= c_4'\, 2^{Mn(1-r)}\left\|\left(\sum_{j=0}^{\infty} 2^{sjq}\,\Psi(2^{-j})^q\left|\left(\check{k}(\varepsilon 2^{-j}.)\phi(2^{-j}.)\widehat{f}\right)^{\vee}(.)\right|^q\right)^{\frac{1}{q}} \,|\, L_p(I\!R^n)\right\|^r, \tag{3.44}$$

where c_4' is independent of M. Because

$$\check{k}(\varepsilon 2^{-j}.)\phi(2^{-j}.) = \check{k}(\varepsilon 2^{-j}.) - \check{k}(\varepsilon 2^{-j}.)\left(1-\phi(2^{-j}.)\right),$$

the right-hand side of (3.44) can be estimated from above by

$$c_5'\, 2^{Mn(1-r)}\left\|\left(\sum_{j=0}^{\infty} 2^{sjq}\,\Psi(2^{-j})^q|(\check{k}(\varepsilon 2^{-j}.)\widehat{f})^{\vee}(.)|^q\right)^{\frac{1}{q}} \,|\, L_p\right\|^r +$$
$$+c_5'\, 2^{Mn(1-r)}\left\|\left(\sum_{j=0}^{\infty} 2^{sjq}\,\Psi(2^{-j})^q|(\check{k}(\varepsilon 2^{-j}.)(1-\phi(2^{-j}.))\widehat{f})^{\vee}(.)|^q\right)^{\frac{1}{q}} \,|\, L_p(I\!R^n)\right\|^r. \tag{3.45}$$

The first term in (3.45) is precisely what we want. The additional term in (3.45) can be treated as in Step 2 and estimated from above by

$$c_6'\, 2^{Mn(1-r)}\, 2^{\frac{(s_0+\theta-s)}{2}Mr}\,\|f \,|\, F_{pq}^{(s,\Psi)}(I\!R^n)\|^r. \tag{3.46}$$

By (3.19), we may choose r such that

$$n\left(\frac{1}{r}-1\right)+\left(\frac{s_0+\theta-s}{2}\right) < 0.$$

Recall that the natural number M is at our disposal. We take M large enough, so that (3.46) can be estimated from above by

$$\frac{1}{2}\,\|f \,|\, F_{pq}^{(s,\Psi)}(I\!R^n)\|^r.$$

This, (3.46) and (3.45) in (3.44) gives

$$\|f\,|\,F^{(s,\Psi)}_{pq}(I\!R^n)\| \le c_7' \Big[\|(\check{k}_0(\varepsilon.)\widehat{f})^\vee\,|\,L_p(I\!R^n)\|+ \left\|\left(\sum_{j=1}^\infty 2^{sjq}\,\Psi(2^{-j})^q|(\check{k}(\varepsilon 2^{-j}.)\widehat{f})^\vee(.)|^q\right)^{\frac{1}{q}}\,|\,L_p(I\!R^n)\right\|\Big]. \tag{3.47}$$

As we said at the beginning of Step 4, we can take $\varepsilon = 2^{-h}$, for some $h \in I\!N_0$ fixed. Therefore

$$\|f\,|\,F^{(s,\Psi)}_{pq}(I\!R^n)\| \le c\,\Big[\|k_0(2^{-h},f)\,|\,L_p(I\!R^n)\|+ \left\|\left(\sum_{j=1}^\infty 2^{sjq}\,\Psi(2^{-j})^q\,|(\check{k}(2^{-j}.)\widehat{f})^\vee(.)|^q\right)^{\frac{1}{q}}\,|\,L_p(I\!R^n)\right\|\Big].$$

(b) We now sketch the modification in the proof in (a) for the case where Ψ is decreasing. In this case, instead of (2.6) one would have (2.7).

In Step 1: we have to notice that, for $j \in I\!N$, we have

$$\begin{cases} \Psi(2^{-j}) \le \Psi(2^{(j+l)}) & \text{if } \ l \in I\!N_0 \\ \Psi(2^{-j}) \le \max\left(\frac{\Psi(2^{-1})}{\Psi(1)}, 2^{\theta'}\right)\, 2^{-\theta' l}\,\Psi(2^{(j+l)}) & \text{if } \ -l \in I\!N \text{ with } j+l \in I\!N_0 \end{cases}$$

Then, instead of (3.11) we have

$$\begin{aligned} &\left|\sum_{l=-\infty}^{M} 2^{sj}\,\Psi(2^{-j})\left(\check{k}(2^{-j}.)\varphi_{j+l}\widehat{f}\right)^\vee(x)\right| \\ &\le \bar{c}\sum_{l=-\infty}^{-1} 2^{(2N-s-\theta')l}\left|\left(\check{k}^0(2^{-j}z)2^{(j+l)s}\Psi(2^{-(j+l)})\widetilde{\varphi}_{j+l}(z)\widehat{f}\right)^\vee(x)\right| \\ &+\sum_{l=0}^{M} 2^{(2N-s)l}\left|\left(\check{k}^0(2^{-j}z)2^{(j+l)s}\Psi(2^{-(j+l)})\widetilde{\varphi}_{j+l}(z)\widehat{f}\right)^\vee(x)\right|. \end{aligned} \tag{3.48}$$

Using (3.13) in (3.48) gives

$$\begin{aligned} &\left|\sum_{l=-\infty}^{M} 2^{sj}\,\Psi(2^{-j})\left(\check{k}(2^{-j}.)\varphi_{j+l}\widehat{f}\right)^\vee(x)\right| \\ &\le \bar{c}\,c_1\sum_{l=-\infty}^{-1} 2^{(2N-s-\theta')l}\,2^{(j+l)s}\,\Psi(2^{-(j+l)})\,\left(\widetilde{\varphi}^*_{j+l}\widehat{f}\right)_a(x) \\ &+c_1\sum_{l=0}^{M} 2^{(2N-s)l}2^{(j+l)s}\,\Psi(2^{-(j+l)})\,\left(\widetilde{\varphi}^*_{j+l}\widehat{f}\right)_a(x). \end{aligned} \tag{3.49}$$

Applying in (3.49) l_q-quasi-norm with respect to j and afterwards L_p-quasi-norm with respect to x, using $2N > s+\theta'$, one would arrive at (3.15). The rest of Step 1 follows like in the case that Ψ is increasing.

In Step 2: One would simply use that

$$\Psi(2^{-j}) \le \Psi(2^{-(j+l)}), \quad \text{for } j \in I\!N_0,\ l \in I\!N : l \ge M+1,$$

since Ψ is decreasing. All the rest in Step 2 will work with $\theta = 0$.

In Step 3: Everything will work with $\theta = 0$.

In Step 4: Everything works without having to consider whether Ψ is decreasing or increasing.
∎

Remark 3.2. *We prove (3.26). Recall that the function $\chi \in \mathcal{D}(I\!R^n)$ satisfies (3.22). Let*

$$\Omega = \{\xi \in I\!R^n : \frac{1}{4} \le |\xi| \le 4\} \qquad \text{and} \qquad \chi_l(x) = \check{k}(2^l x)\,\chi(x), \quad l \in I\!N.$$

If $m \in I\!N$ is sufficiently large in order to $m > 1 + [\lambda]$, then there exists a constant $c_1 > 0$ such that

$$\|\chi_l \,|\, H_2^\lambda(I\!R^n)\| \le c_1 \sum_{|\alpha| \le m} \|D^\alpha \chi_l \,|\, L_\infty(I\!R^n)\|. \tag{3.50}$$

For $|\alpha| \le m$ and $x \in \Omega$

$$\begin{aligned} |(D^\alpha \chi_l)(x)| &\le \sum_{\beta \le \alpha} \binom{\alpha}{\beta} |(D^\beta \check{k})(2^l x)|\, 2^{l|\beta|}\, |(D^{\alpha-\beta}\chi)(x)| \\ &\le c_2\, 2^{l|\alpha|} \left(\sum_{\beta \le \alpha} \binom{\alpha}{\beta} |(D^\beta \check{k})(2^l x)| \right), \end{aligned} \tag{3.51}$$

since $\chi \in \mathcal{D}(I\!R^n)$. Let $m_1 \in I\!N$ be sufficiently large in order to $m - m_1 \le s_0$. Since $\check{k} \in \mathcal{S}(I\!R^n)$, there exists a constant $c_3 > 0$ such that

$$|(D^\beta \check{k})(x)| \le c_3\, (1 + |x|)^{-m_1}, \quad \forall x \in I\!R^n,\ \forall \alpha \in I\!N_0^n : |\alpha| \le m. \tag{3.52}$$

Putting (3.52) in (3.51), we arrive at

$$\begin{aligned} |(D^\alpha \chi_l)(x)| &\le c_2\, c_3\, 2^{l|\alpha|} \left(\max_{|\alpha| \le m} \sum_{\beta \le \alpha} \binom{\alpha}{\beta} \right) (1 + 2^l |x|)^{-m_1} \\ &\le c_4\, 2^{-(l-2)m_1}\, 2^{lm} = c_4\, 2^{2m_1}\, 2^{l(m-m_1)} \le c_4\, 2^{2m_1}\, 2^{ls_0}. \end{aligned}$$

So,

$$\sum_{|\alpha| \le m} \|D^\alpha \chi_l \,|\, L_\infty(I\!R^n)\| \le c_5\, 2^{ls_0}.$$

This in (3.50) gives (3.26).

Theorem 3.3. *Let $0 < p \le \infty$, $0 < q \le \infty$, $s \in I\!R$ and Ψ an admissible function as in Definition 2.1. Let $N \in I\!N_0$ with*

$$\begin{cases} 2N > s, & \text{if } \Psi \text{ is increasing} \\ 2N > s + \theta', & \text{if } \Psi \text{ is decreasing,} \end{cases} \tag{3.53}$$

with $\theta' \in I\!N_0$ as in Remark 2.3. Then, there exists $h \in I\!N_0$ such that

$$\|k_0(2^{-h}, f) \,|\, L_p(I\!R^n)\| + \left(\sum_{j=1}^\infty 2^{jsq}\, \Psi(2^{-j})^q\, \|k(2^{-j}, f)(.) \,|\, L_p(I\!R^n)\|^q \right)^{\frac{1}{q}} \tag{3.54}$$

(modification if $q = \infty$) is an equivalent quasi-norm in $B_{pq}^{(s,\Psi)}(I\!R^n)$.

Proof. This is the counterpart of Theorem 3.1 for $B_{pq}^{(s,\Psi)}(I\!R^n)$, we modify its proof. We do it for the case where the admissible function Ψ is monotone increasing. Then we have (2.6).

Step 1. We have again the splitting (3.9) and the estimate (3.11). But from (3.11) we still have

$$\left|\sum_{l=-\infty}^{M} 2^{sj}\,\Psi(2^{-j})\left(\check{k}(2^{-j}.)\varphi_{j+l}\widehat{f}\right)^{\vee}(x)\right|$$

$$\leq (2\pi)^{-\frac{n}{2}}\,2^{\theta(M+1)}\sum_{l=-\infty}^{M} 2^{(2N-s)l}\,2^{jn}\left|\left[k^{0}(-2^{j}.)*\left(2^{(j+l)s}\,\Psi(2^{-(j+l)})\widetilde{\varphi}_{j+l}\widehat{f}\right)^{\vee}\right](x)\right|. \tag{3.55}$$

Let $1\leq p\leq\infty$. We apply L_p-norm to (3.55), use Young's inequality and obtain

$$\left\|\sum_{l=-\infty}^{M} 2^{sj}\,\Psi(2^{-j})\,\left(\check{k}(2^{-j}.)\varphi_{j+l}\widehat{f}\right)^{\vee}(.)\,|\,L_p(I\!R^n)\right\|$$

$$= c_1\sum_{l=-\infty}^{M} 2^{(2N-s)l}\left\|\left(2^{(j+l)s}\,\Psi(2^{-(j+l)})\widetilde{\varphi}_{j+l}\widehat{f}\right)^{\vee}\,|\,L_p(I\!R^n)\right\|, \tag{3.56}$$

since $k^0\in\mathcal{S}\subset L_1$. Applying l_q-quasi-norm in (3.56), because $2N>s$, we get

$$\left(\sum_{j=1}^{\infty}\left\|\sum_{l=-\infty}^{M} 2^{sj}\,\Psi(2^{-j})\,\left(\check{k}(2^{-j}.)\varphi_{j+l}\widehat{f}\right)^{\vee}(.)|\,L_p(I\!R^n)\right\|^{q}\right)^{\frac{1}{q}}$$

$$\leq c_2\left(\sum_{m=0}^{\infty} 2^{smq}\,\Psi(2^{-m})^{q}\,\|(\widetilde{\varphi}_m\widehat{f})^{\vee}\,|\,L_p(I\!R^n)\|^{q}\right)^{\frac{1}{q}}. \tag{3.57}$$

Let $0<p<1$. For each term on the right hand side of (3.11) we have

$$\left|\left(\check{k}^{0}(2^{-j}.)\,2^{(j+l)s}\,\Psi(2^{-(j+l)})\,\widetilde{\varphi}_{j+l}\widehat{f}\right)^{\vee}(x)\right|$$

$$\leq (2\pi)^{-\frac{n}{2}}\int_{I\!R^n}\left|\left(\check{k}^{0}(2^{-j}.)\alpha(c_3 2^{-j}.)\right)^{\vee}(y)\,\left(2^{(j+l)s}\Psi(2^{-(j+l)})\widetilde{\varphi}_{j+l}\widehat{f}\right)^{\vee}(x-y)\right|dy, \tag{3.58}$$

where $c_3=2^{-(M+1)}$ and $\alpha\in\mathcal{D}(I\!R^n)$ is such that

$$\alpha(x)=1\ \text{ for }\ |x|\leq 1\qquad\text{and}\qquad \operatorname{supp}\alpha\subset\{x\in I\!R^n:|x|\leq 2\}.$$

The Fourier transform of the y-function inside the integral in (3.58) has compact support contained in $\{\xi\in I\!R^n:|\xi|\leq 6\times 2^{j+M}\}$. Since now $0<p<1$, we apply an inequality of Plancherel-Polyá-Nikolskij type, see [Tri83, 1.3.2(5)], and obtain

$$\int_{I\!R^n}\left|\left(\check{k}^{0}(2^{-j}.)\alpha(c_3 2^{-j}.)\right)^{\vee}(y)\,\left(2^{(j+l)s}\Psi(2^{-(j+l)})\widetilde{\varphi}_{j+l}\widehat{f}\right)^{\vee}(x-y)\right|dy$$

$$\leq c_4\,2^{(j+M)n\left(\frac{1}{p}-1\right)}\times$$

$$\left[\int_{I\!R^n}\left|\left(\check{k}^{0}(2^{-j}.)\alpha(c_4 2^{-j}.)\right)^{\vee}(y)\,\left(2^{(j+l)s}\Psi(2^{-(j+l)})\widetilde{\varphi}_{j+l}\widehat{f}\right)^{\vee}(x-y)\right|^{p}dy\right]^{\frac{1}{p}} \tag{3.59}$$

where the constant c_4 is independent of j. Putting (3.59) together with (3.58) in (3.11) and then applying L_p-quasi-norm, we get

$$\left\|\sum_{l=-\infty}^{M} 2^{sj}\,\Psi(2^{-j})\,\left(\check{k}(2^{-j}.)\varphi_{j+l}\widehat{f})\right)^{\vee}(.)|\,L_p(I\!R^n)\right\|$$

$$\leq c_5\,2^{Mn\left(\frac{1}{p}-1\right)}\left[\sum_{l=-\infty}^{M} 2^{(2N-s)pl}\,\left\|2^{(j+l)s}\,\Psi(2^{-(j+l)})(\widetilde{\varphi}_{j+l}\widehat{f})^{\vee}(.)\,|\,L_p(I\!R^n)\right\|^{p}\right]^{\frac{1}{p}}. \tag{3.60}$$

Recall that $\frac{1}{p} > 1$. Let p_1 be its conjugate exponent. Because $2N > s$ and using Hölder inequality we estimate the right hand side of (3.60) by

$$c_6\, 2^{Mn\left(\frac{1}{p}-1\right)} \sum_{l=-\infty}^{M} 2^{\frac{(2N-s)}{2} l} \left\| 2^{(j+l)s}\, \Psi(2^{-(j+l)}) (\widetilde{\varphi}_{j+l} \widehat{f})^\vee(.) \,|\, L_p(I\!R^n) \right\| . \tag{3.61}$$

Putting (3.61) in (3.60) and then applying l_q-quasi-norm with respect to j we arrive at

$$\begin{aligned} &\left(\sum_{j=1}^{\infty} \left\| \sum_{l=-\infty}^{M} 2^{sj}\, \Psi(2^{-j}) \left(\check{k}(2^{-j}.) \varphi_{j+l} \widehat{f} \right)^\vee (.) | L_p(I\!R^n) \right\|^q \right)^{\frac{1}{q}} \\ &\leq c_7\, 2^{Mn\left(\frac{1}{p}-1\right)} \left(\sum_{m=0}^{\infty} 2^{smq}\, \Psi(2^{-m})^q \, \|(\widetilde{\varphi}_m \widehat{f})^\vee \,|\, L_p(I\!R^n)\|^q \right)^{\frac{1}{q}} . \end{aligned} \tag{3.62}$$

This had already been obtained also in the case $1 \leq p \leq \infty$, in (3.57). Recall that

$$(\widetilde{\varphi}_m \widehat{f})^\vee = \left(|2^{-m}.|^{2N}\, H(2^{-m}.)\, [(\varphi_m \widehat{f})^\vee]^\wedge \right)^\vee , \quad m \in I\!N_0,$$

where $H \in \mathcal{D}(I\!R^n)$ is such that

$$H(x) = 1 \quad \text{if} \quad |x| \leq 2. \tag{3.63}$$

By a consequence of Theorem 1.5.2 in [Tri83, (13)], with $\nu > n \left(\frac{1}{\min(p,1)} - \frac{1}{2} \right)$, we have

$$\|(\widetilde{\varphi}_m \widehat{f})^\vee \,|\, L_p(I\!R^n)\| \leq c_8 \, \left\| |2.|^{2N} H(2.) \,|\, H^\nu_2(I\!R^n) \right\| \, \|(\varphi_m \widehat{f})^\vee \,|\, L_p(I\!R^n)\|,$$

where c_8 is independent of $m \in I\!N_0$. Applying this in (3.62) we obtain

$$\begin{aligned} &\left(\sum_{j=1}^{\infty} \left\| \sum_{l=-\infty}^{M} 2^{sj}\, \Psi(2^{-j}) \left(\check{k}(2^{-j}.) \varphi_{j+l} \widehat{f} \right)^\vee (.) | L_p(I\!R^n) \right\|^q \right)^{\frac{1}{q}} \\ &\leq c_9\, 2^{Mn\left(\frac{1}{p}-1\right)} \, \|f \,|\, B^{(s,\Psi)}_{pq}(I\!R^n)\|. \end{aligned} \tag{3.64}$$

Step 2. We estimate the second sum in (3.9) and we have to make sure that (3.9) converges a.e and in some L_r-space with $1 \leq r \leq \infty$. However the latter comes as a by product. Following Step 2 of the proof for $F^{(s,\Psi)}_{pq}(I\!R^n)$, we have also (3.21), where here s_0 will be such that

$$s_0 + \theta + 4\, \sigma_p < s. \tag{3.65}$$

Let $1 \leq p \leq \infty$, and χ as in (3.22). Then we apply L_p-norm to (3.21), and use Young inequality to obtain

$$\begin{aligned} &\left\| \sum_{l=M+1}^{\infty} 2^{sj}\, \Psi(2^{-j}) \left(\check{k}(2^{-j}.) \varphi_{j+l} \widehat{f} \right)^\vee (.) \,|\, L_p(I\!R^n) \right\| \\ &\leq c'_1 \sum_{l=M+1}^{\infty} 2^{(s_0+\theta-s)l} \left\| \left(\frac{\check{k}(2^l.)}{|2^l.|^{s_0}} \chi(.) \right)^\vee \,|\, L_1(I\!R^n) \right\| \left\| \left(2^{(j+l)s} \Psi(2^{-(j+l)}) \varphi'_{j+l} \widehat{f} \right)^\vee \,|\, L_p(I\!R^n) \right\| \\ &\leq c'_2 \sum_{l=M+1}^{\infty} 2^{(s_0+\theta-s)l} \left\| \left(2^{(j+l)s} \Psi(2^{-(j+l)}) \varphi'_{j+l} \widehat{f} \right)^\vee \,|\, L_p(I\!R^n) \right\| , \end{aligned} \tag{3.66}$$

the last inequality is due to

$$\sup_{l \in I\!N} \left\| \left(\frac{\check{k}(2^l.)}{|2^l.|^{s_0}} \chi(.) \right)^\vee \,|\, L_1(I\!R^n) \right\| < \infty,$$

which can be proved introducing inside the Fourier transform inverse the function h in (3.25), applying Theorem 2.2.3 in [Tri92] and using (3.26). Then, applying l_q-quasi-norm to (3.66), because $s_0+\theta-s<0$, we get

$$\left(\sum_{j=1}^{\infty}\left\|\sum_{l=M+1}^{\infty} 2^{sj}\,\Psi(2^{-j})\left(\check{k}(2^{-j})\varphi_{j+l}\widehat{f}\right)^{\vee}(.)|\,L_p(I\!R^n)\right\|^q\right)^{\frac{1}{q}}$$

$$\leq c_3'\,2^{\frac{(s_0+\theta-s)}{2}M}\left(\sum_{m=1}^{\infty} 2^{smq}\,\Psi(2^{-m})^q\,\|(\varphi_m'\widehat{f})^{\vee}|\,L_p(I\!R^n)\|^q\right)^{\frac{1}{q}}. \tag{3.67}$$

Let $0<p<1$. Each term in (3.21) can be estimated from above by

$$\left|\left(\frac{\check{k}(2^{-j}.)}{|2^{-j}.|^{s_0}}\chi(2^{-j-l}.)2^{-(j+l)}\Psi(2^{-(j+l)})\varphi_{j+l}'\widehat{f}\right)^{\vee}(x)\right|$$

$$\leq (2\pi)^{-\frac{n}{2}}\int_{I\!R^n}\left|\left(\frac{\check{k}(2^{-j}.)}{|2^{-j}.|^{s_0}}\chi(2^{-j-l}.)\right)^{\vee}(y)\left(2^{-(j+l)}\Psi(2^{-(j+l)})\varphi_{j+l}'\widehat{f}\right)^{\vee}(x-y)\right|dy. \tag{3.68}$$

But the Fourier transform of the y-function inside the integral in (3.68) has compact support contained in $\{\xi\in I\!R^n:|\xi|\leq 6\times 2^{j+l}\}$, and since now $0<p<1$, we use again an inequality of Plancherel-Polyá-Nikolskij type, see [Tri83, 1.3.2/(5)], and obtain

$$\left|\left(\frac{\check{k}(2^{-j}.)}{|2^{-j}.|^{s_0}}\chi(2^{-j-l}.)2^{-(j+l)}\Psi(2^{-(j+l)})\varphi_{j+l}'\widehat{f}\right)^{\vee}(x)\right|$$

$$\leq c_4'\,2^{(j+l)n\left(\frac{1}{p}-1\right)}\times$$

$$\left[\int_{I\!R^n}\left|\left(\frac{\check{k}(2^{-j}.)}{|2^{-j}.|^{s_0}}\chi(2^{-j-l}.)\right)^{\vee}(y)\left(2^{-(j+l)}\Psi(2^{-(j+l)})\varphi_{j+l}'\widehat{f}\right)^{\vee}(x-y)\right|^p dy\right]^{\frac{1}{p}}, \tag{3.69}$$

where c_4' is independent of l and j. Putting (3.68) and (3.69) in (3.21) and then applying L_p-quasi-norm we get

$$\left\|\sum_{l=M+1}^{\infty} 2^{sj}\,\Psi(2^{-j})\left(\check{k}(2^{-j}.)\varphi_{j+l}\widehat{f}\right)^{\vee}(.)\,|\,L_p(I\!R^n)\right\|$$

$$=(2\pi)^{-\frac{n}{2}}\,c_4'\,2^{\theta}\left[\sum_{l=M+1}^{\infty}2^{(s_0+\theta-s)pl}\int_{I\!R^n}\left|\left(\frac{\check{k}(2^{l}.)}{|2^{l}.|^{s_0}}\chi(.)\right)^{\vee}(\xi)\right|^p d\xi\right.$$

$$\left.\left\|2^{(j+l)s}\Psi(2^{-(j+l)})(\varphi_{j+l}'\widehat{f})^{\vee}|\,L_p(I\!R^n)\right\|^p\right]^{\frac{1}{p}}$$

$$=c_5'\left[\sum_{l=M+1}^{\infty}2^{(s_0+\theta-s)pl}\left\|2^{(j+l)s}\Psi(2^{-(j+l)})(\varphi_{j+l}'\widehat{f})^{\vee}|\,L_p(I\!R^n)\right\|^p\right]^{\frac{1}{p}}, \tag{3.70}$$

we used that

$$\sup_{l\in I\!N}\left\|\left(\frac{\check{k}(2^{l}.)}{|2^{l}.|^{s_0}}\chi(.)\right)^{\vee}|\,L_p(I\!R^n)\right\|^p<\infty,$$

which can be proved introducing inside the Fourier transform inverse the function h in (3.25), applying Theorem 2.2.3 in [Tri92] and using (3.26). Recall that $\frac{1}{p}>1$. Let p_1 be its conjugate exponent. Using Hölder inequality we estimate the right hand side of (3.70) by

$$c_6'\sum_{l=M+1}^{\infty}2^{\frac{(s_0+\theta-s)}{2}l}\left\|2^{(j+l)s}\Psi(2^{-(j+l)})(\varphi_{j+l}'\widehat{f})^{\vee}|\,L_p\right\|. \tag{3.71}$$

Now, by (3.70) and (3.71), applying l_q-quasi-norm and because $s_0+\theta-s<0$, we obtain (3.67) for any value of $0<p\le\infty$. Recall that

$$(\varphi'_m\widehat{f})^\vee = \left(|2^{-m}.|^{s_0}\,\chi(2^{-m}.)\,[(\varphi_m\widehat{f})^\vee]^\wedge\right)^\vee, \quad m\in I\!N,$$

where $\chi\in\mathcal{D}(I\!R^n)$ as in (3.22). By a consequence of Theorem 1.5.2 in [Tri83, (13)], with $\nu>n\left(\frac{1}{\min(p,1)}-\frac{1}{2}\right)$, we have

$$\|(\varphi'_m\widehat{f})^\vee\,|\,L_p(I\!R^n)\|\le c'_7\,\||2.|^{s_0}\chi(2.)\,|\,H_2^\nu(I\!R^n)\|\,\|(\varphi_m\widehat{f})^\vee\,|\,L_p(I\!R^n)\|,$$

where c'_7 is independent of $m\in I\!N$. Applying this in (3.67) we obtain

$$\left(\sum_{j=1}^\infty 2^{sjq}\,\Psi(2^{-j})^q\left\|\sum_{l=M+1}^\infty\left(\check{k}(2^{-j}.)\varphi_{j+l}\widehat{f}\right)^\vee(.)|\,L_p(I\!R^n)\right\|^q\right)^{\frac{1}{q}}$$

$$\le c'_8\,2^{\frac{(s_0+\theta-s)}{4}M}\,\|f\,|\,B_{pq}^{(s,\Psi)}(I\!R^n)\|. \tag{3.72}$$

From (3.9), (3.64) and (3.72)

$$\left(\sum_{j=1}^\infty 2^{sjq}\,\Psi(2^{-j})^q\,\|k(2^{-j},f)\,|\,L_p(I\!R^n)\|^q\right)^{\frac{1}{q}}\le c'_9\,\|f\,|\,B_{pq}^{(s,\Psi)}(I\!R^n)\|. \tag{3.73}$$

One can also prove in a similar way that

$$\|k_0(2^{-h},f)\,|\,L_p(I\!R^n)\|\le c'_{10}\,\|f\,|\,B_{pq}^{(s,\Psi)}(I\!R^n)\|. \tag{3.74}$$

With (3.73) and (3.74) we have proved one of the desired inequalities between quasi-norm (3.54) and $\|.\,|\,B_{pq}^{(s,\Psi)}(I\!R^n)\|$.

Step 3. We prove the convergence in the right-hand side of (3.9) in some space $L_r(I\!R^n)$, $1<r\le\infty$. Let $1\le p\le\infty$. We can rewrite (3.66) as

$$\left\|\sum_{l=M+1}^L 2^{sj}\,\Psi(2^{-j})\,\left(\check{k}(2^{-j}.)\varphi_{j+l}\widehat{f}\right)^\vee(.)\,|\,L_p(I\!R^n)\right\|$$

$$\le c'_2\sum_{l=M+1}^L 2^{(s_0+\theta-s)l}\,\left\|\left(2^{(j+l)s}\Psi(2^{-(j+l)})\varphi'_{j+l}\widehat{f}\right)^\vee|\,L_p(I\!R^n)\right\|, \tag{3.75}$$

with $L>M$. Using $s_0+\theta-s<0$ and $l_q\subset l_1$, if $0<q\le1$, or Hölder inequality if $1<q\le\infty$, we conclude that if M is large enough then the right-hand side of (3.75) can be estimated from above by

$$\varepsilon\left(\sum_{l=M+1}^\infty 2^{slq}\,\Psi(2^{-l})^q\,\|(\varphi'_l\widehat{f})^\vee|\,L_p(I\!R^n)\|^q\right)^{\frac{1}{q}}, \tag{3.76}$$

for given $\varepsilon>0$. Since $f\in B_{pq}^{(s,\Psi)}(I\!R^n)$, as in Step 2,

$$\left(\sum_{m=1}^\infty 2^{smq}\,\Psi(2^{-m})^q\,\|(\varphi'_m\widehat{f})^\vee|\,L_p(I\!R^n)\|^q\right)^{\frac{1}{q}}<\infty.$$

Therefore, by (3.75) and (3.76), we conclude that the right-hand side of (3.9) converges in $L_p(I\!R^n)$. Hence pointwise a.e. and also in $\mathcal{S}'(I\!R^n)$. If $0 < p < 1$, we can rewrite (3.21) as

$$\left| \sum_{l=M+1}^{L} 2^{sj}\, \Psi(2^{-j}) \left(\check{k}(2^{-j}.)\varphi_{j+l}\widehat{f} \right)^{\vee}(x) \right|$$

$$\leq 2^{\theta} \sum_{l=M+1}^{L} 2^{(s_0+\theta-s)l} \left| \left(\check{k^0}(2^{-j}z)|2^{-j}z|^{2N-s_0} 2^{(j+l)s}\, \Psi(2^{-(j+l)})\varphi'_{j+l}\widehat{f} \right)^{\vee}(x) \right|. \tag{3.77}$$

with $L > M$. Applying L_1-norm, using Fubini theorem and change of variables we get

$$\left\| \sum_{l=M+1}^{L} 2^{sj}\, \Psi(2^{-j}) \left(\check{k}(2^{-j}.)\varphi_{j+l}\widehat{f} \right)^{\vee}(.) \,|\, L_1(I\!R^n) \right\|$$

$$\leq 2^{\theta}\, (2\pi)^{-\frac{n}{2}} \sum_{l=M+1}^{L} 2^{(s_0+\theta-s)l} \left\| \left(\frac{\check{k^0}(2^l.)}{|2^l.|^{s_0}} \chi(.) \right)^{\vee} |\, L_1(I\!R^n) \right\|$$

$$\times \left\| 2^{(j+l)s}\, \Psi(2^{-(j+l)}) \left(\varphi'_{j+l}\widehat{f} \right)^{\vee} |\, L_1(I\!R^n) \right\|$$

$$\leq c \sum_{l=M+1}^{L} 2^{(s_0+\theta-s)l} \left\| 2^{(j+l)s}\, \Psi(2^{-(j+l)}) \left(\varphi'_{j+l}\widehat{f} \right)^{\vee} |\, L_1(I\!R^n) \right\|, \tag{3.78}$$

since

$$\sup_{l\in I\!N_0} \left\| \left(\frac{\check{k^0}(2^l.)}{|2^l.|^{s_0}} \chi(.) \right)^{\vee} |\, L_1(I\!R^n) \right\| < \infty.$$

Let $\sigma = s - \sigma_p$. Then we have the embedding $B^{(s,\Psi)}_{pq}(I\!R^n) \subset B^{(\sigma,\Psi)}_{1q}(I\!R^n)$. Putting $s = \sigma + \sigma_p$ in (3.78), we have

$$\left\| \sum_{l=M+1}^{L} 2^{\sigma j}\, \Psi(2^{-j}) \left(\check{k}(2^{-j}.)\varphi_{j+l}\widehat{f} \right)^{\vee}(.) \,|\, L_1(I\!R^n) \right\|$$

$$\leq c \sum_{l=M+1}^{L} 2^{(s_0+\theta-\sigma)l} \left\| 2^{(j+l)\sigma}\, \Psi(2^{-(j+l)}) \left(\varphi'_{j+l}\widehat{f} \right)^{\vee} |\, L_1(I\!R^n) \right\|. \tag{3.79}$$

Since from (3.65), $\sigma > s_0 + \theta$, there exists M large enough such that the right hand side of (3.79) can be estimated from above by

$$\varepsilon \left(\sum_{l=M+1}^{\infty} 2^{\sigma l q}\, \Psi(2^{-l})^q \, \|(\varphi'_l \widehat{f})^{\vee} |\, L_1(I\!R^n)\|^q \right)^{\frac{1}{q}}$$

for given $\varepsilon > 0$. From the embedding mentioned below, and since $f \in B^{(s,\Psi)}_{pq}(I\!R^n)$ (and arguments as in Step 2) we have

$$\left(\sum_{m=1}^{\infty} 2^{\sigma m q}\, \Psi(2^{-m})^q \, \|(\varphi'_m \widehat{f})^{\vee} |\, L_1(I\!R^n)\|^q \right)^{\frac{1}{q}} < \infty.$$

Hence the right hand side of (3.9) converges in $L_1(I\!R^n)$.

Step 4. Let $f \in B^{(s,\Psi)}_{pq}(I\!R^n)$. We want now to prove that the quasi-norm $\|.\,|\, B^{(s,\Psi)}_{pq}(I\!R^n)\|$ can be estimated from above by the quasi-norm in (3.54). We follow the analogous Step 4 of the proof for $F^{(s,\Psi)}_{pq}(I\!R^n)$. We can rewrite everything from (3.37) up to (3.43), but now it will be sufficient that $0 < r < \min(1,p)$. Since $\frac{p}{r} \geq 1$, we apply in (3.43) $L_{\frac{p}{r}}$-norm and use the scalar Hardy-Littlewood maximal inequality as in [Tri83, 1.2.3/(4)]. Then we have

$$\|(\varphi_j \widehat{f})^\vee \,|\, L_p(I\!R^n)\|^r \leq \bar{c}_1 \left\| \left(\check{k}(\varepsilon 2^{-j}.)\phi(2^{-j})\widehat{f}\right)^\vee \,|\, L_p(I\!R^n)\right\|^r, \tag{3.80}$$

always with k_0 instead of k if $j = 0$. Multiplying (3.80) by $2^{sjr}\,\Psi(2^{-j})^r$ and applying $l_{\frac{q}{r}}$-quasi-norm we get

$$\left(\sum_{j=0}^{\infty} 2^{sjq}\,\Psi(2^{-j})^q\,\|(\varphi_j \widehat{f})^\vee \,|\, L_p(I\!R^n)\|^q\right)^{\frac{1}{q}}$$

$$\leq \bar{c}_1^{\frac{1}{r}}\, 2^{Mn\left(\frac{1}{r}-1\right)} \left(\sum_{j=0}^{\infty} 2^{sjq}\,\Psi(2^{-j}) \left\| \left(\check{k}(\varepsilon 2^{-j}.)\phi(2^{-j})\widehat{f}\right)^\vee \,|\, L_p(I\!R^n)\right\|^q\right)^{\frac{1}{q}}. \tag{3.81}$$

Because

$$\check{k}(\varepsilon 2^{-j}.)\phi(2^{-j}.) = \check{k}(\varepsilon 2^{-j}.) - \check{k}(\varepsilon 2^{-j}.)\left(1 - \phi(2^{-j}.)\right),$$

(3.81) can be estimated from above by

$$\bar{c}_2\, 2^{Mn\left(\frac{1}{r}-1\right)} \left(\sum_{j=0}^{\infty} 2^{sjq}\,\Psi(2^{-j})^q \left\| \left(\check{k}(\varepsilon 2^{-j}.)\widehat{f}\right)^\vee \,|\, L_p(I\!R^n)\right\|^q\right)^{\frac{1}{q}}$$

$$+\bar{c}_2\, 2^{Mn\left(\frac{1}{r}-1\right)} \left(\sum_{j=0}^{\infty} 2^{sjq}\,\Psi(2^{-j})^q \left\| \left(\check{k}(\varepsilon 2^{-j}.)(1-\phi(2^{-j}.))\widehat{f}\right)^\vee \,|\, L_p(I\!R^n)\right\|^q\right)^{\frac{1}{q}}. \tag{3.82}$$

The first term in (3.82) is precisely what we want. The additional term in (3.82) can be treated as in Step 2 and estimated from above by

$$\bar{c}_3\, 2^{Mn\left(\frac{1}{r}-1\right)}\, 2^{\frac{(s_0+\theta-s)}{4}M}\, \|f \,|\, B_{pq}^{(s,\Psi)}(I\!R^n)\|. \tag{3.83}$$

By (3.65), we may choose r such that

$$n\left(\frac{1}{r} - 1\right) + \frac{(s_0+\theta-s)}{4} < 0.$$

Recall that the natural number M is at our disposal. We can take M large enough, so that (3.83) can be estimated from above by

$$\frac{1}{2}\,\|f \,|\, B_{pq}^{(s,\Psi)}(I\!R^n)\|.$$

This fact, (3.83) and (3.82) in (3.81) give

$$\|f \,|\, B_{pq}^{(s,\Psi)}(I\!R^n)\| \leq \bar{c}_4 \left[\left\| \left(\check{k}_0(\varepsilon.)\widehat{f}\right)^\vee \,|\, L_p(I\!R^n)\right\| + \right.$$

$$\left.\left(\sum_{j=1}^{\infty} 2^{sjq}\,\Psi(2^{-j})^q \left\| \left(\check{k}(\varepsilon 2^{-j}.)\widehat{f}\right)^\vee \,|\, L_p(I\!R^n)\right\|^q\right)^{\frac{1}{q}}\right]. \tag{3.84}$$

As observed in the proof for $F^{(s,\Psi)}_{pq}(I\!R^n)$, one can take $\varepsilon = 2^{-h}$, for some $h \in I\!N_0$ fixed. As there, from (3.84) we come to

$$\|f \,|\, B^{(s,\Psi)}_{pq}(I\!R^n)\| \leq \bar{c}_5 \; \Bigg[\|k_0(2^{-h}, f) \,|\, L_p(I\!R^n)\| + \left(\sum_{j=1}^{\infty} 2^{sjq}\, \Psi(2^{-j})^q \left\| \left(\check{k}(2^{-j}.)\widehat{f}\right)^{\vee} \,|\, L_p(I\!R^n)\right\|^q\right)^{\frac{1}{q}}\Bigg].$$

■

Remark 3.4. *If we replace k_0 by the new function $2^{hn}\, k_0(2^h.)$, then in (3.7) and (3.54) it would appear simply $\|k_0(1, f) \,|\, L_p(I\!R^n)\|$.*

4. Atomic and Subatomic Decompositions

Recall that $Z\!\!Z^n$ stands for the lattice of all points in $I\!R^n$ with integer-valued components. If $\nu \in I\!N_0$ and $m = (m_1, \dots, m_n) \in Z\!\!Z^n$, $Q_{\nu m}$ denotes the cube in $I\!R^n$ centred at $2^{-\nu}m = (2^{-\nu}m_1 \dots, 2^{-\nu}m_n)$ with side length $2^{-\nu}$, which has sides parallel to the axes of coordinates. Remark that Q_{0m} is a cube with side length 1. Let Q be a cube in $I\!R^n$ and $r > 0$ then $r\,Q$ is a cube in $I\!R^n$ concentric with Q and with side length r times the side length of Q.

Definition 4.1. *(i) Let $K \in I\!N_0$ and $c > 1$. A K times differentiable complex-valued function $a(x)$ in $I\!R^n$ (continuous if $K = 0$) is called an 1_K-atom if*

$$\text{supp}\, a \subset c\, Q_{0m} \quad \text{for some } m \in Z\!\!Z^n \tag{4.1}$$

and

$$|D^\alpha a(x)| \leq 1 \quad \text{for } |\alpha| \leq K. \tag{4.2}$$

(ii) Let $s \in I\!R$, $0 < p \leq \infty$, $K \in I\!N_0$, $L+1 \in I\!N_0$, $c > 1$ and Ψ an admissible function according to Definition 2.1. A K times differentiable complex-valued function $a(x)$ in $I\!R^n$ (continuous if $K = 0$) is called an $(s, p, \Psi)_{K,L}$-atom if for some $\nu \in I\!N_0$

$$\text{supp}\, a \subset c\, Q_{\nu m} \quad \text{for some } m \in Z\!\!Z^n, \tag{4.3}$$

$$|D^\alpha a(x)| \leq 2^{-\nu(s-\frac{n}{p})+|\alpha|\nu}\, \Psi(2^{-\nu})^{-1} \quad \text{for } |\alpha| \leq K, \tag{4.4}$$

and

$$\int_{I\!R^n} x^\beta a(x)\, dx = 0 \quad \text{if } |\beta| \leq L. \tag{4.5}$$

If the atom $a(x)$ is located at $Q_{\nu m}$ (that means $\text{supp}\, a \subset c\, Q_{\nu m}$, with $\nu \in I\!N_0$, $m \in Z\!\!Z^n$, $c > 1$), then we write it $a_{\nu m}(x)$. The value of the number $c > 1$ in (4.1) and (4.3) is unimportant. It simply makes clear that at level ν some overlapping of the supports of $a_{\nu m}(x)$ must be allowed. The moment conditions (4.5) can be reformulated as $(D^\alpha \widehat{a})(0) = 0 \quad \text{if } |\beta| \leq L$, which shows that a sufficiently strong decay of $\widehat{a}(\xi)$ at the origin is required. If $L = -1$ then (4.5) simply means that there are no moment conditions. The reason for the normalising factor in (4.2) and (4.4) is that there exists a constant $c > 0$ such that for all these atoms we have $\|a \,|\, B^{(s,\Psi)}_{pq}(I\!R^n)\| \leq c$, $\|a \,|\, F^{(s,\Psi)}_{pq}(I\!R^n)\| \leq c$. Hence, atoms are normalising building blocks satisfying some moment conditions.

We introduce now the sequence spaces b_{pq} and f_{pq}. If $\nu \in I\!N_0$, $m \in Z\!\!Z^n$ and $Q_{\nu m}$ is a cube as above let $\chi_{\nu m}$ be the characteristic function of $Q_{\nu m}$. If $0 < p \leq \infty$ let $\chi^{(p)}_{\nu m} = 2^{\frac{\nu n}{p}} \chi_{\nu m}$ (obvious modification if $p = \infty$) be the L_p-normalized characteristic function of $Q_{\nu m}$, that means $\|\chi^{(p)}_{\nu m} \,|\, L_p(I\!R^n)\| = 1$.

Definition 4.2. *Let $0 < p \leq \infty$, $0 < q \leq \infty$, and $\lambda = \{\lambda_{\nu m} \in \mathbb{C} : \nu \in I\!N_0, m \in \mathbb{Z}^n\}$. Then*

$$b_{pq} = \left\{ \lambda : \|\lambda \,|\, b_{pq}\| = \left(\sum_{\nu=0}^{\infty} \left(\sum_{m \in \mathbb{Z}^n} |\lambda_{\nu m}|^p \right)^{\frac{q}{p}} \right)^{\frac{1}{q}} < \infty \right\} \tag{4.6}$$

and

$$f_{pq} = \left\{ \lambda : \|\lambda \,|\, f_{pq}\| = \left\| \left(\sum_{\nu=0}^{\infty} \sum_{m \in \mathbb{Z}^n} |\lambda_{\nu m}\, \chi_{\nu m}^{(p)}(.)|^q \right)^{\frac{1}{q}} |L_p(I\!R^n) \right\| < \infty \right\} \tag{4.7}$$

(with the usual modification if $p = \infty$ or/and $q = \infty$).

Remark 4.3. *Observe that*

$$\|\lambda \,|\, b_{pq}\| = \left(\sum_{\nu=0}^{\infty} \left\| \sum_{m \in \mathbb{Z}^n} \lambda_{\nu m}\, \chi_{\nu m}^{(p)}(.) |L_p(I\!R^n) \right\|^q \right)^{\frac{1}{q}},$$

since the $\chi_{\nu m}^{(p)}$'s have disjoint supports, we see that the b and f quasi-norms are obtained from each other interchanging the L_p and l_q quasi-norms (as in the B and F case).

Let $0 < p \leq \infty$ and $0 < q \leq \infty$, then we introduce the abbreviations

$$\sigma_p = n\left(\frac{1}{p} - 1\right)_+ \quad \text{and} \quad \sigma_{pq} = n\left(\frac{1}{\min(p,q)} - 1\right)_+ .$$

Recall $a_+ = \max(a, 0)$, if $a \in I\!R$.

Theorem 4.4. *(i) Let $0 < p < \infty$, $0 < q \leq \infty$, $s \in I\!R$ and Ψ an admissible function according to Definition 2.1. Let $K \in I\!N_0$ and $L + 1 \in I\!N_0$ with*

$$K \geq (1 + [s])_+ \quad \textit{and} \quad L \geq \max(-1, [\sigma_{pq} + \theta - s]), \tag{4.8}$$

if Ψ is increasing, or

$$K \geq (1 + [s + \theta'])_+ \quad \textit{and} \quad L \geq \max(-1, [\sigma_{pq} - s]), \tag{4.9}$$

if Ψ is decreasing (θ and θ' as in Remark 2.3), be fixed. Then $f \in \mathcal{S}'(I\!R^n)$ belongs to $F_{pq}^{(s,\Psi)}(I\!R^n)$ if, and only if, it can be represented as

$$f = \sum_{\nu=0}^{\infty} \sum_{m \in \mathbb{Z}^n} \lambda_{\nu m}\, a_{\nu m}(x), \quad \textit{convergence being in } \mathcal{S}'(I\!R^n), \tag{4.10}$$

where $a_{\nu m}(x)$ are 1_K-atoms ($\nu = 0$) or $(s, p, \Psi)_{K,L}$-atoms ($\nu \in I\!N$) and $\lambda \in f_{pq}$. Furthermore

$$\inf \|\lambda \,|\, f_{pq}\| \tag{4.11}$$

where the infimum is taken over all admissible representations (4.10), is an equivalent quasi-norm in $F_{pq}^{(s,\Psi)}(I\!R^n)$.
(ii) Let $0 < p \leq \infty$, $0 < q \leq \infty$, $s \in I\!R$ and Ψ an admissible function according to Definition 2.1. Let $K \in I\!N_0$ and $L + 1 \in I\!N_0$ with

$$K \geq (1 + [s])_+ \quad \textit{and} \quad L \geq \max(-1, [\sigma_p + \theta - s]), \tag{4.12}$$

if Ψ is increasing, or

$$K \geq (1 + [s + \theta'])_+ \quad \textit{and} \quad L \geq \max(-1, [\sigma_p - s]), \tag{4.13}$$

if Ψ is decreasing (θ and θ' as in Remark 2.3), be fixed. Then $f \in \mathcal{S}'(I\!R^n)$ belongs to $B_{pq}^{(s,\Psi)}(I\!R^n)$ if, and only if, it can be represented by (4.10) where $a_{\nu m}(x)$ are 1_K-atoms ($\nu = 0$) or $(s,p,\Psi)_{K,L}$-atoms ($\nu \in I\!N$) and $\lambda \in b_{pq}$. Furthermore

$$\inf \|\lambda \,|\, b_{pq}\| \tag{4.14}$$

where the infimum is taken over all admissible representations (4.10), is an equivalent quasi-norm in $B_{pq}^{(s,\Psi)}(I\!R^n)$.

We refer to the above theorem as the atomic decomposition theorem. For more references to this subject we refer to [FrJ85], [FJW91], [Far98] and [Tri97]. In Theorem 4.4 no information is given about the possibility to obtain atomic decompositions in which the atoms are constructed with the help of dilatations and translations from one smooth function Φ having compact support. In order to present the subatomic decomposition we need to introduce some special building blocks called quarks.

Definition 4.5. *Let $\Phi \in \mathcal{S}(I\!R^n)$ such that*

$$supp\, \Phi \subset 2\, Q_{00} \quad and \quad \sum_{m \in Z\!\!Z^n} \Phi(x-m) = 1, \quad for \quad x \in I\!R^n, \tag{4.15}$$

and let for any $\beta \in I\!N_0^n$, $\Phi^\beta(x) = x^\beta\, \Phi(x)$. If $0 < p \le \infty$, $s \in I\!R$ and Ψ an admissible function according to Definition 2.1, then

$$(\beta qu)_{\nu m}(x) = 2^{-\nu(s-\frac{n}{p})}\, \Psi(2^{-\nu})^{-1}\, \Phi^\beta(2^\nu x - m) \tag{4.16}$$

is called an (s,p,Ψ)-β-quark related to $Q_{\nu m}$.

Remark 4.6. *Up to normalizing constants the (s,p,Ψ)-β-quarks are $(s,p,\Psi)_{K,L}$-atoms for any $K \in I\!N_0$ and $L = -1$. Moreover, the normalizing constants by which the (s,p,Ψ)-β-quark must be divided to become an $(s,p,\Psi)_{K,L}$-atom can be estimated from above by $c\, 2^{(n+1)|\beta|}$, where $c > 0$ is a constant independent of β:*
I) From (4.16) and (4.15), we get

$$supp\, (\beta qu)_{\nu m} \subset 2\, Q_{\nu m}. \tag{4.17}$$

II) Let $K \in I\!N_0$ and $\alpha \in I\!N_0^n$ be a multi-index with $|\alpha| \le K$. We have

$$|D^\alpha (\beta qu)_{\nu m}(x)| = 2^{-\nu(s-\frac{n}{p})+\nu|\alpha|}\, \Psi(2^{-\nu})^{-1}\, |(D^\alpha \Phi^\beta)(2^\nu x - m)|, \tag{4.18}$$

with $\Phi^\beta(x) = x^\beta\, \Phi(x)$. For $\gamma \in I\!N_0^n$ such that $\gamma \le \beta$ we have

$$D^\gamma(x^\beta) = \frac{\beta!}{(\beta-\gamma)!}\, x^{\beta-\gamma}$$

otherwise, this means $\gamma_i > \beta_i$ for some $i \in \{1,\ldots,n\}$, we have $D^\gamma(x^\beta) = 0$. Notice also that for $\gamma \le \beta$

$$\frac{\beta!}{(\beta-\gamma)!} = \prod_{j=1}^n \frac{\beta_j!}{(\beta_j-\gamma_j)!} \le \prod_{j=1}^n \gamma_j\, 2^{\beta_j} \le |\gamma|^n\, 2^{|\beta|} \le K^n\, 2^{|\beta|}. \tag{4.19}$$

Because

$$\max_{|\delta| \le K} \sup_{x \in I\!R^n} |D^\delta \Phi(x)| < \infty,$$

using Leibniz's rule, (4.18) and (4.19) we arrive at

$$|D^\alpha (\beta qu)_{\nu m}(x)| \le c\, 2^{(n+1)|\beta|}\, 2^{-\nu(s-\frac{n}{p})+\nu|\alpha|}\, \Psi(2^{-\nu})^{-1}. \tag{4.20}$$

We have used the fact that, for any $x \in supp\,(\beta qu)_{\nu m}$, $2^\nu x - m \in 2Q_{00}$, and so $|2^\nu x - m| \leq \sqrt{n} \leq 2^n$. Remark that the constant c does not depend on β, it depends only on n and on K.

We will use below the sequence spaces b_{pq} and f_{pq} with respect to the sequences $\lambda^\beta = \{\lambda^\beta_{\nu m} \in \mathbb{C} : \nu \in I\!N_0, m \in \mathbb{Z}^n\}$ where now $\beta \in I\!N_0^n$.

Theorem 4.7. *(i) Let $0 < p < \infty$, $0 < q \leq \infty$, Ψ an admissible function according to Definition 2.1 and $s \in I\!R$ such that*

$$s > \sigma_{pq} + \theta \qquad or \qquad s > \sigma_{pq}, \tag{4.21}$$

whether Ψ is increasing or Ψ is decreasing, respectively (with $\theta \in I\!N_0$ as in Remark 2.3). There exists a number $\mu_0 > 0$ with the following property: let $\mu > \mu_0$; then $f \in \mathcal{S}'(I\!R^n)$ belongs to $F^{(s,\Psi)}_{pq}(I\!R^n)$ if, and only if, it can be represented as

$$f = \sum_{\beta \in I\!N_0^n} \sum_{\nu=0}^{\infty} \sum_{m \in \mathbb{Z}^n} \lambda^\beta_{\nu m}\,(\beta qu)_{\nu m}(x), \tag{4.22}$$

convergence being in $\mathcal{S}'(I\!R^n)$, where $(\beta qu)_{\nu m}$ are (s,p,Ψ)-β-quarks and

$$\sup_{\beta \in I\!N_0^n} 2^{\mu|\beta|}\,\|\lambda^\beta \mid f_{pq}\| < \infty. \tag{4.23}$$

Furthermore, the infimum in (4.23) over all admissible representations (4.22) is an equivalent quasi-norm in $F^{(s,\Psi)}_{pq}(I\!R^n)$.
(ii) Let $0 < p \leq \infty$, $0 < q \leq \infty$, Ψ an admissible function according to Definition 2.1 and $s \in I\!R$ such that

$$s > \sigma_p + \theta \qquad or \qquad s > \sigma_p, \tag{4.24}$$

whether Ψ is increasing or Ψ is decreasing, respectively (with $\theta \in I\!N_0$ as in Remark 2.3). There exists a number $\mu_0 > 0$ with the following property: let $\mu > \mu_0$; then $f \in \mathcal{S}'(I\!R^n)$ belongs to $B^{(s,\Psi)}_{pq}(I\!R^n)$ if, and only if, it can be represented as in (4.22), convergence being in $\mathcal{S}'(I\!R^n)$, where $(\beta qu)_{\nu m}$ are (s,p,Ψ)-β-quarks and

$$\sup_{\beta \in I\!N_0^n} 2^{\mu|\beta|}\,\|\lambda^\beta \mid b_{pq}\| < \infty. \tag{4.25}$$

Furthermore, the infimum in (4.25) over all admissible representations (4.22) is an equivalent quasi-norm in $B^{(s,\Psi)}_{pq}(I\!R^n)$.

Remark 4.8. *To show that $f \in F^{(s,\Psi)}_{pq}(I\!R^n)$ (respectively, $f \in B^{(s,\Psi)}_{pq}(I\!R^n)$) can be decomposed as (4.22) with (4.23) (respectively, (4.22) with (4.25)), we need not the assumption (4.21) (respectively, (4.24)). These restrictions are needed to prove the converse assertion.*

Proof. We will present only the proof of Theorem 4.4 and Theorem 4.7 part (i); the reason is because the proof of part (ii) is somewhat similar but technically simpler. Nevertheless, occasionally we say a word about the modification corresponding to part (ii).

Step 1. (if-part of the atomic decomposition) The proof relies on the equivalent quasi-norm in $F^{(s,\Psi)}_{pq}(I\!R^n)$ given by (3.7), and the underlying local means according to (3.2)-(3.4), where N in (3.6) may be chosen arbitrarily large. For this Step we follow [Tri97, 13.8] although considering two cases (a) and (b), whether Ψ is increasing or decreasing, respectively.
(a) Let first Ψ be monotone increasing. Hence one have (2.6). Let $a_{\nu m}(x)$ with $\nu \in I\!N_0$ and

$m \in \mathbb{Z}^n$ be an 1_K-atom ($\nu = 0$) or an $(s,p,\Psi)_{K,L}$-atom ($\nu \in \mathbb{N}$) according to Definition 4.1, where K and L are fixed integers satisfying (4.8) such that we have (4.10). We have for $j \in \mathbb{N}$

$$2^{js}\,\Psi(2^{-j})\,k_N(2^{-j},a_{\nu m})(x) = 2^{js}\,\Psi(2^{-j}) \int_{\mathbb{R}^n} a_{\nu m}(x+2^{-j}y)\,\left(\Delta^N k^0\right)(y)\,dy. \tag{4.26}$$

We have to distinguish between $j \geq \nu$ and $j < \nu$. The exceptional values $\nu = 0$ and/or $j = 0$ corresponding to 1_K-atoms and the first summand in (3.7), respectively, can be incorporated in the following considerations after the necessary modifications. Assume in the following that $\nu \in \mathbb{N}$ and $j \in \mathbb{N}$. Let $j \geq \nu$. We put

$$a^{\nu m}(y) = 2^{\nu\left(s-\frac{n}{p}\right)}\,\Psi(2^{-\nu})\,a_{\nu m}(2^{-\nu}(y+m)), \tag{4.27}$$

then $a^{\nu m}(x)$ is a 1_K-atom with respect to the unit cube at the origin. Let $K = 2M$ with $M \in \mathbb{N}_0$ for simplicity. By the arguments below the necessary modifications are clear if K is an odd integer. We insert (4.27) in (4.26), choose $N > M$, and obtain by partial integration

$$\begin{aligned}
&2^{js}\,\Psi(2^{-j})\,k_N(2^{-j},a_{\nu m})(x)\\
&= 2^{js}\,\Psi(2^{-j})\,2^{-\nu\left(s-\frac{n}{p}\right)}\,\Psi(2^{-\nu})^{-1}\int_{\mathbb{R}^n} a^{\nu m}(2^{\nu}(x+2^{-j}y)-m)\,\left(\Delta^N k^0\right)(y)\,dy\\
&= 2^{-(K-s)(j-\nu)}\,2^{\frac{n}{p}\nu}\,\Psi(2^{-j})\,\Psi(2^{-\nu})^{-1}\int_{\mathbb{R}^n}\left(\Delta^{N-M}k^0\right)(y)\,\left(\Delta^M a^{\nu m}\right)(2^{\nu}x-m+2^{\nu-j}y)\,dy.
\end{aligned} \tag{4.28}$$

Since both k^0 and $\Delta^M a^{\nu m}$ have supports in a ball centred at origin of some radius $c_1 > 0$, Ψ is increasing and $j \geq \nu$ it follows that

$$2^{js}\,\Psi(2^{-j})\,|k_N(2^{-j},a_{\nu m})(x)| \leq c_2\,2^{-(K-s)(j-\nu)}\,\widetilde{\chi}^{(p)}_{\nu m}(x), \quad j \geq \nu, \tag{4.29}$$

where $\widetilde{\chi}^{(p)}_{\nu m}(x)$ is the p-normalized characteristic function of the cube $4c_1 Q_{\nu m}$. Let $j < \nu$ and put again $k_N(y) = (\Delta^N k^0)(y)$. Then the integration in

$$2^{js}\,\Psi(2^{-j})\,k_N(2^{-j},a_{\nu m})(x) = 2^{js}\,\Psi(2^{-j})\,2^{jn}\int_{\mathbb{R}^n} k_N(2^j y)\,a_{\nu m}(x+y)\,dy \tag{4.30}$$

can be restricted to $\{y \in \mathbb{R}^n : |y| \leq c_1 2^{-j}\}$. Furthermore with L given by (4.8) we expand $k_N(2^j y)$ up to order L with respect to the off point $2^{-\nu}m - x$ and obtain

$$k_N(2^j y) = \sum_{|\beta|\leq L} c_\beta(x)\,(y-2^{-\nu}m+x)^\beta + 2^{j(L+1)}\,\mathcal{O}(|y-2^{-\nu}m+x|^{L+1}). \tag{4.31}$$

We insert (4.31) in (4.30). By (4.5) the terms with $|\beta| \leq L$ vanish. Since

$$|a_{\nu m}(x+y)| \leq 2^{-\nu\left(s-\frac{n}{p}\right)}\,\Psi(2^{-\nu})^{-1}\,\widetilde{\chi}_{\nu m}(x+y), \tag{4.32}$$

where $\widetilde{\chi}(x)$ is the characteristic function of $c\,Q_{\nu m}$, we obtain

$$\begin{aligned}
&2^{js}\,\Psi(2^{-j})\,|k_N(2^{-j},a_{\nu m})(x)|\\
&\leq 2^{(s+n)j}\,2^{-\nu\left(s-\frac{n}{p}\right)}\,\Psi(2^{-j})\,\Psi(2^{-\nu})^{-1}\int_{\{y:|y|\leq c_1 2^{-j}\}} 2^{j(L+1)}\,\widetilde{\chi}_{\nu m}(x+y)\,\mathcal{O}(|y-2^{-\nu}m+x|^{L+1})\,dy\\
&\leq c_3\,2^{(s+n)j}\,2^{-\nu\left(s-\frac{n}{p}\right)}\,\Psi(2^{-j})\,\Psi(2^{-\nu})^{-1}\,2^{(j-\nu)(L+1)}\int_{\{y:|y|\leq c_1 2^{-j}\}}\widetilde{\chi}_{\nu m}(x+y)\,dy\\
&\leq c_3\,2^{\theta}\,2^{(s+n)j}\,2^{-\nu\left(s-\frac{n}{p}\right)}\,2^{(j-\nu)(L+1-\theta)}\int_{\{y:|y|\leq c_1 2^{-j}\}}\widetilde{\chi}_{\nu m}(x+y)\,dy.
\end{aligned} \tag{4.33}$$

The last inequality comes from $j < \nu$ and (2.6), because $\nu = j + (\nu - j) \leq j + 2^{\nu-j} \leq 2^{\nu-j+1}j$ implies

$$\Psi(2^{-j}) \leq 2^{\theta(\nu-j+1)}\,\Psi(2^{-2^{\nu-j+1}j}) \leq 2^{\theta(\nu-j+1)}\,\Psi(2^{-\nu}).$$

Recall that $j<\nu$. The integral in (4.33) is at most $c_1^n\,2^{-\nu n}$ and it is zero if x is outside a cube $c_4\,2^{\nu-j}\,Q_{\nu m}$ (centred at $2^{-\nu}m$ and with side length $c_4\,2^{-j}$). Hence,

$$\int_{|y|\le c_1 2^{-j}} \widetilde{\chi}_{\nu m}(x+y)\,dy \le c_1^n\,2^{-\nu n}\,\chi(c_4\,2^{\nu-j}\,Q_{\nu m})(x), \tag{4.34}$$

where $\chi(c_4\,2^{\nu-j}\,Q_{\nu m})(x)$ is the characteristic function of the indicated cube. For $x\in c_4\,2^{\nu-j}\,Q_{\nu m}$ we have

$$(\mathcal{M}\chi_{\nu m})\,(x) \ge |c_4\,2^{\nu-j}\,Q_{\nu m}|^{-1}\int_{c_4\,2^{\nu-j}\,Q_{\nu m}} \chi_{\nu m}(y)\,dy \ge c_4^{-n}\,2^{-(\nu-j)n}. \tag{4.35}$$

Let $0<w<\min(1,p,q)$. From (4.35) and (4.34) we get

$$\begin{aligned}\int_{\{y:|y|\le c_1 2^{-j}\}} \widetilde{\chi}_{\nu m}(x+y)\,dy &\le c_1^n\,2^{-\nu n}\,c_4^{\frac{n}{w}}\,2^{(\nu-j)\frac{n}{w}}\,(\mathcal{M}\chi_{\nu m})^{\frac{1}{w}}\,(x)\\ &\le c_5\,2^{-\nu n}\,2^{(\nu-j)\frac{n}{w}}\,(\mathcal{M}\chi_{\nu m})^{\frac{1}{w}}\,(x),\ x\in I\!R^n.\end{aligned} \tag{4.36}$$

Replacing $\chi_{\nu m}$ in (4.36) by $\chi_{\nu m}^{(p)}$ and inserting the estimate (4.36) in (4.33) we obtain

$$\begin{aligned}&2^{sj}\,\Psi(2^{-j})\,|k_N(2^{-j},a_{\nu m})(x)|\\ &\le c_3\,c_5\,2^{\theta}\,2^{-(\nu-j)(L+1+s+n-\theta-\frac{n}{w})}\,\left(\mathcal{M}\chi_{\nu m}^{(p)w}\right)^{\frac{1}{w}}(x),\ x\in I\!R^n\end{aligned} \tag{4.37}$$

Since $L\ge[\sigma_{pq}+\theta-s]$ the number w can be chosen such that $\eta=L+1+s+n-\theta-\frac{n}{w}>0$. Hence

$$2^{sj}\,\Psi(2^{-j})\,|k_N(2^{-j},a_{\nu m})(x)| \le c_6\,2^{-(\nu-j)\eta}\,\left(\mathcal{M}\chi_{\nu m}^{(p)w}\right)^{\frac{1}{w}}(x),\quad j<\nu \tag{4.38}$$

with $\eta>0$. Combining (4.29) and (4.38) we obtain for $q\le 1$,

$$\begin{aligned}&\left|2^{sj}\,\Psi(2^{-j})\,k_N\left(2^{-j},\sum_{\nu,m}\lambda_{\nu m}\,a_{\nu m}\right)(x)\right|^q\\ &\le c_2^q\sum_{\nu\le j}\sum_m |\lambda_{\nu m}|^q\,2^{-\rho(j-\nu)q}\,\widetilde{\chi}_{\nu m}^{(p)q}(x)\\ &\quad+c_6^q\sum_{\nu>j}\sum_m |\lambda_{\nu m}|^q\,2^{-\eta(\nu-j)q}\,\left(\mathcal{M}\chi_{\nu m}^{(p)w}\right)^{\frac{q}{w}}(x),\end{aligned} \tag{4.39}$$

for some $\rho>0$ and $\eta>0$. We sum over j, take the $\frac{1}{q}$ power and afterwards the $L_p(I\!R^n)$-quasi-norm and arrive at

$$\begin{aligned}&\left\|\left(\sum_{j=1}^{\infty}2^{jsq}\,\Psi(2^{-j})^q\left|k_N\left(2^{-j},\sum_{\nu,m}\lambda_{\nu m}\,a_{\nu m}\right)(x)\right|^q\right)^{\frac{1}{q}}\,|\,L_p(I\!R^n)\right\|\\ &\le c_7\left\|\left(\sum_{\nu,m}|\lambda_{\nu m}|^q\,\widetilde{\chi}_{\nu m}^{(p)q}(.)\right)^{\frac{1}{q}}\,|\,L_p(I\!R^n)\right\|+\\ &\quad+c_7\left\|\left(\sum_{\nu,m}|\lambda_{\nu m}|^q\,\left(\mathcal{M}\chi_{\nu m}^{(p)w}\right)^{\frac{q}{w}}(.)\right)^{\frac{1}{q}}\,|\,L_p(I\!R^n)\right\|.\end{aligned} \tag{4.40}$$

The modification of (4.39) if $1<q\le\infty$ is clear, using Hölder inequality. Hence (4.40) holds for any $0<q\le\infty$. The first summand on the right-hand side of is just what we want, since $\widetilde{\chi}_{\nu m}^{(p)}$

can be replaced by $\chi^{(p)}_{\nu m}$. With $g_{\nu m}(x) = \lambda_{\nu m}\, \chi^{(p)}_{\nu m}(x)$ the second summand on the right-hand side can be written as

$$c_7 \left\| \left(\sum_{\nu m} (\mathcal{M} g^w_{\nu m})\,(.)^{\frac{q}{w}} \right)^{\frac{w}{q}} \,|\, L_{\frac{p}{w}}(I\!R^n) \right\|^{\frac{1}{w}}. \tag{4.41}$$

Since $1 < \frac{q}{w} \le \infty$ and $1 < \frac{p}{w} < \infty$ we can apply the vector-valued maximal inequality of Fefferman and Stein, see [Tri92, 2.2.2, p.89]. This gives that (4.41) can be estimated from above by

$$c_8 \left\| \left(\sum_{\nu m} (g^w_{\nu m}(.))^{\frac{q}{w}} \right)^{\frac{w}{q}} \,|\, L_{\frac{p}{w}}(I\!R^n) \right\|^{\frac{1}{w}} = c_8\, \|\lambda \,|\, f_{pq}\|. \tag{4.42}$$

This gives the wanted estimate. As we said, the terms with $\nu = 0$ and/or $j = 0$ are also covered by this technique. Then we obtain

$$\|f \,|F^{(s,\Psi)}_{pq}(I\!R^n)\| \le c\, \|\lambda \,|\, f_{pq}\|. \tag{4.43}$$

(b) In the case that the admissible function is decreasing we have (2.7), and the integers K, L satisfy (4.9). Let us see the necessary modifications in the proof of case (a). Everything is the same up to (4.28). Since (2.7) and $j \ge \nu$, we have $j = (j-\nu)+\nu \le 2^{j-\nu+1}\,\nu$, and then

$$\Psi(2^{-j}) \le \Psi(2^{-2^{j-\nu+1}\nu}) \le 2^{\theta'(j-\nu+1)}\,\Psi(2^{-\nu}).$$

Instead of (4.29) we arrive at

$$2^{js}\,\Psi(2^{-j})\,|k_N(2^{-j}, a_{\nu m})(x)| \le c_2\, 2^{-(K-s-\theta')}\,\widetilde{\chi}^{(p)}_{\nu m}(x), \quad j \ge \nu, \tag{4.44}$$

where $K - s - \theta' > 0$, because now we have (4.9). When $j < \nu$, everything is the same up to (4.32). In this case Ψ is decreasing and $j < \nu$ so

$$\Psi(2^{-j})\,\Psi(2^{-\nu})^{-1} \le 1.$$

Hence, instead of (4.33) we simply have

$$2^{js}\,\Psi(2^{-j})\,|k_N(2^{-j}, a_{\nu m})(x) \le c_3\, 2^{(s+n)j}\, 2^{-\nu\left(s-\frac{n}{p}\right)}\, 2^{(j-\nu)(L+1)} \int_{\{y:|y|\le c_1 2^{-j}\}} \widetilde{\chi}_{\nu m}(x+y)\,dy. \tag{4.45}$$

With this and following (a), leads to

$$2^{sj}\,\Psi(2^{-j})\,|k_N(2^{-j}, a_{\nu m})(x)| \le c_6\, 2^{-(\nu-j)\eta}\, \left(\mathcal{M}\chi^{(p)w}_{\nu m}\right)^{\frac{1}{w}}(x), \quad j < \nu \tag{4.46}$$

with $\eta = s + L + 1 + n - \frac{n}{w} > 0$, because of (4.9). Using now (4.44) and (4.46) instead of (4.29) and (4.38), as in (a), one would arrive in the same way at (4.43).
Let us just mention that in the corresponding proof of Step 1 for the $B^{(s,\Psi)}_{pq}(I\!R^n)$ spaces is sufficient that w satisfies $0 < w < \min(1,p)$, since one only needs the scalar Hardy-Littlewood maximal theorem which holds also for $p = \infty$, see [Tri83, Remark 1.2.3, p.15]. This is the reason for the modification in (4.8), (4.9) to (4.12), (4.13).

Step 2. (only-if part of the subatomic decomposition) The idea on this Step goes back to the proof of the subatomic decomposition for the spaces $B^s_{pq}(I\!R^n)$ presented in [Tri97, 14.15].
Let $f \in F^{(s,\Psi)}_{pq}(I\!R^n)$. Then we have

$$\widehat{f} = \sum_{\nu=0}^{\infty} \varphi_\nu \widehat{f}, \qquad \text{in} \quad \mathcal{S}'(I\!R^n). \tag{4.47}$$

Let Q_ν be a cube in $I\!R^n$ centred at origin and with side length $2\pi 2^\nu$. In particular we have $\operatorname{supp}\varphi_\nu \subset Q_\nu$. We interpret $\varphi_\nu \widehat{f}$ as a periodic distribution and expand it in Q_ν by

$$\varphi_\nu \widehat{f} = \sum_{k\in \mathbb{Z}^n} b_{\nu k}\, \exp(-i2^{-\nu}k\xi), \quad \xi \in Q_\nu, \tag{4.48}$$

with

$$b_{\nu k} = (2\pi)^{-\frac{n}{2}}\, 2^{-\nu n}\, (\varphi_\nu \widehat{f})^\vee(2^{-\nu}k). \tag{4.49}$$

Let $\varrho \in \mathcal{S}(I\!R^n)$ with $\varrho(x) = 1$ if $|x| \le 2$ and $\operatorname{supp}\varrho \subset \pi\, Q_{00}$ and let $\varrho_\nu(\xi) = \varrho(2^{-\nu}\xi)$. Then $\varrho_\nu(\xi) = 1$ if $\xi \in \operatorname{supp}\varphi_\nu$ and $\operatorname{supp}\varrho_\nu \subset Q_\nu$. We multiply (4.48) with ϱ_ν and extend it from Q_ν to $I\!R^n$. Hence

$$(\varphi_\nu \widehat{f})^\vee(x) = \sum_{k\in \mathbb{Z}^n} b_{\nu k}\, [\exp(-i2^{-\nu}k.)\,\varrho_\nu(.)]^\vee(x) = \sum_{k\in \mathbb{Z}^n} d_{\nu k}\, 2^{-\nu\left(s-\frac{n}{p}\right)}\, \check{\varrho}(2^\nu x - k) \tag{4.50}$$

with

$$d_{\nu k} = (2\pi)^{-\frac{n}{2}}\, 2^{\nu\left(s-\frac{n}{p}\right)}\, (\varphi_\nu \widehat{f})^\vee(2^{-\nu}k). \tag{4.51}$$

The entire analytic function $\check{\varrho}(x) \in \mathcal{S}(I\!R^n)$ can be extended from $I\!R^n$ to $\mathbb{C}^n$, furthermore by the Paley-Wiener-Schwartz theorem, see [Tri83, 1.2.1, p.13], for any $\lambda > 0$ and appropriate $c_\lambda > 0$

$$|\check{\varrho}(x+iy)| \le c_\lambda\, e^{c|y|}\, (1+|x|^2)^{-\lambda}, \quad x, y \in I\!R^n. \tag{4.52}$$

Iterative application of Cauchy representation theorem in the complex plane yields

$$\check{\varrho}(z_1,\dots,z_n) = (2\pi i)^{-n} \int_{|\xi_1 - z_1|=1} \cdots \int_{|\xi_n - z_n|=1} \frac{\check{\varrho}(\xi_1,\dots,\xi_n)}{(\xi_1 - z_1)\dots(\xi_n - z_n)} d\xi_1 \dots \xi_n, \tag{4.53}$$

where $z_k \in \mathbb{C}$. By (4.52) we obtain in particular

$$|D^\alpha \check{\varrho}(x)| \le c'_\lambda\, \beta!\, (1+|x|^2)^{-\lambda}, \quad x \in I\!R^n, \tag{4.54}$$

where c'_λ does not depend on $x \in I\!R^n$ and the multi-index β. Let Φ be as in (4.15). With $\mu \in I\!N_0$ fixed, from (4.50)

$$(\varphi_\nu \widehat{f})^\vee = \sum_{k\in \mathbb{Z}^n} d_{\nu k}\, 2^{-\nu\left(s-\frac{n}{p}\right)} \sum_{m\in \mathbb{Z}^n} \check{\varrho}(2^\nu x - k)\, \Phi(2^{(\nu+\mu)}x - m). \tag{4.55}$$

For fixed $k \in \mathbb{Z}^n$ we expand $\check{\varrho}(2^\nu . - k)$ at the point $2^{-(\nu+\mu)}m$ and obtain

$$\check{\varrho}(2^\nu x - k) = \sum_{\beta \in I\!N_0^n} \frac{2^{\nu|\beta|}}{\beta!}\, (D^\beta \check{\varrho})(2^{-\mu}m - k)\, (x - 2^{-(\nu+\mu)}m)^\beta. \tag{4.56}$$

Putting (4.56) in (4.55) gives

$$(\varphi_\nu \widehat{f})^\vee = 2^{-\nu\left(s-\frac{n}{p}\right)} \sum_{k\in \mathbb{Z}^n} d_{\nu k} \sum_{m\in \mathbb{Z}^n} \sum_{\beta \in I\!N_0^n} \frac{2^{-\mu|\beta|}}{\beta!}\, (D^\beta \check{\varrho})(2^{-\mu}x - k)\, \Phi^\beta(2^{(\nu+\mu)}x - m). \tag{4.57}$$

We insert this equality in (4.47) and have

$$f = \sum_{\beta\in \mathbb{N}_0^n} \sum_{\nu=0}^{\infty} \sum_{m\in\mathbb{Z}^n} 2^{\mu\left(s-\frac{n}{p}\right)}\, \Psi(2^{-(\nu+\mu)})^{-1}\, 2^{-(\nu+\mu)\left(s-\frac{n}{p}\right)}\, \Phi^\beta(2^{(\nu+\mu)}x-m) \times$$

$$\times \sum_{k\in\mathbb{Z}^n} d_{\nu k}\, \frac{(D^\beta \check{\varrho})(2^{-\mu}m-k)}{\beta!}\, 2^{-\mu|\beta|}\, \Psi(2^{-(\nu+\mu)})$$

$$= \sum_{\beta\in \mathbb{N}_0^n} \sum_{\nu=0}^{\infty} \sum_{m\in\mathbb{Z}^n} \lambda^\beta_{\nu+\mu,m}\, 2^{\mu\left(s-\frac{n}{p}\right)}\, (\beta qu)_{\nu+\mu,m}(x), \tag{4.58}$$

where $(\beta qu)_{\nu,m}(x)$ are (s,p,Ψ)-β-quarks (related to $Q_{\nu m}$) and

$$\lambda^\beta_{\nu+\mu,m} = 2^{-\mu|\beta|} \sum_{k\in\mathbb{Z}^n} d_{\nu k}\, \frac{(D^\beta \check{\varrho})(2^{-\mu}m-k)}{\beta!}\, \Psi(2^{-(\nu+\mu)}) = 2^{-\mu|\beta|}\, \vartheta^\beta_{\nu+\mu,m}. \tag{4.59}$$

We may replace in (4.58) $\nu+\mu$ by ν and obtain (4.22). Next, we will prove in one hand that there exists a constant $c>0$ such that

$$\|\{d_{\nu k}\, \Psi(2^{-\nu})\}_{\nu,k} \mid f_{pq}\| \le c\, \|f \mid F^{(s,\Psi)}_{pq}(\mathbb{R}^n)\|, \tag{4.60}$$

and on the other hand that there exists a constant $c'>0$, independent of β, such that

$$\|\vartheta^\beta \mid f_{pq}\| \le c'\, \|\{d_{\nu k}\, \Psi(2^{-\nu})\}_{\nu,k} \mid f_{pq}\|. \tag{4.61}$$

This two last inequalities together with (4.59) yields

$$2^{\mu|\beta|}\, \|\lambda^\beta \mid f_{pq}\| \le C\, \|f \mid F^{(s,\Psi)}_{pq}(\mathbb{R}^n)\|, \tag{4.62}$$

where C is a constant independent of β. Let us begin the proof of (4.60). For fixed $\nu\in\mathbb{N}_0$ we have

$$\sum_{k\in\mathbb{Z}^n} |d_{\nu k}\, \Psi(2^{-\nu})\, \chi^{(p)}_{\nu k}(x)|^q \le (2\pi)^{-\frac{n}{2}q}\, 2^{\nu s q}\, \Psi(2^{-\nu})^q \sum_{k\in\mathbb{Z}^n} \chi_{\nu k}(x) \left(\sup_{y\in Q_{\nu k}} |(\varphi_\nu \widehat{f})^\vee(y)| \right)^q$$

$$\le (1+\sqrt{n})^{aq}\, (2\pi)^{-\frac{n}{2}q}\, 2^{\nu s q}\, \Psi(2^{-\nu})^q\, (\varphi^*_\nu f)^q_a(x), \tag{4.63}$$

since for $x,y\in Q_{\nu k}$, $|x-y|\le \sqrt{n}\,2^{-\nu}$, $\sum_{k\in\mathbb{Z}^n}\chi_{\nu k}(x)=1$, and with $a>\frac{n}{\min(p,q)}$, $(\varphi^*_\nu f)_a$ the Peetre maximal function. It follows that

$$\sum_{\nu=0}^{\infty} \sum_{k\in\mathbb{Z}^n} |d_{\nu k}\, \Psi(2^{-\nu})\, \chi^{(p)}_{\nu k}(x)|^q \le c_1 \sum_{\nu=0}^{\infty} 2^{\nu s q}\, \Psi(2^{-\nu})^q\, (\varphi^*_\nu f)^q_a(x).$$

Taking $\frac{1}{q}$-power followed by L_p-quasi-norm and using Theorem 2.7, we get

$$\|\{d_{\nu k}\, \Psi(2^{-\nu})\}_{\nu,k} \mid f_{pq}\| \le c\, \|f \mid F^{(s,\Psi)}_{pq}(\mathbb{R}^n)\|. \tag{4.64}$$

Let us now prove (4.61). Let $\nu\in\mathbb{N}_0$ and $k\in\mathbb{Z}^n$ be fixed. Recall that

$$\Psi(2^{-(\nu+\mu)}) \le \Psi(2^{-\nu}) \qquad \text{or} \qquad \Psi(2^{-(\nu+\mu)}) \le 2^{\theta'\mu}\, \Psi(2^{-\nu})$$

whether Ψ is increasing or decreasing, respectively (with $\theta'\in\mathbb{N}_0$ as in Remark 2.3). By (4.54) there exists a constant c'_λ independent of β with

$$|\vartheta^\beta_{\nu+\mu,m}| \le \sum_{k\in\mathbb{Z}^n} \frac{|(D^\beta\check{\varrho})(2^{-\mu}m-k)|}{\beta!}\, |d_{\nu k}|\, \Psi(2^{-(\nu+\mu)})$$

$$\le c'_\lambda \sum_{k\in\mathbb{Z}^n} \Big(1+|2^\nu(2^{-(\nu+\mu)}m)-k)|\Big)^{-\lambda}\, |d_{\nu k}|\, \Psi(2^{-\nu}). \tag{4.65}$$

We denote $x_m = 2^{-(\nu+\mu)}m$ and let $k_m \in Z\!\!Z^n$ such that $x_m \in Q_{\nu,k_m}$; then clearly $|2^\nu x_m - k_m| \le \frac{\sqrt{n}}{2}$. We decompose $Z\!\!Z^n$ in the sets

$$E_j = \{k \in Z\!\!Z^n : 2^j - 1 \le |k - k_m| \le 2^{j+1} - 1\}, \quad j \in I\!N_0.$$

If $j \in I\!N_0$ is fixed, for $k \in E_j$ we have in one hand

$$2^j \le 1 + |k - k_m| \le 1 + |k - 2^\nu x_m| + |2^\nu x_m - k_m| \le \max(\sqrt{n}, 2)\,(1 + |2^\nu x_m - k|)$$

and so

$$(1 + |2^\nu x_m - k|)^{-\lambda} \le c_1\, 2^{-j\lambda}, \tag{4.66}$$

where $c_1 > 0$ is independent of ν, k and m. On the other hand, if $x \in Q_{\nu+\mu,m}$ and $y \in Q_{\nu k}$

$$\begin{aligned} |y - x| &\le |y - 2^{-\nu}k| + 2^{-\nu}|k - k_m| + |2^{-\nu}k_m - x_m| + |x_m - x| \\ &\le \sqrt{n}\,(1 + 2^{-\mu-1})\,2^{-\nu}(1 + |k - k_m|) \le c_2\, 2^{-\nu+j} \end{aligned} \tag{4.67}$$

where c_2 is independent of ν, k, m but may depend on μ and n. Choose now $0 < w < \min(1, p, q)$. For a fixed $\nu \in I\!N_0$ the cubes $Q_{\nu k}$ have volume $2^{-\nu n}$ and are disjoint so that using the embedding $l_w \subset l_1$ and (4.67) we obtain

$$\begin{aligned} \sum_{k\in E_j} |d_{\nu k}| &\le \left(\sum_{k\in E_j} |d_{\nu k}|^w\right)^{\frac{1}{w}} = \left(2^{\nu n}\int_{|y-x|\le c_2\,2^{j-\nu}} \left(\sum_{k\in E_j} |d_{\nu k}|\,\chi_{\nu k}(y)\right)^w dy\right)^{\frac{1}{w}} \\ &\le c_3\left(2^{jn}\,\mathcal{M}\left(\sum_{k\in Z\!\!Z^n} |d_{\nu k}|\,\chi_{\nu k}\right)^w (x)\right)^{\frac{1}{w}}, \end{aligned} \tag{4.68}$$

for $x \in Q_{\nu+\mu,m}$, where $\mathcal{M}$ stands for the Hardy-Littlewood maximal function and c_3 is a constant independent of ν, m, k. Using (4.68) and (4.66) in (4.65) and assuming $\lambda > \frac{n}{w}$ is sufficiently large we have

$$\begin{aligned} &|\vartheta^\beta_{\nu+\mu,m}\,\chi^{(p)}_{\nu+\mu,m}(x)| \\ &\le c'_\lambda \sum_{k\in Z\!\!Z^n}\left(1 + |2^\nu(2^{-(\nu+\mu)}m) - k)|\right)^{-\lambda} |d_{\nu k}|\,\Psi(2^{-\nu})\,2^{(\nu+\mu)\frac{n}{p}}\,\chi_{\nu+\mu,m}(x) \\ &\le c_4 \sum_{j=0}^{\infty} 2^{-j\lambda} \sum_{k\in E_j} |d_{\nu k}|\,\Psi(2^{-\nu})\,2^{(\nu+\mu)\frac{n}{p}}\,\chi_{\nu+\mu,m}(x) \\ &\le c_5 \sum_{j=0}^{\infty} 2^{-j(\lambda-\frac{n}{w})}\left(\mathcal{M}\left(\sum_{k\in Z\!\!Z^n} |d_{\nu k}|\,\chi^{(p)}_{\nu k}\right)^w (x)\right)^{\frac{1}{w}} \Psi(2^{-\nu})\,\chi_{\nu+\mu,m}(x) \\ &\le c_6\,\Psi(2^{-\nu})\left(\mathcal{M}\left(\sum_{k\in Z\!\!Z^n} |d_{\nu k}|\,\chi^{(p)}_{\nu k}\right)^w (x)\right)^{\frac{1}{w}} \chi_{\nu+\mu,m}(x) \end{aligned} \tag{4.69}$$

where the constant c_6 does not depend upon ν, m but may depend on μ. In (4.69) we take the q-power, sum over $m \in Z\!\!Z^n$ and then over $\nu \in I\!N_0$ and get

$$\sum_{\nu=0}^{\infty}\sum_{m\in Z\!\!Z^n} |\vartheta^\beta_{\nu+\mu,m}\,\chi^{(p)}_{\nu+\mu,m}(x)|^q \le c_6^q \sum_{\nu=0}^{\infty} (h^w_\nu(x))^{\frac{q}{w}}$$

where $h_\nu = \Psi(2^{-\nu})\sum_{k\in Z\!\!Z^n} |d_{\nu k}|\,\chi^{(p)}_{\nu k}$ (with the usual modification if $q = \infty$). Taking the $\frac{1}{q}$-power and the L_p-quasi-norm, and applying the Fefferman-Stein inequality, as in [Tri92, 2.2.2, p.89], since $1 < \frac{p}{w} < \infty$ and $1 < \frac{q}{w} \leq \infty$, we arrive at

$$\begin{aligned}\|\vartheta^\beta \,|\, f_{pq}\| &\leq c_6 \, \|\mathcal{M}\,(h_\nu{}^w(.))^{\frac{1}{w}} \,|\, L_p(l_q)\| = c_6 \, \|\mathcal{M}(h_\nu^w(.)) \,|\, L_{\frac{p}{w}}(l_{\frac{q}{w}})\|^{\frac{1}{w}} \\ &\leq c_7 \, \|h_\nu^w(.) \,|\, L_{\frac{p}{w}}(l_{\frac{q}{w}})\|^{\frac{1}{w}} = c_7 \, \|h_\nu(.) \,|\, L_p(l_q)\| = c_7 \, \|\Psi(2^{-\nu}) \, d_{\nu k} \,|\, f_{pq}\|.\end{aligned}$$

And so the only-if part of the subatomic decomposition for $F_{pq}^{(s,\Psi)}(I\!R^n)$ is complete. In what concerns the corresponding proof for $B_{pq}^{(s,\Psi)}(I\!R^n)$, one has to obtain analogous inequalities as (4.60) and (4.61) (of course with b_{pq} instead of f_{pq}). As for the counterpart of (4.60), it can be proved using the arguments in [Tri97, 14.15, p.102]. As for the counterpart of (4.61), one can use Remark 4.3 and in the proof it is sufficient to choose $0 < w < \min(1, p)$ and use the scalar Hardy-Littlewood maximal theorem which holds also for $p = \infty$.

Step 3. (only-if part of the atomic decomposition) Let $f \in F_{pq}^{(s,\Psi)}(I\!R^n)$. Let first $s > \sigma_{pq} + \theta$ if Ψ is increasing or $s > \sigma_{pq}$ if Ψ is decreasing, and fix $K \in I\!N_0$, $L = -1$ satisfying (4.8) or (4.9) whether Ψ is increasing or decreasing. By Step 2

$$f = \sum_{\beta \in I\!N_0^n} \sum_{\nu=0}^{\infty} \sum_{m \in Z\!\!Z^n} \lambda_{\nu m}^\beta \, (\beta q u)_{\nu m}(x),$$

and

$$\sup_{\beta \in I\!N_0^n} 2^{\mu|\beta|} \, \|\lambda^\beta \,|\, f_{pq}\| < \infty, \tag{4.70}$$

for any $\mu > 0$ sufficiently large. Let

$$a_{\nu m}(x) = \sum_{\beta \in I\!N_0^n} \frac{\lambda_{\nu m}^\beta}{\Lambda_{\nu m}} \, (\beta q u)_{\nu m}(x), \quad \nu \in I\!N_0, \; m \in Z\!\!Z^n, \tag{4.71}$$

with

$$\Lambda_{\nu m} = c_K \sum_{\beta \in I\!N_0^n} |\lambda_{\nu m}^\beta| \, 2^{(n+1)|\beta|} \tag{4.72}$$

(with the addicional factor $\Psi(1)$ on the right hand side, if $\nu = 0$) and c_K the constant c in (4.20) (constant depending only upon n and K, but not depending on β). Then

$$f = \sum_{\nu=0}^{\infty} \sum_{m \in Z\!\!Z^n} \Lambda_{\nu m} \, a_{\nu m}(x). \tag{4.73}$$

It follows by straightforward calculations, using (4.17) and (4.20), that $a_{\nu m}$ are $(s, p, \Psi)_{K,-1}$-atoms if $\nu \in I\!N$ or 1_K-atoms if $\nu = 0$. Finally, we will show that there exists a constant $c > 0$ such that

$$\|\Lambda \,|\, f_{pq}\| \leq c \, \|f \,|\, F_{pq}^{(s,\Psi)}(I\!R^n)\|,$$

where $\Lambda = \{\Lambda_{\nu m} : \nu \in I\!N_0, \; m \in Z\!\!Z^n\}$. This will be done by showing that

$$\|\Lambda \,|\, f_{pq}\| \leq \sup_{\beta \in I\!N_0^n} 2^{\mu|\beta|} \, \|\lambda^\beta \,|\, f_{pq}\|,$$

for some μ sufficiently large. Let $0 < q \le 1$, then with q' the conjugate exponent of $\frac{1}{q}$ and $\varepsilon > n+1$, we have

$$\sum_{\nu=0}^{\infty}\sum_{m\in Z\!\!Z^n} |\Lambda_{\nu m}\,\chi^{(p)}_{\nu m}(x)|^q \le c_K^q \sum_{\nu=0}^{\infty}\sum_{m\in Z\!\!Z^n}\sum_{\beta\in I\!N_0^n} |\lambda^\beta_{\nu m}\,\chi^{(p)}_{\nu m}(x)|^q\, 2^{(n+1)|\beta| q}$$

$$\le c_K^q \left(\sum_{\beta\in I\!N_0^n} 2^{(n+1-\varepsilon)|\beta| q q'}\right)^{\frac{1}{q'}} \left\{\sum_{\beta\in I\!N_0^n}\left(2^{\varepsilon|\beta| q}\sum_{\nu=0}^{\infty}\sum_{m\in Z\!\!Z^n} |\lambda^\beta_{\nu m}\,\chi^{(p)}_{\nu m}(x)|^q\right)^{\frac{1}{q}}\right\}^q$$

$$\le c_1^q \left\{\sum_{\beta\in I\!N_0^n} 2^{\varepsilon|\beta| q}\left(\sum_{\nu=0}^{\infty}\sum_{m\in Z\!\!Z^n} |\lambda^\beta_{\nu m}\,\chi^{(p)}_{\nu m}(x)|^q\right)^{\frac{1}{q}}\right\}^q. \tag{4.74}$$

Let $\overline{p} = \min(1,p)$, from (4.74) we get

$$\left\|\left(\sum_{\nu=0}^{\infty}\sum_{m\in Z\!\!Z^n} |\Lambda_{\nu m}\,\chi^{(p)}_{\nu m}(x)|^q\right)^{\frac{1}{q}} \,|\, L_p(I\!R^n)\right\|^{\overline{p}}$$

$$\le c_1 \sum_{\beta\in I\!N_0^n} 2^{\varepsilon|\beta| q}\left\|\left(\sum_{\nu=0}^{\infty}\sum_{m\in Z\!\!Z^n} |\lambda^\beta_{\nu m}\,\chi^{(p)}_{\nu m}(x)|^q\right)^{\frac{1}{q}} \,|\, L_p(I\!R^n)\right\|^{\overline{p}}.$$

Hence

$$\|\Lambda \,|\, f_{pq}\| \le c_2 \sup_{\beta\in I\!N_0^n} 2^{\varepsilon_1|\beta|}\, \|\lambda^\beta \,|\, f_{pq}\|, \tag{4.75}$$

for some $\varepsilon_1 > \varepsilon$. If $1 < q \le \infty$, let q_1 be its conjugate exponent and $\varepsilon > 0$, then

$$\sum_{\nu=0}^{\infty}\sum_{m\in Z\!\!Z^n} |\Lambda_{\nu m}\,\chi^{(p)}_{\nu m}(x)|^q = c_K^q \sum_{\nu=0}^{\infty}\sum_{m\in Z\!\!Z^n}\left(\sum_{\beta\in I\!N_0^n} 2^{-\varepsilon|\beta|}\, 2^{(n+1-\varepsilon)|\beta|}|\lambda^\beta_{\nu m}\,\chi^{(p)}_{\nu m}(x)|\right)^q$$

$$\le c_K^q \sum_{\nu=0}^{\infty}\sum_{m\in Z\!\!Z^n}\left(\sum_{\beta\in I\!N_0^n} 2^{-\varepsilon|\beta| q_1}\right)^{\frac{q}{q_1}} \sum_{\beta\in I\!N_0^n} 2^{(n+1+\varepsilon)|\beta| q}|\lambda^\beta_{\nu m}\,\chi^{(p)}_{\nu m}(x)|^q$$

$$\le c_2^q \sum_{\beta\in I\!N_0^n} 2^{(n+1+\varepsilon)|\beta| q}\sum_{\nu=0}^{\infty}\sum_{m\in Z\!\!Z^n} |\lambda^\beta_{\nu m}\,\chi^{(p)}_{\nu m}(x)|^q. \tag{4.76}$$

The rest follows as in case $0 < q \le 1$. Therefore we get an inequality as in (4.75) in any case of p and q.

Now let $s \in I\!R$ arbitrary and let $K \in I\!N_0$, $L \in I\!N_0$ satisfying (4.8) or (4.9) whether Ψ is increasing or decreasing, respectively, be fixed. Choose $M \in I\!N$ such that $2M > L$. As remarked in [Tri97, 13.8, p.80] we can change the lift described in 2.4, for $I_{2M}\, f = ((1+|\xi|^{2M})\,\widehat{f})^\vee$ with inverse

$$id + (-\Delta)^M : F^{(s+2M,\Psi)}_{pq}(I\!R^n) \to F^{(s,\Psi)}_{pq}(I\!R^n).$$

So $f \in F^{(s,\Psi)}_{pq}(I\!R^n)$ can be represented as

$$f = g + (-\Delta)^M g \qquad \text{with} \quad \|f \,|\, F^{(s,\Psi)}_{pq}(I\!R^n)\| \sim \|g \,|\, F^{(s+2M,\Psi)}_{pq}(I\!R^n)\|. \tag{4.77}$$

We apply this argument to g with $s+2M$ in place of s. Iteration yields

$$f = f_1 + (-\Delta)^M f_2 \qquad f_1 \in F^{(s+2jM,\Psi)}_{pq}(I\!R^n), \quad f_2 \in F^{(s+2M,\Psi)}_{pq}(I\!R^n), \tag{4.78}$$

with

$$\|f_1 \,|\, F_{pq}^{(s+2jM,\Psi)}(I\!R^n)\| \sim \|f \,|\, F_{pq}^{(s,\Psi)}(I\!R^n)\| \sim \|f_2 \,|\, F_{pq}^{(s+2M,\Psi)}(I\!R^n)\|, \tag{4.79}$$

where j can be chosen arbitrarily large. We choose $\sigma > K$ and iterate as indicated above until the level j such that $s + 2jM - \frac{n}{p} > \sigma$. Hence, by Proposition 2.9 we have the embeddings

$$F_{pq}^{(s+2jM,\Psi)}(I\!R^n) \subset B^\sigma_{\infty\infty}(I\!R^n) = \mathcal{C}^\sigma(I\!R^n), \tag{4.80}$$

and the inequalities

$$\|f_1 \,|\, \mathcal{C}^\sigma(I\!R^n)\| \le c\,\|f_1 \,|\, F_{pq}^{(s+2jM,\Psi)}(I\!R^n)\| \le c'\,\|f \,|\, F_{pq}^{(s,\Psi)}(I\!R^n)\|. \tag{4.81}$$

We decompose

$$f_1(x) = \sum_{m\in Z\!\!Z^n} \lambda_{0m}\, a_{0m}(x) \tag{4.82}$$

with

$$\lambda_{0m} = c_1' \sum_{|\alpha|\le K} \sup_{|y-m|\le\sqrt{n}} |D^\alpha f_1(y)|, \quad m \in Z\!\!Z^n, \tag{4.83}$$

and

$$a_{0m}(x) = \lambda_{0m}^{-1}\, \Phi(x-m)\, f_1(x) \tag{4.84}$$

where

$$c_1' = \sup_{x\in I\!R^n} \sup_{|\delta|\le K} |D^\delta \Phi(x)| \sup_{|\alpha|\le K} \sup_{\beta\le\alpha} \binom{\alpha}{\beta}$$

and Φ as in (4.15). It follows by straightforward calculations that a_{0m} are 1_K-atoms. As proved above, in the first part of the proof of Step 3, $f_2 \in F_{pq}^{(s+2M,\Psi)}(I\!R^n)$ has an atomic decomposition

$$f_2 = \sum_{\nu=0}^{\infty} \sum_{m\in Z\!\!Z^n} \Lambda_{\nu m}\, a_{\nu m},$$

with $a_{\nu m}$ being $(s+2M, p, \Psi)_{K,-1}$-atoms and $\|\Lambda \,|\, f_{pq}\| \le c\,\|f_2 \,|\, F_{pq}^{(s+2M,\Psi)}(I\!R^n)\|$. So,

$$(-\Delta)^M f_2 = \sum_{\nu=0}^{\infty} \sum_{m\in Z\!\!Z^n} \Lambda_{\nu m}\, (-\Delta)^M a_{\nu m}. \tag{4.85}$$

It can be easily seen that $(-\Delta)^M a_{\nu m}$ are $(s, p, \Psi)_{K,L}$-atoms (here one needs $2M > L$). Furthermore

$$\|\Lambda \,|\, f_{pq}\| \le c\,\|f_2 \,|\, F_{pq}^{(s+2M,\Psi)}(I\!R^n)\| \le c'\,\|f \,|\, F_{pq}^{(s,\Psi)}(I\!R^n)\|. \tag{4.86}$$

To complete this Step we still have to prove the inequality

$$\left\| \left(\sum_{m\in Z\!\!Z^n} |\lambda_{0m} \chi_{0m}^{(p)}(\cdot)|^q \right)^{\frac{1}{q}} \,\Big|\, L_p(I\!R^n) \right\| \le c\,\|f \,|\, F_{pq}^{(s,\Psi)}(I\!R^n)\|.$$

We have

$$\sum_{m\in Z\!\!Z^n} |\lambda_{0m} \chi_{0m}^{(p)}(x)|^q = c_1' \sum_{m\in Z\!\!Z^n} \left(\sum_{|\alpha|\le K} \sup_{|y-m|\le\sqrt{n}} |D^\alpha f_1(y)| \right)^q \chi_{0m}(x)$$

$$\le c_2' \sum_{m\in Z\!\!Z^n} \sum_{|\alpha|\le K} \left(\sup_{|x-y|\le 2\sqrt{n}} |D^\alpha f_1(y)| \right)^q \chi_{0m}(x) = c_2' \sum_{|\alpha|\le K} \left(\sup_{|x-y|\le 2\sqrt{n}} |D^\alpha f_1(y)| \right)^q.$$

Taking $\frac{1}{q}$-power in the last inequality and then L_p-quasi-norm we get

$$\left\| \left(\sum_{m\in \mathbb{Z}^n} |\lambda_{0m}\chi^{(p)}_{0m}(x)|^q \right)^{\frac{1}{q}} \mid L_p(I\!R^n) \right\| \le c_3' \sum_{|\alpha|\le K} \left\| \sup_{|x-y|\le 2\sqrt{n}} |D^\alpha f_1(y)| \mid L_p(I\!R^n) \right\|$$

$$\le c_4' \, \|f_1 \mid F^{(s+2jM,\Psi)}_{pq}(I\!R^n)\| \le c_5' \, \|f \mid F^{(s,\Psi)}_{pq}(I\!R^n)\|. \tag{4.87}$$

We made use also of formula (13.62) in [Tri97, p.81]. From (4.78), (4.82), (4.85), (4.86) and (4.87) we get what we wanted.

Step 4. (if-part of the subatomic decomposition) Assume that $f \in \mathcal{S}'(I\!R^n)$ satisfies (4.22) and (4.23). We will show that $f \in F^{(s,\Psi)}_{pq}(I\!R^n)$ and there exists a constant $c' > 0$ such that

$$\|f \mid F^{(s,\Psi)}_{pq}(I\!R^n)\| \le c' \sup_{\beta\in I\!N_0^n} 2^{r|\beta|} \, \|\lambda^\beta \mid f_{pq}\|. \tag{4.88}$$

Since for any $K \in I\!N_0$ and $L = -1$ the (s,p,Ψ)-β-quarks are $(s,p,\Psi)_{K,L}$-atoms multiplied by $c\,2^{(n+1)|\beta|}$, where $c > 0$ is independent of β, it follows from Step 1 that for any $\beta \in I\!N_0^n$

$$f^\beta = \sum_{\nu=0}^{\infty} \sum_{m\in\mathbb{Z}^n} \lambda^\beta_{\nu m} \, (\beta qu)_{\nu m}(x), \tag{4.89}$$

converges in $\mathcal{S}'(I\!R^n)$, $f^\beta \in F^{(s,\Psi)}_{pq}(I\!R^n)$ and

$$\|f^\beta \mid F^{(s,\Psi)}_{pq}(I\!R^n)\| \le c\,2^{(n+1)|\beta|} \, \|\lambda^\beta \mid f_{pq}\|, \tag{4.90}$$

where c is independent of β. So for $r > n+1$

$$\|f^\beta \mid F^{(s,\Psi)}_{pq}(I\!R^n)\| \le c\,2^{(n+1-r)|\beta|} \sup_{\beta\in I\!N_0^n} 2^{r|\beta|} \, \|\lambda^\beta \mid f_{pq}\|, \tag{4.91}$$

where c is independent of β. Applying now the t-triangle inequality, where $t = \min(1,p,q)$ to $\| \sum_{\beta\in I\!N_0^n} f^\beta \mid F^{(s,\Psi)}_{pq}(I\!R^n)\|$ we obtain that $f = \sum_{\beta\in I\!N_0^n} f^\beta$ converges in $F^{(s,\Psi)}_{pq}(I\!R^n)$ and hence we have (4.88).

Remark that in this Step the restriction (4.21) was essential for the use of atomic decomposition with no moment conditions required for the atoms. ■

Remark 4.9. *The coefficients $\lambda^\beta_{\nu m}$ depends linearly on f. This follows from* (4.59) *and* (4.51).

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