FINITE PREORDERS AND TOPOLOGICAL DESCENT

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Abstract

It is shown that the descent constructions of finite preorders provide a simple motivation for those of topological spaces, and new counter-examples to open problems in Topological descent theory are constructed.

Key words: Descent data, (effective) descent map, indexed categories, discrete fibrations, étale-descent.

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0 Introduction

Let $\mathcal{T}op$ be the category of topological spaces. For a given continuous map $p : E \to B$, it might be possible to describe the category $(\mathcal{T}op \downarrow B)$ of bundles over B in terms of $(\mathcal{T}op \downarrow E)$ using the pullback functor

$$p^* : (\mathcal{T}op \downarrow B) \to (\mathcal{T}op \downarrow E), \tag{0.1}$$

in which case p is called an *effective descent morphism*. There are various ways to make this precise (see [5], [6]); one of them is described in Section 3 below.

More generally, the same notion can still be considered when (0.1) is replaced by

$$p^* : \mathbf{A}^B \to \mathbf{A}^E \tag{0.2}$$

where $\mathbf{A}$ is any $\mathcal{T}op$-indexed category, or even also the category $\mathcal{T}op$ replaced by an arbitrary category $\mathcal{C}$; it is still useful to think of the objects of $\mathbf{A}^B$ as a kind of bundles over B, possibly with additional structure.

There is also an "intermediate" level of generality, where each $\mathbf{A}^B$ is a full subcategory in $(\mathcal{T}op \downarrow B)$ determined by a class $\mathbb{E}$ of morphisms in $\mathcal{T}op$. The corresponding effective descent morphisms are called the *effective* $\mathbb{E}$*-descent morphisms*.

The main problem studied in *Topological descent theory* is to find out, for given classes $\mathbb{D}$ and $\mathbb{E}$ of continuous maps, if every $p \in \mathbb{D}$ is an effective $\mathbb{E}$-descent morphism.

Let us recall the main known results of this type (in chronological order):

- A continuous map $p : E \to B$ is said to be locally sectionable if every point in B has an open neigbourhood U such that the map $p^{-1}(U) \to U$ induced by p has a continuous section. Every locally sectionable map is an effective descent morphism ([4]).

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- Every open map is an effective descent morphism (Sobral, see [12]; as observed in [7] it can also be easely deduced from Moerdijk's axioms [9] - just like it is deduced there for locales).

- Every proper map is an effective descent morphism (Moerdijk, Vermeulen [15]; see also [11]).

- Reiterman and Tholen ([11]) finally solved the problem of characterizing the effective descent morphisms in $\mathcal{T}op$ and gave a first example of *non-effective* descent morphism.

- Every effective descent morphism is also an effective étale-descent morphism ([5]). (As T. Plewe observed later, there is a simple purely-categorical proof of this fact.)

- Every triquotient map is an effective descent morphism (T.Plewe [10]), but there are counter-examples for the converse; yet, the class of triquotient maps contains all locally sectionable, all open surjections and all proper maps.

- Effective descent morphisms are stable under pullback in categories with pullbacks and coequalizers of certain naturally arising equivalence relations ([12]). This result was generalized to effective $\mathbb{E}$-descent morphisms in [13].

- A morphism is an effective descent morphism if and only if every pullback of it is an effective bijective-descent map ([14]).

- There are simple examples of non-effective descent morphisms ([14]).

Analysing the finite counter-examples of [13] and [14] we arrived to the conclusion that most of the phenomena which occur in difficult problems and proofs of Topological descent are easely detectable and easely understandable already on the level of finite topological spaces - and since those are just finite preorders, a lot of standard arguments can be used!

Accordingly, in this paper we develop the very simple descent theory of (finite) preorders, and then explain that Topological descent theory is just an infinite extension of it. We also construct new finite counter-examples to some problems of Topological descent theory.

In order to convince the readers interested in topological descent that "they must immediatly interrupt their work and read our paper" let us point out the following:

The Reiterman - Tholen's characterization of effective descent topological maps mentioned above says:

Theorem 0.1 *The map $p : E \to B$ is an effective descent morphism if and only if every crest of ultrafilters in B has a lifting along p (see [11] for details).*

In the case of finite topological spaces, which are exactly the finite preorders, it reduces to

Theorem 0.2 *The map $p : E \to B$ is an effective descent morphism if and only if for every for every chain $b_2 \leq b_1 \leq b_0$ in B there exists $e_2 \leq e_1 \leq e_0$ in E with $p(e_i) = b_i$, for $i = 0, 1, 2$.*

The paper is organized as follows:

0. Introduction
1. Finite topological spaces
2. Quotient and Day - Kelly maps
3. Effective descent morphisms
4. Generalized descent
5. Bijective descent
6. Étale descent
7. Triquotient maps

8. Counter-examples
9. Remarks on infinite topological spaces

Note that the results of Sections 2 and 3 in some sense go back to J. Giraud [2], and are closely related to the similar results for categories (although they are not straightforward consequences of those). A general approach to descent constructions for internal category-like structures is developed by M. Gran [3] ("Maltsev case") and I. Le Creurer [8] ("lextensive" case). Since the category of sets is lextensive, the results of [8] could be used here; however we give independent proofs in order to make the paper self-contained.

1 Finite topological spaces

Finite topological spaces have the "open closure operator". That is, for every subset X of a finite topological space A, there is a smallest open set $\downarrow X$ containing X. Moreover,

$$\downarrow X = \cup_{x \in X} \downarrow x, \tag{1.1}$$

where $\downarrow x = \downarrow \{x\}$.

We write

$$y \to x \Leftrightarrow y \in \downarrow x; \tag{1.2}$$

in classical notation our $y \to x$ would be just $y, y, \cdots \to x$.

Proposition 1.1 *If A is a finite topological space, then $\to$ is a preorder, i.e. it is reflexive and transitive:*

$$x \to x \tag{1.3}$$

$$z \to y \to x \Rightarrow z \to x \tag{1.4}$$

for every $x, y, z \in A$. This determines an isomorphism

$$FinTop \cong FinPreord \tag{1.5}$$

between the category of finite topological spaces and the category of finite preordered sets.

It is also well-known that 1.5 extends to an isomorphism between $Preord$ and the category of topological spaces for which the set of open subsets is closed under intersection.

Since

$$\downarrow x = \{y \in A | y \to x\} \tag{1.6}$$

we also introduce

$$\uparrow x = \{y \in A | x \to y\} \tag{1.7}$$

and we have $\uparrow x = \overline{\{x\}}$, the closure of $\{x\}$.

Proposition 1.2 *Let A and A' be topological spaces with the same underlying set. The following conditions are equivalent:*

(a) a subset X is open in A if and only if it is closed in A';

(b) the preorders in A and A' are opposite to each other, i.e. $y \to x$ in A if and only if $x \to y$ in A'.

According to (1.5), a map $\alpha : A \to B$ of finite topological spaces is continuous if and only if it is a monotone map (i.e. $y \to x \Rightarrow \alpha(y) \to \alpha(x)$) of the corresponding preordered sets.

For such a map α we also have

Proposition 1.3 *The following conditions are equivalent:*

(a) α is a proper map;

(b) α is a closed map;

(c) $\alpha(\uparrow x)$ is closed in B for every $x \in A$;

(d) $\alpha(\uparrow x) = \uparrow\alpha(x)$ for every $x \in A$;

(e) for every $x \in A$ and $\alpha(x) \to b$ in B there exists $a \in A$ with $x \to a$ and $\alpha(a) = b$.

Proposition 1.4 *The following conditions are equivalent:*

(a) α is an open map;

(b) $\alpha(\downarrow x)$ is open in B for every $x \in A$;

(c) $\alpha(\downarrow x) = \downarrow\alpha(x)$ for every $x \in A$;

(d) for every $x \in A$ and $b \to \alpha(x)$ in B there exists $a \in A$ with $a \to x$ and $\alpha(a) = b$.

Proposition 1.5 *The following conditions are equivalent:*

(a) α is an étale map (i.e. a local homeomorphism);

(b) α is an open map and its restriction to $\downarrow x$ is injective for every $x \in A$;

(c) the map $\downarrow x \to \downarrow\alpha(x)$ induced by α is bijective for every $x \in A$;

(d) for every $x \in A$ and $b \to \alpha(x)$ in B there exists a unique $a \in A$ with $a \to x$ and $\alpha(a) = b$.

2 Quotient and Day - Kelly maps

Let *Rel* be the category of pairs $A = (A, R_A)$, where $R_A \subseteq A \times A$ is an arbitrary binary relation on A. The "quotient maps" in this category have a simple description:

Proposition 2.1 *For a morphism $\alpha : A \to B$ with $\alpha(A) = B$, the following conditions are equivalent:*

(a) α is a regular epimorphism, i.e.

$$\begin{tikzcd} A \times_B A \arrow[r, shift left, "\pi_1"] \arrow[r, shift right, "\pi_2"'] & A \arrow[r, "\alpha"] & B \end{tikzcd} \tag{2.1}$$

is a coequalizer diagram in Rel;

(b) R_B is the smallest relation on B which makes $\alpha : A \to B$ a morphism in Rel;

(c) $R_B = (\alpha \times \alpha)(R_A)$, the image of $R_A \subseteq A \times A$ under the map $\alpha \times \alpha : A \times A \to B \times B$;

(d) $b' R_B b$ if and only if there exist $a', a \in A$ with $\alpha(a') = b'$, $\alpha(a) = b$ and $a' R_A a$.

Exactly the same is true in the category $ReflRel$ of sets equipped with a reflexive relation, but not in $Preord$ - since transitivity of R_A in (2.1) does not imply transitivity of R_B. However, given such a coequalizer diagram in $ReflRel$ with transitive R_A, we obtain a coequalizer diagram in $Preord$ just by taking the transitive closure of R_B. Therefore we have

Proposition 2.2 *For a morphism $\alpha : A \to B$ in Preord (or in FinPreord) with $\alpha(A) = B$, the following conditions are equivalent:*

(a) α is a regular epimorphism;

(b) R_B is the smallest relation on B which makes $\alpha : A \to B$ a morphism in Preord;

(c) R_B is the transitive closure of $(\alpha \times \alpha)(R_A)$;

(d) $b' R_B b$ if and only if there exists a (finite) sequence $(a'_1, a_1), \cdots, (a'_n, a_n) \in R_A$ with $b' = \alpha(a'_1), \alpha(a_i) = \alpha(a'_{i+1})$, for $i = 1, \cdots, n-1$, and $\alpha(a_n) = b$.

The fact that regular epimorphisms in *Rel* and *ReflRel* are "better" than in *Preord* can also be expressed categorically:

Proposition 2.3 *(a) The regular epimorphisms in Rel and ReflRel are pullback stable, i.e. if*

$$\begin{tikzcd} E \times_B A \arrow[r, "\pi_2"] \arrow[d, "\pi_1"'] & A \arrow[d, "\alpha"] \\ E \arrow[r, "p"'] & B \end{tikzcd} \tag{2.2}$$

is a pullback (in one of these categories) and p is a regular epimorphism, then so is π_2;

(b) a morphism $p : E \to B$ in Preord is a pullback stable regular epimorphism if and only if it is a regular epimorphism in Rel (or, equivalently, in ReflRel).

Proof. (a) follows from 2.1(a) $\Leftrightarrow$ (c) and the fact that the regular epimorphisms in *Sets* are pullback stable.

Since every morphism in *Preord*, which is a regular epimorphism in *Rel*, must be a regular epimorphism in *Preord*, the "if" part of (b) follows from (a).

In order to prove the "only if" part of (b) we take:

- an arbitrary pair $(b', b) \in R_B$;
- $A = \{b, b'\}$ with the induced preorder;
- $\alpha : A \to B$ the inclusion map.

Since $\pi_2 : E \times_B A \to A$ is a regular epimorphism, there exists a sequence

$$(x'_1, x_1), \cdots, (x'_n, x_n) \in R_{E \times_B A}$$

with $b' = \pi_2(x'_1), \pi_2(x_i) = \pi_2(x'_{i+1})$, for $i = 1, \cdots, n-1$, and $\pi_2(x_n) = b$. However, since there are no elements in A different from b and b', this means that $b' = \pi_2(x'_k)$ and $\pi_2(x_k) = b$ for some k $(1 \le k \le n)$. Therefore, the pair $(\pi_1(x'_k), \pi_1(x_k)) \in R_E$ has $p(\pi_1(x'_k)) = b'$ and $p(\pi_1(x_k)) = b$ as desired. $\square$

Remark 2.4 According to topological terminology, we say that $p : E \to B$ is a hereditary quotient map if, for every $B' \subseteq B$ with the induced preorder, the map $p^{-1}(B') \to B'$ induced by p is a quotient map (i.e. a regular epimorphism). Since in the proof of the "only if" part of 2.3(b) the morphism $\alpha : A \to B$ was an inclusion map with the induced order in A, we conclude that the pullback stable regular epimorphisms in *Preord* are the same as the hereditary quotient maps.

Now we return to finite topological spaces.

A continuous map $p : E \to B$ is said to be a Day - Kelly map if for every $b \in B$ and every open covering family $(E_i)_{i \in I}$ of $p^{-1}(b)$ in E, there exists a finite set $\{i_1, \cdots, i_n\}$ with

$$b \in Int(p(E_{i_1}) \cup \cdots \cup p(E_{i_n})) \tag{2.3}$$

In the finite case this simplifies in the obvious way: we can just take I to be a one element set.

Since the Day - Kelly maps are known to be precisely the pullback stable regular epimorphisms of topological spaces (see [1] and [4]) - or directly from the results above - we obtain:

Proposition 2.5 *For a morphism $p : E \to B$ in $FinTop$ the following conditions are equivalent:*

(a) p is a pullback stable regular epimorphism (in $FinTop$, or equivalently in $FinPreord$);

(b) for every $b' \to b$ in B there exists $e' \to e$ in E with $p(e') = b'$ and $p(e) = b$;

(c) p is a Day - Kelly map;

(d) p is a hereditary quotient map;

(e) for every $b \in B$ and open set $E' \subseteq E$ containing $p^{-1}(b)$, we have $b \in Int(p(E'))$.

3 Effective descent morphisms

Various definitions of effective descent morphism are compared in [5] and [6]; one of them says that a morphism $p : E \to B$ in a category $\mathcal{C}$ is an effective descent morphism if the pullback functor

$$p^* : (\mathcal{C} \downarrow B) \to (\mathcal{C} \downarrow E) \tag{3.1}$$

is monadic.

However, we will only need to know that the class of effective descent morphisms satisfies the following (see [5] for details):

Proposition 3.1 *(a) If $\mathcal{C}$ has pullbacks and coequalizers (of equivalence relations), then every effective descent morphism in $\mathcal{C}$ is a pullback stable regular epimorphism.*

(b) If $\mathcal{C}$ is exact then the class of effective descent morphisms in $\mathcal{C}$ coincides with the class of regular epimorphisms.

Proposition 3.2 *Let $\mathcal{C}$ and $\mathcal{C}'$ be categories satisfying*

(a) $\mathcal{C}'$ has pullbacks and coequalizers;

(b) every regular epimorphism in $\mathcal{C}'$ is an effective descent morphism;

(c) $\mathcal{C}$ is a full subcategory of $\mathcal{C}'$ closed under pullback;

(d) every pullback stable regular epimorphism in $\mathcal{C}$ is a regular epimorphism in $\mathcal{C}'$.

Then a pullback stable regular epimorphism $p : E \to B$ in $\mathcal{C}$ is an effective descent morphism if and only if

$$E, B, E \times_B A \in \mathcal{C} \Rightarrow A \in \mathcal{C} \tag{3.2}$$

for every pullback (2.2) in $\mathcal{C}'$.

Using these two propositions, it is easy to characterize the effective descent morphisms in *Rel*, *ReflRel*, and *Preord*:

Proposition 3.3 *Every regular epimorphism in Rel is an effective descent morphism, and the same is true for ReflRel.*

Proof. An object (A, R_A) in *Rel* can be considered as a graph

$$R_A \rightrightarrows A \tag{3.3}$$

and we take $\mathcal{C} = Rel$ and $\mathcal{C}'$ to be the category of graphs. The conditions 3.2(a)-(d) obviously hold (just note that 3.2(b) follows from 3.1(b) since now $\mathcal{C}$ is a topos). Since the implication (3.2) obviously holds as soon as p is an epimorphism in $\mathcal{C}'$, we conclude that every pullback stable regular epimorphism in *Rel* is an effective descent morphism - and then we apply 2.3(a).

The same arguments, but with reflexive graphs instead of graphs, can be used for reflexive relations. □

Proposition 3.4 *For a morphism $p : E \to B$ in Preord (or in FinPreord) the following conditions are equivalent:*

(a) p is an effective descent morphism;

(b) p is a pullback stable regular epimorphism and, for every pullback (2.2) in Rel (or in ReflRel) with $E, B, E \times_B A$ preorders, A also is a preorder;

(c) for every $b_2 \to b_1 \to b_0$ in B there exists $e_2 \to e_1 \to e_0$ in E with $p(e_i) = b_i$, for $i = 0, 1, 2$.

Proof. $(a) \Leftrightarrow (b)$ follows from the previous results. More precisely, we can apply Proposition 3.2 to $\mathcal{C} = Preord$, $\mathcal{C}' = Rel$ (or *ReflRel*) since in that case:

- 3.2(a) and 3.2(c) are obvious;
- 3.2(b) follows from Proposition 3.3;
- 3.2(d) follows from 2.3(b).

$(c) \Rightarrow (b)$: Suppose p satisfies (c). Then p is a pullback stable regular epimorphism by 2.5$(b) \Leftrightarrow (a)$, and we only need to show that, for every pullback (2.2) with transitive $R_E, R_B, R_{E \times_B A}$, the relation R_A is also transitive. However this is clear: given $a_2 \to a_1 \to a_0$ in A, there exists $e_2 \to e_1 \to e_0$ in E with $p(e_2) = \alpha(a_2)$, $p(e_1) = \alpha(a_1)$, $p(e_0) = \alpha(a_0)$ and hence $(e_2, a_2) \to (e_1, a_1) \to (e_0, a_0)$. Therefore $(e_2, a_2) \to (e_0, a_0)$, since $R_{E \times_B A}$ is transitive, which gives $a_2 \to a_0$ since $\pi_2 : E \times_B A \to A$ is a morphism in *Rel*.

$(b) \Rightarrow (c)$: Suppose p satisfies (b) and take:

- an arbitrary $b_2 \to b_1 \to b_0$ in B;
- $A = \{a_0, a_1, a_2\}$ any three element set with $R_A = \Delta_A \cup \{(a_2, a_1), (a_1, a_0)\}$;
- $\alpha : A \to B$ with $\alpha(a_0) = b_0, \alpha(a_1) = b_1$ and $\alpha(a_2) = b_2$ (note that α need not be an injection!).

Since E and B are preorders, but A is not, $E \times_B A$ must not be a preorder. That is, there exist $x_0, x_1, x_2 \in E \times_B A$ with $(x_2, x_1), (x_1, x_0) \in R_{E \times_B A}$ but $(x_2, x_0) \notin R_{E \times_B A}$.

We have

$$E \times_B A = (p^{-1}(b_2) \times \{a_2\}) \cup (p^{-1}(b_1) \times \{a_1\}) \cup (p^{-1}(b_0) \times \{a_0\}) \tag{3.4}$$

and since $\pi_2 : E \times_B A \to A$ must be a regular epimorphism in Rel, it is easy to see that we must have $x_i \in p^{-1}(b_i) \times \{a_i\}$, for $i = 0, 1, 2$. After that, we take $e_2 = \pi_1(x_2)$, $e_1 = \pi_1(x_1)$ and $e_2 = \pi_1(x_2)$. $\square$

4 Generalized descent

Let $\mathcal{C}$ be a category. Recall that a $\mathcal{C}$-indexed category $\mathbf{A}$ consists of

- categories $\mathbf{A}^B$, defined for all objects B in $\mathcal{C}$,
- functors $p^* : \mathbf{A}^B \to \mathbf{A}^E$, for all morphisms $p : E \to B$ in $\mathcal{C}$, and
- natural isomorphisms $\varphi_{p,q} : q^* p^* \to (pq)^*$ and $\psi_B : (1_B)^* \to 1_{\mathbf{A}^B}$, for all $q : F \to E$ and $p : E \to B$ in $\mathcal{C}$, with the standard coherence conditions.

For a given morphism $p : E \to B$ in a category $\mathcal{C}$ with pullbacks and a $\mathcal{C}$-indexed category $\mathbf{A}$, the category $Des_{\mathbf{A}}(p)$ of $\mathbf{A}$-descent data for p is defined as a suitable 2-equalizer

$$\begin{tikzcd} Des_{\mathbf{A}}(p) \arrow[r] & \mathbf{A}^E \arrow[r, shift left] \arrow[r, shift right] & \mathbf{A}^{E \times_B E} \arrow[l] \arrow[r, shift left=2] \arrow[r] \arrow[r, shift right=2] & \mathbf{A}^{E \times_B E \times_B E} \end{tikzcd} \tag{4.1}$$

(described in [6] in the language of internal actions). The functor p^* has a canonical factorization

$$\begin{tikzcd} \mathbf{A}^B \arrow[rr, "p^*"] \arrow[dr, "K^p{}_{\mathbf{A}}"'] & & \mathbf{A}^E \\ & Des_{\mathbf{A}}(p) \arrow[ur, "U^p{}_{\mathbf{A}}"'] & \end{tikzcd} \tag{4.2}$$

and p is said to be an effective $\mathbf{A}$-descent morphism if K^p is a category equivalence.

In particular, any pullback stable class $\mathbb{E}$ of morphisms in $\mathcal{C}$ can be regarded as a $\mathcal{C}$-indexed category: we take

- $\mathbb{E}^B = \mathbb{E}(B)$ to be the full subcategory in $(\mathcal{C} \downarrow B)$ with objects all (A, α) with $\alpha : A \to B$ in $\mathbb{E}$;
- $p^* : \mathbb{E}^B \to \mathbb{E}^E$ the pullback functor $(A, \alpha) \mapsto (E \times_B A, \pi_1)$ along $p : E \to B$;
- $\varphi_{p,q}$ and ψ_B the canonical isomorphisms $F \times_E (E \times_B (-)) \cong F \times_B (-)$ and $B \times_B (-) \cong (-)$ respectively.

The category $Des_{\mathbb{E}}(p)$ can be described as the category of triples $(C, \gamma, \xi) =$

$$E \times_B C \xrightarrow{\quad \xi \quad} C \xrightarrow{\quad \gamma \quad} E \tag{4.3}$$

such that $\gamma \in \mathbb{E}$ and the diagram

$$\begin{tikzcd}
E \times_B E \times_B C \arrow[r, "1_E \times \xi"] \arrow[d, "< \pi_1, \pi_3 >"'] & E \times_B C \arrow[d, "\xi"'] & C \arrow[l, "< \gamma, 1_C >"'] \arrow[dl, "1_C"] \\
E \times_B C \arrow[r, "\xi"] \arrow[d, "\pi_1"'] & C \arrow[dl, "\gamma"] & \\
E & &
\end{tikzcd} \tag{4.4}$$

commutes (we use the standard notation, writing π_i - here i= 1, 2 or 3 - for all kinds of pullback projections; note also that the commutativity of the bottom triangle is already used in the square to make $1_E \times \xi$ well defined).

If $\mathcal{C} = Rel$ or $\mathcal{C}$ is any other concrete category considered in the previous sections, then we write

$$\xi(e, c) = e \cdot c, \tag{4.5}$$

and the commutativity of (4.4) translates as

$$\begin{aligned} e \cdot (e' \cdot c) &= e \cdot c \\ \gamma(c) \cdot c &= c \\ \gamma(e \cdot c) &= e. \end{aligned} \tag{4.6}$$

The functor $K^p{}_{\mathbb{E}} : \mathbb{E}(B) \to Des_{\mathbb{E}}(p)$ is defined by

$$K^p{}_{\mathbb{E}}(A, \alpha) = (E \times_B A, \pi_1, < \pi_1, \pi_3 >); \tag{4.7}$$

using the elements, $< \pi_1, \pi_3 >: E \times_B (E \times_B A) \to E \times_B A$ would be written as

$$e \cdot (e', a) = (e, a) \tag{4.8}$$

If every $\delta : D \to E$ in $\mathbb{E}$ gives $p \cdot \delta : E \to B$ in $\mathbb{E}$, then the diagram (4.2) (for $\mathbf{A} = \mathbb{E}$) can be identified with the standard diagram

$$\begin{tikzcd}
\mathbb{E}(B) \arrow[rr, "p^*"] \arrow[dr, "\text{comparison}"'] & & \mathbb{E}(E) \\
& \mathbb{E}(E)^{\mathbb{T}} \arrow[ur, "\text{forgetful}"'] &
\end{tikzcd} \tag{4.9}$$

for the monad $\mathbb{T}$ of the adjunction $p! \dashv p^*$.

And, of course, if $\mathbb{E}$ is the class of all morphisms in $\mathcal{C}$ then an effective $\mathbb{E}$-descent morphism is the same as an effective descent morphism, as defined in the previous section.

5 Bijective descent

In this section $\mathbb{E}$ denotes the class of morphisms in *Preord* which are bijections.

Proposition 5.1 *For a morphism $p : E \to B$ in Preord, the following conditions are equivalent:*

(a) p is a regular epimorphism in Rel;

(b) for every pullback (2.2) with $\alpha \in \mathbb{E}$, the projection $\pi_2 : E \times_B A \to A$ is a regular epimorphism in Preord.

Proof. $(a) \Rightarrow (b)$ follows from 2.3(b).

$(b) \Rightarrow (a)$ can be proved with the same arguments as the "only if" part of 2.3(b), but the $\alpha : A \to B$ from 2.3(b) now has to be a bijection - and we just take $A = B$ as a set, with R_A the smallest preorder under which $\{b, b'\}$ has the preorder induced by R_B. □

Proposition 5.2 *For a morphism $p : E \to B$ in Preord, the following conditions are equivalent:*

(a) p is an effective $\mathbb{E}$-descent morphism;

(b) p satisfies the equivalent conditions of Proposition 5.1, and for every pullback (2.2) in Rel (or in ReflRel) with $\alpha \in \mathbb{E}$ and $E, B, E \times_B A$ preorders, A also is a preorder;

(c) p is surjective, and for every $b_2 \to b_1 \to b_0$ in B with $b_2 \neq b_1$ there exists $e_2 \to e_1 \to e_0$ in E with $p(e_i) = b_i$, for $i = 0, 1, 2$.

Proof. $(a) \Leftrightarrow (b)$ can be easily proved similarly to $(a) \Leftrightarrow (b)$ of Proposition 3.4, with suitable generalizations of 3.1 and 3.2.

We can also repeat the proof of $(c) \Rightarrow (b)$ from 3.4, since we do not need to consider the case $\alpha(a_2) = \alpha(a_0)$: if $\alpha(a_2) = \alpha(a_0)$ then $a_2 = a_0$, and then $a_2 \to a_0$ since R_A, being the image of $R_{E\times_B A}$, is reflexive.

And finally, in order to use the proof of 3.4 $(b) \Rightarrow (c)$, we just modify it as we did for 2.3(b) in order to prove 5.1$(b) \Rightarrow (a)$. That is, we take $A = B$ as a set (so now $\alpha = 1_B$ is a bijection) with

$$R_A = \Delta_B \cup \{(b_2, b_1), (b_1, b_0)\}, \tag{5.1}$$

excluding the trivial cases $b_2 = b_1$ and $b_1 = b_0$; since $b_2 \neq b_0$, the set $\{b_2, b_1, b_0\}$ has exactly three elements as needed in the proof of 3.4$(b) \Rightarrow (c)$. □

Note that the same results are true in *FinPreord* or if $\mathbb{E}$ is the class of all injections.

6 Étale descent

As follows from Proposition 1.5, a continuous map $\alpha : A \to B$ of finite topological spaces is étale if and only if it is a discrete fibration of the corresponding preorders (considered as categories). Accordingly, in order to investigate the étale descent, we will take $\mathbb{E}$ to be the class of discrete fibrations of preorders.

On the other hand, the discrete fibrations $A \to B$ correspond to the functors $B^{op} \to Sets$, and, moreover, the standard equivalence

$$\mathbb{E}(B) \sim Sets^{B^{op}} \tag{6.1}$$

is in fact an equivalence of *Preord*-indexed categories.

Using the equivalence (6.1) and the 2-equalizer (4.1) we can describe $Des_{\mathbb{E}}(p)$ (for a given $p : E \to B$ in *Preord*) as the 2-equalizer

$$Des_{\mathbb{E}}(p) \longrightarrow Sets^{E^{op}} \begin{array}{c}\longrightarrow\\ \longleftarrow\\ \longrightarrow\end{array} Sets^{(E\times_B E)^{op}} \begin{array}{c}\longrightarrow\\ \longrightarrow\\ \longrightarrow\end{array} Sets^{(E\times_B E\times_B E)^{op}} \tag{6.2}$$

and then a straightforward calculation gives

Proposition 6.1 *Let $\mathcal{X}$ be the category of pairs (X, ξ), where $X : E^{op} \to Sets$ is a functor, and*

$$\xi = (\xi_{e,e'})_{(e,e')\in E\times_B E}$$

a family of maps $\xi_{e,e'} : X(e') \to X(e)$ such that

$$\xi_{e,e'}\xi_{e',e"} = \xi_{e',e"}, \quad \xi_{e,e} = 1_{X(e)} \tag{6.3}$$

and, for every $(e_1, e'_1) \to (e_0, e'_0)$ in $E \times_B E$, the diagram

$$\begin{tikzcd} X(e'_0) \arrow[r, "X(e'_0{,}e'_1)"] \arrow[d, "\xi_{e_0,e_1}"'] & X(e'_1) \arrow[d, "\xi_{e_1,e'_1}"] \\ X(e_0) \arrow[r, "X(e_0{,}e_1)"'] & X(e_1) \end{tikzcd} \tag{6.4}$$

commutes.

Let $\phi : Sets^{B^{op}} \to \mathcal{X}$ be the functor defined by $A \mapsto (p^{op}A, 1)$, where 1 is the family of identity morphisms

$$1_{e,e'} = 1_{A(p(e))} : A(p(e')) \to A(p(e)). \tag{6.5}$$

Then there exists a category equivalence $Des_{\mathbb{E}}(p) \sim \mathcal{X}$ making the diagram

$$\begin{tikzcd} \mathbb{E}(B) \arrow[r, "K^p{}_{\mathbb{E}}"] \arrow[d, "\sim"'] & Des_{\mathbb{E}}(p) \arrow[d, "\sim"] \\ Sets^{B^{op}} \arrow[r, "\phi"'] & \mathcal{X} \end{tikzcd} \tag{6.6}$$

commute, up to isomorphism.

Corollary 6.2 *A morphism $p : E \to B$ in Preord is an effective $\mathbb{E}$-descent morphism if and only if the functor ϕ described in Proposition 6.1 is a category equivalence.*

We point out that Corollary 6.2 should be used to obtain an elementary characterization of effective étale-descent morphisms of finite topological spaces, which itself should suggest such a characterization for all spaces (although this project does not seem to be straightforward!).

Note also that the category $\mathcal{X}$ of Proposition 6.1 can be described as the category of "double functors" $Eq(p) \to Sets_2$, where $Eq(p)$ is the equivalence relation

$$E \times_B E \rightrightarrows E \tag{6.7}$$

(= kernel pair of p) considered as a double category, and $Sets_2$ the double category of $Sets$, maps and commutative squares. Accordingly, there is a natural description of the functor $\phi : Sets^{B^{op}} \to \mathcal{X}$.

7 Triquotient maps

A continuous map $p : E \to B$ of topological spaces is said to be a triquotient map if there exists a map $q : Open(E) \to Open(B)$ of the sets of open subsets in E and in B respectively, satisfying the following conditions:

- $q(U) \subseteq p(U)$, for every $U \in Open(E)$;
- $q(E) = B$;
- q is monotone, i.e. $U \subseteq V \Rightarrow q(U) \subseteq q(V)$;
- for every $U \in Open(E)$, $b \in q(U)$, and covering family $(E_i)_{i \in I}$ of $p^{-1}(b) \cap U$, there exists a finite set $\{i_1, \cdots i_n\} \subseteq I$ with
$$b \in q(E_{i_1} \cup \cdots \cup E_{i_n}). \tag{7.1}$$

The q above is called a triquotiency-assignement for p.

In the finite case, just like for the Day - Kelly maps, we could take I to be a one element set. That is, in the finite case, the last condition above is equivalent to

- If U and V are open subsets in E and b is an element in B, then
$$b \in q(U),\ p^{-1}(b) \cap U \subseteq V \Rightarrow b \in q(V). \tag{7.2}$$

However, even in the finite case we are very far from being able to find a nice characterization of the triquotient maps. A surprising result is (compare with $(c) \Leftrightarrow (b)$ in Proposition 2.5!):

Proposition 7.1 *If $p : E \to B$ is a triquotient map of topological spaces, then for every natural number n and every $b_n \to b_{n-1} \to \cdots \to b_1 \to b_0$ in B there exists $e_n \to e_{n-1} \to \cdots \to e_1 \to e_0$ in E with $p(e_i) = b_i$, for each $i = 0, \cdots, n$.*

Proof. For a fixed $b_n \to b_{n-1} \to \cdots \to b_1 \to b_0$ in B we introduce, for $i = 0, \cdots, n$, the sets E_i defined by

$$E_i = \{e \in E |\ \text{there exists } e = e_i \to e_{i-1} \to \cdots \to e_0 \text{ in } E \text{ with } p(e_{i-1}) = b_{i-1}, \cdots, p(e_0) = b_0\} \tag{7.3}$$

and we are going to prove that each E_i is open and

$$b_i \in q(E_i) \tag{7.4}$$

for each i. This will give $b_n \in p(E_n)$, and therefore there exists $e_n \to \cdots \to e_1 \to e_0$ with the required property.

The fact that each E_i is open follows from the obvious equalities

$$E_0 = E, \quad E_i = \downarrow(p^{-1}(b_{i-1}) \cap E_{i-1}) \ (i > 0). \tag{7.5}$$

In order to prove (7.4) we use the induction by $i = 0, \cdots, n$.

For $i = 0$ we have $b_0 \in B = q(E) = q(E_0)$.

Suppose $b_{i-1} \in q(E_{i-1})$. Since $b_i \in \downarrow b_{i-1}$ and $q(E_i)$ is open, in order to prove that $b_i \in q(E_i)$ it suffices to prove that $b_{i-1} \in q(E_i)$. However this follows from Condition 7.1 applied to $U = E_{i-1}$, $V = E_i$, and $b = b_{i-1}$, since $p^{-1}(b_{i-1}) \cap E_{i-1} \subseteq E_i$ by (7.5). □

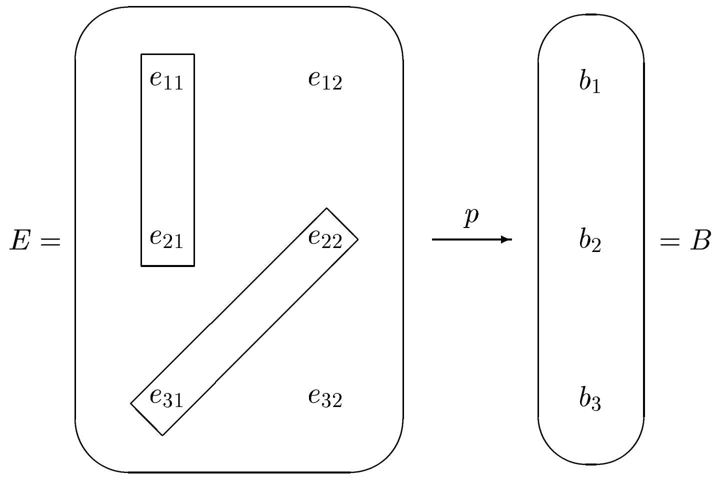
Now it is easy to construct effective descent morphisms of finite topological spaces which are not triquotient maps. Thus, the fact that the class of triquotient maps in $\mathcal{T}op$ is a proper subclass of the one of effective descent morphisms already appears for the finite spaces.

8 Counter-examples

So far we have never mentioned the (non-effective) $\mathbb{E}$-descent morphisms. They are those which have the comparison functor of (4.9) full and faithfull. If $\mathbb{E}$ (of (4.9)) is the class of all morphisms in the ground category $\mathcal{C}$, and $\mathcal{C}$ has (pullbacks and) coequalizers of equivalence relations, then they are the same as the pullback stable regular epimorphisms. In particular, the descent morphisms in $\mathcal{T}op$ are the same as the Day - Kelly maps - which brings the following

Problem 8.1 *Is every Day - Kelly map an effective descent morphism in $\mathcal{T}op$?*

The first counter-example was described in [11]; it uses ultrafilters, and the proof uses pseudotopological spaces. However, as shown in [14], there is even a finite counter-example; it can be displayed as



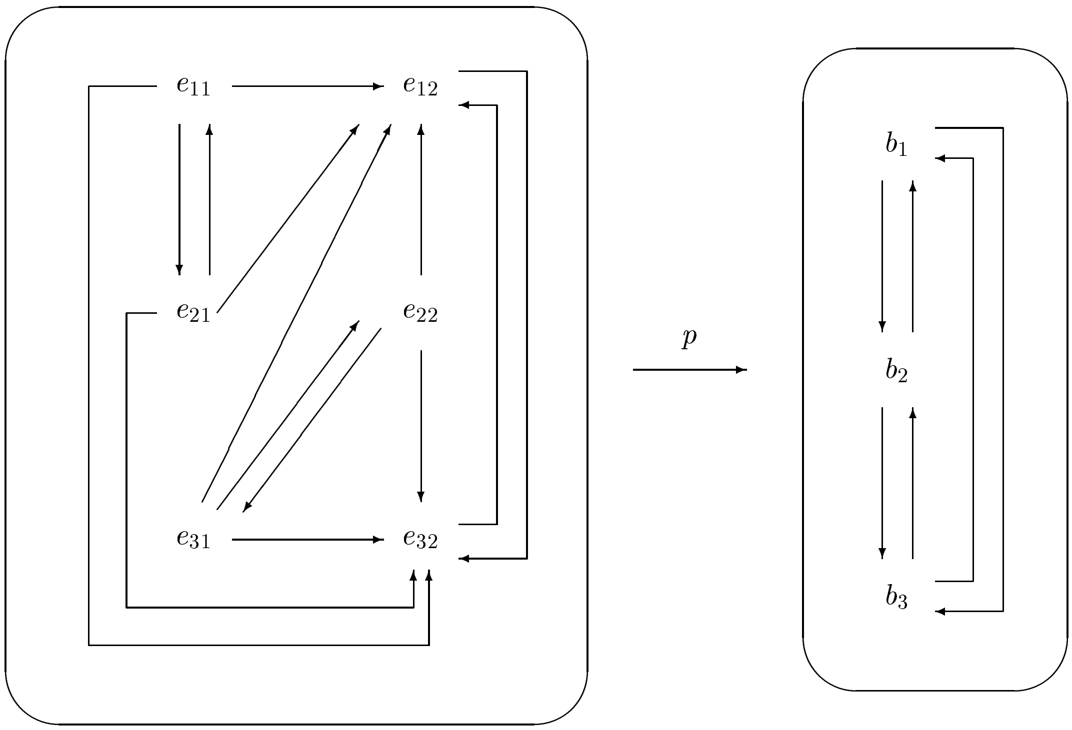
(8.1)

where:

- B has the codiscrete topology;
- the nontrivial open sets in E are $\{e_{11}, e_{21}\}$, $\{e_{22}, e_{31}\}$ and their union;
- p is defined by $p(e_{ij}) = b_i$.

The preorder approach of the present paper makes the whole story trivial: the Day - Kelly maps which are not effective descent morphisms are those maps $p : E \to B$ which satisfy 2.5(b) but not 3.4(c). Briefly, they are those which are surjective on arrows, but not on composable pairs of arrows.

The preorder translation of (8.1) is

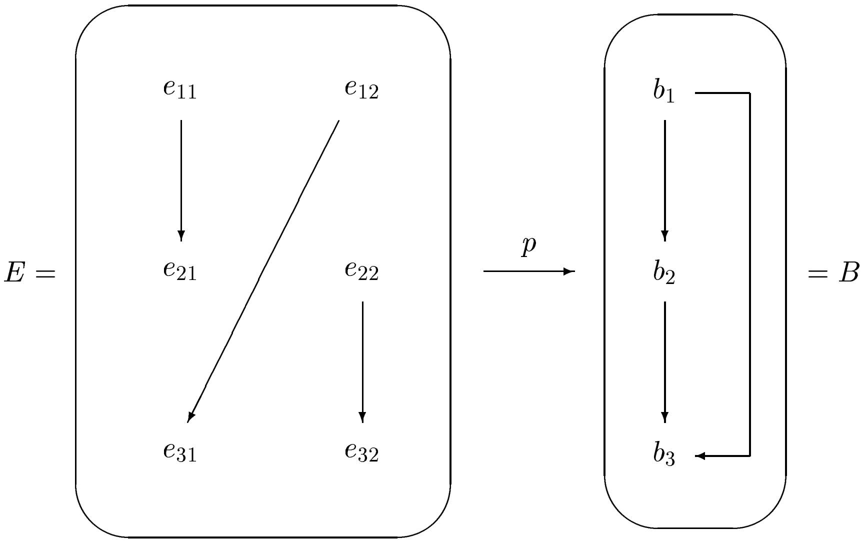


(8.2)

where the identity arrows are omitted. It is easy to see here that p is surjective on arrows but there is no $e" \to e' \to e$ in E whose image in B is $b_3 \to b_1 \to b_2$ and so p is not surjective on composable pairs.

Note also that the preorder approach suggests to consider the following two (counter -)examples, the first of which is more straightforward, and the second gives the smallest possible spaces:

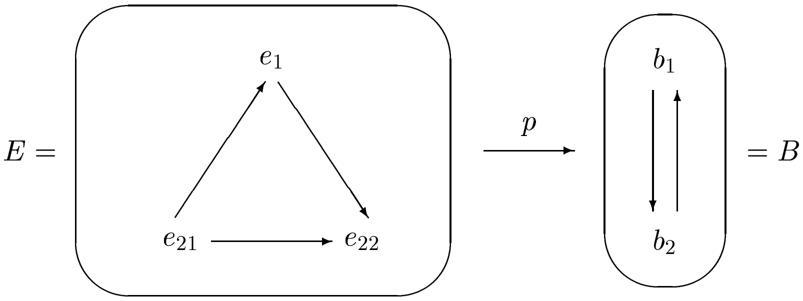
Example 8.2 *Take*



(8.3)

- in fact this is exactly the finite version of the original counter-example from [11].

Example 8.3 *Take*



(8.4)

- there is no $e'' \to e' \to e$ whose image in B is $b_1 \to b_2 \to b_1$

Remark 8.4 As we see from 5.2 $(a) \Leftrightarrow (c)$ we have

(a) $p : E \to B$ of Example 8.2 is not even an effective bijective-descent morphism (as well as the one from [14] mentioned above - see (8.2));

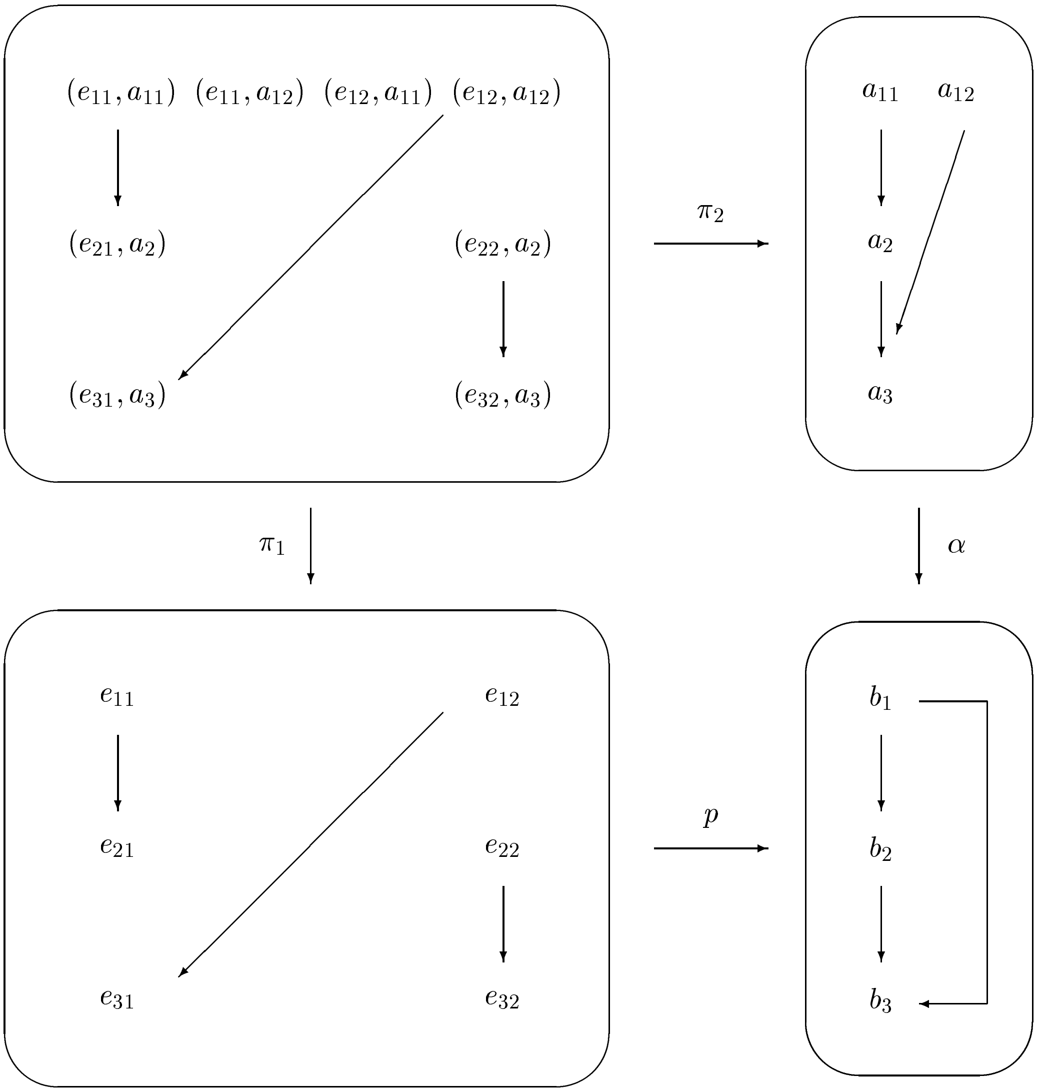
(b) $p : E \to B$ of Example 8.3 is an effective bijective-descent morphism.

Consider a further problem, which is suggested by the fact that every effective descent morphism in $\mathcal{T}op$ is an effective étale-descent morphism:

Problem 8.5 *Does one of the following two classes of maps contain the other:*

- *Day - Kelly maps,*
- *effective étale-descent morphisms ?*

Using $p : E \to B$ of Example 8.2, consider the pullback in $ReflRel$:



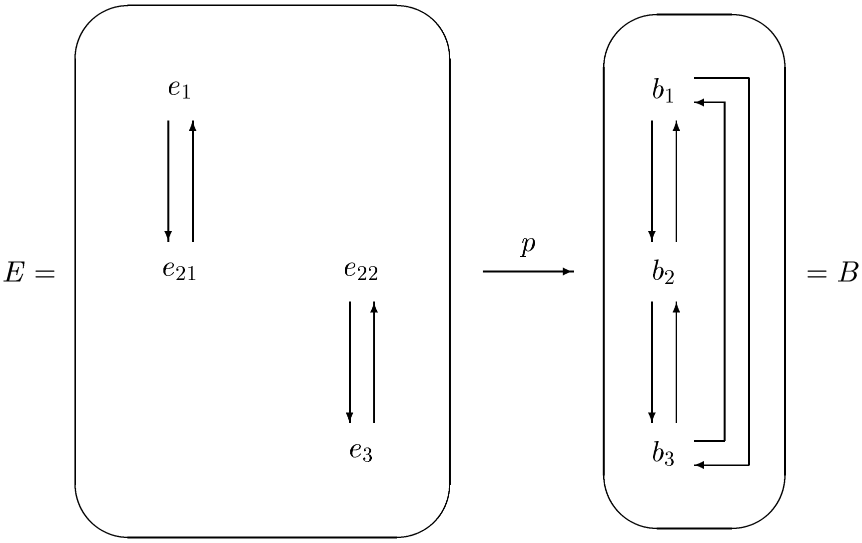
(8.5)

where again the display shows all arrows except the identities. Clearly $E \times_B A$ is a preorder, and using 1.5 $(a) \Leftrightarrow (c)$ it is easy to see that $\pi_1 : E \times_B A \to E$ is étale. Therefore, $(E \times_B A, \pi_1, < \pi_1, \pi_3 >)$ constructed as in (4.7) belongs to $Des_{\mathbb{E}}(p)$, where $\mathbb{E}$ is the class of étale maps in the category of topological spaces. Since p is an effective descent morphism in $ReflRel$, and A is not a preorder, there is no object in $\mathbb{E}(B)$ corresponding to $(E \times_B A, \pi_1, < \pi_1, \pi_3 >)$. That is we obtain

Proposition 8.6 *The Day - Kelly map $p : E \to B$ described in Example 8.2 is not an effective étale-descent morphism.*

Finally, let us consider

Example 8.7 *Take*



(8.6)

- clearly this is not a Day - Kelly map, but a simple calculation using Corollary 6.2 (or directly, using the fact that B is a codiscrete space and E is a coproduct of two codiscrete spaces) shows that it is an effective étale-descent morphism.

Together with Proposition 8.6 this gives the negative answer to Problem 8.5.

9 Remarks on infinite spaces

In this section we list the questions and results of Topological descent theory, which became much more clear to us as soon as we understood their finite versions using the preorder approach.

(9.1) Our simple characterization of the effective descent morphisms of preorders, which A. Grothendieck and J. Giraud would probably consider as an obvious fact already 35 years ago (see [2]), can however be considered as a basic result whose "infinite filter generalization" is the J. Reiterman - W. Tholen's complete characterization of the effective descent morphisms of topological spaces (see Theorem 0.1). Just observe that:

(a) The preorder on a finite topological space corresponds to the convergency structure on an infinite one; we will write $\mathcal{F} \to x$ when $\mathcal{F}$ is a filter converging to a point x. In a finite space $\mathcal{F} \to x$ if and only if $y \to x$ for every y which belongs to the intersection of the elements of $\mathcal{F}$. Moreover, the passage from the topologies to the corresponding convergency structures determines a category isomorphism which extends the isomorphism (1.5).

(b) Since ultrafilters on a finite set are principal filters generated by the one point subsets, the "relevant part" of a crest of ultrafilters $((\mathcal{F}_i \to b_i)_{i \in I}, \mathcal{U}, b)$ (in the sense of [11]) is the composable pair $b" \to b' \to b$ in which $b' = b_i$ and $b"$ have i generating $\mathcal{U}$ and $\{b"\}$ generating the corresponding $\mathcal{F}_i$.

(c) Recall that the isomorphism $FinTop \cong FinPreord$ extends to an isomorphism $FinPsTop \cong FinReflRel$, where $FinPsTop$ is the category of finite pseudotopological spaces. And the results of [11] use the embedding $\mathcal{T}op \to PsTop$ exactly in the same way as we use $Prord \to ReflRel$.

(9.2) The three classes of morphisms in $\mathcal{T}op$ which were known to be classes of effective descent morphisms before [11], are

1. (locally) sectionable maps,
2. open surjections,
3. proper surjections.

Why? In the finite case (although "proper" reduces to "closed") these three classes naturally occur as the three simple cases. Indeed, in order to find $e_2 \to e_1 \to e_0$ for a given $b_2 \to b_1 \to b_0$ as in 3.4(c), one could either

- use a section $B \to E$ (or a local section);
- or first find e_0 with $p(e_0) = b_0$, then e_1 using e_0 and $b_1 \to b_0$ via 1.4(d) (which is equivalent to openness), and then e_2 using $b_2 \to b_1$ via 1.4(d) again;
- or first find e_2 with $p(e_2) = b_2$, then e_1 using e_2 and $b_2 \to b_1$ via 1.3(e) (which is equivalent to closeness), and then e_0 using e_1 and $b_1 \to b_0$ via 1.3(e) again.

(9.3) See Problem 8.5; the negative answer is provided by finite counter-examples (see (8.5) and (8.6)):

(9.4) Proposition 5.2 clearly shows the difference between the (ordinary) effective descent morphisms and the effective bijective-descent morphisms: compare 5.2(c) with 3.4(c). Note also that our proof of 5.1 is in fact the finite version of the proof of Theorem 4.2 in [14].

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