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A THEOREM OF FR. FABRICIUS-BJERRE ON HELICES

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Abstract: We give an alternative proof of a theorem proved in [1].

1. In what follows $f : I \to R^n$, where I is an interval with more than one point, will be a C^1 regular curve, parametrised by arc-length. We say that f is a *helix* with respect to a unit vector $\vec{a}$ if there is $c \in R$ such that, for $s \in I, (f'(s) \mid \vec{a}) = c$, where $(\ldots \mid \ldots)$ stands for the usual inner product. The unit vector $\vec{a}$ is a *reference vector* for the helix f. If f happened not to be parametrised by arc-length we would require $(\frac{f'(s)}{\|f'(s)\|} \mid \vec{a}) = c$.

Let $(\vec{a_1}, \ldots, \vec{a_n})$ be an orthonormal basis for R^n and write $f(s) = f_1(s)\vec{a_1} + \ldots + f_n(s)\vec{a_n}$, for $s \in I$. We then have the following observation.

f is a helix with respect to $\vec{a} = \vec{a_n}$ if and only if $f_n(s) = cs + d, s \in I$, where c, d are constants with $\mid c \mid \leq 1$.

It follows that if f is a non-injective helix with respect to $\vec{a} = \vec{a_n}$ then c must be zero and, consequently, the image of f lies in a hyperplane. Therefore a twisted, that is to say not contained in a plane, closed curve in R^3 is not a helix. For an interesting article on a related matter see [2].

We recall that two hyperplanes Y_1, Y_2 are *parallel* (*perpendicular*) if the underlying vector subspaces $D(Y_1), D(Y_2)$ are equal $(D(Y_1)^{\perp} \subset D(Y_2))$, where $\perp$ means orthogonal complement as usual.

Let $f : I \to R^n$ be a helix with respect to $\vec{a}$ and constant c. We will denote by $F : I \to R^n$ its orthogonal projection into a hyperplane Π normal to $\vec{a}$ and will always assume that F is regular or, equivalently, that $c \neq \pm 1$. In the present context the choice of Π is not relevant since any two such projections are related by a translation determined by a scalar multiple of $\vec{a}$. Below we let $\Pi =< \vec{a} >^{\perp}$, where $< \ldots >$ means vector subspace spanned by.

If $n = 3$ then the helices in the plane Π, with respect to unit vectors in $D(\Pi)$, are precisely those curves which have a straight line segment as image and we have

Let $f : I \to R^3$ be a helix. Then F is a helix in Π if and only if $f(I)$ is a straight line segment.

a statement which can be regarded as a motivation for Fabricius-Bjerre's result.

From now on the word helix will refer to helices with constant $c \neq 0, \pm 1$ and we will exclude the case of f having image contained in a straight line.

Theorem (Fr. Fabricius-Bjerre)**:** *Let $f : I \to R^n$, $n > 3$, be a helix with $\vec{a}$ as a reference vector and denote by $F : I \to R^n$ its orthogonal projection into a hyperplane Π normal to $\vec{a}$. Then F is a helix in Π if and only if $f(I) \subset \Pi_1$, where Π_1 is a hyperplane neither parallel nor perpendicular to Π.*

Proof: Let us assume that F is a helix with respect to $\vec{b} \in D(\Pi)$.

We consider an orthonormal basis $(\vec{a_1}, \ldots, \vec{a_{n-1}}, \vec{a_n})$ such that $\vec{a_{n-1}} = \vec{b}, \vec{a_n} = \vec{a}$, write $f(s) = f_1(s)\vec{a_1} + \ldots + f_n(s)\vec{a_n}$, for $s \in I$, and, as mentioned above, let Π be the vector subspace $< \vec{a_1}, \ldots, \vec{a_{n-1}} >$. It follows that $f(s) = F(s) + (cs + d)\ \vec{a_n}$ and $\| F'(s) \| = \sqrt{1 - c^2} = c_1$, with $0 < c_1 < 1$. The map $\lambda(t) = \frac{t}{c_1}$ is a change of parameter such that $F \circ \lambda$ is parametrised by arc-length. Since F is an helix with respect to $\vec{a_{n-1}}$ we have

$$(F \circ \lambda)(s) = f_1(\lambda(s))\ \vec{a_1} + \ldots + f_{n-2}(\lambda(s))\ \vec{a_{n-2}} + (c_2 s + d_2)\ \vec{a_{n-1}}$$

and, consequently,

$$F(s) = f_1(s)\ \vec{a_1} + \ldots + f_{n-2}(s)\ \vec{a_{n-2}} + (c_1 c_2 s + d_2)\ \vec{a_{n-1}}.$$

Therefore

$$f(s) = f_1(s)\ \vec{a_1} + \ldots + f_{n-2}(s)\ \vec{a_{n-2}} + c_1 c_2 s\ \vec{a_{n-1}} + cs\ \vec{a_n} + d_2\ \vec{a_{n-1}} + d\ \vec{a_n} =$$
$$= (T \circ \phi)(F(s)),$$

where T is the translation determined by $(d - \frac{cd_2}{c_1 c_2})\ \vec{a_n}$ and $\phi : R^n \to R^n$ is the linear map with matrix

$$\begin{bmatrix} 1 & 0 & \ldots & 0 & 0 & 0 \\ 0 & 1 & \ldots & 0 & 0 & 0 \\ \vdots & & & & & \\ 0 & 0 & \ldots & 1 & 0 & 0 \\ 0 & 0 & \ldots & 0 & 1 & 0 \\ 0 & 0 & \ldots & 0 & \frac{c}{c_1 c_2} & 0 \end{bmatrix}$$

with respect to the basis $(\vec{a_1}, \ldots, \vec{a_{n-1}}, \vec{a_n})$. Since the image of Π by $T \circ \phi$ is a hyperplane Π_1 which is neither parallel nor perpendicular to Π the conclusion follows.

Let us assume now that $f(I) \subset \Pi_1$, where Π_1 is a hyperplane neither parallel nor perpendicular to Π. Let $(\vec{a_1}, \ldots, \vec{a_n})$ be an orthonormal basis with $\vec{a_n} = \vec{a}$ and take a unit vector $\vec{u} = u_1\vec{a_1} + \ldots + u_n\vec{a_n}$ in $D(\Pi_1)^\perp$. We then have $(f'(s) \mid \vec{a_n}) = c$ and $(f'(s) \mid \vec{u}) = 0$ from where it follows $(F'(s) \mid u_1\vec{a_1} + \ldots + u_{n-1}\vec{a_{n-1}}) = -\ c\ u_n$. Since Π_1 is a hyperplane neither parallel nor perpendicular to Π we can conclude that F is a helix in Π_1 with reference vector $\vec{v} = \frac{u_1\vec{a_1} + \ldots + u_{n-1}\vec{a_{n-1}}}{\|u_1\vec{a_1} + \ldots + u_{n-1}\vec{a_{n-1}}\|}$ and constant $C = \frac{-cu_n}{\sqrt{1-c^2}\ \|u_1\vec{a_1} + \ldots + u_{n-1}\vec{a_{n-1}}\|}$.

It only remains in fact to show that $C \neq \pm 1$. If $C = \pm 1$ then $F(I)$ would be contained in a straight line with $< \vec{v} >$ as underlying vector subspace and, consequently, $f(I)$ would be contained in a plane τ such that $D(\tau) = < \vec{a}, \vec{v} >$. The plane τ cannot be contained in Π_1 since Π_1 and Π are perpendicular. Therefore $\tau \cap \Pi_1$ is a straight line which contains the image $f(I)$. This has been ruled out from the beginning. $\bowtie$

Going through the proof one sees that we can remove the restriction *the image of f is not contained in a straight line* if in the statement of the theorem we take helix to mean helix with non-zero constant.

References

[1] Fr. Fabricius-Bjerre, *On helices in the euclidean n-space*, Math. Scand. 35 (1974), 159-164.

[2] Joel L. Weiner, *How helical can a closed, twisted space curve be?*, Amer. Math. Monthly, 107 (2000), 327-333.

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