# On Coregular Closure Operators

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**Abstract**

Among closure operators -in the sense of Dikranjan and Giuli [5]- the regular ones have a relevant role and have been widely investigated. On the contrary, the coregular closure operators were introduced only recently in [3] and they need to be further investigated. In this paper we study (co)regular closure operators -in connection connectednesses and disconnectednesses- in the realm of topological spaces and modules.

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## 0 Introduction

Regular closure operators were introduced by Salbany [12] in 1976 in the category of topological spaces and have been investigated and used by several authors, namely because they play an important role in the study of epimorphisms. They have also been used in the context of the "Diagonal Theorem", that is, the characterization of delta subcategories (see for instance [9], [10] and [7]).

The recent study of nabla subcategories by Clementino and Tholen [3] led these authors to the definition of coregular closure operator which turned out to play exactly the role of regular one in this context. Besides some interesting examples presented in [3] not much is known about these closure operators, even in the category of topological spaces.

In this paper we investigate the behaviour nabla subcategories and their respective coregular closure operators in the category $\mathcal{T}$op of topologoical spaces (section 3) and $\mathcal{M}$od$_R$ of modules over a ring $R$ (section 4).

In $\mathcal{T}$op we study in particular the least coregular closure operators and obtain a proper class of coregular closure operators that do not form a chain (Proposition 3.14).

In $\mathcal{M}$od$_R$we show in Theorem 4.6 that the regular and coregular closure operators are exactly the maximal and the minimal closure operators defined by radicals and idempotent radicals, respectively.

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## 1 Preliminaries

We will first introduce some notions and techniques that will be used througout.

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## 1.1 Factorization systems

In the category $\mathcal{T}op$ of topological spaces and continuous maps the class $\mathcal{M}$ of embeddings has some special features that can be formulated in a categorical way. For each space $X$ the class $\mathcal{M}/X$ of embeddings with codomain $X$ can be preordered by $\leq$, where $(m : M \hookrightarrow X) \leq (n : N \hookrightarrow X)$ if there exists $t : M \to N$ such that $n \cdot t = m$. Considering in $\mathcal{M}/X$ the equivalence relation defined by: $m \cong n$ if $m \leq n$ and $n \leq m$, it is obtains that each equivalence class corresponds exactly to an inclusion of a subspace of $X$. In $\mathcal{M}/X$ one can form arbitrary meets (and so also arbitrary joins) and the class $\mathcal{M}$ is stable under pullback.

Furthermore, every morphism $f : X \longrightarrow Y$ can be factorized as follows

$$\begin{tikzcd} X \arrow[rr, "f"] \arrow[dr, "e"'] & & Y \\ & f(X) \arrow[ur, hook, "a"'] & \end{tikzcd}$$

where $a$ is an embedding and $e$ is a continuous surjection. Moreover, this factorization is unique, up to isomorphism (that is, if $f = a \cdot e = a' \cdot e'$ with $a, a'$ embeddings and $e, e'$ surjections, there exists an isomorphism $h$ such that $h \cdot e = e'$ and $a' \cdot h = a$. The direct image of $M \subseteq X$ under $f$ is obtained by factorizing $M \overset{m}{\hookrightarrow} X \xrightarrow{f} Y$ as above, while the inverse image of $N \subseteq Y$ under $f$ is exactly the pullback (=fibred product) of $N \overset{n}{\hookrightarrow} Y$ along $f$.

This construction can be generalized straightforward to a general category $\mathcal{X}$: given two classes of morphisms $\mathcal{E}$ and $\mathcal{M}$ closed under composition and containing the isomorphisms of $\mathcal{X}$, one says that $(\mathcal{E}, \mathcal{M})$ is a *factorization system for morphisms in* $\mathcal{X}$ if every $\mathcal{X}$-morphism has a unique (up to isomorphism) $(\mathcal{E}, \mathcal{M})$-factorization. That is, for each morphism $f : X \longrightarrow Y$, there exists $e : X \longrightarrow M$ and $m : M \longrightarrow Y$ in $\mathcal{M}$ such that $f = m \cdot e$.

A factorization system is called *proper* if $\mathcal{E}$ is a class of epimorphisms and $\mathcal{M}$ is a class of monomorphisms. Consequently, $\mathcal{E}$ contains the regular epimorphisms and $\mathcal{M}$ the regular monomorphisms (cf. [8]).

The category $\mathcal{X}$ is said to be $\mathcal{M}$-*complete* if $\mathcal{X}$ has $\mathcal{M}$-pullbacks (i.e, the pullback of a morphism in $\mathcal{M}$ along any morphism exists and belongs to $\mathcal{M}$) and $\mathcal{M}$-intersections (i.e, $\mathcal{X}$ has limits of families of morphisms in $\mathcal{M}$ with common codomain , which belong to $\mathcal{M}$). We remark that the $\mathcal{M}$-completeness of $\mathcal{X}$ guarantees the existence of a factorization system $(\mathcal{E}, \mathcal{M})$ for morphisms (see [1]).

Having in mind the behaviour in $\mathcal{T}op$ of the factorization described above, in an $\mathcal{M}$-complete category $\mathcal{X}$, with the factorization system $(\mathcal{E}, \mathcal{M})$, a *subobject* of $X \in \mathcal{X}$ is a morphism $m : M \longrightarrow X$ in $\mathcal{M}$. Denoting by sub$X$ the class of subobjects of $X$, sub$X$ is a (possibly large) complete lattice, with its preorder defined as in the topological setting. Every morphism $f : X \longrightarrow Y$ in $\mathcal{X}$ induces an *image-preimage* adjunction $f(-) \dashv f^{-1}(-) : \mathrm{sub}Y \longrightarrow \mathrm{sub}X$, with $f^{-1}(n)$ the pullback of $n \in \mathrm{sub}Y$ along $f$, and $f(m)$ the $\mathcal{M}$-part of the factorization of $f \cdot m$. One always has $m \leq f^{-1}(f(m))$ and $f(f^{-1}(n)) \leq n$.

For more details on factorization systems see [1].

## 1.2 Closure Operators

From now on we work in an $\mathcal{M}$-complete category $\mathcal{X}$ with finite limits and a proper factorization system $(\mathcal{E}, \mathcal{M})$.

A closure operator $c$ of the category $\mathcal{X}$ with respect to the factorization system $(\mathcal{E}, \mathcal{M})$ is given by a family of maps $(c_X : \mathrm{sub}X \longrightarrow \mathrm{sub}X)_{X \in \mathcal{X}}$ such that:

1. $m \leq c_X(m)$ for all $m \in \text{sub}X$;
2. if $m_1 \leq m_2$ then $c_X(m_1) \leq c_X(m_2)$ for all $m_1, m_2 \in \text{sub}X$;
3. $f(c_X(m)) \leq c_Y(f(m))$ for all $m \in \text{sub}X$ and $f : X \longrightarrow Y$ in $\mathcal{X}$.

Condition 3 can equivalently be expressed as $c_X(f^{-1}(n)) \leq f^{-1}(c_Y(n))$ for all $n$ in sub$Y$ and $f : X \longrightarrow Y$ in $\mathcal{X}$.

A subobject $m$ of $X$ is *c-closed* if $c_X(m) \cong m$, and it is *c-dense* if $c_X(m) \cong 1_X$.

A closure operator $c$ is *idempotent* if $c(m)$ is $c$-closed for every $m \in \mathcal{M}$, and is *weakly hereditary* if $j_m$ is $c$-dense with $m = c(m) \cdot j_m$.

The preorders of the classes sub$X$ induce in a natural way a partial order in the conglomerate $CL(\mathcal{X})$ of all closure operators in $\mathcal{X}$ (w.r.t.$(\mathcal{E}, \mathcal{M})$), that has meets and joins formed pointwise.

For additional information on closure operators see [7].

# 2 Regular and coregular closure operators versus connectedness

Given a closure operator $c$ in $\mathcal{X}$, an object $X$ of $\mathcal{X}$ is called *c-separated* if its diagonal $\delta_X := <1_X, 1_X>: X \longrightarrow X \times X$ is $c$-closed, and it is called *c-connected* if $\delta_X$ is $c$-dense. This way one defines the subcategories $\Delta(c)$ of $c$-separated objects and $\nabla(c)$ of $c$-connected objects (all the subcategories of $\mathcal{X}$ we consider are full and closed under isomorphisms and we denote its conglomerate by $SUB(\mathcal{X})$). The objects that belong to $\Delta(c) \cap \nabla(c)$ are those whose diagonal is an isomorphism, which are exactly the *preterminal* objects.

The $\Delta$ and $\nabla$ assignments give rise to the functors

$$\Delta : CL(\mathcal{X}) \longrightarrow SUB(\mathcal{X})^{op}$$

$$\nabla : CL(\mathcal{X}) \longrightarrow SUB(\mathcal{X})$$

where the partially ordered conglomerates $CL(\mathcal{X})$ and $SUB(\mathcal{X})$ are considered as categories.

On the other hand, each subcategory of $\mathcal{X}$ defines two special closure operators, a regular and a coregular closure operator we describe below.

**Definitions 2.1** ([3]) Let $\mathcal{A}$ be a subcategory of $\mathcal{X}$. The *regular* and *coregular closure operators induced by* $\mathcal{A}$ are locally defined by:

$$\text{reg}_X^{\mathcal{A}}(m) := \bigwedge\{h^{-1}(\delta_A) \,|\, h : X \longrightarrow A^2, A \in \mathcal{A} \text{ and } h(m) \leq \delta_A\},$$

$$\text{coreg}_X^{\mathcal{A}}(m) := m \vee \bigvee\{h(1_{A^2}) \,|\, h : A^2 \longrightarrow X, A \in \mathcal{A} \text{ and } h(\delta_A) \leq m\},$$

for every $m \in \text{sub}X$ and every $X \in \mathcal{X}$.

We remark that every regular closure operator is idempotent and every coregular closure operator is weakly hereditary.

Regular closure operators were introduced by Salbany in [12] – with a different (but equivalent) description – and were widely used in the literature. Coregular closure operators were introduced by Clementino and Tholen in [3] in order to describe $\nabla$-subcategories.

Let $\mathcal{A}$, $\mathcal{B}$ be subcategories of $\mathcal{X}$. If $\mathcal{A} \subseteq \mathcal{B}$ then $\text{coreg}^{\mathcal{A}} \leq \text{coreg}^{\mathcal{B}}$ and $\text{reg}^{\mathcal{A}} \geq \text{reg}^{\mathcal{B}}$, hence reg and coreg may be interpreted as functors.

**Proposition 2.2** ([3])

1. *The functor* reg : $SUB(\mathcal{X})^{op} \longrightarrow CL(\mathcal{X})$ *is right adjoint to* $\Delta$.
2. *The functor* coreg : $SUB(\mathcal{X}) \longrightarrow CL(\mathcal{X})$ *is right adjoint to* $\nabla$.

**Corollary 2.3** *Let $\mathcal{A}$ be a subcategory of $\mathcal{X}$ and $c$ a closure operator in $\mathcal{X}$. Then:*

1. *(a)* $\mathcal{A} \subseteq \Delta(\mathrm{reg}^{\mathcal{A}})$ *and* $c \leq \mathrm{reg}^{\Delta(c)}$,
   *(b)* $\mathcal{A} \subseteq \Delta(c) \Longleftrightarrow c \leq \mathrm{reg}^{\mathcal{A}}$*;*
2. *(a)* $\mathcal{A} \subseteq \nabla(\mathrm{coreg}^{\mathcal{A}})$ *and* $c \geq \mathrm{coreg}^{\nabla(c)}$,
   *(b)* $\mathcal{A} \subseteq \nabla(c) \Longleftrightarrow c \geq \mathrm{coreg}^{\mathcal{A}}$.

From this proposition one has that there is a bijection between (co)regular closure operators and delta(nabla) subcategories.

# 3 Coregular closure operators in $\mathcal{T}$op

In this section we will present examples of coregular closure operators and $\nabla$-subcategories in the category of topological spaces.

It was proved in [3] that $\nabla$ and $\Delta$ subcategories in $\mathcal{T}$op extend disconnectednesses and connectednesses as studied by Arhangel'skiĭ and Wiegandt.

**Proposition 3.1** *Let $\mathcal{A}$ be a subcategory of $\mathcal{T}op$. Then*

$$\Delta(\mathrm{coreg}^{A}) = r(\mathcal{A}) := \{X \in \mathcal{T}\mathrm{op} \,|\, (\forall A \in \mathcal{A})\ g : A \longrightarrow X \ \Rightarrow g \text{ is constant } \},$$

$$\nabla(\mathrm{reg}^{A}) = l(\mathcal{A}) := \{X \in \mathcal{T}\mathrm{op} \,|\, (\forall A \in \mathcal{A})\ f : X \longrightarrow A \ \Rightarrow f \text{ is constant } \}.$$

The subcategories of the type $l(\mathcal{A})$ and $r(\mathcal{A})$ are called *left-constant* and *right-constant*, respectively and in the particular case of topological spaces, they are also called connectednesses and disconnectednesses.

The following examples were studied in [3].

**Example 3.2** Let $k$ be the Kuratowski closure operator. The subcategory $\nabla(k)$ is the class of Hausdorff spaces and $\Delta(k)$ is the class of irreducible spaces. A topological space $X$ is *irreducible* if for $U, V \subseteq X$, open sets and $U \cap V = \emptyset$, $U = \emptyset$ or $V = \emptyset$.

The class $\mathcal{C}$on of connected spaces is not the nabla subcategory of the usual closure operator $k$ but, as we will see, is a nabla subcategory.

**Example 3.3** Let conn be the *connected component closure operator*, defined by $\mathrm{conn}_X(M) := \bigcup_{x \in M} \mathrm{comp}_X(x)$, where $\mathrm{comp}_X(x)$ is the connected component of $x$. The nabla subcategory of conn is the subcategory of connected spaces. We do not know if the connected component closure operator is the coregular closure operator of $\mathcal{C}$on.

**Example 3.4** The *path-connected component closure operator*, defined like the connected component closure operator is the coregular closure operator defined by the unit interval $[0, 1]$. Its nabla subcategory is the subcategory of path-connected spaces.

Below we outline the behaviour of some relevant coregular closure operators and the respective $\nabla$-subcategories. From this study it turns out that $\nabla$-subcategories cover a much richer range of subcategories than the connectednesses. We focus our study in the least and largest of these closure operators and subcategories.

We will denote by $\mathsf{D}$, $\mathsf{E}$ and $\mathsf{S}$ the discrete space with two points 0 and 1, the indiscrete space with two points and the Sierpinski space, respectively; (in)disc denotes the (in)discrete closure operator.

The discrete closure operator is obviously the coregular closure induced by the subcategory $\mathcal{S}$gl. The indiscrete closure operator is also coregular as we show next.

We remark that in $\mathcal{T}$op each nabla subcategory is closed under continuous images.

**Proposition 3.5** *For a subcategory $\mathcal{A}$ of $\mathcal{T}op$ closed under images, the following conditions are equivalent:*

(i) $\nabla(\mathrm{coreg}^{\mathcal{A}}) = \mathcal{T}op$;

(ii) $\mathrm{coreg}^{\mathcal{A}} = \mathrm{indisc}$;

(iii) $\mathsf{D} \in \mathcal{A}$.

*Proof*: (i)$\Rightarrow$(ii) Obvious.

(ii)$\Rightarrow$(iii) If $\mathrm{coreg}^{\mathcal{A}} = \mathrm{indisc}$ then $\mathrm{coreg}^{\mathcal{A}}_{\mathsf{D}}(0) = \mathsf{D}$. So, there is a continuous map $h : A^2 \longrightarrow \mathsf{D}$ with $A \in \mathcal{A}$, $h(a,a) = 0$ for all $a \in A$ and $h(b,c) = 1$ for two distinct points $b, c$ of $A$. For $g : A \longrightarrow A^2$ defined by $g(x) = (b, x)$, the function $h \cdot g$ is continuous and $h \cdot g(A) = \mathsf{D}$. So $\mathsf{D}$ is in $\mathcal{A}$ because $\mathcal{A}$ is closed under images.

(iii)$\Rightarrow$(i) Let $X$ be a topological space. For $(x,y) \in X \times X$, one defines $h : \mathsf{D}^2 \longrightarrow X^2$ with $h(\delta_{\mathsf{D}}) \subseteq \delta_X$ and $h(0,1) = (x,y)$. The function $h$ is continuous because its domain is a discrete space. Hence $\mathrm{coreg}^{\mathcal{A}}(\delta_X) = 1_{X^2}$ and so $X \in \nabla(\mathrm{coreg}^{\mathcal{A}})$. ∎

**Corollary 3.6** *If $\mathcal{A}$ is a nabla subcategory and $\mathcal{A} \neq \mathcal{T}$op then $\mathcal{A} \subseteq \mathcal{C}$on.*

*Proof*: Every nabla subcategory containing a disconnected space must contain $\mathsf{D}$ since it is closed under images. ∎

**Proposition 3.7** *Let $X$ be a topological space and $M \subseteq X$. Then*

$$\mathrm{coreg}^{\mathsf{E}}_X(M) = \{x \in X \mid (\exists y \in M) : k(x) = k(y)\}.$$

*Proof*: Let $x$ be an element of $\mathrm{coreg}^{\mathsf{E}}_X(M)$. There is $f : \mathsf{E} \times \mathsf{E} \longrightarrow X$ with $f(0,1) = x$ and $\{f(0,0), f(1,1)\} \subseteq M$. Since $\mathsf{E} \times \mathsf{E}$ is indiscrete and $f$ continuous, $f(\mathsf{E} \times \mathsf{E})$ is indiscrete, and so $k(x) = k(f(0,0))$.

Conversely, if for $x \in X$ exists $y \in M$ such that $k(x) = k(y)$ then the function $f : \mathsf{E} \times \mathsf{E} \longrightarrow X$ defined by $f(0,0) = f(1,1) = y$ and $f(0,1) = f(1,0) = x$ is continuous. ∎

**Corollary 3.8** *The nabla subcategory induced by* $\mathrm{coreg}^{\mathsf{E}}$ *is the subcategory of indiscrete spaces.*

**Corollary 3.9** *If $X$ is a $T_0$-space and $M \subseteq X$, then $\mathrm{coreg}^{\mathsf{E}}_X(M) = M$.*

**Proposition 3.10** *Let $X \in \mathcal{T}$op and $M \subseteq X$. Then*

$$\mathrm{coreg}^{\mathsf{S}}_X(M) = \{x \in X \mid (\exists z, w \in M) : z \in k(x) \text{ and } x \in k(w)\}.$$

*Proof*: Let $c := \mathrm{coreg}^{\mathsf{S}}$ and $x$ be an element of $c(M)$. There is $f : \mathsf{S} \times \mathsf{S} \longrightarrow X$ with $f(0,1) = x$, $f(0,0) = w$ and $f(1,1) = z$ ($z, w \in M$).

From $k((0,0)) = \mathsf{S} \times \mathsf{S}$, we know that $(0,1) \in k((0,0))$ and, by continuity of $f$, $f(0,1) = x \in k(w)$. In the same way $k((0,1)) = \{(0,1),(1,1)\}$ implies that $(1,1) \in k((0,1))$ and finally that $z \in k(x)$.

Conversely, we have $z \in k(x)$ and $x \in k(w)$ with $z, w \in M$ and $x \in X$, and we want prove that $x \in c(M)$. So, it is enough to show that the function $f : \mathsf{S} \times \mathsf{S} \longrightarrow X$ with $f(0,1) = f(1,0) = x$, $f(0,0) = w$ and $f(1,1) = z$ is continuous. Let $F \subseteq X$ be a closed set:

$$f^{-1}(F) = \begin{cases} \emptyset & \text{if } w \notin F, x \notin F \text{ e } z \notin F \\ \mathsf{S} \times \mathsf{S} & \text{if } w \in F (\Rightarrow x \in F \Rightarrow z \in F) \\ \mathsf{S} \times \mathsf{S} \backslash \{(0,0)\} & \text{if } w \notin F \text{ e } x \in F (\Rightarrow z \in F) \\ \{(1,1)\} & \text{if } w \notin F, x \notin F \text{ e } z \in F \end{cases}$$

Since the inverse image of a closed set is closed, then $f$ is continuous, and the proof is complete. ∎

From the definition of $\mathrm{coreg}^{\mathsf{S}}$ we may conclude immediately the following results.

**Corollary 3.11** *If $X$ is a $T_1$-space and $M \subseteq X$, then,* $\mathrm{coreg}^{\mathsf{S}}_X(M) = M$.

**Corollary 3.12** *A space $X$ belongs to $\nabla(\mathrm{coreg}^{\mathsf{S}})$ if and only if*

$$(\forall x, y \in X)\, (\exists z, w \in X) : z \in k(x) \cap k(y) \text{ and } \{x, y\} \subseteq k(w).$$

From the results above *we have the following chain of coregular closure operators:*

$$\mathrm{disc} = \mathrm{coreg}^{\mathcal{S}\mathrm{gl}} < \mathrm{coreg}^{\mathsf{E}} < \mathrm{coreg}^{\mathsf{S}} < \mathrm{coreg}^{\mathcal{C}\mathrm{on}} < \mathrm{coreg}^{\mathsf{D}} = \mathrm{indisc}.$$

*Moreover, if $c$ is a coregular closure operator different from these, then*

$$\mathrm{coreg}^{\mathsf{S}} < c < \mathrm{coreg}^{\mathcal{C}\mathrm{on}}.$$

In fact, if $c = \mathrm{coreg}^{\mathcal{A}}$ with $c \neq \mathrm{disc}$ and $c \neq \mathrm{coreg}^{\mathsf{E}}$ then there exists $X \in \mathcal{A}$ such that X has a non trivial open set because $X$ can not be a singleton space or an indiscrete space. Since nabla subcategories are closed under images, $\mathsf{s} \in \nabla(\mathrm{coreg}^{\mathcal{A}})$ which implies that $\mathrm{coreg}^{\mathsf{S}} \leq \mathrm{coreg}^{\mathcal{A}}$.

Note that *the trivial closure operator is not a coregular closure operator.*

Since $\mathrm{coreg}^{\mathsf{E}}$ and $\mathrm{coreg}^{\mathsf{S}}$ are discrete in $T_0$ spaces and in $T_1$ spaces, respectively, we could conjecture that the next coregular closure operator would be the largest one discrete in $T_2$-spaces, $\mathrm{coreg}^{\nabla(k)}$, but that is not true. On the contrary, there are plenty of coregular closure operators.

For an infinite cardinal $\alpha$, let $X_\alpha$ be a topological space $(X, \mathcal{T})$, where $X$ has cardinal $\alpha$ and $\mathcal{T}$ is the cofinite topology.

**Proposition 3.13** *If $\alpha$ and $\beta$ are two infinite cardinals and $\alpha < \beta$, then:*

$$\mathrm{coreg}^{X_\beta} < \mathrm{coreg}^{X_\alpha}.$$

*Proof*: First, we will prove that $X_\alpha \notin \nabla(\mathrm{coreg}^{X_\beta})$, which implies that $\mathrm{coreg}^{X_\alpha} \neq \mathrm{coreg}^{X_\beta}$. Let $h : X_\beta \longrightarrow X_\alpha$ be a continuous map. Then $X_\beta = \bigcup\limits_{x \in X_\alpha} h^{-1}(x)$, with $h^{-1}(x)$ a closed set for each $x$. But we know that $X_\beta$ is not the union of $\alpha$ finite sets because $\alpha < \beta$. And so, one of the sets $h^{-1}(x)$ has to be equal to $X_\beta$, and then $h$ is constant.

Hence $X_\alpha \in r(\{X_\beta\}) = \Delta(\mathrm{coreg}^{X_\beta})$ and therefore it cannot belong to $\nabla(\mathrm{coreg}^{X_\beta})$, since only the singleton spaces and the empty space are in $\Delta(\mathrm{coreg}^{X_\beta}) \cap \nabla(\mathrm{coreg}^{X_\beta})$.

Next, we will prove that $\mathrm{coreg}^{X_\beta} \leq \mathrm{coreg}^{X_\alpha}$. If $x \in \mathrm{coreg}_Y^{X_\beta}(M) \setminus M$, for a subspace $M$ of $Y$, then there is $h : X_\beta \times X_\beta \longrightarrow Y$ such that $h(\delta_{X_\beta}) \subseteq M$ and $h(a,b) = x$ for $a, b$ in $X_\beta$. Now, let us consider a subspace $X_\alpha$ of $X_\beta$ such that $a, b \in X_\alpha$. Then $x \in h|_{X_\alpha \times X_\alpha}(X_\alpha \times X_\alpha)$, and so $x \in \mathrm{coreg}_Y^{X_\alpha}(M)$. ■

The construction of the cofinite topology can be generalized. In fact, for two infinite cardinals $\gamma < \alpha$, we define the topological space $X_\alpha^\gamma$, where the cardinal of $X_\alpha^\gamma$ is $\alpha$ and $A \subseteq X_\alpha^\gamma$ is closed if its cardinal is less than $\gamma$ or $A = X_\alpha^\gamma$. For $\gamma = \aleph_0$ the topology defined this way is the cofinite one.

**Proposition 3.14** *Let $\alpha$, $\beta$, $\gamma$ and $\eta$ be infinite cardinals. If $\eta < \gamma \leq \alpha < \beta$ then:*

*1.* $\mathrm{coreg}^{X_\alpha^\eta} < \mathrm{coreg}^{X_\alpha^\gamma}$;

*2.* $\mathrm{coreg}^{X_\beta^\gamma} < \mathrm{coreg}^{X_\alpha^\gamma}$.

*Proof*: 1. If $\eta < \gamma$, then the identity map $f : X_\alpha^\gamma \longrightarrow X_\alpha^\eta$ is continuous, therefore $\mathrm{coreg}^{X_\alpha^\eta} \leq \mathrm{coreg}^{X_\alpha^\gamma}$ because the nabla subcategories are closed under images.

Next we will show that $X_\alpha^\gamma \in r(\{X_\alpha^\eta\})$. Let $g : X_\alpha^\eta \longrightarrow X_\alpha^\gamma$ be a continuous map. If $|g(X_\alpha^\eta)| \geq \eta$, then $g(X_\alpha^\eta)$ has a proper subset $F$ of cardinal larger or equal to $\eta$. But $|g^{-1}(F)| < \eta$, and so $|F| \leq |g^{-1}(F)| < \eta$. This implies that $|g(X_\alpha^\eta)| < \gamma$ and so $g(X_\alpha^\eta)$ is a discrete subspace of $X_\alpha^\gamma$. A discrete space which is image of $X_\alpha^\eta$ is a singleton.

In conclusion $X_\alpha^\gamma \notin \nabla(\mathrm{coreg}^{X_\alpha^\eta})$ and then $\mathrm{coreg}^{X_\alpha^\eta} \neq \mathrm{coreg}^{X_\alpha^\gamma}$.

The proof of 2 is similar to the case $\gamma = \aleph_0$. ■

**Corollary 3.15** *Between* $\mathrm{coreg}^{\mathsf{S}}$ *and* $\mathrm{coreg}^{\nabla(k)}$ *there is a proper class of coregular closure operators.*

**Remark 3.16** For $\mathcal{A} = \{X_\alpha^\alpha \mid \alpha \text{ is an infinite cardinal}\}$, $\mathrm{coreg}^{\mathcal{A}} \leq \mathrm{coreg}^{\nabla(k)}$. We do not know if they are equal.

# 4 Coregular closure operators in $\mathcal{M}\mathrm{od}_R$

Let $\mathcal{M}\mathrm{od}_R$ be the category of $R$-modules with its (surjective homomorphisms, injective homomorphisms) factorization system (i.e, (epi, mono)-factorization)). So, in this case a subobject is, up to isomorphism, a submodule.

**Definition 4.1** A *preradical* $r$ in $\mathcal{M}\mathrm{od}_R$ is a subfunctor of the identity functor in $\mathcal{M}\mathrm{od}_R$; that is $r : \mathcal{M}\mathrm{od}_R \longrightarrow \mathcal{M}\mathrm{od}_R$ is a map such that $r(M)$ is a submodule of $M$ and $f(r(M)) \subseteq r(f(M))$, for each $M, N \in \mathcal{M}\mathrm{od}_R$ and each homomorphism $f : M \longrightarrow N$.

A preradical $r$ is *idempotent* if $r(r(M)) = r(M)$ for every $M \in \mathcal{M}\mathrm{od}_R$, and it is a *radical* if $r(M/r(M)) = \mathbb{O}$ for every $M \in \mathcal{M}\mathrm{od}_R$.

To each preradical $r$ a *torsion-free subcategory* $\mathcal{F}_r = \{M : r(M) = \mathbb{O}\}$ and a *torsion subcategory* $\mathcal{T}_r = \{M : r(M) = M\}$ are associated.

Preradicals and closure operators in $\mathcal{M}\mathrm{od}_R$ are closely connected: each closure operator induces a preradical $r$ by $r(M) := c_M(\mathbb{O})$; on the other hand, each preradical defines in a natural way two closure operators, $\min^r$ and $\max^r$, the least and the largest one such that $c_M(\mathbb{O}) = r(M)$ for every $R$-module $M$. They are called the *minimal* and the *maximal closure operators*, respectively, and defined by

$$\min_M^r(N) = N + r(M)$$

$$\max_M^r(N) = \pi^{-1}(r(M/N)),$$

where $N$ is a submodule of $M$ and $\pi : M \longrightarrow M/N$ is the canonical projection.

The next results are partially in [3], [7].

**Proposition 4.2** *Let $r$ be a preradical in $\mathcal{M}\mathrm{od}_R$. Then:*

1. $\nabla(\min^r) = \nabla(\max^r) = \mathcal{T}_r$;
2. $\Delta(\min^r) = \Delta(\max^r) = \mathcal{F}_r$.

*Proof*: 1. We already know that $\nabla(\min^r) \subseteq \nabla(\max^r)$, because $\min^r \leq \max^r$. A module $N$ is in $\nabla(\max^r)$ if and only if $\max^r_{N^2}(\delta_N) = N \times N$. The equality $\max^r_{N^2}(\delta_N) = N \times N$ means that $\pi^{-1}(r(N \times N/\delta_N)) = \pi^{-1}(N \times N/\delta_N)$ and, consequently, $r(N \times N/\delta_N) = N \times N/\delta_N$ because $\pi$ is surjective. By the isomorphism $N \times N/\delta_N \cong N$, $N \in \nabla(\max^r)$ if and only if $N \in \mathcal{T}_r$.

At last, we show that $\mathcal{T}_r \subseteq \nabla(\min^r)$. If $N \in \mathcal{T}_r$, then $r(N) = N$. A preradical is finitely productive, and so

$$\min^r_{N^2}(\delta_N) = \delta_N + r(N \times N) = \delta_N + r(N) \times r(N) = \delta_N + N \times N = N \times N.$$

2. The proof of $\mathcal{F}_r = \Delta(\max^r) \subseteq \Delta(\min^r)$ is similar to the first part of the proof of 1. To show the remaining inclusion, if $N$ is in $\Delta(\min^r)$, then $\delta_N + r(N \times N) = \delta_N$, and consequently $r(N) \times r(N) \subseteq \delta_N$. From this fact we have that $r(N)$ is a singleton and so $r(N) = \mathbb{O}$. ∎

**Corollary 4.3** *If $c$ is a closure operator in $\mathcal{M}\mathrm{od}_R$ and $r$ is the preradical induced by $c$, then*

$$\nabla(c) = \mathcal{T}_r \;\; e \;\; \Delta(c) = \mathcal{F}_r.$$

From this result, we have that the torsion subcategories and the nabla subcategories are exactly the same, and, at the same time, the free-torsion and the delta subcategories coincide.

Now we investigate the (co)regular closure operators in $\mathcal{M}\mathrm{od}_R$.

**Proposition 4.4** *If $r$ is a radical then $reg^{\mathcal{F}_r} = \max^r$.*

*Proof*: It is true in general that $\mathrm{reg}^{\Delta(c)} \geq c$. In particular for $c = \max^r$, we know that $\Delta(\max^r) = \mathcal{F}_r$ from Proposition 4.2, and so $\mathrm{reg}^{\mathcal{F}_r} \geq \max^r$. To proof the other inequality is enough to show that $\mathrm{reg}^{\mathcal{F}_r}_M(\mathbb{O}) = r(M)$. From the former inequality we have $r(M) = \max^r_M(\mathbb{O}) \subseteq \mathrm{reg}^{\mathcal{F}_r}_M(\mathbb{O})$. In $\mathcal{M}\mathrm{od}_R$ the regular closure operator may be computed by

$$\mathrm{reg}^{\mathcal{F}_r}_M(\mathbb{O}) = \bigcap\{\ker f \mid f : M \longrightarrow X,\ X \in \mathcal{F}_r\}.$$

The quotient module $M/r(M)$ is in $\mathcal{F}_r$, because $r$ is a radical. Hence, for $\pi : M \to M/r(M)$ the canonical homomorphism, we have $\mathrm{reg}^{\mathcal{F}_r}_M(\mathbb{O}) \subseteq \ker\pi = r(M)$. ∎

**Proposition 4.5** *If $r$ is an idempotent preradical, then $coreg^{\mathcal{T}_r} = \min^r$.*

*Proof*: That $coreg^{\mathcal{T}_r} \leq \min^r$ may be concluded analogously to the preceding proposition. We only have to show $\min^r_M(\mathbb{O}) = r(M) \subseteq coreg^{\mathcal{T}_r}(\mathbb{O})$. By definition of coregular closure operator, we have

$$\mathrm{coreg}^{\mathcal{T}_r}_M(\mathbb{O}) = \bigcup\{h(X^2) \mid h : X^2 \longrightarrow M, X \in \mathcal{T}_r \text{ and } h(\delta_X) = \mathbb{O}\}.$$

Let $g : r(M) \times r(M) \longrightarrow M$ be the homomorphism defined by $g(x, y) = x - y$. Since $r(M) \in \mathcal{T}_r$ because $r$ is idempotent, $g(\delta_{r(M)}) = \mathbb{O}$ and $g(r(M) \times r(M)) = r(M)$, we conclude that $r(M)) \subseteq \mathrm{coreg}^{\mathcal{T}_r}_M(\mathbb{O})$ as claimed ∎

Since every torsion-free (torsion) subcategoriy of $\mathcal{M}\mathrm{od}_R$ is induced by a radical (idempotent preradical) (cf. [6]) and every delta (nabla) subcategory is torsion-free(torsion), we have:

**Theorem 4.6** *Let $c$ be a closure operator in $\mathcal{M}\mathrm{od}_R$.*

1. *$c$ is a regular closure operator if and only if $c = \max^r$ for a unique radical $r$.*
2. *$c$ is a coregular closure operator if and only if $c = \min^r$ for a unique idempotent preradical $r$.*

We point out that in 1(2) the radical (idempotent preradical) is unique because there is a one-to-one correspondence between maximal (minimal) closure operators and preradicals. Hence, from the results above, it follows that there is a one-to-one correspondence between the conglomerates of regular closure operators, radicals and torsion-free subcategories as well as a one-to-one correspondence between coregular closure operators, idempotent preradicals and torsion subcategories in $\mathcal{M}\mathrm{od}_R$.

In [6], it is stated that every subcategory $\mathcal{A}$ of $\mathcal{M}\mathrm{od}_R$ induces a preradical $t_{\mathcal{A}}$ defined by:

$$t_{\mathcal{A}}(M) := \bigcap\{\ker f \mid f : M \longrightarrow A,\ A \in \mathcal{A}\},$$

which is exactly the preradical associated to $\mathrm{reg}^{\mathcal{A}}$.

For a subcategory $\mathcal{A}$ of $\mathcal{M}\mathrm{od}_R$, we define a preradical $s_{\mathcal{A}}$ by

$$s_{\mathcal{A}}(M) := \mathrm{coreg}^{\mathcal{A}}_M(\mathbb{O}).$$

**Proposition 4.7** *Let $\mathcal{A}$ be a subcategory of $\mathcal{M}\mathrm{od}_R$ and $r$ be a preradical of $\mathcal{M}\mathrm{od}_R$. Then:*

1. *$\mathcal{A} \subseteq \mathcal{F}_{t_{\mathcal{A}}}$ and $r \leq t_{\mathcal{F}_r}$;*
2. *$\mathcal{A} \subseteq \mathcal{T}_{s_{\mathcal{A}}}$ and $r \geq s_{\mathcal{T}_r}$.*

The proof follows directly from the definitions.

**Proposition 4.8** *For every subcategory $\mathcal{A}$ of $\mathcal{M}\mathrm{od}_R$, we have:*

1. *$t_{\mathcal{A}}$ is a radical;*
2. *$s_{\mathcal{A}}$ is an idempotent preradical.*

*Proof*: 1. Since every regular closure operator is maximal, and by [6] we know that a maximal closure operator is idempotent if and only if the preradical it induces is a radical, the preradical $t_{\mathcal{A}}$ is a radical for every subcategory $\mathcal{A}$ of $\mathcal{M}\mathrm{od}_R$.

For 2, we use a similar result of [6] which says that a minimal closure operator is weakly hereditary if and only if it induces an idempotent preradical. ■

**Proposition 4.9** *Let $\mathcal{A}$ be a subcategory of $\mathcal{M}\mathrm{od}_R$. Then:*

1. *$\mathcal{F}_{s_{\mathcal{A}}} = r(\mathcal{A}) := \{M \in \mathcal{M}\mathrm{od}_R \mid (\forall A \in \mathcal{A})\ f : A \longrightarrow M\ \Rightarrow f(A) = \mathbb{O}\}$;*
2. *$\mathcal{T}_{t_{\mathcal{A}}} = l(\mathcal{A}) := \{M \in \mathcal{M}\mathrm{od}_R \mid (\forall A \in \mathcal{A})\ g : M \longrightarrow A \Rightarrow g(M) = \mathbb{O}\}$.*

*Proof*: 1. Let $X$ be in $r(\mathcal{A})$, so that for every homomorphism $f : A \longrightarrow X$ with $A \in \mathcal{A}$, we have $f(A) = \mathbb{O}$.

Let $h : A^2 \longrightarrow X$ be a homomorphism with $A \in \mathcal{A}$. If we define $f_1, f_2 : A \longrightarrow X$ by $f_1(a) := h(a, 0)$ and $f_2(b) := h(0, b)$, then $f_1(A) = f_2(A) = \mathbb{O}$, which implies $h(A \times A) = \mathbb{O}$. From this fact, we have that $\mathrm{coreg}^{\mathcal{A}}_X(\mathbb{O}) = s_{\mathcal{A}}(X) = \mathbb{O}$, and so $X \in \mathcal{F}_{s_{\mathcal{A}}}$.

Conversely if $X \in \mathcal{F}_{s_{\mathcal{A}}}$, then for all $h : A^2 \longrightarrow X$, with $A \in \mathcal{A}$ and $h(\delta_{\mathcal{A}}) = \mathbb{O}$, we have $h(A^2) = \mathbb{O}$.

Let $f : A \longrightarrow X$ be a homomorphism with $A \in \mathcal{A}$, and consider $g : A \times A \longrightarrow X$ defined by $g(a, b) := f(a) - f(b)$. Since $g(\delta_A) = \mathbb{O}$, $g(A^2) = \mathbb{O}$, and consequently $f$ is constant.

2. Analogously for the left-constant subcategories. ■

**Corollary 4.10** *For every subcategory $\mathcal{A}$ of $\mathcal{M}\mathrm{od}_R$ :*

1. $r(\mathcal{A}) = \Delta(\mathrm{coreg}^{\mathcal{A}})$;
2. $l(\mathcal{A}) = \nabla(\mathrm{reg}^{\mathcal{A}})$.

*Proof*: 1. From the preceding proposition $r(\mathcal{A}) = \mathcal{F}_{s_{\mathcal{A}}}$, and by Corollary 4.3 $\mathcal{F}_{s_{\mathcal{A}}} = \Delta(c)$ for every closure operator $c$ such that $c_M(0) = s_{\mathcal{A}}(M)$ for $M \in \mathcal{M}\mathrm{od}_R$. In particular, $\mathcal{F}_{s_{\mathcal{A}}} = \Delta(\mathrm{coreg}^{\mathcal{A}})$.

The proof of 2 is similar. ■

If $r$ is an idempotent radical, then the pair $(\mathcal{T}_r, \mathcal{F}_r)$ is a torsion theory in sense of [4]. The torsion and torsion-free subcategories of a torsion theory are the left and the right constant subcategories, respectively. Each pair $(l\mathcal{A}, rl\mathcal{A})$ determines an idempotent radical $r$ such that $\mathcal{T}_r = l\mathcal{A}$ and $\mathcal{F}_r = rl\mathcal{A}$. This idempotent radical is exactly the one induced by $\mathrm{reg}^{rl\mathcal{A}}$ and by $\mathrm{coreg}^{l\mathcal{A}}$.

# References


[1] J.Adámek, H.Herrlich and G.Strecker, *Abstract and concrete categories*, Wiley, New York (1990).

[2] A.Arhangel'skiĭ and R.Wiegandt, Connectedness and disconnectedness in topology, *General Topology and Applications* **5** (1975) 9-33.

[3] M.M.Clementino and W.Tholen, Separation versus connectedness, *Topology and its Applications* **75** (1997) 143-181.

[4] S.Dickson, A thorsion theory for abelian categories, *Trans. Amer. Soc.* **121** (1966) 223-235.

[5] D.Dikranjan and E.Giuli, Closure operators I, *Topology Appl.* **27** (1987) 129-143.

[6] D.Dikranjan and E.Giuli, Factorizations, injectivity and compactnes in categories of modules, *Communications in Algebra* **19** (1991) 45-83.

[7] D.Dikranjan and W.Tholen, *Categorical structures of closure operators, with applications to topology, algebra and discrete mathematics*, Kluwer Academic Publishers (1995).

[8] P.J.Freyd e G.M.Kelly, Categories of continuous functors, I, *J.Pure Appl. Algebra* **2** (1972) 169-191.

[9] E.Giuli and M.Hušek, A diagonal theorem for epireflective subcategories of $\mathcal{T}$op and cowellpoweredness, *Ann. Pure ed appl.* **145** (1986) 337-346.

[10] E.Giuli, S.Mantovani and W.Tholen, Objects with closed diagonals, *J.Pure Appl. Algebra* **51** (1988) 129-140.

[11] G.Gutierres, Conexidade em categorias, *Master thesis* (Universidade de Coimbra, 1998).

[12] S.Salbany, Reflective subcategories and closure operators, *Lecture Notes in Math.* (Springer, Berlin 1976) 548-565.