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LOCALLY SIERPINSKI QUOTIENTS

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Abstract: Given any non-trivial, connected topological space X, it is possible to define an equivalence relation $\sim$ on it such that the topological quotient space $X/\sim$ is the Sierpinski space. Locally Sierpinski spaces are generalizations of the Sierpinski space and here we address the following question. Does a statement like the one above hold if *Sierpinski* is replaced by (proper) *locally Sierpinski*? The answer is no and we will give below a few counterexamples. The situation where a homeomorphism group acts on a topological n-manifold will also be analysed, the conclusion being that the cases $n = 1, n > 1$ are radically different.

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1. Locally Sierpinski spaces

A topological space X is said to be a *locally Sierpinski space*, *l. S. space* in short, if every point $x \in X$ has an open neighbourhood U_x homeomorphic to the Sierpinski space $(\{0,1\}, \{\emptyset, \{0\}, \{0,1\}\})$.

If in an l. S. space X a unitary set $\{x\}$ is open we say that x is a *centre*. The remaining points will be called *satellites*. The centres and the open sets homeomorphic to the Sierpinski space form a base for the topology of X. Therefore X is locally (path-)connected and its (path-) connected components are open.

Moreover there is a bijection between the set of centres and the set of components of X. Such a bijection associates to each centre the component which contains it.

Examples of connected l. S. spaces are often referred to in topology textbooks [3]. They are obtained as follows. Let X be a set with, at least, two points. Fix $p \in X$ and define a set to be open if it is either the empty set or contains p.

L. S. spaces can be characterized as locally (path-)connected spaces whose (path-)components are subspaces of the same type as X.

2. Existence of quotient maps

Proposition 2.1. *Let X be a connected topological space.*

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If X can be partitioned as $X = A \cup (\bigcup_{i \in I} A_i)$, $I \neq \emptyset$, where A is open and, for each $i \in I$, A_i is closed and $A \cup A_i$ is open, then there is a quotient map $p : X \to S$, where S is an l. S. space with $\#I$ as the number of satellites.

Conversely, if there is a quotient map $p : X \to S$, where S is l. S., then X can be partitioned as $X = A \cup (\bigcup_{i \in I} A_i)$, with A open, for each $i \in I$, A_i closed, $A \cup A_i$ open and $\#I$ the number of satellites.

Proof. Assume that there is a quotient map $p : X \to S$, c is the centre of S and $x_i, i \in I$, are the satellites. Let then $A = p^{-1}(c)$ and $A_i = p^{-1}(\{c, x_i\}), i \in I$.

Assume now that $X = A \cup (\bigcup_{i \in I} A_i)$, $I \neq \emptyset$, where A is open and, for each $i \in I$, A_i is closed and $A \cup A_i$ is open.

Add an extra point c to I to obtain a new set J and make this new set into an l. S. space by taking as a topological basis the subsets $\{c\}, \{c, i\}, i \in I$. Then define $p : X \to S$ by $p(A) = c, p(A_i) = i, i \in I$. This map is obviously surjective and continuous. It is also a quotient map.

Suppose that $U \subset J$, nonempty and different from J, is such that $p^{-1}(U)$ is open. Then $c \in U$ and therefore U is open in J. If c were not in U then U would be closed and $p^{-1}(U)$ would also be closed. This contradicts the connectedness of X. $\bowtie$

By assuming that X is locally connected, we can actually formulate the first part of the proposition, but not the second, in terms of the components X_λ, $\lambda \in \Lambda$, of X. The proof would follow as above. For each $\lambda \in \Lambda$, we would have a map $p : X_\lambda \to S_\lambda$ and would use these maps to obtain $p : X \to S$, where S is the disjoint union of the S_λ's.

As to a counterexample for the converse consider the disjoint union of a Sierpinski space and a trivial two point space. That is, consider $X = \{a, b, c, d\}$, with topology $\tau_X = \{\emptyset, X, \{a, b\}, \{c\}, \{a, b, c\}, \{c, d\}\}$, and the Sierpinski space $\{c, d\}$, with $\{c\}$ open. The map $p : X \to S$ given by $p(\{a, b, d\}) = \{d\}, p(c) = c$ is a quotient map.

3. Examples

1- Let X be a T_1 connected space and r be an integer such that $1 \leq r < \#X$. Then a connected l. S. space with r satellites is obtainable from

X. One fixes r points in X, one takes for A their complement and each A_i is formed by just one of the fixed points.

2- Let X be a set with at least 2 points and let us fix $p \in X$. Define $U \subset X$ to be open if $U = X$ or $p \notin U$. This space is connected and the only l. S. space which is obtainable from it is the Sierpinski space itself.

3- Let X now be the real numbers and define U to be open if it is empty or an open interval of the form $(-\infty, a)$ or $(-\infty, +\infty)$. Again only the Sierpinski space is obtainable.

4. S^1 and other spheres

As mentioned earlier any finite, connected l. S. space can be obtained from the circle S^1. However if the quotient space is a space of orbits originated by the action of a subgroup G of the homeomorphism group of S^1 the possibilities are severely restricted. For R this problem was solved in [1]. From now on we will assume that a group of homeomorphisms G acts on S^1.

Proposition 4.1. *No free action on S^1 gives rise to an l. S. space.*

Proof. Assume that S^1/G is l. S., $p : X \to S^1/G$ is the projection and c is the centre of S^1/G.

If $p^{-1}(c)$ is S^1 with one point removed then that point would be a fixed point for every element of G.

Suppose then that $p^{-1}(c)$ is not S^1 with one point removed. Take a component $\widehat{x_0 y_0}$ of $p^{-1}(c)$ and choose distinct x_1, y_1 in it. There is $f \in G$ which maps x_1 to y_1. Since f maps the closed arc $\widehat{x_0 y_0}$ to itself, it must have a fixed point. ⋈

Proposition 4.2. *If S^1/G is an l. S. space then it has one or two satellites. Both cases can occur.*

Proof. Let c be the centre of S^1/G, p the projection which, since we are dealing with a group action, is open, and $p^{-1}(c) = \bigcup_{i \in I} C_i$, where the C_i's are the connected components. We will show that the inverse image of the set of satellites is precisely the set of end points of the open arcs C_i. Obviously the end points of the C_i's project to satellites.

If $x \in p^{-1}(s)$, where s is a satellite, consider the open set $\{c, s\}$ and its inverse image. Let C denote the connected component in that inverse image which contains x.

If no point other than x lies in $C \cap p^{-1}(s)$, x is an end point of one of the C_i's. On the other hand, if there is another x' in $C \cap p^{-1}(s)$ then there must exist a component C_i such that $C_i \subset \overset{\frown}{xx'}$. The end points of C_i are in the orbit of x and, consequently, x must be an end point of a component C_j.

If the inverse image of the set of satellites is a singleton then S^1/G is the Sierpinski space. If not the C_i's are arcs with distinct end points. These end points give rise to one or two orbits and therefore the quotient is the Sierpinski space or has two satellites.

The Sierpinski space can be obtained by letting G be the group of S^1 homeomorphisms which fix, for instance, the north pole or, alternatively, fix the subset formed by the north and south poles. An l. S. space with two satellites can be obtained by letting G be the group of S^1 homeomorphisms which fix, for instance, the north pole and the south pole. $\bowtie$

Combining the corresponding result for R [1] and the next lemma gives us another way of recovering Proposition 4.2.

Lemma 4.3. *Let X be a connected, locally path-connected, semilocally simply connected topological space and let $(\widetilde{X}, \pi)$ be its universal covering. Assume there is a homeomorphism group G such that X/G is l. S. Then there is also a group $\widetilde{G}$ such that $\widetilde{X}/\widetilde{G}$ and X/G are homeomorphic.*

Proof. Let $\widetilde{G} = \{\tilde{f} : \widetilde{X} \to \widetilde{X} \mid \pi \circ \tilde{f} = f \circ \pi, \text{for some } f \in G\}$ and let it act on $\widetilde{X}$. Then $x_1, x_2 \in \widetilde{X}$ are in the same orbit iff the same happens to $\pi(x_1), \pi(x_2)$.

Consider now the projection $p : X \to X/G$ and compose it with π. This composition is a quotient map which induces exactly the same equivalence relation on $\widetilde{X}$ as the $\widetilde{G}$-action. Therefore the spaces $\widetilde{X}/\widetilde{G}$ and X/G are homeomorphic. $\bowtie$

Let now X be a topological space and G be a homeomorphism group. If Y is a homogeneous topological space and H is its homeomorphism group then the quotient spaces X/G and $X \times Y/G \times H$ are homeomorphic. Hence, for instance, the torus $S^1 \times S^1$ can be acted upon by a group of homeomorphisms such that the quotient is an l. S. space with one or two satellites.

In the case of $S^n, n \geq 2$, we have

Proposition 4.4. *Given a positive integer r, there is a group G of S^n homeomorphisms, $n \geq 2$, such that S^n/G is an l. S. space with r satellites.*

Proof. The result follows from the following facts:

S^n is the 1-point compactification of R^n and any homeomorphism of R^n extends naturally to a homeomorphism of S^n.

It is known [2] that given two n-sequences $(x_1, \ldots, x_r)$, $(y_1, \ldots, y_r)$ of distinct points in $R^n, n \geq 2$, there is a homeomorphism $f : R^n \to R^n$ such that $f(x_i) = y_i, i = 1, \ldots, r$.

We can therefore take the group G formed by the S^n homeomorphisms which fix r points. $\bowtie$

5. Topological n-manifolds

In view of the results of the previous section it is only natural to ask what sort of l. S. spaces can be obtained if one starts with a topological n-manifold, $n \geq 2$.

We start with an independent proof of a particular case of Proposition 5 in [2]. In what follows each time we refer to a chart for an n-manifold it will be understood that the codomain is R^n.

Proposition 5.1. *Let F be a finite subset of $R^n, n \geq 2$, and let $x, y \in R^n \setminus F$. Then there is a homeomorphism $h : R^n \to R^n$ which is pointwise fixed on F, is the identity outside a compact set and such that $h(x) = y$.*

Proof. We will write $\overline{xy}$ for the line segment determined by x and y.

Case 1: $\overline{xy} \cap F = \emptyset$

There is no loss of generality in supposing $x = (\underline{0}, -1), y = (\underline{0}, 1) \in R^{n-1} \times R$. Then there exist $r, s > 0$ such that $(D^{n-1}(r) \times [-1-s, 1+s]) \cap F = \emptyset$, where $D^{n-1}(r) = \{p \in R^{n-1} \mid \parallel p \parallel \leq r\}$.

Let $f : [-1-s, 1+s] \to [-1-s, 1+s]$ be a homeomorphism which fixes the end-points and maps -1 to 1 and define $h : D^{n-1}(r) \times [-1-s, 1+s] \to D^{n-1}(r) \times [-1-s, 1+s]$ by $h(p,t) = (p, (1 - \frac{\|p\|}{r})f(t) + \frac{\|p\|}{r}t)$. This h is a continuous bijection and, due to compactness and Hausdorffness, it is a homeomorphism. On the boundary of $D^{n-1}(r) \times [-1-s, 1+s]$ the homeomorphism h is the identity and we get the required homeomorphism for R^n if we extend it by taking it to be the identity outside $D^{n-1}(r) \times [-1-s, 1+s]$.

Case 2: $\overline{xy} \cap F \neq \emptyset$

We can find $z \in R^n$ such that $\overline{xz} \cap F = \emptyset$ and $\overline{zy} \cap F = \emptyset$. By case 1, there are R^n-homeomorphisms h_1 and h_2, both of which are pointwise fixed on F, are the identity outside compact subsets K_1, K_2, respectively, and such that $h_1(x) = z$, $h_2(z) = y$. If we take $h = h_2 \circ h_1$ we have a homeomorphism as required. $\bowtie$

Then we get

Corollary 5.2. *Let F be a finite subset of a topological n-manifold, $n \geq 2$, and let x, y belong to the same chart domain. Then there exists a homeomorphism h of M which is pointwise fixed on F and such that $h(x) = y$.*

Proof. Let $\phi : U \to R^n$ be a chart with $x, y \in U$. Use it and Proposition 5.1 to obtain a homeomorphism $h : U \to U$ which maps x to y, is pointwise fixed on $U \cap F$ and is the identity outside a compact subset of U. Then extend h to M by defining it to be the identity outside U. $\bowtie$

It is now easy to show that n-dimensional manifolds behave quite differently from 1-dimensional ones as it was to be expected from previous results in this paper and [1].

Proposition 5.3. *Let M be a connected, topological n-manifold, $n \geq 2$. For any positive integer r, there is a homeomorphism group G such that M/G is an l. S. space with r satellites.*

Proof. Let F be a subset of M with r elements and let G be the group of homeomorphisms of M which fix each point of F. We need to show that, for $x, y \in M \setminus F$, there is $h \in G$ such that it maps x to y. It will then follow that M/G is an l. S. space with r satellites. We can find a sequence $x = x_1, x_2, \ldots, x_{n-1}, x_n = y$ in M such that, for $i = 1, \ldots, n-1$, x_i, x_{i+1} are both in a chart domain [3]. By the previous corollary, there are homeomorphisms $h_i \in G$ such that $h_i(x_i) = x_{i+1}$. Then $h = h_{n-1} \circ \ldots \circ h_2 \circ h_1$ is the required homeomorphism. $\bowtie$

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