Hölder continuity of local weak solutions for parabolic equations exhibiting two degeneracies*

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Abstract

We consider equations of the form

$$\partial_t v - \operatorname{div}(\alpha(v)\nabla v) = 0 \ ,$$

where $v \in [0,1]$ and $\alpha(v)$ degenerates for $v=0$ and $v=1$. We show that local weak solutions are locally Hölder continuous provided α behaves like a power near the two degeneracies. We adopt the technique of intrinsic rescaling developed by DiBenedetto.

Key words. Degenerate parabolic equations, intrinsic rescaling, degeneracies like powers.

1. Introduction

The analysis of the local behaviour of solutions of degenerate parabolic equations has deserved a considerable ammount of attention in the last two decades giving rise to an extensive literature (see [5] and [6] for a quite complete account). The introduction by DiBenedetto in [3] of the degeneracy rescaling technique opened up a powerfull way of approach that has been quite successfull; this paper is a further contribution to the subject in this spirit.

We will be concerned with a parabolic equation with two degeneracies, namely

$$\partial_t v - \operatorname{div}(\alpha(v)\nabla v) = 0 \ , \tag{1}$$

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where $v \in [0,1]$ and $\alpha(v)$ degenerates for $v = 0$ and $v = 1$. An equation of this type is physically relevant since it shows up in a model describing the flow of two immiscible fluids through a porous medium. Following the important pioneering work in [9], that deals with the case of a strictly parabolic equation, a mathematical analysis of this model for an equation of the type of (1) (v being in that setting the saturation) was developed in [1], where it is established the existence of a weak solution for the problem and the continuity of the saturation v under the assumption that α degenerates at most at one side, although no restrictions are made on the nature of the degeneracy. The result was extended to a setting where two degeneracies for α are allowed, provided some information on the nature of one of the degeneracies is assumed. First, in the same paper, the decay of α at one side was taken at most logarithmic and later, in [4], α was allowed to behave like a power. In this article we show that v is locally Hölder continuous if α decays like a power at *both* degeneracies, a nontrivial result that was announced in [4] but for which a proof was still missing. We stress that the novelty lies in the Hölder character of the solution since the continuity is well known. In fact, equation (1) can be written in the form

$$\partial_t \beta(w) - \Delta w = 0 \qquad \text{with} \qquad w = \int_0^v \alpha(s)\, \mathrm{d}s \ ; \ \ v = \beta(w) \ ,$$

where β is an increasing function, for which the results of [7] apply.

Some further physical motivation for the study of this type of equations, arising in polymer chemistry and combustion models, may be found in [8], where a numerical scheme is used to compute approximate solutions and interface curves for the Cauchy problem associated to (1) in a one dimensional case.

We decided to write the paper in a purely abstract setting since the equation is interesting enough from the mathematical point of view in that it exhibits a strong smoothing effect that prevents the possible development of singularities due to the degeneracies. The ultimate goal would be to obtain the continuity for a general quasilinear equation, irrespective of the nature of the two degeneracies. This is also relevant in physical terms since the available experimental information about α is only of a qualitative nature, which makes all assumptions on the degeneracies quite restrictive (see [6]). We hope to address this problem successfully in the future.

2. Statement of the problem and main result

We denote with Ω a bounded and regular domain in $\mathbb{R}^N$, with smooth boundary $\partial\Omega$. Let $Q = \Omega \times (0,T)$, for $T > 0$ be the space-time domain with lateral boundary $\Sigma = \partial\Omega \times (0,T)$ and parabolic boundary $\partial_p Q = \Sigma \cup \big(\Omega \times \{0\}\big)$.

We will consider here local weak solutions of equation (1), assuming that they exist; for the existence theory see [1]. A local weak solution of (1) is a measurable function $v(x,t)$ defined in Q and such that

(i) $v \in [0,1]$ and $v \in C\big(0,T;L^2(\Omega)\big)$;

(ii) $\displaystyle\int_0^v \alpha(s)\,\mathrm{d}s \;\in\; L^2\big(0,T;H^1_{\mathrm{loc}}(\Omega)\big)$;

(iii) for every subset K of Ω and every subinterval $[t_1,t_2]$ of $(0,T]$,

$$\int_{K\times\{t_2\}} v\,\phi - \int_{K\times\{t_1\}} v\,\phi - \int_{t_1}^{t_2}\int_K v\,\partial_t\phi + \int_{t_1}^{t_2}\int_K \alpha(v)\,\nabla v\cdot\nabla\phi = 0\ ,$$

$$\forall\phi\in H^1_{\mathrm{loc}}\big(0,T;L^2(K)\big)\cap L^2_{\mathrm{loc}}\big(0,T;H^1_0(K)\big)\ .$$

We can write **(iii)** in an equivalent way that is technically more convenient and involves the discrete time derivative. Recalling the definition of the Steklov average of a function $u\in L^1(Q)$

$$u_h = \begin{cases} \dfrac{1}{h}\displaystyle\int_t^{t+h} u(\cdot,\tau)\,\mathrm{d}\tau & \text{if} \quad t\in(0,T-h] \\ \\ 0 & \text{if} \quad t\in(T-h,T) \end{cases} \quad ,\quad 0<h<T\ ,$$

the equivalent formulation reads (see [5] for details)

(iii)$'$ for every subset K of Ω,

$$\int_{K\times\{t\}} \partial_t[v_h]\,\varphi + \int_{K\times\{t\}} \big(\alpha(v)\nabla v\big)_h\cdot\nabla\varphi = 0\ ,$$

$$\forall\varphi\in W_0^{1,p}(K)\ ,\quad \forall 0<t<T-h\ . \tag{2}$$

The regularity result will be obtained under the following assumptions on the diffusion coefficient α:

(A1) α is a continuous function and $\alpha(v)>0$ for $v\in(0,1)$.

(A2) $\exists\delta_0\in(0,\frac{1}{2})$ such that, for constants $0<C_0<C_1$,

$$C_0\,\phi(v)\le\alpha(v)\le C_1\,\phi(v)\ ,\quad \forall v\in[0,\delta_0]$$

and

$$C_0\,\psi(1-v)\le\alpha(v)\le C_1\,\psi(1-v)\ ,\quad \forall v\in[1-\delta_0,1]\ .$$

(A3) $\psi(s)=s^{p_1}$; $\phi(s)=s^{p_2}$; $p_1>p_2>0$.

Although we are working with powers, the proof carries through with power-like ϕ and ψ, i.e., functions that satisfy a condition of the type

$$\phi(v) \geq C\, v\, \phi'(v) \ .$$

Our main result is

Theorem 1 *Under assumptions* **(A1)-(A3)***, any local weak solution of (1) is locally Hölder continuous.*

The proof of the local Hölder continuity is based in constructing for every point in the interior of Q a sequence of nested and shrinking cylinders with vertex at that point, such that the essential oscillation of the function in those cylinders converges to zero when the cylinders shrink to zero, in a way quantitavely determined by the data. This can be achieved, roughly speaking, by considering the equation in a geometry dictated by its own stucture. This means that, instead of the usual cylinders, we have to work in cylinders whose dimensions take the degeneracies of the equation into account. In the next section we'll make this idea precise.

We close this section by recalling some notation and a few facts that will be useful later. We use the usual Sobolev spaces and define

$$V_0^2(Q) = L^\infty\big(0,T;L^2(\Omega)\big) \cap L^2\big(0,T;W_0^{1,2}(\Omega)\big)$$

endowed with the norm

$$\|u\|^2_{V_0^2(Q)} = \operatorname*{ess\ sup}_{0\leq t\leq T} \|u(\cdot,t)\|^2_{2,\Omega} + \|\nabla u\|^2_{2,Q} \ .$$

Given a point $x_0 \in \mathbb{R}^N$, $B_\rho(x_0)$ denotes the N-dimensional ball with centre at x_0 and radius ρ:

$$B_\rho(x_0) := \Big\{ x \in \mathbb{R}^N : |x - x_0| < \rho \Big\} \ ;$$

given a point $(x_0,t_0) \in \mathbb{R}^{N+1}$, the cylinder of radius ρ and height $\tau > 0$ is

$$(x_0,t_0) + Q(\tau,\rho) := B_\rho(x_0) \times (t_0 - \tau, t_0) \ .$$

We next introduce a now standard logarithmic function that is used as a recurrent tool in the proof of results concerning the local behaviour of solutions of degenerate and singular PDE's. Given constants a, b, c, with $0 < c < a$, define the nonnegative function

$$\psi^{\pm}_{\{a,b,c\}}(s) \equiv \left(\ln\left\{\frac{a}{(a+c)-(s-b)_{\pm}}\right\}\right)_{+}$$
$$= \begin{cases} \ln\left\{\frac{a}{(a+c)\pm(b-s)}\right\} & \text{if} \quad b\pm c \lessgtr s \lessgtr b\pm(a+c) \\ \\ 0 & \text{if} \quad s \gtreqless b\pm c \end{cases}$$

whose first derivative is

$$\left(\psi^{\pm}_{\{a,b,c\}}\right)'(s) = \begin{cases} \frac{1}{(b-s)\pm(a+c)} & \text{if} \quad b\pm c \lessgtr s \lessgtr b\pm(a+c) \\ \\ 0 & \text{if} \quad s \lessgtr b\pm c \end{cases} \quad \gtreqless 0 \ ,$$

and second derivative $\left(\psi^{\pm}_{\{a,b,c\}}\right)'' = \left\{\left(\psi^{\pm}_{\{a,b,c\}}\right)'\right\}^2 \geq 0$. Now, given a bounded function u in a cylinder $(x_0,t_0)+Q(\tau,\rho)$ and a number k, define the constant

$$H^{\pm}_{u,k} \equiv \operatorname*{ess\ sup}_{(x_0,t_0)+Q(\tau,\rho)} \left|(u-k)_{\pm}\right|$$

and the function

$$\Psi^{\pm}\left(H^{\pm}_{u,k},(u-k)_{\pm},c\right) \equiv \psi^{\pm}_{\left\{H^{\pm}_{u,k},k,c\right\}} \circ u \ , \quad 0<c<H^{\pm}_{u,k} \ . \tag{3}$$

From now on, when referring to this function we will write $\psi^{\pm}(u)$, ommiting the subscripts; it will be clear what they are in every particular situation.

Finally, we recall a technical lemma that will be used in the course of the proof.

Lemma 1 (DeGiorgi, [2]) *Let $v \in W^{1,1}\big(B_\rho(x_0)\big) \cap C\big(B_\rho(x_0)\big)$, with $\rho > 0$ and $x_0 \in \mathbb{R}^N$, and k_1 and k_2 real numbers such that $k_1 < k_2$. There exists a constant C, depending only on N and p, and independent of ρ, x_0, v, k_1 and k_2, such that*

$$(k_2-k_1)\left|[v>k_2]\right| \leq C\,\frac{\rho^{N+1}}{\left|[v<k_1]\right|}\int_{[k_1<v<k_2]} |\nabla v| \ . \tag{4}$$

Remark 1 The conclusion of the lemma remains true for $v \in W^{1,1}(\Omega)\cap C(\Omega)$, provided Ω is convex. Also, the continuity is not a necessary condition for the result. If $v \in W^{1,1}(\Omega)$ define the sets in (4) through any representative in the equivalence class; it can be shown that the conclusion of the lemma is independent of that choice.

Throughout the paper, the letter C denotes a constant that depends only on the data. The same C will be used to denote different constants.

3. The rescaled cylinders

Consider a point $(x_0, t_0) \in Q$ and, by translation and to simplify, assume $(x_0, t_0) = (0, 0)$. Consider a small positive number $\epsilon > 0$ and $R > 0$ such that $Q((2R)^{2-\epsilon}, 2R) \subset Q$ and define

$$\mu_- := \operatorname*{ess\,inf}_{Q((2R)^{2-\epsilon},2R)} \theta \ ; \quad \mu_+ := \operatorname*{ess\,sup}_{Q((2R)^{2-\epsilon},2R)} \theta \ ; \quad \omega := \operatorname*{ess\,osc}_{Q((2R)^{2-\epsilon},2R)} \theta = \mu_+ - \mu_- \ .$$

Construct the cylinder

$$Q(\theta R^2, R) \ , \qquad \text{with} \quad \theta^{-1} = \phi\Big(\frac{\omega}{2^m}\Big) \ ,$$

where the number m will be chosen large later in the proof, independently of ω. We assume $\mu_+ = 1$ and $\mu_- = 0$, since the other possibilities are clearly more favorable.

We may assume that $Q(\theta R^2, R) \subset Q((2R)^{2-\epsilon}, 2R)$, which means that

$$-\theta R^2 \geq -(2R)^{2-\epsilon} \quad \Leftrightarrow \quad \theta^{-1} \geq 2^{\epsilon-2} R^{\epsilon} \ . \tag{5}$$

If this does not hold then we have

$$\phi\Big(\frac{\omega}{2^m}\Big) < C R^{\epsilon} \ ,$$

and then the oscillation would go to zero with R and there would be nothing to prove. We then have the relation

$$\operatorname*{ess\,osc}_{Q(\theta R^2,R)} \theta \leq \omega \tag{6}$$

which will be the starting point of the iteration process that leads to our main result. Note that we had to consider the cylinder $Q((2R)^{2-\epsilon}, 2R)$ and assume (5), so that (6) would hold for the rescalled cylinder $Q(\theta R^2, R)$. This is in general not true for a given cylinder since its dimensions would have to be intrinsically defined in terms of the essential oscillation of the function within it. Observe also that when the oscillation ω is small, and for m very large, then the cylinder $Q(\theta R^2, R)$ is very long in the t direction. It's this feature that will allow us to accommodate the two degeneracies in the problem. We will also assume, without loss of generality, that $\omega < \delta_0$, where δ_0 was introduced in **(A2)**.

We now consider subcylinders of $Q(\theta R^2, R)$ of the form

$$Q_R^{t^*} \equiv (0, t^*) + Q\Big(\frac{R^2}{\psi(\frac{\omega}{4})}, R\Big) \ , \qquad \text{with} \quad t^* < 0 \ .$$

They are contained in $Q(\theta R^2, R)$ if $\theta R^2 \geq -t^* + \frac{R^2}{\psi(\frac{\omega}{4})}$, which holds if $\phi(\frac{\omega}{2^m}) \leq \psi(\frac{\omega}{4})$ and t^* is chosen such that

$$t^* \in \Big(\frac{R^2}{\psi(\frac{\omega}{4})} - \frac{R^2}{\phi(\frac{\omega}{2^m})}, 0 \Big) \ . \tag{7}$$

We will assume further, and for thechnical reasons, that

$$\phi\Big(\frac{\omega}{2^m}\Big) \leq \frac{1}{2}\psi\Big(\frac{\omega}{4}\Big) \tag{8}$$

emphasizing that it is the choice of a big cylinder $Q(\theta R^2, R)$, by choosing m very large, that makes (8) possible.

The proof of our main result follows from the analysis of two complementary cases and the achievement of the same type of conclusion for both. We can briefly describe them in the following way: in the first case we assume that there is a cylinder of the type $(0, t^*) + Q(dR^p, R)$ where θ is essentially away from its infimum. We show that going down to a smaller cylinder the oscillation decreases by a small factor that we can exhibit and that does not depend on the oscillation. If that cylinder can not be found then θ is essentially away from its supremum in all cylinders of that type and we can add up this information to reach the same conclusion as in the previous case. We state this in a precise way in the form of an alternative, studied separetely in the next two sections.

4. The first alternative

For a constant $\nu_0 \in (0,1)$, that will be determined depending only on the data, we will assume in this section that there is a cylinder of the type $Q_R^{t^*}$ for which

$$\Big|\Big\{(x,t) \in Q_R^{t^*} \ : \ v(x,t) > 1 - \frac{\omega}{2}\Big\}\Big| \leq \nu_0 \ |Q_R^{t^*}| \tag{9}$$

leaving for the next section the analysis of the complementary case. We start by showing that if (9) holds then v is away from the degeneracy at 1 in a smaller cylinder of the same type. The next lemma specifies what this means.

Lemma 2 *There exists a constant $\nu_0 \in (0,1)$, depending only on the data, such that if (9) holds then*

$$v(x,t) < 1 - \frac{\omega}{4} \qquad a.e. \qquad (x,t) \in Q_{\frac{R}{2}}^{t^*} \ .$$

Proof. Let $v_\omega \equiv \min\{v, 1 - \frac{\omega}{4}\}$. Take the cylinder for which (9) holds, define

$$R_n = \frac{R}{2} + \frac{R}{2^{n+1}} \ , \quad n = 0, 1, \ldots ,$$

and construct the family of nested and shrinking cylinders

$$Q_{R_n}^{t^*} = B_{R_n} \times \Big(t^* - \frac{R_n^2}{\psi(\frac{\omega}{4})}, t^* \Big) \ .$$

Consider piecewise smooth cutoff functions $0 < \zeta_n \le 1$, defined in these cylinders, and satisfying the following set of assumptions

$$\zeta_n = 1 \ \text{ in } \ Q_{R_{n+1}}^{t^*} \qquad\qquad \zeta_n = 0 \ \text{ on } \ \partial_p Q_{R_n}^{t^*}$$

$$|\nabla \zeta_n| \le \frac{2^{n+1}}{R}$$

$$|\Delta \zeta_n| \le \frac{2^{2(n+1)}}{R^2} \qquad\qquad 0 \le \partial_t \zeta_n \le 2^{2(n+1)} \frac{\psi(\frac{\omega}{4})}{R^2} \ ,$$

and let

$$k_n = 1 - \frac{\omega}{4} - \frac{\omega}{2^{n+2}} \ , \quad n = 0, 1, \ldots .$$

Choose as test function in (2) $\varphi = \big[(v_\omega)_h - k_n \big]_+ \xi_n^2$ and integrate in time over $(t^* - \frac{R_n^2}{\psi(\frac{\omega}{4})}, t)$ for $t \in (t^* - \frac{R_n^2}{\psi(\frac{\omega}{4})}, t^*)$. Putting

$$\tau_n \equiv t^* - \frac{R_n^2}{\psi(\frac{\omega}{4})}$$

the first term gives (for $K = B_{R_n}$)

$$\int_{\tau_n}^t \int_{B_{R_n}} \partial_t \{ v_h \} \ \big[(v_\omega)_h - k_n \big]_+ \xi_n^2 = \frac{1}{2} \int_{\tau_n}^t \int_{B_{R_n}} \partial_t \Big\{ \big[(v_\omega)_h - k_n \big]_+^2 \Big\} \xi_n^2$$

$$+ \Big(1 - \frac{\omega}{4} - k_n \Big) \int_{\tau_n}^t \int_{B_{R_n}} \partial_t \Big\{ \Big(\Big[v - (1 - \frac{\omega}{4}) \Big]_+ \Big)_h \Big\} \xi_n^2 \ .$$

Next, integrate by parts and let $h \to 0$. Making use of Lemma 3.2. in [5, Ch.I], for example, we get

$$\frac{1}{2} \int_{B_{R_n} \times \{t\}} (v_\omega - k_n)_+^2 \, \xi_n^2 - \frac{1}{2} \int_{B_{R_n} \times \{\tau_n\}} (v_\omega - k_n)_+^2 \, \xi_n^2$$

$$- \int_{\tau_n}^t \int_{B_{R_n}} \big(v_\omega - k_n \big)_+^2 \xi_n \partial_t \xi_n + \Big(1 - \frac{\omega}{4} - k_n \Big) \Bigg\{ \int_{B_{R_n} \times \{t\}} \Big[v - (1 - \frac{\omega}{4}) \Big]_+ \xi_n^2$$

$$- \int_{B_{R_n} \times \{\tau_n\}} \Big[v - (1 - \frac{\omega}{4}) \Big]_+ \xi_n^2 - 2 \int_{\tau_n}^t \int_{B_{R_n}} \Big[v - (1 - \frac{\omega}{4}) \Big]_+ \xi_n \partial_t \xi_n \Bigg\} = (*) \ .$$

Since the second and the fifth terms vanish, due to the fact that ξ_n was chosen such that it vanishes on the parabolic boundary of $Q_{R_n}^{t^*}$, and the fourth term is positive, we get, using the other assumptions on ξ_n,

$$(*) \ge \frac{1}{2} \int_{B_{R_n} \times \{t\}} (v_\omega - k_n)_+^2 \, \xi_n^2 - \psi\Big(\frac{\omega}{4} \Big) \frac{2^{2(n+1)}}{R^2} \Big(\frac{\omega}{4} \Big)^2 \int_{\tau_n}^t \int_{B_{R_n}} \chi_{\{v_\omega \ge k_n\}}$$

$$-2\psi\Big(\frac{\omega}{4}\Big)\frac{2^{2(n+1)}}{R^2}\Big(\frac{\omega}{4}\Big)^2\int_{\tau_n}^t\int_{B_{R_n}}\chi_{\{v\geq 1-\frac{\omega}{4}\}} = (**)\ .$$

Observe that $(v_\omega - k_n)_+ \leq \frac{\omega}{4}$, $1-\frac{\omega}{4}-k_n \leq \frac{\omega}{4}$ and $\big[v-(1-\frac{\omega}{4})\big]_+ \leq \frac{\omega}{4}$. Finally, remarking that $v \geq 1-\frac{\omega}{4} \Rightarrow v_\omega \geq k_n$, we obtain

$$(**) \geq \frac{1}{2}\int_{B_{R_n}\times\{t\}}(v_\omega-k_n)_+^2\,\xi_n^2 - 3\psi\Big(\frac{\omega}{4}\Big)\frac{2^{2(n+1)}}{R^2}\Big(\frac{\omega}{4}\Big)^2\int_{\tau_n}^t\int_{B_{R_n}}\chi_{\{v_\omega\geq k_n\}}\ .$$

Concerning the diffusion term, we first pass to the limit in h, obtaining

$$\int_{\tau_n}^t\int_{B_{R_n}}\big(\alpha(v)\nabla v\big)_h\cdot\nabla\Big\{\big[(v_\omega)_h-k_n\big]_+\xi_n^2\Big\}$$

$$\longrightarrow \int_{\tau_n}^t\int_{B_{R_n}}\alpha(v)\nabla v\cdot\big\{\xi_n^2\nabla(v_\omega-k_n)_+ + 2(v_\omega-k_n)_+\xi_n\nabla\xi_n\big\}$$

$$= \int_{\tau_n}^t\int_{B_{R_n}}\alpha(v)\Big|\xi_n\nabla(v_\omega-k_n)_+\Big|^2 + 2\int_{\tau_n}^t\int_{B_{R_n}}\alpha(v)(v_\omega-k_n)_+\xi_n\nabla v\cdot\nabla\xi_n = (*)\ .$$

Next, we estimate the second term:

$$\left|2\int_{\tau_n}^t\int_{B_{R_n}}\alpha(v)(v_\omega-k_n)_+\xi_n\,\nabla v\cdot\nabla\xi_n\right|$$

$$\leq 2\int_{\tau_n}^t\int_{B_{R_n}}|\alpha(v)|\,(v_\omega-k_n)_+\,\xi_n\,|\nabla\xi_n|\big|\,\nabla(v_\omega-k_n)_+\big|$$

$$+\left|2\Big(1-\frac{\omega}{4}-k_n\Big)\int_{\tau_n}^t\int_{B_{R_n}}\xi_n\,\nabla\Big\{\Big(\int_{1-\frac{\omega}{4}}^v\alpha(s)\,\mathrm{d}s\Big)_+\Big\}\cdot\nabla\xi_n\right|$$

$$\leq C_1\psi\Big(\frac{\omega}{2}\Big)\left\{\epsilon\int_{\tau_n}^t\int_{B_{R_n}}\Big|\xi_n\nabla(v_\omega-k_n)_+\Big|^2 + \frac{1}{\epsilon}\int_{\tau_n}^t\int_{B_{R_n}}(v_\omega-k_n)_+^2\,|\nabla\xi_n|^2\right\}$$

$$+2\Big(\frac{\omega}{4}\Big)\left|-\int_{\tau_n}^t\int_{B_{R_n}}\Big(\int_{1-\frac{\omega}{4}}^v\alpha(s)\,\mathrm{d}s\Big)_+\big(|\nabla\xi_n|^2+\xi_n\Delta\xi_n\big)\right|$$

$$\leq C_1\epsilon\psi\Big(\frac{\omega}{2}\Big)\int_{\tau_n}^t\int_{B_{R_n}}\Big|\xi_n\nabla(v_\omega-k_n)_+\Big|^2 + \frac{C_1 2^{2(n+1)}\psi(\frac{\omega}{2})}{\epsilon R^2}\Big(\frac{\omega}{4}\Big)^2\int_{\tau_n}^t\int_{B_{R_n}}\chi_{\{v_\omega\geq k_n\}}$$

$$+2\Big(\frac{\omega}{4}\Big)2\frac{2^{2(n+1)}}{R^2}\psi\Big(\frac{\omega}{4}\Big)\Big(\frac{\omega}{4}\Big)\int_{\tau_n}^t\int_{B_{R_n}}\chi_{\{v_\omega\geq k_n\}}\ ,$$

since

$$\Big(\int_{1-\frac{\omega}{4}}^v\alpha(s)\,\mathrm{d}s\Big)_+ \leq \psi\Big(1-(1-\frac{\omega}{4})\Big)\Big(1-(1-\frac{\omega}{4})\Big)\ .$$

Observing that $\nabla(v_\omega - k_n)_+$ is only nonzero in the set $\{k_n < v < 1-\frac{\omega}{4}\}$, and that in this set

$$\alpha(v) \ge C_0\,\psi(1-v) \ge C_0\,\psi\Big(1-(1-\frac{\omega}{4})\Big) = C_0\,\psi\Big(\frac{\omega}{4}\Big)\ ,$$

we conclude, choosing $\epsilon = \dfrac{(C_0-\frac12)\psi(\frac{\omega}{4})}{C_1\psi(\frac{\omega}{2})}$, that

$$\int_{\tau_n}^{t}\int_{B_{R_n}} \alpha(v)\ \nabla v\cdot\nabla\Big[(v_\omega-k_n)_+\xi_n^2\Big] \ge \frac12\,\psi\Big(\frac{\omega}{4}\Big)\int_{\tau_n}^{t}\int_{B_{R_n}} \Big|\xi_n\nabla(v_\omega-k_n)_+\Big|^2$$

$$-\left\{\frac{\big[C_1\psi(\frac{\omega}{2})\big]^2}{(C_0-\frac12)\psi(\frac{\omega}{4})} + 4\psi\Big(\frac{\omega}{4}\Big)\right\}\frac{2^{2(n+1)}}{R^2}\Big(\frac{\omega}{4}\Big)^2\int_{\tau_n}^{t}\int_{B_{R_n}} \chi_{\{v_\omega\ge k_n\}}\ .$$

Now, putting both estimates together, we arrive at

$$\operatorname*{ess\ sup}_{\tau_n\le t\le t^*}\int_{B_{R_n}\times\{t\}} (v_\omega-k_n)_+^2\,\xi_n^2 + \psi\Big(\frac{\omega}{4}\Big)\int_{\tau_n}^{t^*}\int_{B_{R_n}}\Big|\xi_n\nabla(v_\omega-k_n)_+\Big|^2$$

$$\le 2\left\{\frac{\big[C_1\psi(\frac{\omega}{2})\big]^2}{(C_0-\frac12)\psi(\frac{\omega}{4})} + 7\psi\Big(\frac{\omega}{4}\Big)\right\}\frac{2^{2(n+1)}}{R^2}\Big(\frac{\omega}{4}\Big)^2\int_{\tau_n}^{t^*}\int_{B_{R_n}} \chi_{\{v_\omega\ge k_n\}}\ .$$

Next we perform a change in the time variable, putting $\overline{t} = (t-t^*)\psi(\frac{\omega}{4})$ and define

$$\overline{v_\omega}(\cdot,\overline{t}) = v_\omega(\cdot,t) \qquad \text{and} \qquad \overline{\xi_n}(\cdot,\overline{t}) = \xi_n(\cdot,t)\ ,$$

obtaining the simplified inequality

$$\Big\|(\overline{v_\omega}-k_n)_+\ \overline{\xi_n}\Big\|^2_{V_0^2\left(Q(R_n^2,R_n)\right)}$$

$$\le 2\left\{\frac{C_1^2}{(C_0-\frac12)}\left[\frac{\psi(\frac{\omega}{2})}{\psi(\frac{\omega}{4})}\right]^2 + 7\right\}\frac{2^{2(n+1)}}{R^2}\Big(\frac{\omega}{4}\Big)^2\int_{-R_n^2}^{0}\int_{B_{R_n}} \chi_{\{\overline{v_\omega}\ge k_n\}}\ . \tag{10}$$

Define, for each n,

$$A_n = \int_{-R_n^2}^{0}\int_{B_{R_n}} \chi_{\{\overline{v_\omega}\ge k_n\}}\mathrm{d}x\ \mathrm{d}\overline{t}$$

and observe that the following estimates hold

$$\frac{1}{2^{2(n+2)}}\ \Big(\frac{\omega}{4}\Big)^2\ A_{n+1} \le \big|k_{n+1}-k_n\big|^2\ A_{n+1}$$

$$\le \big\|(\overline{v_\omega}-k_n)_+\big\|^2_{2,Q(R_{n+1}^2,R_{n+1})}$$

$$\le \big\|(\overline{v_\omega}-k_n)_+\overline{\xi_n}\big\|^2_{2,Q(R_n^2,R_n)}$$

$$\leq C \left\| (\overline{v_\omega} - k_n)_+ \overline{\xi_n} \right\|^2_{V_0^2\left(Q(R_n^2, R_n)\right)} \; A_n^{\frac{2}{N+2}}$$

$$\leq 2C \left\{ \frac{C_1^2}{(C_0 - \frac{1}{2})} \left[\frac{\psi(\frac{\omega}{2})}{\psi(\frac{\omega}{4})} \right]^2 + 7 \right\} \frac{2^{2(n+1)}}{R^2} \left(\frac{\omega}{4} \right)^2 \; A_n^{1+\frac{2}{N+2}} \; . \tag{11}$$

In fact, the first and the third inequalities are obvious; the second one holds due to the fact that $k_n < k_{n+1}$; the fourth inequality is a consequence of a well known imbedding theorem (see, e.g., corollary 3.1 on chapter I of [5]) and the last one follows from (10). Next, define the numbers

$$X_n = \frac{A_n}{\left|Q(R_n^2, R_n)\right|} \; ,$$

divide (11) by $\left|Q(R_{n+1}^2, R_{n+1})\right|$ and obtain the recursive relation

$$X_{n+1} \leq \gamma \, 4^{2n} \; X_n^{1+\frac{2}{N+2}} \; ,$$

where

$$\gamma = C \left\{ \frac{C_1^2}{(C_0 - \frac{1}{2})} \left[\frac{\psi(\frac{\omega}{2})}{\psi(\frac{\omega}{4})} \right]^2 + 7 \right\} .$$

We can use a lemma on the fast geometric convergence of sequences (see lemma 4.1 on chapter I of [5]) to conclude that if

$$X_0 \leq \gamma^{-\frac{N+2}{2}} 4^{-2(\frac{N+2}{2})^2} \equiv \nu_0 \tag{12}$$

then

$$X_n \longrightarrow 0 \; . \tag{13}$$

But (12) is nothing but the assumption (9) of the lemma and the conclusion easily follows from (13). In fact, observe that

$$R_n \searrow \frac{R}{2} \qquad \text{and} \qquad k_n \nearrow 1 - \frac{\omega}{4} \; ,$$

and since (13) implies that $A_n \to 0$, we conclude that

$$\left| \left\{ (x, \overline{t}) \in Q\left(\left(\tfrac{R}{2}\right)^2, \tfrac{R}{2} \right) \; : \; \overline{v_\omega}(x, \overline{t}) \geq 1 - \tfrac{\omega}{4} \right\} \right|$$

$$= \left| \left\{ (x, t) \in Q^{t^*}_{\frac{R}{2}} \; : \; v(x,t) \geq 1 - \frac{\omega}{4} \right\} \right| = 0$$

and the lemma is proved.

Let's just show why ν_0 is independent of ω, since this is crucially related to the fact that ψ is a power. In fact, we have

$$\nu_0 = \gamma^{-\frac{N+2}{2}} 4^{-2(\frac{N+2}{2})^2} = 4^{-2(\frac{N+2}{2})^2} C \left\{ \frac{C_1^2}{(C_0 - \frac{1}{2})} \left[\frac{\psi(\frac{\omega}{2})}{\psi(\frac{\omega}{4})} \right]^2 + 7 \right\}^{-\frac{N+2}{2}} \; ,$$

and so we conclude showing that

$$\frac{\psi(\frac{\omega}{2})}{\psi(\frac{\omega}{4})} = \left(\frac{\omega}{2}\right)^{p_1} \left(\frac{4}{\omega}\right)^{p_1} = 2^{p_1}$$

is independent of ω.

$\blacksquare$

Our next aim is to show that the conclusion of Lemma 2 holds in a full cylinder of the type $Q(\tau,\rho)$. The idea is to use the fact that at the time level

$$-\widehat{t} := t^* - \frac{\left(\frac{R}{2}\right)^2}{\psi\left(\frac{\omega}{4}\right)} \tag{14}$$

the function $v(x)$ is strictly below the level $1-\frac{\omega}{4}$ in the ball $B_{\frac{R}{2}}$ and look at this time level as an initial condition to make the conclusion hold up to $t=0$, eventually shrinking the ball. Again this is a sophisticated way of showing that somehow the equation behaves like the heat equation. As an intermediate step we need the following lemma.

Lemma 3 *Given $\nu_1 \in (0,1)$, there exists $s_1 \in \mathbb{N}$, depending only on the data, such that*

$$\left|\left\{x \in B_{\frac{R}{4}} \ : \ v(x,t) \geq 1 - \frac{\omega}{2^{s_1}}\right\}\right| \leq \nu_1 \left|B_{\frac{R}{4}}\right| \ , \quad \forall t \in (-\widehat{t},0) \ .$$

Proof. We use a logarithmic estimate for the function $(v-k)_+$ in the cylinder $Q(\widehat{t},\frac{R}{2})$, with the choices

$$k = 1-\frac{\omega}{4} \qquad \text{and} \qquad c = \frac{\omega}{2^{n+2}} \ ,$$

where $n \in \mathbb{N}$ will be chosen later, as parameters in the standard logarithmic function (3). We have

$$v-k \leq H^+_{v,k} = \operatorname*{ess\ sup}_{Q(\widehat{t},\frac{R}{2})} \left|\left(v-1+\frac{\omega}{4}\right)_+\right| \leq \frac{\omega}{4} \tag{15}$$

and, since if $H^+_{v,k}=0$ the result is trivial, we may assume it is strictly positive and choose n big enough so that $c < H^+_{v,k}$. We recall that in this case the logarithmic function is defined in the whole domain of v, $Q(\widehat{t},\frac{R}{2})$ (since it is obvious that $H^+_{v,k}-v+k+c>0$), and given by

$$\Psi^+ = \psi^+_{\left\{H^+_{v,k},k,\frac{\omega}{2^{n+2}}\right\}}(v) = \begin{cases} \ln\left\{\dfrac{H^+_{v,k}}{H^+_{v,k}-v+k+\frac{\omega}{2^{n+2}}}\right\} & \text{if} \quad v > k+\frac{\omega}{2^{n+2}} \\ 0 & \text{if} \quad v \leq k+\frac{\omega}{2^{n+2}} \end{cases} .$$

From (15), we can easily estimate

$$\Psi^+ \leq n \ln 2 \qquad \text{since} \qquad \frac{H^+_{v,k}}{H^+_{v,k} - v + k + \frac{\omega}{2^{n+2}}} \leq \frac{\frac{\omega}{4}}{\frac{\omega}{2^{n+2}}} = 2^n$$

and the derivative (here in the nonvanishing case $v > k + c$)

$$\left|\left(\psi^+\right)'(v)\right| = \left|\frac{-1}{H^+_{v,k} - v + k + c}\right| \leq \left|\frac{1}{c}\right| = \frac{2^{n+2}}{\omega} \ .$$

To obtain the estimate, choose a piecewise smooth cutoff function $0 < \zeta(x) \leq 1$, defined on $B_{\frac{R}{2}}$ and such that

$$\zeta = 1 \ \text{ in } \ B_{\frac{R}{4}} \qquad \text{and} \qquad |\nabla\zeta| \leq \tfrac{C}{R} \ ,$$

and multiply (2) by $2\psi^+(v_h)\left(\psi^+\right)'(v_h)\,\zeta^2$. Integrating in time in $(-\widehat{t}, t)$, with $t \in (-\widehat{t}, 0)$, and performing the usual integrations by parts and passages to the limit in h for the Steklov averages, we obtain concerning the time part

$$\int_{B_{\frac{R}{2}} \times \{t\}} \left[\psi^+(v)\right]^2 \zeta^2 - \int_{B_{\frac{R}{2}} \times \{-\widehat{t}\}} \left[\psi^+(v)\right]^2 \zeta^2 \ .$$

Now observe that as a consequence of Lemma 2, we have $v(x, -\widehat{t}) < k$ in the ball $B_{\frac{R}{2}}$, which implies that

$$\left[\psi^+(v)\right](x, -\widehat{t}) = 0 \ , \quad x \in B_{\frac{R}{2}} \ .$$

As for the space part, we start by passing to the limit in h, thus getting

$$\int_{-\widehat{t}}^{t}\int_{B_{\frac{R}{2}}} \alpha(v)\nabla v \cdot \nabla\left\{2\psi^+(v)\left(\psi^+\right)'(v)\,\zeta^2\right\}$$

$$= \int_{-\widehat{t}}^{t}\int_{B_{\frac{R}{2}}} \alpha(v)|\nabla v|^2\left\{2\left(1 + \psi^+(v)\right)\left[(\psi^+)'(v)\right]^2 \zeta^2\right\}$$

$$+2\int_{-\widehat{t}}^{t}\int_{B_{\frac{R}{2}}} \alpha(v)\nabla v \cdot \nabla\zeta\left\{2\psi^+(v)\left(\psi^+\right)'(v)\,\zeta\right\} = (*) \ .$$

Using Young's inequality, we estimate the second term as

$$\left|2\int_{-\widehat{t}}^{t}\int_{B_{\frac{R}{2}}} \alpha(v)\nabla v \cdot \nabla\zeta\left\{2\psi^+(v)\left(\psi^+\right)'(v)\,\zeta\right\}\right|$$

$$\leq 2\int_{-\hat t}^{t}\int_{B_{\frac{R}{2}}} \alpha(v)|\nabla v|^2\zeta^2\psi^+(v)\big[(\psi^+)'(v)\big]^2 + 2\int_{-\hat t}^{t}\int_{B_{\frac{R}{2}}} \alpha(v)|\nabla \zeta|^2\psi^+(v)\ ,$$

and as a consequence we get

$$(*) \geq 2\int_{-\hat t}^{t}\int_{B_{\frac{R}{2}}} \alpha(v)|\nabla v|^2\zeta^2\big[(\psi^+)'(v)\big]^2 - 2\int_{-\hat t}^{t}\int_{B_{\frac{R}{2}}} \alpha(v)|\nabla \zeta|^2\psi^+(v)$$

$$\geq -2n\ln 2\,\frac{C}{R^2}\int_{-\hat t}^{t}\int_{B_{\frac{R}{2}}} \alpha(v)\chi_{\{v>1-\frac{\omega}{4}\}} \geq -2n\ln 2\,\frac{C}{R^2}\,\hat t\,\big|B_{\frac{R}{2}}\big|\,C_1\,\psi\Big(\frac{\omega}{4}\Big)\ ,$$

because in the relevant set

$$\alpha(v) \leq C_1\,\psi(1-v) \leq C_1\,\psi\Big(1-\big(1-\frac{\omega}{4}\big)\Big) = C_1\,\psi\Big(\frac{\omega}{4}\Big)\ .$$

Now we observe that, due to our choice of t^*,

$$\hat t = -t^* + \frac{\big(\frac{R}{2}\big)^2}{\psi\big(\frac{\omega}{4}\big)} \leq \theta R^2\ ,$$

and so we can condensate the two estimates in the logarithmic estimate

$$\sup_{-\hat t\leq t\leq 0}\int_{B_{\frac{R}{2}}\times\{t\}} \big[\psi^+(v)\big]^2\,\zeta^2 \leq C\,n\,\theta\,\big|B_{\frac{R}{2}}\big|\,\psi\Big(\frac{\omega}{4}\Big)\ . \tag{16}$$

Now, since the integrand is nonnegative, we estimate below the left hand side of (16) integrating over the smaller set

$$S = \Big\{x\in B_{\frac{R}{4}}\ :\ v(x,t)\geq 1-\frac{\omega}{2^{n+2}}\Big\} \subset B_{\frac{R}{2}}$$

and, observing that in S, $\zeta = 1$ and

$$\big[\psi^+(v)\big]^2 \geq \Big[\ln\big(2^{n-1}\big)\Big]^2 = (n-1)^2(\ln 2)^2\ ,$$

we get $(n-1)^2(\ln 2)^2|S| \leq C\,n\,\theta\,\big|B_{\frac{R}{2}}\big|\,\psi\big(\frac{\omega}{4}\big)$, i.e.,

$$\Big|\Big\{x\in B_{\frac{R}{4}}\ :\ v(x,t)\geq 1-\frac{\omega}{2^{n+2}}\Big\}\Big| \leq C\,\frac{n}{(n-1)^2}\,\frac{\psi\big(\frac{\omega}{4}\big)}{\phi\big(\frac{\omega}{2^m}\big)}\,\big|B_{\frac{R}{4}}\big|\ .$$

To prove the lemma we just need to choose

$$s_1 = n+2 \qquad\text{with}\qquad n \geq 1 + \frac{2C}{\nu_1}\,\frac{\psi\big(\frac{\omega}{4}\big)}{\phi\big(\frac{\omega}{2^m}\big)}\ ,$$

since if $n \geq 1 + \frac{2}{\alpha}$ then $\frac{n}{(n-1)^2} \leq \alpha$, $\alpha > 0$. Concerning the independence of ω, observe that

$$\frac{\psi\left(\frac{\omega}{4}\right)}{\phi\left(\frac{\omega}{2^m}\right)} = \frac{\left(\frac{\omega}{4}\right)^{p_1}}{\left(\frac{\omega}{2^m}\right)^{p_2}} = 2^{mp_2 - p_1}\ \omega^{p_1 - p_2} \leq 2^{mp_2 - p_1}\ ,$$

because $p_1 > p_2$ and $\omega < 1$, and this expression does not depended on ω. We strongly emphasize that it is crucial at this point that both degeneracies are powers and that $p_1 > p_2$.

■

We now arrive at the main result of this section that establishes the first alternative.

Proposition 1 *There exist constants $\nu_0 \in (0,1)$ and $1 < s_1 \in \mathbb{N}$, depending only on the data, such that if (9) holds then*

$$v(x,t) < 1 - \frac{\omega}{2^{s_1+1}}\ , \quad a.e. \quad (x,t) \in Q\left(\widehat{t}, \frac{R}{8}\right)\ .$$

Proof. Consider the cylinder for which (9) holds, let

$$R_n = \frac{R}{8} + \frac{R}{2^{n+3}}\ , \quad n = 0, 1, \ldots$$

and construct the family of nested and shrinking cylinders $Q(\widehat{t}, R_n)$, where $\widehat{t}$ is given by (14). Take piecewise smooth cutoff functions $0 < \zeta_n(x) \leq 1$, not depending on t, defined in B_{R_n} and satisfying the assumptions

$$\zeta_n = 1 \ \text{ in } \ B_{R_{n+1}}\ , \qquad |\nabla \zeta_n| \leq \frac{2^{n+4}}{R} \qquad \text{and} \qquad |\Delta \zeta_n| \leq \frac{2^{2(n+4)}}{R^2}\ .$$

Take also

$$k_n = 1 - \frac{\omega}{2^{s_1+1}} - \frac{\omega}{2^{s_1+1+n}}\ , \quad n = 0, 1, \ldots,$$

(where $s_1 > 1$ is to be chosen) and derive local energy inequalities similar to those obtained in Lemma 2, now for the functions

$$(v_\omega - k_n)_+ \zeta_n^2\ , \qquad \text{with} \qquad v_\omega \equiv \min\left(v, 1 - \frac{\omega}{2^{s_1}}\right)$$

and in the cylinders $Q(\widehat{t}, R_n)$. Observing that, due to Lemma 2, we have $v(x, -\widehat{t}) < 1 - \frac{\omega}{4} \leq k_n$ in the ball $B_{\frac{R}{2}} \supset B_{R_n}$, which implies that

$$(v - k_n)_+(x, -\widehat{t}) = 0\ , \quad x \in B_{R_n}\ , \quad n = 0, 1, \ldots\ ,$$

and performing the same type of reasoning used in Lemma 2 (which we shall not repeat here), we get

$$\sup_{-\widehat{t}<t<0} \int_{B_{R_n}\times\{t\}} (v_\omega - k_n)_+^2 \zeta_n^2 + \psi\Big(\frac{\omega}{2^{s_1}}\Big) \int_{-\widehat{t}}^0 \int_{B_{R_n}} \big|\zeta_n \nabla (v_\omega - k_n)_+\big|^2$$

$$\leq \left\{ \frac{4{C_1}^2\, \psi\big(\frac{\omega}{2^{s_1-1}}\big)^2}{(2C_0-1)\,\psi\big(\frac{\omega}{2^{s_1}}\big)} + 8\,\psi\Big(\frac{\omega}{2^{s_1}}\Big) \right\} \frac{2^{2(n+4)}}{R^2} \Big(\frac{\omega}{2^{s_1}}\Big)^2 \int_{-\widehat{t}}^0 \int_{B_{R_n}} \chi_{\{v_\omega \geq k_n\}} \;.$$

The change in the time variable $\overline{t} = \left(\frac{R}{2}\right)^2 \frac{t}{\widehat{t}}$, with the new function

$$\overline{v_\omega}(\cdot, \overline{t}) = v_\omega(\cdot, t) \;,$$

leads to

$$\frac{\big(\frac{R}{2}\big)^2}{\psi\big(\frac{\omega}{2^{s_1}}\big)\,\widehat{t}} \sup_{-(\frac{R}{2})^2<t<0} \int_{B_{R_n}\times\{t\}} (\overline{v_\omega} - k_n)_+^2 \zeta_n^2 + \int_{-(\frac{R}{2})^2}^0 \int_{B_{R_n}} \big|\zeta_n \nabla (\overline{v_\omega} - k_n)_+\big|^2$$

$$\leq \left\{ \frac{4{C_1}^2\, \psi\big(\frac{\omega}{2^{s_1-1}}\big)^2}{(2C_0-1)\,\psi\big(\frac{\omega}{2^{s_1}}\big)^2} + 8 \right\} \frac{2^{2(n+4)}}{R^2} \Big(\frac{\omega}{2^{s_1}}\Big)^2 \int_{-(\frac{R}{2})^2}^0 \int_{B_{R_n}} \chi_{\{\overline{v_\omega} \geq k_n\}} \;,$$

after multiplication of the whole expression by $\dfrac{\big(\frac{R}{2}\big)^2}{\psi\big(\frac{\omega}{2^{s_1}}\big)\,\widehat{t}}$. Now, if s_1 is chosen sufficiently large, we have

$$\frac{\big(\frac{R}{2}\big)^2}{\psi\big(\frac{\omega}{2^{s_1}}\big)\,\widehat{t}} \geq 1 \tag{17}$$

and get, with $\gamma = \left\{ \frac{4{C_1}^2\, \psi(\frac{\omega}{2^{s_1-1}})^2}{(2C_0-1)\,\psi(\frac{\omega}{2^{s_1}})^2} + 8 \right\}$,

$$\Big\| (\overline{v_\omega} - k_n)_+\, \zeta_n \Big\|^2_{V_0^2\left(Q((\frac{R}{2})^2, R_n)\right)} \leq \gamma\, \frac{2^{2(n+4)}}{R^2} \Big(\frac{\omega}{2^{s_1}}\Big)^2 \int_{-(\frac{R}{2})^2}^0 \int_{B_{R_n}} \chi_{\{\overline{v_\omega} \geq k_n\}} \;.$$

Define, for each n,

$$A_n = \int_{-(\frac{R}{2})^2}^0 \int_{B_{R_n}} \chi_{\{\overline{v_\omega} \geq k_n\}}\; \mathrm{d}x\; \mathrm{d}\overline{t}$$

and observe that the following estimates hold, by a reasoning similar to the one that led to (11):

$$\frac{1}{2^{2(n+2)}} \Big(\frac{\omega}{2^{s_1}}\Big)^2 A_{n+1} = \big|k_{n+1} - k_n\big|^2 A_{n+1}$$

$$\leq \left\|(\overline{v_\omega}-k_n)_+\right\|^2_{2,Q\left((\frac{R}{2})^2,R_{n+1}\right)}$$

$$\leq \left\|(\overline{v_\omega}-k_n)_+\zeta_n\right\|^2_{2,Q\left((\frac{R}{2})^2,R_n\right)} \leq C \left\|(\overline{v_\omega}-k_n)_+ \zeta_n\right\|^2_{V_0^2\left(Q\left((\frac{R}{2})^2,R_n\right)\right)} A_n^{\frac{2}{N+2}}$$

$$\leq \gamma \frac{2^{2(n+4)}}{R^2}\left(\frac{\omega}{2^{s_1}}\right)^2 A_n^{1+\frac{2}{N+2}} .$$

Next, define the numbers

$$X_n = \frac{A_n}{\left|Q\left((\frac{R}{2})^2,R_n\right)\right|} ,$$

divide the inequality by $\left|Q\left((\frac{R}{2})^2,R_{n+1}\right)\right|$ and obtain the recursive relation

$$X_{n+1} \leq \gamma\, 4^{2n}\, X_n^{1+\frac{2}{N+2}} .$$

Using again the lemma on the fast geometric convergence of sequences (see lemma 4.1 on chapter I of [5]) we conclude that if

$$X_0 \leq \gamma^{-\frac{N+2}{2}} 4^{-\frac{(N+2)^2}{2}} \equiv \nu_1 \in (0,1) \tag{18}$$

then

$$X_n \longrightarrow 0 . \tag{19}$$

Apply Lemma 3 with this ν_1 and conclude that there exists s_1, depending only on the data, such that

$$\left|\left\{x\in B_{\frac{R}{4}} \;:\; v(x,t)\geq 1-\frac{\omega}{2^{s_1}}\right\}\right| \leq \nu_1 \left|B_{\frac{R}{4}}\right| , \quad \forall t\in(-\widehat{t},0) ,$$

and obtain (18) as follows

$$\begin{aligned} X_0 &= \frac{\int_{-(\frac{R}{2})^2}^0 \int_{B_{\frac{R}{4}}} \chi_{\left\{\overline{v_\omega}\ \geq 1-\frac{\omega}{2^{s_1}}\right\}}}{\left|Q\left((\frac{R}{2})^2,\frac{R}{4}\right)\right|} \\ &= \frac{(\frac{R}{2})^2}{\widehat{t}} \frac{\int_{-\widehat{t}}^0 \int_{B_{\frac{R}{4}}} \chi_{\left\{v\ \geq 1-\frac{\omega}{2^{s_1}}\right\}}}{\left|Q\left((\frac{R}{2})^2,\frac{R}{4}\right)\right|} \\ &\leq \frac{(\frac{R}{2})^2}{\widehat{t}} \frac{\widehat{t}\left|\left\{x\in B_{\frac{R}{4}} \;:\; v(x,t)\geq 1-\frac{\omega}{2^{s_1}}\right\}\right|}{(\frac{R}{2})^2\left|B_{\frac{R}{4}}\right|} \leq \nu_1 . \end{aligned}$$

Now the conclusion easily follows from (19). In fact, observe that

$$R_n \searrow \frac{R}{8} \qquad \text{and} \qquad k_n \nearrow 1 - \frac{\omega}{2^{s_1+1}} \ ,$$

and since (19) implies that $A_n \to 0$, we conclude that

$$\begin{aligned} &\left| \left\{ (x,\bar{t}) \in Q\Big(\big(\tfrac{R}{2}\big)^2, \tfrac{R}{8} \Big) \ : \ \overline{v_\omega}(x,\bar{t}) \geq 1 - \tfrac{\omega}{2^{s_1+1}} \right\} \right| \\ &= \left| \left\{ (x,t) \in Q\Big(\widehat{t}, \frac{R}{8} \Big) \ : \ v(x,t) \geq 1 - \frac{\omega}{2^{s_1+1}} \right\} \right| \quad = 0 \end{aligned}$$

and the proposition is proved.

■

Remark 2 It is very important to remark that ν_1 in the proof is independent of s_1, since otherwise the reasoning would be falatious. In fact, we have

$$\nu_1 = \gamma^{-\frac{N+2}{2}} 4^{-\frac{(N+2)^2}{2}}$$

and with our assumption that ψ is a power, we have

$$\gamma = \left\{ \frac{4{C_1}^2}{2C_0 - 1} \left[\frac{\psi(\frac{\omega}{2^{s_1-1}})}{\psi(\frac{\omega}{2^{s_1}})} \right]^2 + 8 \right\} = \left\{ \frac{4{C_1}^2}{2C_0 - 1} \, 2^{2p_1} + 8 \right\} \ ,$$

which clearly shows what was claimed. This also proves the independence of s_1 with respect to ω, through Lemma 3, together with the remark that concerning the choice (17), we have

$$\frac{\big(\frac{R}{2}\big)^2}{\psi\big(\frac{\omega}{2^{s_1}}\big)\, \widehat{t}} \geq \frac{\phi\big(\frac{\omega}{2^m}\big)}{4\psi\big(\frac{\omega}{2^{s_1}}\big)} \qquad \text{because} \qquad \widehat{t} \leq \theta R^2$$

and so it can be made larger than 1 if

$$\frac{\phi\big(\frac{\omega}{2^m}\big)}{4\psi\big(\frac{\omega}{2^{s_1}}\big)} = \frac{\big(\frac{\omega}{2^m}\big)^{p_2}}{4\big(\frac{\omega}{2^{s_1}}\big)^{p_1}} = \omega^{p_2 - p_1} 2^{p_1 s_1 - m p_2 - 2} \geq 1 \ .$$

Since $p_1 > p_2$ and $\omega < 1$, it is enough to choose

$$s_1 \geq \frac{2 + m p_2}{p_1} \ ,$$

that clearly doesn't depend on ω.

Corollary 1 *There exist constants $\nu_0, \sigma_0 \in (0,1)$, depending only on the data, such that if (9) holds then*

$$\operatorname*{ess\,osc}_{Q(\widehat{t}, \frac{R}{8})} v \ \leq \ \sigma_0 \ \omega \ . \tag{20}$$

Proof. We can use Proposition 1 to obtain $s_1 \in \mathbb{N}$ such that

$$\operatorname*{ess\,sup}_{Q(\widehat{t},\frac{R}{8})} v \le 1 - \frac{\omega}{2^{s_1+1}}$$

and from this we get

$$\operatorname*{ess\,osc}_{Q(\widehat{t},\frac{R}{8})} v = \operatorname*{ess\,sup}_{Q(\widehat{t},\frac{R}{8})} v - \operatorname*{ess\,inf}_{Q(\widehat{t},\frac{R}{8})} v \le 1 - \frac{\omega}{2^{s_1+1}} - 0 = \Big(1 - \frac{1}{2^{s_1+1}}\Big)\omega$$

and the corollary follows with $\sigma_0 = \big(1 - \frac{1}{2^{s_1+1}}\big)$.

■

5. The second alternative

Let's now suppose that (9) does not hold. In that case we have the complementary condition, and since $1 - \frac{\omega}{2} \ge \frac{\omega}{2}$, it means that for every cylinder of the type $Q_R^{t^*}$, we have

$$\left|\left\{(x,t) \in Q_R^{t^*} \ : \ v(x,t) < \frac{\omega}{2}\right\}\right| \le (1-\nu_0)\ \left|Q_R^{t^*}\right| \ . \tag{21}$$

We are in the case for which we have to analyze the behaviour of v near 0, the other point where $\alpha(v)$ degenerates. We will show that also in this case a conclusion similar to (20) can be taken. Recall that the constant ν_0 has already been determined in the previous section and is given by (12).

Fix a cylinder $Q_R^{t^*} \subset Q(\theta R^2, R)$ for which (21) holds. There exists a time level

$$t^\circ \in \left[t^* - \frac{R^2}{\psi\big(\frac{\omega}{4}\big)}, t^* - \frac{\nu_0}{2}\frac{R^2}{\psi\big(\frac{\omega}{4}\big)}\right]$$

such that

$$\left|\left\{x \in B_R \ : \ v(x,t^\circ) < \frac{\omega}{2}\right\}\right| < \left(\frac{1-\nu_0}{1-\frac{\nu_0}{2}}\right) \left|B_R\right| \ . \tag{22}$$

In fact, if this is not true, then

$$\begin{aligned}
\left|\left\{x \in B_R \ : \ v(x,t^\circ) < \frac{\omega}{2}\right\}\right| \ &\ge \ \int_{t^*-\frac{R^2}{\psi(\frac{\omega}{4})}}^{t^*-\frac{\nu_0}{2}\frac{R^2}{\psi(\frac{\omega}{4})}} \left|\left\{x \in B_R \ : \ v(x,t) < \frac{\omega}{2}\right\}\right| \,\mathrm{d}t \\
&\ge \ \left\{t^* - \frac{\nu_0}{2}\frac{R^2}{\psi\big(\frac{\omega}{4}\big)} - t^* + \frac{R^2}{\psi\big(\frac{\omega}{4}\big)}\right\}\left(\frac{1-\nu_0}{1-\frac{\nu_0}{2}}\right)|B_R| \\
&= \ (1-\nu_0)\ \frac{R^2}{\psi\big(\frac{\omega}{4}\big)}\ |B_R| \\
&= \ (1-\nu_0)\ |Q_R^{t^*}| \ ,
\end{aligned}$$

which contradicts our assumption. The next lemma asserts that the set where $v(x)$ is close to its infimum is small, not only at a specific time level, but for all time levels near the top of the cylinder $Q_R^{t^*}$.

Lemma 4 *There exists $1 < s_2 \in \mathbb{N}$, depending only on the data, such that*

$$\left|\left\{x \in B_R \ : \ v(x,t) < \frac{\omega}{2^{s_2}}\right\}\right| \leq \left(1 - \left(\frac{\nu_0}{2}\right)^2\right) |B_R| \ ,$$

for all $t \in \left[t^* - \frac{\nu_0}{2}\frac{R^2}{\psi(\frac{\omega}{4})}, t^*\right]$.

Proof. We use a logarithmic estimate for the function $(v-k)_-$ in the cylinder $B_R \times (t^\circ, t^*)$, with the choices

$$k = \frac{\omega}{2} \quad \text{ and } \quad c = \frac{\omega}{2^{n+1}} \ ,$$

where $n \in \mathbb{N}$ will be chosen later, as parameters in the standard logarithmic function (3). We have

$$k - v \leq H_{v,k}^- = \operatorname*{ess\,sup}_{B_R \times (t^\circ, t^*)} \left|\left(v - \frac{\omega}{2}\right)_-\right| \leq \frac{\omega}{2} \tag{23}$$

and, since if $H_{v,k}^- = 0$ the result is trivial, we may assume it is strictly positive and choose n big enough so that $c < H_{v,k}^-$. We recall that in this case the logarithmic function is defined in the whole domain of v, $B_R \times (t^\circ, t^*)$ (since it is obvious that $H_{v,k}^- + v - k + c > 0$), and given by

$$\Psi^- = \psi^-_{\left\{H_{v,k}^-, k, \frac{\omega}{2^{n+1}}\right\}}(v) = \begin{cases} \ln\left\{\dfrac{H_{v,k}^-}{H_{v,k}^- + v - k + \frac{\omega}{2^{n+1}}}\right\} & \text{if} \quad v < k - \frac{\omega}{2^{n+1}} \\ 0 & \text{if} \quad v \geq k - \frac{\omega}{2^{n+1}} \end{cases} .$$

From (23), we can easily estimate

$$\Psi^- \leq n \ln 2 \qquad \text{since} \qquad \frac{H_{v,k}^-}{H_{v,k}^- + v - k + \frac{\omega}{2^{n+1}}} \leq \frac{\frac{\omega}{2}}{\frac{\omega}{2^{n+1}}} = 2^n$$

and the derivative (here in the nonvanishing case $v < k - c$)

$$\left|(\psi^-)'(v)\right| = \left|\frac{-1}{H_{v,k}^- + v - k + c}\right| \leq \left|\frac{1}{c}\right| = \frac{2^{n+1}}{\omega} \ .$$

To obtain the estimate, choose a piecewise smooth cutoff function $0 < \zeta(x) \leq 1$, defined on B_R and such that, for a certain $\sigma \in (0,1)$,

$$\zeta = 1 \ \text{ in } \ B_{(1-\sigma)R} \qquad \text{and} \qquad |\nabla\zeta| \leq \frac{C}{\sigma R} \ ,$$

and multiply (2) by $2\psi^-(v_h)\big(\psi^-\big)'(v_h)\,\zeta^2$. Integrating in time in (t°,t), with $t\in(t^\circ,t^*)$, and performing the usual integrations by parts and passages to the limit in h for the Steklov averages, we obtain concerning the time part

$$\int_{B_R\times\{t\}}\big[\psi^-(v)\big]^2\,\zeta^2-\int_{B_R\times\{t^\circ\}}\big[\psi^-(v)\big]^2\,\zeta^2\ .$$

As for the space part, we start by passing to the limit in h, thus getting

$$\begin{aligned}&\int_{t^\circ}^t\int_{B_R}\alpha(v)\nabla v\cdot\nabla\Big\{2\psi^-(v)\big(\psi^-\big)'(v)\,\zeta^2\Big\}\\ =&\int_{t^\circ}^t\int_{B_R}\alpha(v)|\nabla v|^2\Big\{2\big(1+\psi^-(v)\big)\big[(\psi^-)'(v)\big]^2\,\zeta^2\Big\}\\ &+2\int_{t^\circ}^t\int_{B_R}\alpha(v)\nabla v\cdot\nabla\zeta\Big\{2\psi^-(v)\big(\psi^-\big)'(v)\,\zeta\Big\}=(*)\ .\end{aligned}$$

Using Young's inequality to estimate the second term as in the proof of Lemma 3, we get

$$(*)\geq 2\int_{t^\circ}^t\int_{B_R}\alpha(v)|\nabla v|^2\zeta^2\big[(\psi^-)'(v)\big]^2-2\int_{t^\circ}^t\int_{B_R}\alpha(v)|\nabla\zeta|^2\psi^-(v)$$

$$\geq -2n\ln 2\,\frac{C}{\sigma^2R^2}\int_{t^\circ}^t\int_{B_R}\alpha(v)\chi_{\{v<\frac{\omega}{2}\}}\geq -2n\ln 2\,\frac{C}{\sigma^2R^2}\,(t^*-t^\circ)\,\big|B_R\big|\,C_1\,\phi\Big(\frac{\omega}{2}\Big)\ ,$$

because in the relevant set

$$\alpha(v)\leq C_1\,\phi(v)\leq C_1\,\phi\Big(\frac{\omega}{2}\Big)\ .$$

Now we observe that,

$$t^*-t^\circ\leq t^*-t^*+\frac{R^2}{\psi\big(\frac{\omega}{4}\big)}=\frac{R^2}{\psi\big(\frac{\omega}{4}\big)}\ ,$$

and so we can condensate the two estimates in the logarithmic estimate

$$\begin{aligned}\sup_{t^\circ\leq t\leq t^*}\int_{B_R\times\{t\}}\big[\psi^-(v)\big]^2\,\zeta^2\ &\leq\ \int_{B_R\times\{t^\circ\}}\big[\psi^-(v)\big]^2\,\zeta^2+C\,n\,\frac{1}{\sigma^2}\,\frac{\phi\big(\frac{\omega}{2}\big)}{\psi\big(\frac{\omega}{4}\big)}\,|B_R|\\ &\leq\ C\,n^2\left(\frac{1-\nu_0}{1-\frac{\nu_0}{2}}\right)|B_R|+C\,n\,\frac{1}{\sigma^2}\,\frac{\phi\big(\frac{\omega}{2}\big)}{\psi\big(\frac{\omega}{4}\big)}\,|B_R|\ ,\end{aligned}$$

because $\psi^-(v)=0$ in the set $\{v>k=\frac{\omega}{2}\}$ and using (22). Now, since the integrand is nonnegative, we estimate below the left hand side in the expression above integrating over the smaller set

$$S=\Big\{x\in B_{(1-\sigma)R}\ :\ v(x,t)<\frac{\omega}{2^{n+1}}\Big\}\subset B_R\ ,\quad t\in(t^\circ,t^*)\ ,$$

and, observing that in S, $\zeta = 1$ and

$$\big[\psi^-(v)\big]^2 \geq \Big[\ln\big(2^{n-1}\big)\Big]^2 = (n-1)^2(\ln 2)^2\ ,$$

we get

$$\begin{aligned}
|S| \quad &\leq \quad C\left(\frac{n}{n-1}\right)^2\left(\frac{1-\nu_0}{1-\frac{\nu_0}{2}}\right)|B_R| + C\,\frac{n}{(n-1)^2}\,\frac{1}{\sigma^2}\,\frac{\phi\big(\frac{\omega}{2}\big)}{\psi\big(\frac{\omega}{4}\big)}\,|B_R| \\
&\leq \quad C\left(\frac{n}{n-1}\right)^2\left(\frac{1-\nu_0}{1-\frac{\nu_0}{2}}\right)|B_R| + C\,\frac{1}{n}\,\frac{1}{\sigma^2}\,\frac{\phi\big(\frac{\omega}{2}\big)}{\psi\big(\frac{\omega}{4}\big)}\,|B_R|\ .
\end{aligned}$$

On the other hand, we have

$$\begin{aligned}
&\left|\left\{x \in B_R\ :\ v(x,t) < \frac{\omega}{2^{n+1}}\right\}\right| \\
\leq &\left|\left\{x \in B_{(1-\sigma)R}\ :\ v(x,t) < \frac{\omega}{2^{n+1}}\right\}\right| + \big|B_R \setminus B_{(1-\sigma)R}\big| \\
&\leq |S| + N\sigma|B_R|\ ,
\end{aligned}$$

so the conclusion is that for all $t \in (t^\circ, t^*)$,

$$\begin{aligned}
&\left|\left\{x \in B_R\ :\ v(x,t) < \frac{\omega}{2^{n+1}}\right\}\right| \\
&\leq C\left\{\left(\frac{n}{n-1}\right)^2\left(\frac{1-\nu_0}{1-\frac{\nu_0}{2}}\right) + \frac{1}{n}\,\frac{1}{\sigma^2}\,\frac{\phi\big(\frac{\omega}{2}\big)}{\psi\big(\frac{\omega}{4}\big)} + N\sigma\right\}|B_R|\ .
\end{aligned}$$

Now, choose σ so small that $CN\sigma \leq \frac{3}{8}\nu_0^2$ and n so large that

$$C\left(\frac{n}{n-1}\right)^2 \leq \left(1-\frac{\nu_0}{2}\right)(1+\nu_0) \equiv \beta > 1 \quad \text{ and } \quad \frac{C}{n\sigma^2}\,\frac{\phi\big(\frac{\omega}{2}\big)}{\psi\big(\frac{\omega}{4}\big)} \leq \frac{3}{8}\nu_0^2$$

and the lemma follows with $s_2 = n+1$.

Concerning the independence of s_2 with respect to ω, and since ν_0 has already been chosen respecting that independence, it is enough to note that, due to (8),

$$\frac{\phi\big(\frac{\omega}{2}\big)}{\psi\big(\frac{\omega}{4}\big)} \leq \frac{\phi\big(\frac{\omega}{2}\big)}{\phi\big(\frac{\omega}{2^m}\big)} = 2^{(m-1)p_2}\ ,$$

expression does not depended on ω.

∎

Now we want to show that the same type of conclusion holds in an upper portion of the full cylinder $Q(\theta R^2, R)$, say for all $t \in \big(-\frac{\theta}{2}R^2, 0\big)$. We just have to use the fact that (21) holds for *all* cylinders of the type $Q_R^{t^*}$ and so, recalling (7), the conclusion of the previous lemma holds true for all time levels

$$t \geq \frac{R^2}{\psi(\frac{\omega}{4})} - \frac{R^2}{\phi(\frac{\omega}{2^m})} - \frac{\nu_0}{2}\frac{R^2}{\psi(\frac{\omega}{4})} \ .$$

Using assumption (8), we have

$$\begin{aligned}
\frac{R^2}{\psi(\frac{\omega}{4})} - \frac{R^2}{\phi(\frac{\omega}{2^m})} - \frac{\nu_0}{2}\frac{R^2}{\psi(\frac{\omega}{4})} &\leq \left\{\frac{1}{2}\big(1-\frac{\nu_0}{2}\big) - 1\right\}\frac{R^2}{\phi(\frac{\omega}{2^m})} \\
&= -\left\{1+\frac{\nu_0}{2}\right\}\frac{R^2}{2\phi(\frac{\omega}{2^m})} \\
&\leq -\frac{\theta}{2}R^2
\end{aligned}$$

and obtain

Corollary 2 *For all* $t \in \big(-\frac{\theta}{2}R^2, 0\big)$,

$$\left|\left\{x \in B_R \ : \ v(x,t) < \frac{\omega}{2^{s_2}}\right\}\right| \leq \Big(1 - \big(\frac{\nu_0}{2}\big)^2\Big)\, |B_R| \ .$$

The previous result means that the set of degeneracy at 0 doesn't fill the entire ball B_R, for all $t \in \big(-\frac{\theta}{2}R^2, 0\big)$. Our next goal is to use this stability information to show that

Proposition 2 *For every* $\lambda_0 \in (0,1)$, *there exist constants* $s_2 < s_3 \in \mathbb{N}$ *and* $m_0 \in \mathbb{N}$, *depending only on the data, such that, if* $m > m_0$ *then*

$$\left|\left\{(x,t) \in Q\big(\tfrac{\theta}{2}R^2, R\big) \ : \ v(x,t) < \frac{\omega}{2^{s_3}}\right\}\right| \leq \lambda_0 \left|Q\big(\tfrac{\theta}{2}R^2, R\big)\right| \ .$$

Instead of proving this proposition directly for v we will formulate an equivalent result for a new unknown function defined by

$$w = A(v) \equiv \int_0^v \alpha(s)\,\mathrm{d}s \ , \quad v \in [0, \delta_0] \ .$$

We find that w satisfies the equation

$$\partial_t B(w) - \Delta w = 0 \ , \quad \text{in } \mathcal{D}'(Q) \ , \tag{24}$$

where $B = A^{-1}$ is the inverse of A; both functions are monotone increasing. Define also

$$\overline{\omega} \equiv \int_0^\omega \alpha(s)\,\mathrm{d}s \ ,$$

and rephrase Corollary 2 in terms of w as follows.

Corollary 3 *There exists a number $s_4 \in \mathbb{N}$, depending only on the data, such that, for all $t \in \big(-\frac{\theta}{2}R^2, 0\big)$,*

$$\left|\Big\{x \in B_R \ : \ w(x,t) < \frac{\overline{\omega}}{2^{s_4}}\Big\}\right| \le \Big(1 - (\frac{\nu_0}{2})^2\Big) \, |B_R| \ .$$

This fact is a consequence of the following simple reasoning:

$$\begin{aligned} v < \frac{\omega}{2^{s_2}} \Longleftrightarrow w < \int_0^{\frac{\omega}{2^{s_2}}} \alpha(s)\,\mathrm{d}s &= \int_0^{\frac{\omega}{2^{s_2}}} s^{p_2}\,\mathrm{d}s \\ &= \frac{\Big(\frac{\omega}{2^{s_2}}\Big)^{p_2+1}}{p_2+1} = \frac{\int_0^{\omega} \alpha(s)\,\mathrm{d}s}{2^{s_2}} \\ &= \frac{\overline{\omega}}{2^{s_4}} \ , \quad \text{with} \quad s_4 = s_2(p_2+1) \ . \end{aligned}$$

We will next prove, using the previous Corollary, that

Proposition 3 *For every $\lambda_0 \in (0,1)$, there exist constants $s_5 \in \mathbb{N}$ and $m_0 \in \mathbb{N}$, depending only on the data, such that, if $m > m_0$ then*

$$\left|\Big\{(x,t) \in Q\big(\tfrac{\theta}{2}R^2, R\big) \ : \ w(x,t) < \frac{\overline{\omega}}{2^{s_5}}\Big\}\right| \le \lambda_0 \, \big|Q\big(\tfrac{\theta}{2}R^2, R\big)\big| \ .$$

Proof. We derive an energy inequality from equation (24) and use again the iteration technique. To be strictly correct, we must argue as before with the Steklov averages, but to simplify we do only the formal computations this time. Consider a piecewise smooth cutoff function $0 < \zeta \le 1$, defined in $Q(\theta R^2, 2R)$, and satisfying the following set of assumptions

$$\zeta = 1 \ \text{ in } \ Q(\tfrac{\theta}{2}R^2, R) \qquad\qquad \zeta = 0 \ \text{ on } \ \partial_p Q(\theta R^2, 2R)$$

$$|\nabla \zeta| \le \frac{1}{R} \qquad\qquad 0 \le \partial_t \zeta \le C\,\frac{\phi(\frac{\omega}{2^m})}{R^2} \ ,$$

and let

$$k = \frac{\overline{\omega}}{2^l} \ , \quad (l > s_4 \text{ to be chosen}) \ .$$

Multiply (24) by $-(w-k)_-\zeta^2$ and integrate in time over $(-\theta R^2, 0)$. Concerning the time part, we get (formally)

$$\int_{-\theta R^2}^0 \int_{B_{2R}} \partial_t B(w) \left[-(w-k)_- \zeta^2\right] = \int_{B_{2R}} \int_{-\theta R^2}^0 \partial_t \Big(\int_0^{(w-k)_-} B'(k-s) s \,\mathrm{d}s \Big) \zeta^2$$

$$= \int_{B_{2R}\times\{0\}}\left(\int_0^{(w-k)_-} B'(k-s)s\,\mathrm{d}s\right)\zeta^2 - \int_{B_{2R}\times\{-\theta R^2\}}\left(\int_0^{(w-k)_-} B'(k-s)s\,\mathrm{d}s\right)\zeta^2$$

$$-2\int_{-\theta R^2}^0\int_{B_{2R}}\left(\int_0^{(w-k)_-} B'(k-s)s\,\mathrm{d}s\right)\zeta\partial_t\zeta$$

$$\geq -C\,\frac{\overline{\omega}}{2^l}B\Big(\frac{\overline{\omega}}{2^l}\Big)\,\frac{\phi(\frac{\omega}{2^m})}{R^2}\,\left|Q\Big(\frac{\theta}{2}R^2,R\Big)\right|\ ,$$

as a consequence of the following three facts: $\zeta(x,-\theta R^2)=0$, due to our choice of ζ,

$$\int_0^{(w-k)_-} B'(k-s)s\,\mathrm{d}s \geq 0\ ,$$

since $B'\geq 0$, and

$$\begin{aligned}\int_0^{(w-k)_-} B'(k-s)s\,\mathrm{d}s &\leq (w-k)_-\int_0^{(w-k)_-} B'(k-s)\,\mathrm{d}s\\ &= (w-k)_-\Big[-B\big(k-(w-k)_-\big)+B(k)\Big]\\ &= (w-k)_-\big[B(k)-B(w)\big]\\ &\leq kB(k)=\frac{\overline{\omega}}{2^l}B\Big(\frac{\overline{\omega}}{2^l}\Big)\ .\end{aligned}$$

From the space part, comes

$$\int_{-\theta R^2}^0\int_{B_{2R}}\nabla w\cdot\nabla\Big[-(w-k)_-\,\zeta^2\Big] = \int_{-\theta R^2}^0\int_{B_{2R}}\Big|\zeta\,\nabla(w-k)_-\Big|^2$$

$$-2\int_{-\theta R^2}^0\int_{B_{2R}}(w-k)_-\,\zeta\,\nabla w\cdot\nabla\zeta$$

$$\geq \frac{1}{2}\int_{-\theta R^2}^0\int_{B_{2R}}\Big|\zeta\,\nabla(w-k)_-\Big|^2 - C\,\Big(\frac{\overline{\omega}}{2^l}\Big)^2\,\frac{1}{R^2}\,\left|Q\Big(\frac{\theta}{2}R^2,R\Big)\right|\ ,$$

and we combine both estimates in

$$\int_{-\theta R^2}^0\int_{B_{2R}}\Big|\zeta\,\nabla(w-k)_-\Big|^2$$

$$\leq C\,\frac{\overline{\omega}}{2^l}B\Big(\frac{\overline{\omega}}{2^l}\Big)\,\frac{\phi(\frac{\omega}{2^m})}{R^2}\,\left|Q\Big(\frac{\theta}{2}R^2,R\Big)\right| + C\,\Big(\frac{\overline{\omega}}{2^l}\Big)^2\,\frac{1}{R^2}\,\left|Q\Big(\frac{\theta}{2}R^2,R\Big)\right|\ .$$

Integrating over the smaller set $Q\Big(\frac{\theta}{2}R^2,R\Big)$, where $\zeta=1$, we arrive at

$$\Big\|\nabla(w-k)_-\Big\|^2_{2,Q(\frac{\theta}{2}R^2,R)}$$

$$\leq \left\{1+\frac{2^l}{\overline{\omega}} B\Big(\frac{\overline{\omega}}{2^l}\Big)\, \phi\Big(\frac{\omega}{2^m}\Big)\right\} \frac{C}{R^2} \left(\frac{\overline{\omega}}{2^l}\right)^2 \left|Q\Big(\frac{\theta}{2}R^2, R\Big)\right| . \qquad (25)$$

Let's now use Lemma 1, with the function $w(x,t)$ defined for $t \in [-\frac{\theta}{2}R^2, 0]$, and the levels

$$k_1 = k = \frac{\overline{\omega}}{2^l} \quad \text{and} \quad k_2 = \frac{\overline{\omega}}{2^{l+1}} \ ; \quad l = s_4, s_4+1, \ldots$$

We know from Corollary 3, and since $l \geq s_4$, that

$$\begin{aligned}
\left|\left\{x \in B_R \ : \ w(x,t) > \frac{\overline{\omega}}{2^l}\right\}\right| &\geq \big|B_R\big| - \left|\left\{x \in B_R \ : \ w(x,t) < \frac{\overline{\omega}}{2^{s_4}}\right\}\right| \\
&\geq \big|B_R\big| - \Big(1 - \big(\frac{\nu_0}{2}\big)^2\Big) \big|B_R\big| \\
&= \big(\frac{\nu_0}{2}\big)^2 \big|B_R\big| ,
\end{aligned}$$

for all $t \in [-\frac{\theta}{2}R^2, 0]$. Next, define

$$A_l(t) = \left\{x \in B_R \ : \ w(x,t) < \frac{\overline{\omega}}{2^l}\right\} \qquad \text{and} \qquad A_l = \int_{-\frac{\theta}{2}R^2}^{0} \big|A_l(t)\big| \, \mathrm{d}t ,$$

and using Lemma 1, obtain

$$\Big(\frac{\overline{\omega}}{2^l} - \frac{\overline{\omega}}{2^{l+1}}\Big) \big|A_{l+1}(t)\big| \leq \frac{C\, R^{N+1}}{\big(\frac{\nu_0}{2}\big)^2 \big|B_R\big|} \int_{A_l(t)\setminus A_{l+1}(t)} |\nabla w| \ .$$

Integrate this inequality in time over $[-\frac{\theta}{2}R^2, 0]$, use Hölder's inequality and square both sides, to get

$$\begin{aligned}
\Big(\frac{\overline{\omega}}{2^l}\Big)^2 A_{l+1}^2 &\leq C\, \frac{R^2}{\nu_0^4} \left(A_l - A_{l+1}\right) \int_{-\frac{\theta}{2}R^2}^{0} \int_{A_l(t)\setminus A_{l+1}(t)} |\nabla w|^2 \\
&\leq C\, \frac{R^2}{\nu_0^4} \left(A_l - A_{l+1}\right) \int_{-\frac{\theta}{2}R^2}^{0} \int_{B_R} \big|\nabla (w-k)_-\big|^2 \\
&\leq \frac{C}{\nu_0^4} \left(A_l - A_{l+1}\right) \left\{1+\frac{2^l}{\overline{\omega}} B\Big(\frac{\overline{\omega}}{2^l}\Big)\, \phi\Big(\frac{\omega}{2^m}\Big)\right\} \Big(\frac{\overline{\omega}}{2^l}\Big)^2 \left|Q\Big(\frac{\theta}{2}R^2, R\Big)\right|
\end{aligned}$$

using also inequality (25).

Adding these inequalities for $l = s_4, s_4+1, \ldots, s_5-1$, where s_5 is to be chosen, we get

$$\sum_{l=s_4}^{s_5-1} A_{l+1}^2 \leq \frac{C}{\nu_0^4} \left(A_{s_4} - A_{s_5}\right) \left\{1+\frac{2^{s_5}}{\overline{\omega}} B\Big(\frac{\overline{\omega}}{2^{s_4}}\Big)\, \phi\Big(\frac{\omega}{2^m}\Big)\right\} \left|Q\Big(\frac{\theta}{2}R^2, R\Big)\right| ,$$

and since $A_{s_4} - A_{s_5} \le \left|Q(\frac{\theta}{2}R^2, R)\right|$ and

$$\sum_{l=s_4}^{s_5-1} A_{l+1}^2 \ge (s_5 - s_4) A_{s_5}^2 \ ,$$

we finally conclude that

$$A_{s_5} \le \frac{C}{\nu_0^2} (s_5 - s_4)^{-\frac{1}{2}} \left\{ 1 + \frac{2^{s_5}}{\overline{\omega}} B\Big(\frac{\overline{\omega}}{2^{s_4}}\Big) \, \phi\Big(\frac{\omega}{2^m}\Big) \right\}^{\frac{1}{2}} \left| Q\Big(\frac{\theta}{2}R^2, R\Big) \right| .$$

To prove the result we choose s_5 so large that

$$\frac{C}{\nu_0^2} (s_5 - s_4)^{-\frac{1}{2}} \le \frac{\lambda_0}{\sqrt{2}}$$

and m_0 so large that

$$\frac{2^{s_5}}{\overline{\omega}} \, B\Big(\frac{\overline{\omega}}{2^{s_4}}\Big) \, \phi\Big(\frac{\omega}{2^{m_0}}\Big) \le 1 \ . \tag{26}$$

Concerning the dependence on ω, we immediately conclude that, since ν_0 and s_4 are independent of ω (Lemma 2 and Corollary 3), so is s_5. The independence of m_0 on ω is more delicate and again crucially related to the fact that ϕ is a power. Observe that as a consequence of this fact

$$B(s) = \Big\{ (p_2 + 1) s \Big\}^{\frac{1}{p_2+1}} \qquad \text{and} \qquad \overline{\omega} = \frac{\omega^{p_2+1}}{p_2 + 1} \ .$$

So, from (26), we must choose m_0 such that

$$\Big(\frac{\omega}{2^{m_0}}\Big)^{p_2} \le \frac{\omega^{p_2+1}}{(p_2+1) 2^{s_5}} \, \frac{2^{\frac{s_4}{p_2+1}}}{\omega}$$

that is

$$2^{m_0 p_2} \ge (p_2 + 1) \, 2^{s_5 - \frac{s_4}{p_2+1}} \ ,$$

and now it is clear that m_0 can be chosen independently of ω.

■

We can now obtain Proposition 2 rephrasing the contents of Proposition 3. In fact,

$$w(x,t) < \frac{\overline{\omega}}{2^{s_5}} \Longleftrightarrow v(x,t) < \frac{\omega}{2^{\frac{s_5}{p_2+1}}}$$

and we conclude putting $s_3 = \frac{s_5}{p_2+1}$.

We next explain how to choose the constant m (independently of ω) and consequently fix the height of the cylinder $Q(\theta R^2, R)$. We follow closely the idea in [4]. Let

$$\phi_1 = \phi\Big(\frac{\omega}{2^{s_3}}\Big) \ , \qquad \phi_2 = \phi\Big(\frac{\omega}{2^{s_3+2}}\Big) \ , \qquad \mu = \phi_2 \Big(\frac{\phi_1}{\phi_2}\Big)^{\frac{N+2}{2}} \ ,$$

and choose $m > m_0$ as the smallest real number such that

$$\frac{\mu}{\phi(\frac{\omega}{2^m})} = n_0 \ , \quad \text{for some integer } n_0 \in \mathbb{N} \ . \tag{27}$$

Since ϕ is a power, it is clear that m is independent of ω.

Then break $Q(\theta R^2, R)$ again, this time into n_0 subcylinders of the form

$$Q_R^j = B_R \times \Big(-j\frac{R^2}{\mu}, -(j-1)\frac{R^2}{\mu} \Big) \ ; \quad j = 1, 2, \ldots, n_0 \ .$$

Since these cylinders are disjoint and they exhaust $Q(\theta R^2, R)$, from Proposition 2 it follows that

$$\exists j_0 \in \{1, \ldots, n_0\} \quad : \quad \left| \Big\{ (x,t) \in Q_R^{j_0} \ : \ v(x,t) < \frac{\omega}{2^{s_3}} \Big\} \right| \le \lambda_0 \left| Q_R^{j_0} \right| \ . \tag{28}$$

We now use this information to show that in a smaller cylinder the function v is strictly away from the degeneracy point 0.

Lemma 5 *The number s_3 can be chosen such that*

$$v(x,t) > \frac{\omega}{2^{s_3+1}} \ , \quad a.e. \quad (x,t) \in Q_{\frac{R}{2}}^{j_0} \ .$$

Proof. Let $v_\omega \equiv \max\{v, \frac{\omega}{2^{s_3+2}}\}$. Define

$$R_n = \frac{R}{2} + \frac{R}{2^{n+1}} \ , \quad n = 0, 1, \ldots,$$

and construct the family of nested and shrinking cylinders

$$Q_{R_n}^{j_0} = B_{R_n} \times \Big(-j_0\frac{R_n^2}{\mu}, -(j_0-1)\frac{R_n^2}{\mu} \Big) \ .$$

Consider piecewise smooth cutoff functions $0 < \zeta_n \le 1$, defined in these cylinders, and satisfying the following set of assumptions

$$\begin{array}{ccc} \zeta_n = 1 \ \text{ in } \ Q_{R_{n+1}}^{j_0} & & \zeta_n = 0 \ \text{ on } \ \partial_p Q_{R_n}^{j_0} \\ & |\nabla \zeta_n| \le \dfrac{2^{n+1}}{R} & \\ |\Delta \zeta_n| \le \dfrac{2^{2(n+1)}}{R^2} & & 0 \le \partial_t \zeta_n \le 2^{2(n+1)}\dfrac{\mu}{R^2} \ , \end{array}$$

and let

$$k_n = \frac{\omega}{2^{s_3+1}} + \frac{\omega}{2^{s_3+1+n}} \ , \quad n = 0, 1, \ldots .$$

Choose as test function in (2) $\varphi = -\big[(v_\omega)_h - k_n\big]_- \xi_n^2$ and proceed as in the proof of Lemma 2 to get

$$\text{ess sup} \int_{B_{R_n}\times\{t\}} (v_\omega - k_n)_-^2\ \xi_n^2 + \phi\Big(\frac{\omega}{2^{s_3+2}}\Big) \int_{-j_0 \frac{R_n^2}{\mu}}^{-(j_0-1)\frac{R_n^2}{\mu}} \int_{B_{R_n}} \Big|\xi_n \nabla (v_\omega - k_n)_-\Big|^2$$

$$\leq C\, \frac{2^{2(n+1)}}{R^2} \Big(\frac{\omega}{2^{s_3+2}}\Big)^2 \phi\Big(\frac{\omega}{2^{s_3}}\Big) \int_{-j_0 \frac{R_n^2}{\mu}}^{-(j_0-1)\frac{R_n^2}{\mu}} \int_{B_{R_n}} \chi_{\{v_\omega \leq k_n\}}\ .$$

Now perform a change in the time variable, putting $\bar{t} = \big\{t + (j_0 - 1)\frac{R_n^2}{\mu}\big\}\, \mu$ and define

$$\overline{v_\omega}(\cdot, \bar{t}) = v_\omega(\cdot, t) \qquad \text{and} \qquad \overline{\xi_n}(\cdot, \bar{t}) = \xi_n(\cdot, t)\ ,$$

and obtain the inequality, with $\Gamma = \frac{\phi_1}{\phi_2}$,

$$\Gamma^{\frac{N+2}{2}}\, \text{ess sup} \int_{B_{R_n}\times\{t\}} (\overline{v_\omega} - k_n)_-^2\ \overline{\xi_n}^2 + \int_{-R_n^2}^{0} \int_{B_{R_n}} \Big|\overline{\xi_n}\ \nabla(\overline{v_\omega} - k_n)_-\Big|^2$$

$$\leq C\, \frac{2^{2(n+1)}}{R^2} \Big(\frac{\omega}{2^{s_3}}\Big)^2 \Gamma \int_{-R_n^2}^{0} \int_{B_{R_n}} \chi_{\{\overline{v_\omega} \leq k_n\}}\ .$$

The conclusion of the proof follows from a refinement of the iteration technique used in the previous results; it can be find in [4].

■

Remark 3 Observe that (28) plays the same role as the assumption in the first alternative and that Lemma 5 is the corresponding analogue of Lemma 2. The next two results, leading to the conclusion of the second alternative, reproduce the ideas in Lemma 3 and Proposition 1, i.e., we first use the logarithmic estimates to extend the result to a full cylinder in time and then, with the aid of the energy estimates, conclude that v is strictly away from 0 in a cylinder of the type $Q(\tau, \rho)$.

Lemma 6 *Given $\nu_1 \in (0,1)$, there exists $s_6 \in \mathbb{N}$, depending only on the data, such that*

$$\Big|\Big\{x \in B_{\frac{R}{4}}\ :\ v(x,t) \leq \frac{\omega}{2^{s_6}}\Big\}\Big| \leq \nu_1\, \big|B_{\frac{R}{4}}\big|\ ,\quad \forall t \in \Big(-j_0 \frac{\big(\frac{R}{2}\big)^2}{\mu}, 0\Big)\ .$$

Proof. To simplify, put

$$\tilde{t} = j_0 \frac{\big(\frac{R}{2}\big)^2}{\mu}\ .$$

We use a logarithmic estimate for the function $(v-k)_-$ in the cylinder $Q(\tilde{t},\frac{R}{2})$, with the choices

$$k=\frac{\omega}{2^{s_3+1}} \qquad \text{and} \qquad c=\frac{\omega}{2^{n+1}}\ ,$$

where $n\in\mathbb{N}$ will be chosen later, as parameters in the standard logarithmic function Ψ^- (see (3)).

To obtain the estimate, choose a piecewise smooth cutoff function $0<\zeta(x)\leq 1$, defined on $B_{\frac{R}{2}}$ and such that

$$\zeta=1 \ \text{ in } \ B_{\frac{R}{4}} \qquad \text{and} \qquad |\nabla\zeta|\leq\tfrac{C}{R}\ ,$$

and multiply (2) by $2\psi^-(v_h)\big(\psi^-\big)'(v_h)\,\zeta^2$. As in the proof of Lemma 3, with the obvious changes, we get

$$\sup_{-\tilde{t}\leq t\leq 0}\int_{B_{\frac{R}{2}}\times\{t\}}\big[\psi^-(v)\big]^2\,\zeta^2\leq C\,(n-s_3)\,\phi\Big(\frac{\omega}{2^{s_3+1}}\Big)\,\frac{\tilde{t}}{R^2}\,\big|B_{\frac{R}{2}}\big|$$

$$\leq C\,(n-s_3)\,\frac{\phi\big(\frac{\omega}{2^{s_3+1}}\big)}{\phi\big(\frac{\omega}{2^m}\big)}\,\big|B_{\frac{R}{2}}\big|$$

since, recalling (27),

$$\tilde{t}=j_0\frac{\big(\frac{R}{2}\big)^2}{\mu}\leq\frac{n_0}{\mu}\big(\frac{R}{2}\big)^2=\frac{\big(\frac{R}{2}\big)^2}{\phi\big(\frac{\omega}{2^m}\big)}\ .$$

Now, since the integrand is nonnegative, we estimate below the left hand side of the inequality integrating over the smaller set

$$S=\Big\{x\in B_{\frac{R}{4}}\ :\ v(x,t)\leq\frac{\omega}{2^{n+1}}\Big\}\subset B_{\frac{R}{2}}$$

and, observing that in S, $\zeta=1$ and

$$\big[\psi^-(v)\big]^2\geq(n-s_3)^2(\ln 2)^2\ ,$$

we get

$$\Big|\Big\{x\in B_{\frac{R}{4}}\ :\ v(x,t)\leq\frac{\omega}{2^{n+1}}\Big\}\Big|\leq C\,\frac{1}{n-s_3}\,\frac{\phi\big(\frac{\omega}{2^{s_3+1}}\big)}{\phi\big(\frac{\omega}{2^m}\big)}\,\big|B_{\frac{R}{4}}\big|\ .$$

To prove the lemma choose

$$s_6=n+1 \qquad \text{with} \qquad n\geq s_3+\frac{C}{\nu_1}\,\frac{\phi\big(\frac{\omega}{2^{s_3+1}}\big)}{\phi\big(\frac{\omega}{2^m}\big)}\ .$$

Concerning the independence of ω, observe that s_3 was chosen independently of ω and

$$\frac{\phi\left(\frac{\omega}{2^{s_3+1}}\right)}{\phi\left(\frac{\omega}{2^m}\right)} = \frac{\left(\frac{\omega}{2^{s_3+1}}\right)^{p_2}}{\left(\frac{\omega}{2^m}\right)^{p_2}} = 2^{(m-s_3-1)p_2} \ ,$$

expression that does not depended on ω.

■

Proposition 4 *There exists a constant* $1 < s_6 \in \mathbb{N}$*, that can be chosen depending only on the data, such that*

$$v(x,t) > \frac{\omega}{2^{s_6}} \ , \quad a.e. \quad (x,t) \in Q\Big(\tilde{t}, \frac{R}{8}\Big) \ .$$

Proof. Define

$$R_n = \frac{R}{8} + \frac{R}{2^{n+3}} \ , \quad n = 0, 1, \ldots$$

and construct the family of nested and shrinking cylinders $Q(\tilde{t}, R_n)$. Take piecewise smooth cutoff functions $0 < \zeta_n(x) \le 1$, independent of t, defined in B_{R_n} and satisfying the assumptions

$$\zeta_n = 1 \ \text{ in } \ B_{R_{n+1}} \ , \qquad |\nabla \zeta_n| \le \tfrac{2^{n+4}}{R} \qquad \text{and} \qquad |\Delta \zeta_n| \le \tfrac{2^{2(n+4)}}{R^2} \ .$$

Take also

$$k_n = \frac{\omega}{2^{s_6+1}} + \frac{\omega}{2^{s_6+1+n}} \ , \quad n = 0, 1, \ldots,$$

(where $s_6 > 1$ is to be chosen) and derive local energy inequalities similar to those obtained in Proposition 1, now for the functions

$$-(v_\omega - k_n)_- \zeta_n^2 \ , \qquad \text{with} \qquad v_\omega \equiv \max\big(v, \frac{\omega}{2^{s_6+1}}\big)$$

and in the cylinders $Q\big(\tilde{t}, R_n\big)$. Using the same reasoning, we get

$$\sup_{-\tilde{t}<t<0} \int_{B_{R_n}\times\{t\}} (v_\omega - k_n)_+^2 \zeta_n^2 + \phi\Big(\frac{\omega}{2^{s_6}}\Big) \int_{-\tilde{t}}^0 \int_{B_{R_n}} \big|\zeta_n \nabla (v_\omega - k_n)_+\big|^2$$

$$\le C\, \phi\Big(\frac{\omega}{2^{s_6}}\Big) \, \frac{2^{2(n+4)}}{R^2} \Big(\frac{\omega}{2^{s_6+1}}\Big)^2 \int_{-\tilde{t}}^0 \int_{B_{R_n}} \chi_{\{v_\omega \le k_n\}} \ .$$

The change in the time variable $\bar{t} = \Big(\dfrac{R}{2}\Big)^2 \dfrac{t}{\tilde{t}}$, with the new function

$$\overline{v_\omega}(\cdot, \bar{t}) = v_\omega(\cdot, t) \ ,$$

leads to

$$\frac{\left(\frac{R}{2}\right)^2}{\phi\left(\frac{\omega}{2^{s_6}}\right)\tilde{t}} \sup_{-(\frac{R}{2})^2<t<0} \int_{B_{R_n}\times\{t\}} (\overline{v_\omega} - k_n)_-^2 \zeta_n^2 + \int_{-(\frac{R}{2})^2}^0 \int_{B_{R_n}} \big|\zeta_n \nabla (\overline{v_\omega} - k_n)_-\big|^2$$

$$\leq C\,\frac{2^{2(n+4)}}{R^2}\Big(\frac{\omega}{2^{s_6+1}}\Big)^2\int_{-(\frac{R}{2})^2}^{0}\int_{B_{R_n}}\chi_{\{\overline{v_\omega}\leq k_n\}}\ ,$$

after multiplication of the whole expression by $\dfrac{\big(\frac{R}{2}\big)^2}{\phi\big(\frac{\omega}{2^{s_6}}\big)\,\tilde{t}}$. Now, if s_6 is chosen sufficiently large, we have

$$\frac{\big(\frac{R}{2}\big)^2}{\phi\big(\frac{\omega}{2^{s_6}}\big)\,\tilde{t}}\geq 1$$

since

$$\frac{\big(\frac{R}{2}\big)^2}{\phi\big(\frac{\omega}{2^{s_6}}\big)\,\tilde{t}}=\frac{\mu}{j_0\,\phi\big(\frac{\omega}{2^{s_6}}\big)}\geq\frac{\mu}{n_0\,\phi\big(\frac{\omega}{2^{s_6}}\big)}=\frac{\phi\big(\frac{\omega}{2^m}\big)}{\phi\big(\frac{\omega}{2^{s_6}}\big)}\ ,$$

which shows also that it suffices to pick $s_6>m$.

The rest of the proof follows the same lines of the proof of Proposition 1, that consist basically in using the iteration technique and then Lemma 6. We ommit the details.

$\blacksquare$

Corollary 4 *There exists a constant* $\sigma_1\in(0,1)$*, depending only on the data, such that*

$$\operatorname*{ess\,osc}_{Q\left(\tilde{t},\frac{R}{8}\right)} v\ \leq\ \sigma_1\ \omega\ .$$

Proof. It is similar to the proof of Corollary 1. We find $\sigma_1=\big(1-\frac{1}{2^{s_6}}\big)$.

$\blacksquare$

An imediate consequence of Corollaries 1 and 4 is

Proposition 5 *There exists a constant* $\sigma\in(0,1)$*, that depends only on the data, and a cylinder* $Q\big(t^{\diamondsuit},\frac{R}{8}\big)$*, such that*

$$\operatorname*{ess\,osc}_{Q(t^{\diamondsuit},\frac{R}{8})} v\ \leq\ \sigma\ \omega\ .$$

Proof. Since one of (9) or (21) has to be true, the conclusion of at least one of Corollaries 1 or 4 holds. Choosing

$$\sigma=\max\ \big\{\sigma_0,\sigma_1\big\}\ ,$$

and

$$t^{\diamondsuit}=\min\ \Big\{\,\widehat{t}\,,\,\tilde{t}\,\Big\}\ ,$$

we obtain the conclusion.

$\blacksquare$

The proof of Theorem 1 now follows from Proposition 5 through the usual method of defining recursively a sequence of nested and shrinking cylinders Q_n and a sequence ω_n converging to zero such that

$$\operatorname*{ess\,osc}_{Q_n} v \ \leq \ \omega_n \ .$$

See [5, Prop. 3.1 and Lemma 3.1] and [11] for the details. We stress that the Hölder continuity is obtained since σ in Proposition 5 is independent of the oscillation ω.

Remark 4 Although we restrict ourselves to an equation without lower order terms, the analysis can be extended to that (physically more interesting) case. For problems dealing with the simultaneous flow of two immiscible fluids in a porous medium the lower order terms present difficulties that are technically analogous to those arising from a Hadamard growth condition; the proof can be adapted, reproducing the techniques used in [1]. Allthough we are aware of this fact, we decided to keep the proof in its present form, bringing to light what we consider to be the essential new difficulties relative to previous works.

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