ALGEBRAICALLY LINEARIZABLE DYNAMICAL SYSTEMS

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Summary

The main result of this paper is the evidence of an explicit linearization of dynamical systems of Ruijsenaars-Schneider (RS) type and of the perturbations introduced by F. Calogero of these systems with all orbits periodic of same period. Several other systems share the existence of this explicit linearization, among them, the Calogero-Moser system (with and without external potential) and the Calogero-Sutherland system. This explicit linearization is compared with the notion of maximal superintegrability which has been discussed in several articles (to quote few of them, Hietarinta [12], Henon [11], Harnad-Winternitz [10], S. Wojchiechowsky [15]).

Short title: Superintegrability

Introduction

Let $H : V^{2m} \to R$ be a Hamiltonian system defined on a symplectic manifold V^{2m}, of dimension $2m$, equipped with a symplectic form ω of dimension $2m$. Recall that H is said to be integrable in Arnol'd-Liouville sense if H displays m generically independent first integrals (one of these maybe the Hamiltonian itself) which are in involution for the Poisson bracket associated with the symplectic form ω. A vector field X on a manifold V of dimension n defines a flow and a dynamical system. The vector field (not necessarily Hamiltonian) is classically said to be maximally superintegrable if it has $n-1$ generically independent globally defined first integrals $f_1, ..., f_{n-1}$. The orbits of X are then contained in the connected components of the common level sets of the functions $f_i, i = 1, ..., n-1$. Some Hamiltonian systems are known to be maximally superintegrable and so they display $2m-1$ first integrals. This is so for instance of the rational Calogero-Moser system, the Kepler problem, the isotropic oscillator (cf. [1], [10], [11], [12], [15]). Recently, this specific class of Hamiltonian systems has deserved interest in several articles (cf. for instance, [10]). In this article the definition of algebraic linearization is proposed in a slightly broader sense (also more precise sense).

Definition

A differential system is algebraically (resp. analytically) linearizable if there are n globally defined functions (rational, resp. meromorphic) which are generically independent so that the time evolution of the flow expressed in these functions is linear (in time) and algebraic in the initial coordinates.

First purpose of this article is to prove that the Hyperbolic and the Rational Ruijsenaars-Schneider systems are algebraically linearizable. The perturbations recently considered by Calogero [3], [4] of the Hyperbolic and Rational Ruijsenaars-Schneider systems which display only periodic orbits of the same period are algebraically linearizable as well. This is proved rather easily using the Lax matrix first introduced by Bruschi-Calogero [2] and the extra-equation which implements the integrability of these systems first introduced in the article [8].

I. Algebraic linearization of the rational Calogero-Moser system and of the rational Calogero-Moser system with an external quadratic potential

This first paragraph is devoted to the proof that the rational Calogero-Moser system (with or without) external quadratic potential is algebraically linearizable. The usefullness of this (apparently) new notion is displayed on these classical examples. The rational Calogero-Moser system is represented by the Hamiltonian

$$H = (1/2)\sum_{i=1}^{m} y_i^2 + g^2 \sum_{ij} (x_i - x_j)^{-2} \tag{1.1}$$

where the constant g is a parameter. J. Moser introduced the matrix function:

$$L(x,y)/L_{ij} = y_i \delta_{ij} + g\mathrm{i}(x_i - x_j)^{-1}(1 - \delta_{ij}), \tag{1.2}$$

and observed that the time evolution of this matrix function $L(x, y)$ along the flow is of Lax pair type:

$$\dot{L} = [L, M]. \tag{1.3}$$

This Lax pair equation is supplemented with the equation:

$$\dot{X} = [X, M] + L, \tag{1.4}$$

displayed by the diagonal matrix X:

$$X(x,y)/X_{ij} = x_i \delta_{ij}. \tag{1.5}$$

Following the classical approach, introduce the rational functions:

$$F_k = tr(L^k). \tag{1.6}$$

The Lax matrix equation yields:

$$\dot{F}_k = 0. \tag{1.7}$$

Introduce then the functions:

$$G_k = tr(XL^k), \tag{1.8}$$

which undergo the time evolution:

$$\dot{G}_k = F_{k+1}. \tag{1.9}$$

Clearly, the whole collection of the rational functions F_k, G_k provide the algebraic linearization of the system.

Indeed, classical superintegrability can be recovered as follows:

$$F_k \ (k = 1, \ldots, m), \quad H_k = F_k G_k - F_{k+1} G_{k-1} \ (k = 1, \ldots, m-1)$$

provide $2m - 1$ integrals of motion.

The next classical example to be considered is the rational Calogero-Moser system with an external quadratic potential. The system is described by the Hamiltonian:

$$H = (1/2) \sum_{i=1}^{m} y_i^2 + g^2 \sum_{ij} (x_i - x_j)^{-2} + (\lambda^2/2) \sum_{i=1}^{m} y_i^2. \tag{1.10}$$

The equations (1.3) and (1.4), with the same matrices L and X get modified as follows:

$$\dot{L} = [L, M] - \lambda^2 X, \tag{1.11a}$$

$$\dot{X} = [X, M] + L. \tag{1.11b}$$

The classical approach consists in (cf [9]) introducing matrices:

$$Z = L + \mathrm{i}\lambda X, \tag{1.12a}$$

$$W = L - \mathrm{i}\lambda X. \tag{1.12b}$$

These matrices undergo the time evolution:

$$\dot{Z} = \mathrm{i}\lambda Z + [Z, M], \tag{1.13a}$$

$$\dot{W} = -\mathrm{i}\lambda W + [W, M]. \tag{1.13b}$$

It was then observed ([9]) that the matrix $P = ZW$ defines Lax matrix for the system:

$$\dot{P} = [P, M]. \tag{1.14}$$

Here, we note that the functions:

$$F_k = tr(ZP^k), \tag{1.15a}$$

$$G_k = tr(WP^k), \tag{1.15b}$$

yield:

$$\dot{F}_k = \mathrm{i}\lambda F_k, \tag{1.16a}$$

$$\dot{G}_k = \mathrm{i}\lambda G_k. \tag{1.16b}$$

Thus these functions provide the algebraic linearization of the system.

II. Algebraic linearization of the Calogero-Sutherland system

The Calogero-Sutherland system is defined by the Hamiltonian

$$H(x,y) = \frac{1}{2}\sum_{i=1}^{m} y_i^2 + \frac{g^2}{2}\sum_{\substack{i,j=1 \\ i\neq j}}^{m} \sinh^{-2}(x_i - x_j) \tag{2.1}$$

and has a Lax pair $\dot{L} = [L, M]_-$ with Lax matrix

$$L_{ij} = y_i\delta_{ij} + \frac{\sqrt{-1}g}{sinh(x_i - x_j)}(1 - \delta_{ij}). \tag{2.2}$$

Defining the matrix X by

$$X_{ij} = \exp(2x_i)\delta_{ij}, \tag{2.3}$$

we get the dynamical equation

$$\dot{X} = [X, L]_+ + [X, M]_-. \tag{2.4}$$

Above and throughout of course $[A, B]_- \equiv AB - BA$ and $[A, B]_+ \equiv AB + BA$.

Consider the functions

$$F_k = Tr(L^k), \quad k = 1, \ldots, m \tag{2.5a}$$

$$G_k = Tr(XL^k), \quad k = 1, \ldots .m \tag{2.5b}$$

The functions F_k are first integrals of the dynamical system defined by (2.1). Newton's formulae relate these constant of motion with the coefficients $A_0, ..., A_{n-1}$ of the characteristic polynomial of the matrix L. The theorem yields:

$$L^n = A_{n-1}L^{n-1} + A_{n-2}L^{n-2} + ... + A_0 I. \qquad (2.6)$$

Theorem II-1

The functions G_k undergo a linear evolution under the time evolution of the system defined by (2.1).

Proof

Once (L,M) is a Lax pair of the system and X satisfyies (2.3),

$$\dot{G}_k = tr(XL^{k+1}) = G_{k+1}. \qquad (2.7)$$

Thus, the vector $G = (G_0, ..., G_{n-1})$ displays the time evolution:

$$\dot{G} = AG, \qquad (2.8)$$

where the matrix A is with coefficients first integrals of the differential system:

$$A_{ij} = \delta_{i+1,j} + A_{j-1}\delta_{in}. \qquad (2.9)$$

So, the Calogero-Sutherland system is algebraically linearizable.

III. Algebraic linearization of Hyperbolic and Rational Ruijsenaars-Schneider systems

The dynamical systems of Ruijsenaars-Schneider (RS) type characterized by the equations of motion

$$\ddot{z}_j = \sum_{k=1,k\neq j}^{n} \dot{z}_j \dot{z}_k f(z_j - z_k), \;\; j = 1, ..., n, \tag{3.1}$$

are "integrable" or "solvable" [4], if

$$f(z) = 2/z \qquad \text{“}case(i)\text{”}, \tag{3.2a}$$

$$f(z) = 2/[z(1 + r^2 z^2)] \quad \text{“}case(ii)\text{”}, \tag{3.2b}$$

$$f(z) = 2a\mathrm{cotgh}(az) \quad \text{“}case(iii)\text{”}, \tag{3.2c}$$

$$f(z) = 2a/\sinh(az) \quad \text{“}case(iv)\text{”}, \tag{3.2d}$$

$$f(z) = 2a\mathrm{cotgh}(az)/[1 + r^2 \sinh^2(az)] \quad \text{“}case(v)\text{”}, \tag{3.2e}$$

$$f(z) = -a\mathcal{P}'(az)/[\mathcal{P}(az) - \mathcal{P}(ab)] \quad \text{“}case(vi)\text{”}. \tag{3.2f}$$

Of course the solutions $z_j(t)$ of (3.1) move in the complex plane; and indeed all the constants appearing in (3.2), namely r, a and b, as well as the constants ω and ω' implicit in the definition of the Weierstrass function $\mathcal{P}(z) \equiv \mathcal{P}(z|\omega, \omega')$, might be complex.

Indeed, the main contribution of this paper is to solve explicitly several systems following a scheme which may be of broader interest.

The starting point of the analysis is the observation [2] that (3.1) with (3.2e) is equivalent to the following "Lax-type" ($n\times n$)-matrix equation:

$$\dot{L} = [L, M]_-, \tag{3.3}$$

with

$$L_{jk} = \delta_{jk}\dot{z}_j + (1 - \delta_{jk})(\dot{z}_j\dot{z}_k)^{1/2}\alpha(z_j - z_k), \tag{3.4}$$

$$M_{jk} = \delta_{jk} \sum_{m=1, m\neq j}^{n} \dot{z}_m\beta(z_j - z_m) + (1 - \delta_{jk})(\dot{z}_j\dot{z}_k)^{1/2}\gamma(z_j - z_k), \tag{3.5}$$

and

$$\alpha(z) = \sinh(a\mu)/\sinh[a(z + \mu)], \tag{3.6a}$$

$$\beta(z) = -a\mathrm{cotgh}(a\mu)/[1 + r^2\sinh^2(az)], \tag{3.6b}$$

$$\gamma(z) = -a\mathrm{cotgh}(az)\alpha(z), \tag{3.6c}$$

where

$$\sinh(a\mu) = \mathrm{i}/r. \tag{3.7d}$$

It was furthermore recently noted [8] that the diagonal matrix

$$X(t) = \mathrm{diag}\{\exp[2az_j(t)]\}, \tag{3.8}$$

undergoes the following time evolution:

$$\dot{X} = [X, M]_- + a[X, L]_+. \tag{3.9}$$

Let F_k and G_k be the functions defined as:

$$F_k = tr(L^k), G_k = tr(XL^k). \tag{3.10}$$

The functions F_k are first integrals of the dynamical system defined by (3.1). Newton's formulae relate these constant of motion with the coefficients $A_0, ..., A_{n-1}$ of the characteristic polynomial of the matrix L. The theorem yields:

$$L^n = A_{n-1}L^{n-1} + A_{n-2}L^{n-2} + ... + A_0 I, \tag{3.11}$$

Theorem III-1

The functions G_k undergo a linear evolution under the time evolution of the system (3.1).

Proof

The equations (3.3) and (3.9) yield:

$$\dot{G}_k = 2atr(XL^{k+1}) = 2aG_{k+1}. \tag{3.12}$$

Thus,the vector $G = (G_0, ..., G_{n-1})$ displays the time evolution:

$$\dot{G} = AG, \tag{3.13}$$

where the matrix A is with coefficients first integrals of the differential system:

$$A_{ij} = 2a\delta_{i+1,j} + 2aA_{j-1}\delta_{i,n}. \tag{3.14}$$

F. Calogero introduced the following perturbation of the trigonometric and rational Ruijsenaars-Schneider systems characterized by the equations of motion:

$$\ddot{z}_j + \mathrm{i}\Omega\dot{z}_j = \sum_{k=1,k\neq j}^{n} \dot{z}_j\dot{z}_k f(z_j - z_k), \;\; j = 1, ..., n. \tag{3.15}$$

F. Calogero made the remarkable conjecture [3], now proved in the trigonometric and rational cases, that all the orbits of the dynamical system defined by (3.15) are periodic of period Ω. The equations under consideration here are modified due to the presence of the perturbation. The Lax equation (3.3) gets modified into (cf. [8]):

$$\dot{L} = [L, M]_- + \mathrm{i}\Omega L, \tag{3.16}$$

and the time evolution of the matrix X is not modified. This yields new time evolution for the functions F_k and G_k:

$$\dot{F}_k = \mathrm{i}\Omega k F_k \qquad (3.17a).$$

$$\dot{G}_k = 2atr(XL^{k+1}) + \mathrm{i}\Omega ktr(XL^k) = 2aG_{k+1} + \mathrm{i}\Omega kG_k \qquad (3.17b).$$

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