Some remarks on the Poisson-Nijenhuis and the Jacobi structures

J. M. Nunes da Costa *
Departamento de Matemática,
Universidade de Coimbra,
Apartado 3008,
3000 Coimbra, Portugal
Email: jmcosta@mat.uc.pt

Abstract

We study the relationship between two (compatible) Jacobi structures on a manifold M, using their associated homogeneous Poisson structures on $\mathbb{R} \times M$, in the case where these Poisson tensors are related by a Nijenhuis tensor.

M. S. Classification (1991): 58F05.
Key words: Poisson-Nijenhuis manifold, Jacobi manifold.

Introduction

All objects considered in this paper (manifolds, maps, differential forms, vector and tensor fields) are assumed to be differentiable of class C^∞.

Let M be a smooth manifold equipped with a *Poisson tensor* Λ and a *Nijenhuis tensor* N, that is, a tensor field of type $(1,1)$ whose *Nijenhuis torsion* $\tau(N)$ vanishes everywhere. The Nijenhuis torsion $\tau(N)$ of N is given by the following formula, where X and Y are vector fields on M,

$$\tau(N)(X,Y) = [NX,NY] - N([NX,Y] + [X,NY] - N[X,Y]).$$

We denote by $\Lambda^\# : T^*M \to TM$ the vector bundle map such that, for any $x \in M$, $\alpha, \beta \in T^*_xM$,

$$< \beta, \Lambda^\#(\alpha) >= \Lambda_x(\alpha,\beta).$$

With Λ and N we can define a tensor field $\mathcal{R}(\Lambda,N)$ of type $(2,1)$, called the *Magri-Morosi concomitant* [Ma-Mo 84] of Λ and N, that is defined, for any pair of a 1-form α and a vector field X, by

*This work was partially supported by CMUC-FCT and PRAXIS

$$\mathcal{R}(\Lambda, N)(\alpha, X) = (\mathcal{L}_{\Lambda^\#(\alpha)}N)X - \Lambda^\#(\mathcal{L}_X({}^tN\alpha)) + \Lambda^\#(\mathcal{L}_{NX}\alpha), \tag{1}$$

where ${}^tN$ is the transpose of $N : TM \to TM$.

Let Λ and N be, respectively, a Poisson tensor and a Nijenhuis tensor on M. The triple (M, Λ, N) is called a *Poisson-Nijenhuis manifold* [KSch-Ma 90] if

$$N\Lambda^\# = \Lambda^\# {}^tN \quad \text{and} \quad \mathcal{R}(\Lambda, N) = 0.$$

If (M, Λ, N) is a Poisson-Nijenhuis manifold, there exists a sequence $(\Lambda_k)_{k\in\mathbb{N}}$ of Poisson tensors on M, with $\Lambda_k = N^k\Lambda$. Moreover, these Poisson tensors are pairwaise compatible, that is, $[\Lambda_l, \Lambda_k] = 0$, for all $l, k \in \mathbb{N}$.

Besides the Poisson-Nijenhuis structure, we are going to use, in what follows, the notion of Jacobi manifold.

A *Jacobi manifold* [Lich 78] is a triple (M, C, E) where C and E are respectively a bivector and a vector field on M, such that

$$[E, C] = 0 \quad \text{and} \quad [C, C] = 2E \wedge C.$$

When $E = 0$, the Jacobi manifold is a Poisson manifold. The *Jacobi bracket* of $f, g \in C^\infty(M, \mathbb{R})$ is given by

$$\{f, g\} = C(df, dg) + f(E.g) - g(E.f),$$

and it defines a *local Lie algebra structure* on $C^\infty(M, \mathbb{R})$.

If (M, C, E) is a Jacobi manifold and h is a nowhere vanishing function on M, then the pair $(hC, C^\#(dh) + hE) = (C_h, E_h)$ defines a new Jacobi structure on M, which is said to be *conformally equivalent* to (C, E).

With each Jacobi manifold (M, C, E), we may associate a homogeneous Poisson structure $(\Lambda, \dfrac{\partial}{\partial t})$ on $\mathbb{R} \times M$, with Λ given by

$$\Lambda = \exp(-t)(C + \frac{\partial}{\partial t} \wedge E), \tag{2}$$

where t is the canonical coordinate on $\mathbb{R}$.

This paper is divided into two sections. Section 1 is devoted to the subject of compatible Jacobi manifolds. We show how the compatibility is related with the Lichnerowicz-Jacobi cohomology and we present a way of generating compatible Jacobi structures on a manifold. In section 2, we establish the conditions on the Poisson-Nijenhuis structure of $\mathbb{R} \times M$ to ensure the compatibility of the corresponding Jacobi structures on M.

1 Compatible Jacobi manifolds

The notion of compatibility of two Jacobi structures on a manifold was introduced in [NdC 98]. We recall that two Jacobi structures (C_1, E_1) and (C_2, E_2) on a manifold M are said to be *compatible* if $(C_1 + C_2, E_1 + E_2)$ is again a Jacobi structure on M.

Proposition 1.1 [NdC 98] *Let M be a manifold endowed with two Jacobi structures (C_1, E_1) and (C_2, E_2). Then, (C_1, E_1) and (C_2, E_2) are compatible if and only if*

$$[E_1, C_2] + [E_2, C_1] = 0 \quad \textit{and} \quad [C_1, C_2] = E_1 \wedge C_2 + E_2 \wedge C_1.$$

There are some equivalent ways of expressing the compatibility of two Jacobi structures on a manifold [NdC 98]. But the study of the compatibility of two Jacobi structures on a manifold, can also be done using the Lichnerowicz-Jacobi cohomology. If (M, C, E) is a Jacobi manifold, let us denote by $A^k(M)$ the space of skew-symmetric contravariant tensor fields of order k (k-tensors) on M and define the differential operator

$$\sigma : A^k(M) \to A^{k+1}(M), \quad \sigma(P) = -[C, P] + kE \wedge P. \tag{3}$$

The restriction of σ to the subspace

$$A^k_I(M) = \{P \in A^k(M) : \mathcal{L}_E P = 0\}$$

of the invariant k-tensors with respect to the vector field E is a *cohomology operator* on the Jacobi manifold and the resultant cohomology is called the *Lichnerowicz-Jacobi cohomology* of M [Le-Ma-Pa 97].

Proposition 1.2 *Two Jacobi structures (C_1, E_1) and (C_2, E_2) on a manifold M are compatible if and only if*

$$\sigma_1(C_2) = -\sigma_2(C_1) \quad \textit{and} \quad \sigma_1(E_2) = -\sigma_2(E_1),$$

where $\sigma_i, i = 1, 2$, are the cohomology operators of the Lichnerowicz-Jacobi cohomology of M, with respect to both Jacobi structures.

Proof.
A direct computation using Proposition 1.1 and the definition (3) of the cohomology operators $\sigma_i, i = 1, 2$, gives the desired result. □

In [NdC 98] it was proved that two conformally equivalent Jacobi structures on M are compatible. Another way of obtaining compatible Jacobi structures uses the Lie derivative on the direction of some vector field.

Proposition 1.3 *Let X be a vector field on the Jacobi manifold (M, C, E) such that*

$$\mathcal{L}_X(\mathcal{L}_X C) = 0 \quad \textit{and} \quad \mathcal{L}_X(\mathcal{L}_X E) = 0.$$

Then the pair $(C_1, E_1) = (\mathcal{L}_X C, \mathcal{L}_X E)$ defines a new Jacobi structure on M which is compatible with (C, E).

Proof.
With $E_1 = \mathcal{L}_X E = [X, E]$, we have $\mathcal{L}_{E_1} C = -\mathcal{L}_E(\mathcal{L}_X C)$, that is

$$\mathcal{L}_{E_1} C + \mathcal{L}_E C_1 = 0. \tag{4}$$

So,

$$\begin{aligned}\mathcal{L}_{E_1}C_1 &= \mathcal{L}_X(\mathcal{L}_{E_1}C)\\ &= -\mathcal{L}_X(\mathcal{L}_E C_1)\\ &= -\mathcal{L}_{[X,E]}(\mathcal{L}_X C)\\ &= -\mathcal{L}_{E_1}C_1,\end{aligned}$$

and

$$\mathcal{L}_{E_1}C_1 = 0. \tag{5}$$

On the other hand,

$$\begin{aligned}\mathcal{L}_X([C,C]) &= [\mathcal{L}_X C, C] + [C, \mathcal{L}_X C]\\ &= 2[C_1, C]\end{aligned}$$

and

$$\mathcal{L}_X(E \wedge C) = E_1 \wedge C + E \wedge C_1;$$

so,

$$[C_1, C] = E_1 \wedge C + E \wedge C_1. \tag{6}$$

Because $\mathcal{L}_X E_1 = \mathcal{L}_X C_1 = 0$, if we take the Lie derivative on the direction of X of both members of (6), we obtain

$$[C_1, C_1] = 2E_1 \wedge C_1. \tag{7}$$

The equalities (5) and (7) prove that (C_1, E_1) is a Jacobi structure, while (4) and (6) show the compatibility. □

We can also prove the following result.

Proposition 1.4 *Let X_1 and X_2 be two vector fields on the Jacobi manifold (M, C, E) such that*

$$[X_1, X_2] = 0, \quad [X_1, \mathcal{L}_{X_2}C] = 0, \quad [X_1, \mathcal{L}_{X_2}E] = 0$$

and

$$\mathcal{L}_{X_i}(\mathcal{L}_{X_i}C) = 0, \quad \mathcal{L}_{X_i}(\mathcal{L}_{X_i}E) = 0, \quad i = 1, 2.$$

Then

$$(C_1, E_1) = (\mathcal{L}_{X_1}C, \mathcal{L}_{X_1}E) \quad \textit{and} \quad (C_2, E_2) = (\mathcal{L}_{X_2}C, \mathcal{L}_{X_2}E)$$

are compatible Jacobi structures on M.

Proof.
We only have to check the compatibility. By Proposition 1.3, we know that the Jacobi structures (C, E) and (C_2, E_2) are compatible. Then,

$$[C, C_2] = E \wedge C_2 + E_2 \wedge C \quad \text{and} \quad [E, C_2] + [E_2, C] = 0. \tag{8}$$

If we take the Lie derivative, on the direction of X_1, of both members of equalities (8), we obtain

$$[C_1, C_2] = E_1 \wedge C_2 + E_2 \wedge C_1 \quad \text{and} \quad [E_1, C_2] + [E_2, C_1] = 0. \tag{9}$$

□

2 Homogeneous Poisson-Nijenhuis manifolds

Let M be a manifold endowed with two Jacobi structures (C_1, E_1) and (C_2, E_2). Take the corresponding homogeneous Poisson structures on $\mathbb{R} \times M$, $(\Lambda_i = \exp(-t)(C_i + \frac{\partial}{\partial t} \wedge E_i), \frac{\partial}{\partial t})$, $i = 1, 2$.

Proposition 2.1 [NdC 98] *The Jacobi structures (C_1, E_1) and (C_2, E_2) on M are compatible if and only if Λ_1 and Λ_2 are compatible Poisson tensors on $\mathbb{R} \times M$.*

Using this result, we want to study the relationship between two compatible Jacobi structures on M and their associated (compatible) homogeneous Poisson strucutres on $\mathbb{R} \times M$, in the case where these Poisson structures are related by a Nijenhuis tensor.

Let $\bar{N}$ be a $(1,1)$-tensor on $\mathbb{R} \times M$ such that $\mathcal{L}_{\frac{\partial}{\partial t}} \bar{N} = 0$. Then, $\bar{N}$ is given by

$$\bar{N} = N + Y \otimes dt + \frac{\partial}{\partial t} \otimes \gamma + g \frac{\partial}{\partial t} \otimes dt, \tag{10}$$

where N is a $(1,1)$-tensor on M, Y is a vector field on M, γ is a 1-form on M and $g \in C^\infty(M, \mathbb{R})$. Reciprocally, if $\bar{N}$ is given by (10), then $\mathcal{L}_{\frac{\partial}{\partial t}} \bar{N} = 0$.

Remarks

1. The image of a vector field on $\mathbb{R} \times M$ of the form $\bar{X} = \exp(-t)(f \frac{\partial}{\partial t} + X)$, where X is a vector field on M and $f \in C^\infty(M, \mathbb{R})$, by the $(1,1)$-tensor given by (10) is a vector field on $\mathbb{R} \times M$ of the same type of $\bar{X}$.
2. If Λ is the homogeneous Poisson tensor on $\mathbb{R} \times M$, given by (2), then $\bar{N}\Lambda$ is a homogeneous bivector on $\mathbb{R} \times M$: $\mathcal{L}_{\frac{\partial}{\partial t}}(\bar{N}\Lambda) = -\bar{N}\Lambda$

Proposition 2.2 *Let $\bar{N}$ be a $(1,1)$-tensor on $\mathbb{R} \times M$ given by (10). Then $\bar{N}$ is a Nijenhuis tensor on $\mathbb{R} \times M$ if and only if*

$$\begin{aligned} &i) \quad \tau(N) = Y \otimes d\gamma; \\ &ii) \quad \mathcal{L}_N \gamma = g d\gamma; \\ &iii) \quad \mathcal{L}_Y N = -Y \otimes dg; \\ &iv) \quad {}^t N(dg) = \mathcal{L}_Y \gamma + g dg. \end{aligned}$$

Proof.
First, remark that if X is a vector field on M, then $\bar{N}X = NX + < \gamma, X > \frac{\partial}{\partial t}$ and that $\bar{N}(\frac{\partial}{\partial t}) = Y + g \frac{\partial}{\partial t}$. If X_1 and X_2 are vector fields on M, then

$$\tau(\bar{N})(X_1, X_2) \quad = \quad \tau(N)(X_1, X_2) + (X_2 . < \gamma, X_1 > - X_1 . < \gamma, X_2 >$$

$$
\begin{aligned}
&+ <\gamma,[X_1,X_2]>)\bar{N}(\frac{\partial}{\partial t}) + ((NX_1).<\gamma,X_2> -(NX_2).<\gamma,X_1> \\
&- <\gamma,[NX_1,X_2]> - <\gamma,[X_1,NX_2]> - <\gamma,N[X_1,X_2]>)\frac{\partial}{\partial t} \\
= \;& \tau(N)(X_1,X_2) - d\gamma(X_1,X_2)Y - (gd\gamma(X_1,X_2) - \mathcal{L}_N\gamma(X_1,X_2))\frac{\partial}{\partial t}.
\end{aligned}
$$

So, $\tau(\bar{N})(X_1,X_2)=0$ if and only if

$$\tau(N)(X_1,X_2) = (Y\otimes d\gamma)(X_1,X_2) \tag{11}$$

and

$$\mathcal{L}_N\gamma(X_1,X_2) = gd\gamma(X_1,X_2). \tag{12}$$

Let X be a vector field on M. Then,

$$
\begin{aligned}
\tau(\bar{N})(X,\frac{\partial}{\partial t}) &= [NX,Y] + ((NX).g)\frac{\partial}{\partial t} - (Y.<\gamma,X>)\frac{\partial}{\partial t} \\
&\quad -N[X,Y] - <\gamma,[X,Y]>\frac{\partial}{\partial t} - (X.g)(Y+g\frac{\partial}{\partial t}) \\
&= -(\mathcal{L}_YN)(X) - (Y\otimes dg)(X) \\
&\quad +(<{}^tNdg,X> - <\mathcal{L}_Y\gamma,X> - <gdg,X>)\frac{\partial}{\partial t}.
\end{aligned}
$$

So, $\tau(\bar{N})(X,\frac{\partial}{\partial t})=0$ if and only if

$$(\mathcal{L}_YN)(X) = -(Y\otimes dg)(X) \tag{13}$$

and

$$<{}^tNdg - \mathcal{L}_Y\gamma - gdg, X>=0. \tag{14}$$

Equalities (11), (12), (13) and (14) end the proof. □

Let us take the homogeneous Poisson tensor $\Lambda = \exp(-t)(C+\frac{\partial}{\partial t}\wedge E)$ on $(\mathbb{R}\times M)$.

Lemma 2.1 *With the notations of Proposition 2.2, $\bar{N}\Lambda = \Lambda\,{}^t\bar{N}$ if and only if*

$$
\begin{aligned}
&i) \quad NE = C^{\#}(\gamma) + gE; \\
&ii) \quad NC - C\,{}^tN = E\otimes Y + Y\otimes E; \\
&iii) \quad <\gamma,E>=0.
\end{aligned}
$$

Proof.
Let α be any 1-form on M. Then,

$$
\begin{aligned}
\bar{N}(\Lambda^{\#}(\alpha)) &= \exp(-t)(\bar{N}(C^{\#}(\alpha) - <\alpha,E>\frac{\partial}{\partial t})) \\
&= \exp(-t)(N(C^{\#}(\alpha)) - <\alpha,E>Y - (<\alpha,C^{\#}(\gamma)> + <\alpha,E>g)\frac{\partial}{\partial t})
\end{aligned}
$$

and, taking account that

$$^t\bar{N}(\alpha) = {}^tN\alpha + < \alpha, Y > dt,$$

we compute

$$\Lambda^{\#}({}^t\bar{N}\alpha) = \exp(-t)(C^{\#}({}^tN\alpha) + < \alpha, Y > E - < \alpha, NE > \frac{\partial}{\partial t})$$

and we conclude that $\bar{N}\Lambda^{\#}(\alpha) = \Lambda^{\#}({}^tN\alpha)$ if and only if

$$< \alpha, NE > = < \alpha, C^{\#}(\gamma) + gE > \tag{15}$$

and

$$(NC^{\#} - Y \otimes E)\alpha = (C^{\#}\,{}^tN + E \otimes Y)\alpha. \tag{16}$$

Equalities (15) and (16) give conditions *i*) and *ii*).

On the other hand,

$$\bar{N}(\Lambda^{\#}(dt)) = \exp(-t)(NE + < \gamma, E > \frac{\partial}{\partial t}) \tag{17}$$

and, because $^t\bar{N}(dt) = \gamma + gdt$,

$$\Lambda^{\#}({}^t\bar{N}(dt)) = \exp(-t)(C^{\#}(\gamma) + gE). \tag{18}$$

Once the right members of (17) and (18) are equal, we obtain condition *iii*).

□

Let us take the Magri-Morosi concomitant $\mathcal{R}(\Lambda, \bar{N})$ of Λ and $\bar{N}$, (1).

Lemma 2.2 *With the notations of Proposition 2.2, $\mathcal{R}(\Lambda, \bar{N}) = 0$ if and only if*

i) $NE + \mathcal{L}_Y E = gE - C^{\#}(dg)$
ii) $\mathcal{L}_Y C = gC - NC + Y \otimes E$
iii) $(\mathcal{L}_E N)X + < \gamma, X > E = C^{\#}(i_X d\gamma) + (X.g)E$
iv) $\mathcal{R}(C, N)(df, X) = (X.(Y.f))E - (X.(E.f))Y - < \gamma, X > C^{\#}(df)$,

where X is any vector field on M and f is any function of $C^{\infty}(M, \mathbb{R})$.

Proof.

Let f be any function of $C^{\infty}(M, \mathbb{R})$. The component of the vector field $\mathcal{R}(\Lambda, \bar{N})(df, \frac{\partial}{\partial t})$ on $\frac{\partial}{\partial t}$ is

$$\exp(-t)(< df, C^{\#}(dg) + \mathcal{L}_Y E + NE - gE >). \tag{19}$$

The other components (without $\frac{\partial}{\partial t}$) of $\mathcal{R}(\Lambda, \bar{N})(df, \frac{\partial}{\partial t})$, obtained from the computation of $< dh, \mathcal{R}(\Lambda, \bar{N})(df, \frac{\partial}{\partial t}) >$, for any $h \in C^{\infty}(M, \mathbb{R})$, are

$$\exp(-t)(< dh, (-\mathcal{L}_Y C)^{\#}(df) + gC^{\#}(df) - N(C^{\#}(df)) + (E.f)Y >). \tag{20}$$

From (19) and (20), we conclude that $\mathcal{R}(\Lambda,\bar{N})(df,\frac{\partial}{\partial t})=0$ gives conditions $i)$ and $ii)$ of the Lemma.

On the other hand, if X is any vector field on M, the component of the vector field $\mathcal{R}(\Lambda,\bar{N})(df,X)$ on $\frac{\partial}{\partial t}$ is

$$\exp(-t)(<df,-(\mathcal{L}_E N)X-<\gamma,X>E+C^{\#}(i_X d\gamma)+(X.g)E>, \tag{21}$$

while the components without $\frac{\partial}{\partial t}$, obtained from the computation of $<dh,\mathcal{R}(\Lambda,\bar{N})(df,X)>$, for any $h\in C^{\infty}(M,\mathbb{R})$, are

$$\exp(-t)(<dh,\mathcal{R}(C,N)(df,X)+<\gamma,X>C^{\#}(df)+(X.(E.f))Y-(X.(Y.f))E>. \tag{22}$$

From (21) and (22), we conclude that $\mathcal{R}(\Lambda,\bar{N})(df,X)=0$ gives conditions $iii)$ and $iv)$ of the Lemma. □

Theorem 2.1 *Let* $\Lambda=\exp(-t)(C+\frac{\partial}{\partial t}\wedge E)$ *and* $\bar{N}=N+Y\otimes dt+\frac{\partial}{\partial t}\otimes\gamma+g\frac{\partial}{\partial t}\otimes dt$, *be respectively the homogeneous Poisson tensor and the Nijenhuis tensor on* $\mathbb{R}\times M$. *Then, the triple* $(\mathbb{R}\times M,\Lambda,\bar{N})$ *is a Poisson-Nijenhuis manifold if and only if conditions i), ii) and iii) of Lemma (2.1) and i) - iv) of Lemma (2.2) hold. Moreover, the homogeneous Poisson tensor* $\bar{N}\Lambda$ *is given by* $\exp(-t)(C_1+\frac{\partial}{\partial t}\wedge E_1)$, *where*

$$(C_1,E_1)=(gC-\mathcal{L}_Y C,gE-C^{\#}(dg)-\mathcal{L}_Y E) \tag{23}$$

is a Jacobi structure on M, *compatible with* (C,E).

Proof.
We only have to find the expressions of C_1 and E_1. For any pair (f,h) of functions on M,

$$<dh,(\bar{N}\Lambda)^{\#}(df)>=\exp(-t)<dh,NC^{\#}(df)-(Y\otimes E)(df)> \tag{24}$$

and

$$<dh,(\Lambda^{\#}({}^t\bar{N})(df)>=\exp(-t)<dh,C^{\#}({}^tN(df))+(E\otimes Y)(df)>. \tag{25}$$

Since $\bar{N}\Lambda=\frac{1}{2}(\bar{N}\Lambda+\Lambda\,{}^t\bar{N})$, we obtain from (24) and (25), using conditions $ii)$ of Lemma 2.1 and $ii)$ of Lemma 2.2,

$$\begin{aligned}(\bar{N}\Lambda)(df,dh) &= \exp(-t)(NC-Y\otimes E)(df,dh)\\ &= \exp(-t)(gC-\mathcal{L}_Y C)(df,dh).\end{aligned} \tag{26}$$

Also,

$$<dh,\bar{N}(\Lambda^{\#}(dt))>=\exp(-t)<dh,NE> \tag{27}$$

and

$$< dh, \Lambda^{\#}({}^t\bar{N}(dt)) >= \exp(-t) < dh, C^{\#}(\gamma) + gE > . \tag{28}$$

From (27) and (28), and taking account conditions i) of Lemma 2.1 and i) of Lemma 2.2, we conclude that

$$\begin{aligned} (\bar{N}\Lambda)(dt, dh) &= \exp(-t)NE(dt, dh) \\ &= \exp(-t)(gE - C^{\#}(dg) - \mathcal{L}_Y E)(dt, dh). \end{aligned} \tag{29}$$

□

This paper is in final form and no version of it will be submitted for publication elsewhere.

Acknowledgements. The author wishes to thank Professors C.-M. Marle and J. Monterde for helpful discussions.

References

[KSch-Ma 90] Kosmann-Schwarzbach Y. and Magri F., *Poisson-Nijenhuis structures*, Ann. Inst. Henri Poincaré **53**, 35 (1990).

[Le-Ma-Pa 97] León M. de, Marrero J. C. and Padrón E., *Lichnerowicz-Jacobi cohomology*, J. Phys. A: Math. Gen. **30**, 6029 (1997).

[Lich 78] Lichnerowicz A., *Les variétés de Jacobi et leurs algèbres de Lie associées*, J. Math. pures et appl. **57**, 453 (1978).

[Ma-Mo 84] Magri F. and Morosi C., *A Geometrical characterization of integrable Hamiltonian systems through the theory of Poisson-Nijenhuis manifolds*, Quaderno S19, Univ. of Milan, 1984.

[NdC 98] Nunes da Costa J. M., *Compatible Jacobi manifolds: geometry and reduction*, J. Phys. A: Math. Gen. **31**, 1025 (1998).