Pré-Publicações do Departamento de Matemática
Universidade de Coimbra
Preprint Number 03–11

FUNCTORIAL QUASI-UNIFORMITIES ON FRAMES

MARIA JOÃO FERREIRA AND JORGE PICADO

ABSTRACT: In this paper we present a unified study of functorial frame quasi-uniformities by means of Weil entourages and frame congruences. In particular, we use the pointfree version of the Fletcher construction, introduced by the authors in a previous paper, to describe all functorial transitive quasi-uniformities.

KEYWORDS: frame, biframe, strictly zero-dimensional biframe, Skula biframe, Skula functor, Weil entourage, quasi-uniform frame, functorial quasi-uniformity, T-section.
AMS SUBJECT CLASSIFICATION (2000): 06D22, 54B30, 54E05, 54E15, 54E55.

1. Introduction

The method of constructing compatible quasi-uniformities for an arbitrary frame, introduced in [8], naturally raises the question of its functoriality. The purpose of the present paper is to address this question, together with a unified treatment of functorial quasi-uniformities on frames.

To put this in perspective, we recall that a topological space $(X, \mathcal{T})$ is *uniformizable* if there exists a uniformity $\mathcal{E}$ on X such that the corresponding induced topology $\mathcal{T}(\mathcal{E})$ coincides with the given topology $\mathcal{T}$. As it is well-known, the topological spaces that are uniformizable are precisely the completely regular ones. This result has a perfect analog in the two-sided theory of quasi-uniform spaces (where they are considered over their induced bitopologies): a bitopological space $(X, \mathcal{T}_1, \mathcal{T}_2)$ is *quasi-uniformizable*, i.e. there exists a quasi-uniformity $\mathcal{E}$ on X such that $\mathcal{T}(\mathcal{E}) = \mathcal{T}_1$ and $\mathcal{T}(\mathcal{E}^{-1}) = \mathcal{T}_2$, if and only if it is pairwise completely regular. However, in the one-sided theory, where a quasi-uniformity is considered over a single underlying topology, the resemblance with the symmetric case is over and one gets a striking result: every topological space is quasi-uniformizable, that is, every topological space $(X, \mathcal{T})$ gives rise to a (transitive) quasi-uniformity $\mathcal{E}_P(\mathcal{T})$ on X which generates as one of its topologies the given topology $\mathcal{T}$. This result was

The authors acknowledge partial financial assistance by the *Centro de Matemática da Universidade de Coimbra/FCT*. The second author also acknowledges support from the *FCT* through grant FCT POCTI/1999/MAT/33018 “Quantales”.

firstly proved by Pervin [14] and $\mathcal{E}_P(\mathcal{T})$ is nowadays called the *(Császár-)Pervin quasi-uniformity*. So, every topological space $(X, \mathcal{T})$ gives rise to a bitopological space $(X, \mathcal{T}(\mathcal{E}_P(\mathcal{T})), \mathcal{T}(\mathcal{E}_P(\mathcal{T})^{-1}))$, where $\mathcal{T}(\mathcal{E}_P(\mathcal{T})) = \mathcal{T}$. The join $\mathcal{T}(\mathcal{E}_P(\mathcal{T})) \vee \mathcal{T}(\mathcal{E}_P(\mathcal{T})^{-1})$ of the two topologies is called the *Skula topology* and the above bitopological space is referred to as the *Skula bitopological space.*

Let T denote the forgetful functor from the category QUnif of quasi-uniform spaces and uniformly continuous maps to the category Top of topological spaces and continuous maps which assigns to each $(X, \mathcal{E}) \in \mathsf{QUnif}$ its first topology $\mathcal{T}(\mathcal{E})$. A *functorial quasi-uniformity* [4] on the topological spaces is a *T-section*, that is, a functor $F : \mathsf{Top} \to \mathsf{QUnif}$ such that $TF = 1_{\mathsf{Top}}$. In other words, F assigns a compatible quasi-uniformity to each topological space in such a way that continuous maps become uniformly continuous.

In [4], Brümmer proved that the Pervin quasi-uniformity defines the coarsest T-section $\mathcal{C}_1^* : \mathsf{Top} \to \mathsf{QUnif}$. In [18], Salbany proved that, for any T-section F, the join of the two topologies generated by the quasi-uniformity of $F(X, \mathcal{T})$ is precisely the Skula topology $\mathcal{T}(\mathcal{E}_P(\mathcal{T})) \vee \mathcal{T}(\mathcal{E}_P(\mathcal{T})^{-1})$.

Transitive quasi-uniform spaces form an important subcategory of QUnif and they play a role almost as general as that of quasi-uniform spaces in the study of topological properties. The most striking aspect of transitive functorial quasi-uniformities, as Brümmer proved in [6], is that they can all be obtained by a construction due to Fletcher [9], considering the interior-preserving open covers of their associated topological spaces.

The present paper is devoted to placing these results in a pointfree context. It is part of a larger program started in [8], motivated by Problem 3 of Brümmer [7], asking for a pointfree formulation of the classical theory of functorial transitive quasi-uniformities. After recalling some basics on frames and quasi-uniform frames (Section 2), we study general functorial frame quasi-uniformities (Section 3). In the remaining sections we apply the general method of constructing compatible transitive quasi-uniformities on an arbitrary frame, introduced in [8], to describe all functorial transitive quasi-uniformities.

2. Preliminaries

2.1. Frames and biframes. Pointfree topology is part of the study of *frames* (or *locales*), that is, complete lattices L satisfying the infinite distributive law

$$x \wedge \bigvee S = \bigvee \{x \wedge s \mid s \in S\}$$

for every $x \in L$ and every $S \subseteq L$. This notion generalizes both the lattice of open sets of a topological space and that of a Boolean algebra. A *frame homomorphism* $f : L \to M$ is a map between frames which preserves finite meets (including the top element 1) and arbitrary joins (including the bottom element 0). The corresponding category will be denoted by Frm. If L is a frame and $x \in L$ then

$$x^* := \bigvee \{a \in L \mid a \wedge x = 0\}$$

is the *pseudocomplement* of x. Obviously, if $x \vee x^* = 1$, x is complemented and we denote the complement x^* by $\neg x$. Note that, in any frame, the first De Morgan law

$$(\bigvee_{i \in I} x_i)^* = \bigwedge_{i \in I} x_i^*$$

holds but for infima we have only the trivial inequality

$$\bigvee_{i \in I} x_i^* \leq (\bigwedge_{i \in I} x_i)^*.$$

Recall also that a *biframe* is a triple (L_0, L_1, L_2) where L_1 and L_2 are subframes of the frame L_0, which together generate L_0. A *biframe homomorphism,* $f : (L_0, L_1, L_2) \longrightarrow (M_0, M_1, M_2)$, is a frame homomorphism $f : L_0 \longrightarrow M_0$ which maps L_i into $M_i\,(i = 1, 2)$ and BiFrm denotes the resulting category.

Further, a biframe (L_0, L_1, L_2) is *strictly zero-dimensional* [1] if it satisfies the following condition or its counterpart with L_1 and L_2 reversed: each $x \in L_1$ is complemented in L_0, with complement in L_2, and L_2 is generated by these complements. Along this paper, we always assume that strictly zero-dimensional biframes satisfy this condition, not its counterpart with L_1 and L_2 reversed.

For general facts concerning frames we refer to Johnstone [12] or Vickers [19]. Additional information concerning biframes may be found in [1] and [3].

2.2. The Skula biframe $Sk(L)$ of a frame L. The lattice of frame congruences on L under set inclusion is a frame, denoted by $\mathfrak{C}L$. A good presentation of the congruence frame is given by Frith [11]. Here, we shall need the following properties:

(1) For any $x \in L$, ∇_x and Δ_x are, respectively, the congruences defined by $\{(a,b) \in L \times L \mid a \vee x = b \vee x\}$ and $\{(a,b) \in L \times L \mid a \wedge x = b \wedge x\}$.
(2) Each ∇_x is complemented in $\mathfrak{C}L$ with complement Δ_x.
(3) $\nabla L = \{\nabla_x \mid x \in L\}$ is a subframe of $\mathfrak{C}L$. Let ΔL denote the subframe of $\mathfrak{C}L$ generated by $\{\Delta_x \mid x \in L\}$. Since $\theta = \bigvee\{\nabla_y \wedge \Delta_x \mid (x,y) \in \theta, x \leq y\}$, for every $\theta \in \mathfrak{C}L$, the triple $(\mathfrak{C}L, \nabla L, \Delta L)$ is a biframe (usually referred to as the *Skula biframe* of L [11]). This is the analogue, for frames, of the Skula bitopological space and it is, clearly, a strictly zero-dimensional biframe.
(4) The correspondence $x \mapsto \nabla_x$ defines an epimorphism and a monomorphism $\nabla_L : L \to \mathfrak{C}L$ and gives an isomorphism $L \to \nabla L$, whereas the map $x \mapsto \Delta_x$ is a dual poset embedding $L \to \Delta L$ taking finitary meets to finitary joins and arbitrary joins to arbitrary meets.

The following result from [11] will be helpful in the sequel.

Lemma. (J. Frith [11]) *Let $h : L \to M$ be a frame homomorphism. If each element of $h[L]$ is complemented then there exists a unique frame homomorphism $\overline{h}$ such that the diagram*

$$\begin{tikzcd} L \arrow[r, "\nabla_L"] \arrow[dr, "h"'] & \mathfrak{C}L \arrow[d, dashed, "\overline{h}"] \\ & M \end{tikzcd}$$

commutes.

Proof: Clearly, if there exists such an $\overline{h}$, we must have

$$\overline{h}(\nabla_x) = h(x) \tag{2.2.1}$$

$$\overline{h}(\Delta_x) = \overline{h}(\neg\nabla_x) = \neg h(x). \tag{2.2.2}$$

Then, for any $\theta \in \mathfrak{C}L$,

$$\begin{aligned} \overline{h}(\theta) &= \overline{h}(\bigvee\{\Delta_x \wedge \nabla_y \mid (x,y) \in \theta, x \leq y\}) \\ &= \bigvee\{\neg h(x) \wedge h(y) \mid (x,y) \in \theta, x \leq y\}. \end{aligned}$$

This defines a frame homomorphism $\overline{h} : \mathfrak{C}L \to M$ (for a proof see [11], Theorem 5.17). The uniqueness follows from the fact that ∇_L is an epimorphism. ■

For any frame homomorphism $h : L \to M$, consider the map $\overline{h} := \overline{\nabla_M \cdot h}$

$$\begin{tikzcd} L \arrow[r, "\nabla_L"] \arrow[d, "h"'] & \mathfrak{C}L \arrow[d, dashed, "\overline{h}"] \\ M \arrow[r, "\nabla_M"'] & \mathfrak{C}M \end{tikzcd} \tag{2.2.3}$$

given by the Lemma. Clearly, by (2.2.1) and (2.2.2), $\overline{h}$ is a biframe map $Sk(L) \to Sk(M)$. We refer to the functor

$$\begin{array}{rcc} Sk : \mathsf{Frm} & \longrightarrow & \mathsf{BiFrm} \\ L & \longmapsto & Sk(L) \\ (h : L \to M) & \longmapsto & (\overline{h} : Sk(L) \to Sk(M)) \end{array}$$

as the *Skula functor*.

2.3. Weil entourages. For a frame L consider the frame $\mathcal{D}(L \times L)$ of all non-void decreasing subsets of $L \times L$, ordered by inclusion. The coproduct $L \oplus L$ will be represented as usual (cf. [12]), as the subset of $\mathcal{D}(L \times L)$ consisting of all C-*ideals*, that is, of sets A for which

$$\{x\} \times S \subseteq A \Rightarrow (x, \bigvee S) \in A$$

and

$$S \times \{y\} \subseteq A \Rightarrow (\bigvee S, y) \in A.$$

Since the premise is trivially satisfied if $S = \emptyset$, each C-ideal A contains $\mathbf{O} := \{(0,a),(a,0) \mid a \in L\}$, and $\mathbf{O}$ is the bottom element of $L \oplus L$. Obviously, each $x \oplus y = \downarrow (x,y) \cup \mathbf{O}$ is a C-ideal. The coproduct injections $u_i^L : L \to L \oplus L$ are defined by $u_1^L(x) = x \oplus 1$ and $u_2^L(x) = 1 \oplus x$ so that $x \oplus y = u_1^L(x) \wedge u_2^L(y)$.

For any frame homomorphism $h : L \longrightarrow M$, the definition of coproduct ensures us the existence (and uniqueness) of a frame homomorphism $h \oplus h : L \oplus L \longrightarrow M \oplus M$ such that $(h \oplus h) \cdot u_i^L = u_i^M \cdot h$ $(i = 1, 2)$.

A *Weil entourage* [15] on L is just an element E of $L \oplus L$ for which $\bigvee\{x \in L \mid (x,x) \in E\} = 1$. The collection $WEnt(L)$ of all Weil entourages of L with the inclusion is a partially ordered set with finitary meets (including a unit $1 = L \oplus L$).

If E and F are elements of $L \oplus L$ then

$$E \circ F := \bigvee \{x \oplus y \mid \exists z \in L \setminus \{0\} : (x, z) \in E,\ (z, y) \in F\}.$$

A Weil entourage E is called *transitive* if $E \circ E = E$.

2.4. Quasi-uniform frames. Let $\mathcal{E} \subseteq L \oplus L$ and $x, y \in L$. If

$$E \circ (x \oplus x) \subseteq y \oplus y \text{ for some } E \in \mathcal{E},$$

we write $x \overset{\mathcal{E}}{\triangleleft}_1 y$. Similarly, we define

$$x \overset{\mathcal{E}}{\triangleleft}_2 y \equiv (x \oplus x) \circ E \subseteq y \oplus y, \text{ for some } E \in \mathcal{E}.$$

A filter $\mathcal{E}$ of $WEnt(L)$ is a *quasi-uniformity* on the frame L if it satisfies the following conditions:

(QU1) For every $E \in \mathcal{E}$ there exists $F \in \mathcal{E}$ such that $F \circ F \subseteq E$.

(QU2) For every $x \in L$, $x = \bigvee \{y \in L \mid y \overset{\overline{\mathcal{E}}}{\triangleleft}_1 x\}$, where $\overline{\mathcal{E}} := \mathcal{E} \cup \mathcal{E}^{-1}$.

Note that, since $\overline{\mathcal{E}}$ is a symmetric filter, the partial orders $\overset{\overline{\mathcal{E}}}{\triangleleft}_1$ and $\overset{\overline{\mathcal{E}}}{\triangleleft}_2$ do coincide.

A *quasi-uniform frame* is just a pair $(L, \mathcal{E})$ where L is a frame and $\mathcal{E}$ is a quasi-uniformity on L. If $(L, \mathcal{E}_1)$ and $(M, \mathcal{E}_2)$ are quasi-uniform frames, $f : (L, \mathcal{E}_1) \to (M, \mathcal{E}_2)$ is a *uniform homomorphism* if $f : L \to M$ is a frame homomorphism such that $(f \oplus f)(E) \in \mathcal{E}_2$, for all $E \in \mathcal{E}_1$. The resulting category is denoted by QUFrm.

A quasi-uniform frame $(L, \mathcal{E})$ is called *transitive* if $\mathcal{E}$ has a base consisting of transitive entourages. For more information on transitive quasi-uniformities we refer to [13].

We note further that the partial orders $\overset{\overline{\mathcal{E}}}{\triangleleft}_1$ and $\overset{\overline{\mathcal{E}}}{\triangleleft}_2$ induce the following important subframes of L:

$$\mathcal{L}_1(\mathcal{E}) := \left\{x \in L \mid x = \bigvee \{y \in L \mid y \overset{\mathcal{E}}{\triangleleft}_1 x\}\right\}$$

$$\mathcal{L}_2(\mathcal{E}) := \left\{x \in L \mid x = \bigvee \{y \in L \mid y \overset{\mathcal{E}}{\triangleleft}_2 x\}\right\}.$$

It is worth pointing that the *admissibility condition* (QU2) is equivalent to saying that the triple $(L, \mathcal{L}_1(\mathcal{E}), \mathcal{L}_2(\mathcal{E}))$ is a biframe [16]. This is the pointfree expression of the classical fact that each quasi-uniform space $(X, \mathcal{E})$ induces a bitopological structure $(\mathcal{T}_1(\mathcal{E}), \mathcal{T}_2(\mathcal{E})) = (\mathcal{T}(\mathcal{E}), \mathcal{T}(\mathcal{E}^{-1}))$ on X.

We also note that $\overset{\mathcal{E}}{\triangleleft}_1$ and $\overset{\mathcal{E}}{\triangleleft}_2$ may be characterized in the following way [17]:

- $x \overset{\mathcal{E}}{\triangleleft}_1 y$ if and only if there exists $E \in \mathcal{E}$ such that

$$st_1(x,E) := \bigvee\{\alpha \in L \mid (\alpha,\beta) \in E, \beta \wedge x \neq 0\} \leq y; \tag{2.4.1}$$

- $x \overset{\mathcal{E}}{\triangleleft}_2 y$ if and only if there exists $E \in \mathcal{E}$ such that

$$st_2(x,E) := \bigvee\{\beta \in L \mid (\alpha,\beta) \in E, \alpha \wedge x \neq 0\} \leq y. \tag{2.4.2}$$

The elements $st_i(x,E)$, $i = 1,2$, satisfy the following properties, for every $x,y \in L$ [15]:

(S1) $x \leq y \Rightarrow st_i(x,E) \leq st_i(y,E)$, for every $E \in L \oplus L$;
(S2) For every Weil entourage E, $x \leq st_1(x,E) \wedge st_2(x,E)$;
(S3) For every $E, F \in L \oplus L$, $st_i(x, E \cap F) \leq st_i(x,E) \wedge st_i(x,F)$;
(S4) For every $E, F \in L \oplus L$,

$$st_1(st_1(x,E),F) \leq st_1(x, F \circ E)$$

and

$$st_2(st_2(x,E),F) \leq st_2(x, E \circ F);$$

(S5) For every quasi-uniformity $\mathcal{E}$, $st_i(x,E) \leq y$ for some $E \in \mathcal{E}$ implies the existence of $z \in \mathcal{L}_j(\mathcal{E})$, $j \neq i$, such that $z \wedge x = 0$ and $z \vee y = 1$;
(S6) For every $E \in L \oplus L$, $st_i(\bigvee_J x_j, E) = \bigvee_J st_i(x_j, E)$;
(S7) For every $E \in L \oplus L$ and every frame homomorphism $h : L \to M$,

$$st_i(h(x), (h \oplus h)(E)) \leq h(st_i(x,E)).$$

3. Functorial compatible quasi-uniformities

3.1. The forgetful functor $T : \mathsf{QUFrm} \to \mathsf{Frm}$. For each quasi-uniform frame $(L,\mathcal{E})$ consider the first part $\mathcal{L}_1(\mathcal{E})$ of the biframe $(L, \mathcal{L}_1(\mathcal{E}), \mathcal{L}_2(\mathcal{E}))$ associated to $(L,\mathcal{E})$. This correspondence defines a forgetful functor $T : \mathsf{QUFrm} \longrightarrow \mathsf{Frm}$. Indeed, for any uniform homomorphism $h : (L,\mathcal{E}_1) \to (M,\mathcal{E}_2)$, h maps $\mathcal{L}_1(\mathcal{E}_1)$ into $\mathcal{L}_1(\mathcal{E}_2)$: for any $x \in \mathcal{L}_1(\mathcal{E}_1)$, $x = \bigvee\{y \in L \mid y \overset{\mathcal{E}_1}{\triangleleft}_1 x\}$, so $h(x) = \bigvee\{h(y) \mid y \overset{\mathcal{E}_1}{\triangleleft}_1 x\}$; but, by property (S7), $h(y) \overset{\mathcal{E}_2}{\triangleleft}_i h(x)$ whenever $y \overset{\mathcal{E}_1}{\triangleleft}_i x$ $(i = 1,2)$, thus

$$\begin{aligned} h(x) &= \bigvee\{h(y) \mid y \overset{\mathcal{E}_1}{\triangleleft}_1 x\} \\ &\leq \bigvee\{h(y) \mid h(y) \overset{\mathcal{E}_2}{\triangleleft}_1 h(x)\} \end{aligned}$$

$$\leq \bigvee\{z \in M \mid z \overset{\mathcal{E}_2}{\triangleleft_1} h(x)\} \leq h(x)$$

and, consequently, $h(x) \in \mathcal{L}_1(\mathcal{E}_2)$.

Note that, similarly, h maps $\mathcal{L}_2(\mathcal{E}_1)$ into $\mathcal{L}_2(\mathcal{E}_2)$ and thus h is even a biframe map from $(L, \mathcal{L}_1(\mathcal{E}_1), \mathcal{L}_2(\mathcal{E}_1))$ into $(M, \mathcal{L}_1(\mathcal{E}_2), \mathcal{L}_2(\mathcal{E}_2))$.

3.2. The Frith quasi-uniformity. Let (L_0, L_1, L_2) be a strictly zero-dimensional biframe. For any $a \in L_1$ let

$$E_a = (a \oplus 1) \vee (1 \oplus \neg a).$$

This is obviously a transitive Weil entourage of L_0. It is also worth pointing that, since $(a \oplus 1) \cup (1 \oplus a)$ is already a C-ideal, E_a is simply $(a \oplus 1) \cup (1 \oplus \neg a)$. The following result, which is a particular case of Theorem 5.5 of [13], is of central importance in the sequel.

Theorem. (Hunsaker and Picado [13]) *For any strictly zero-dimensional biframe (L_0, L_1, L_2), the family $\mathcal{S} = \{E_a \mid a \in L_1\}$ is a subbase for a transitive, totally bounded, quasi-uniformity $\mathcal{F}$ on L_0, for which $\mathcal{L}_i(\mathcal{F}) = L_i$ $(i = 1, 2)$.* $\square$

The quasi-uniformity $\mathcal{F}$ is called the *Frith quasi-uniformity* on L_0.

3.3. The functor $\mathcal{C}_1^* : \mathsf{Frm} \to \mathsf{QUFrm}$. Following [8], we say that a quasi-uniformity $\mathcal{E}$ on $\mathfrak{C}L$ is *compatible with* L whenever $\mathcal{L}_1(\mathcal{E}) = \nabla L \cong L$. More generally, we say that a quasi-uniformity $\mathcal{E}$ on a frame M is *compatible with* L if $\mathcal{L}_1(\mathcal{E}) \cong L$. For any frame L, the Skula biframe $Sk(L)$ is clearly strictly zero-dimensional. Therefore, by Theorem 3.2, $\{E_{\nabla_a} \mid a \in L\}$ is a subbase for a transitive, totally bounded, quasi-uniformity $\mathcal{F}_{\mathfrak{C}L}$ on $\mathfrak{C}L$, compatible with L.

Remark. Note that this is the pointfree counterpart of the Pervin quasi-uniformity: starting with a frame L we have a quasi-uniformity on $\mathfrak{C}L$ which generates, as its first subframe, an isomorphic copy of the given frame L.

Let us show that the correspondence $L \longmapsto (\mathfrak{C}L, \mathcal{F}_{\mathfrak{C}L})$ defines a functor $\mathcal{C}_1^* : \mathsf{Frm} \longrightarrow \mathsf{QUFrm}$. For any frame homomorphism $h : L \to M$, take the map $\overline{h}$ given by (2.2.3). It suffices to check that

$$\overline{h} : (\mathfrak{C}L, \mathcal{F}_{\mathfrak{C}L}) \to (\mathfrak{C}M, \mathcal{F}_{\mathfrak{C}M})$$

is a uniform homomorphism, which is easy:

$$\begin{aligned}(\overline{h} \oplus \overline{h})(E_{\nabla_a}) &= (\overline{h} \oplus \overline{h})(\nabla_a \oplus 1) \vee (\overline{h} \oplus \overline{h})(1 \oplus \Delta_a) \\ &= (\overline{h}(\nabla_a) \oplus \overline{h}(1)) \vee (\overline{h}(1) \oplus \overline{h}(\Delta_a)) \\ &= (\nabla_{h(a)} \oplus 1) \vee (1 \oplus \Delta_{h(a)}) \in \mathcal{F}_{\mathfrak{C}M}.\end{aligned}$$

In conclusion,

$$\begin{array}{rccc} \mathcal{C}_1^* : & \mathsf{Frm} & \longrightarrow & \mathsf{QUFrm} \\ & L & \longmapsto & (\mathfrak{C}L, \mathcal{F}_{\mathfrak{C}L}) \\ & (h : L \to M) & \longmapsto & (\overline{h} : (\mathfrak{C}L, \mathcal{F}_{\mathfrak{C}L}) \to (\mathfrak{C}M, \mathcal{F}_{\mathfrak{C}M})) \end{array}$$

is a functor such that $T\mathcal{C}_1^*(L) = \mathcal{L}_1(\mathcal{F}_{\mathfrak{C}L}) = \nabla L \cong L$, that is, $T\mathcal{C}_1^* \cong 1_{\mathsf{Frm}}$. This suggests the following definition.

3.4. T-sections. We say that a functor $F : \mathsf{Frm} \longrightarrow \mathsf{QUFrm}$ is a *section* of T (briefly, *T-section*) if $TF \cong 1_{\mathsf{Frm}}$, that is, if there is a natural isomorphism $i_F : 1_{\mathsf{Frm}} \Rightarrow TF$. In other words, T-sections correspond exactly to quasi-uniformities on frames which are functorial in the sense that any frame homomorphism $L \to M$ is uniform relative to the quasi-uniformities assigned to L and M respectively.

If F and G are T-sections, we say that F is *coarser* than G, written $F \leq G$, if there is a natural transformation $\overline{i} : F \Rightarrow G$ such that $T(\overline{i}_L) \cdot i_{F(L)} = i_{G(L)}$ for every frame L. This is a reflexive and transitive relation, that is, a preorder, and so it can be made a partial order in the standard way.

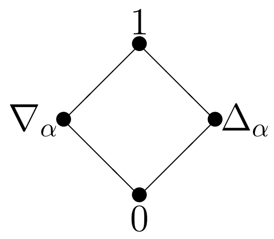
A T-section F is *transitive* if $F(L)$ is a transitive quasi-uniform frame for every frame L.

3.5. T-sections induce strictly zero-dimensional biframes. Let F be a T-section and let $F(L) = (F_0(L), \mathcal{E}_{F(L)})$ for each frame L. We denote by

$$\mathcal{B}_F(L) = (F_0(L), \mathcal{L}_1(\mathcal{E}_{F(L)}), \mathcal{L}_2(\mathcal{E}_{F(L)}))$$

the biframe associated to the quasi-uniform frame $F(L)$ and by $\mathcal{B}_F$ the corresponding functor $\mathsf{Frm} \to \mathsf{BiFrm}$.

Let $\mathsf{3}$ denote the three-element frame $\{0 < \alpha < 1\}$. It is clear that $\mathfrak{C}\mathsf{3}$ is just the Boolean algebra with four elements



It is also an easy exercise to conclude that $\mathfrak{C}\mathsf{3}$ has a unique quasi-uniform structure, generated by the entourage E_{∇_α}. We refer to it as the *Sierpiński quasi-uniform frame.*

Lemma. *For each T-section F, $\mathcal{B}_F(\mathsf{3}) \cong Sk(\mathsf{3})$.*

Proof: Clearly $\mathcal{L}_1(\mathcal{E}_{F(\mathsf{3})}) \cong \mathsf{3} \cong \nabla\mathsf{3}$. Let $x = i_{F(\mathsf{3})}(\alpha)$ denote the non-trivial element of the frame $\mathcal{L}_1(\mathcal{E}_{F(\mathsf{3})})$. Since

$$x = \bigvee\{y \in \mathcal{L}_1(\mathcal{E}_{F(\mathsf{3})}) \mid y \overset{\mathcal{E}_{F(L)}}{\lhd_1} x\}$$

and $\mathcal{L}_1(\mathcal{E}_{F(\mathsf{3})}) \cong \mathsf{3}$, then $x \overset{\mathcal{E}_{F(\mathsf{3})}}{\lhd_1} x$. By (S5), this means that there is some $b \in \mathcal{L}_2(\mathcal{E}_{F(\mathsf{3})})$ such that $b \wedge x = 0$ and $b \vee x = 1$. This shows that $\mathcal{L}_1(\mathcal{E}_{F(\mathsf{3})})$ is complemented by elements of $\mathcal{L}_2(\mathcal{E}_{F(\mathsf{3})})$.

Now consider $y \in \mathcal{L}_2(\mathcal{E}_{F(\mathsf{3})})$. Similarly

$$y = \bigvee\{z \in \mathcal{L}_2(\mathcal{E}_{F(\mathsf{3})}) \mid z \overset{\mathcal{E}_{F(\mathsf{3})}}{\lhd_2} y\}$$

and, for each such z, there exists $w \in \mathcal{L}_1(\mathcal{E}_{F(\mathsf{3})})$ satisfying $z \wedge w = 0$ and $y \vee w = 1$. We know already that w has a complement $\neg w \in \mathcal{L}_2(\mathcal{E}_{F(\mathsf{3})})$. This complement satisfies $z \leq \neg w \leq y$ thus y is a join of complements of members of $\mathcal{L}_1(\mathcal{E}_{F(\mathsf{3})})$. In conclusion $\mathcal{B}_F(\mathsf{3})$ is a strictly zero-dimensional biframe. This implies that $\mathcal{L}_2(\mathcal{E}_{F(\mathsf{3})}) = \{0, \neg x, 1\} \cong \Delta\mathsf{3}$. ■

More generally, we have the following important result:

Proposition. *For each T-section F, $\mathcal{B}_F(L) \cong Sk(L)$.*

Proof: First, let us show that $\mathcal{B}_F(L)$ is strictly zero-dimensional, that is,

(1) $\mathcal{L}_1(\mathcal{E}_{F(L)})$ is complemented with complements in $\mathcal{L}_2(\mathcal{E}_{F(L)})$;
(2) Every element of $\mathcal{L}_2(\mathcal{E}_{F(L)})$ is a join of complements of members of $\mathcal{L}_1(\mathcal{E}_{F(L)})$.

(1) For each $a \in L$ consider $f_a : \mathsf{3} \to L$ defined by $f_a(\alpha) = a$. Since the diagram

$$\begin{tikzcd} \mathsf{3} \arrow[r, "f_a"] \arrow[d, "i_{F(\mathsf{3})}"'] & L \arrow[d, "i_{F(L)}"] \\ \mathcal{L}_1(\mathcal{E}_{F(\mathsf{3})}) \arrow[r, "TF(f_a)"'] & \mathcal{L}_1(\mathcal{E}_{F(L)}) \end{tikzcd} \tag{3.5.1}$$

commutes, we have $i_{F(L)}(a) = (TF(f_a) \cdot i_{F(\mathsf{3})})(\alpha) = TF(f_a)(x)$. By the Lemma we know that $i_{F(\mathsf{3})}(\alpha) = x$ is complemented with complement in $\mathcal{L}_2(\mathcal{E}_{F(\mathsf{3})})$. Then, obviously, $TF(f_a)(x)$ has a complement $TF(f_a)(\neg x)$. Since $i_{F(L)}$ is an isomorphism we may conclude that every element of $\mathcal{L}_1(\mathcal{E}_{F(L)})$ has a complement in $\mathcal{L}_2(\mathcal{E}_{F(L)})$.

(2) Let $y \in \mathcal{L}_2(\mathcal{E}_{F(L)})$. For each $z \in \mathcal{L}_2(\mathcal{E}_{F(L)})$ satisfying $z \overset{\mathcal{E}_{F(L)}}{\lhd_2} y$ there exists $w \in \mathcal{L}_1(\mathcal{E}_{F(L)})$ such that $z \wedge w = 0$ and $y \vee w = 1$. By (1), w is complemented with complement in $\mathcal{L}_2(\mathcal{E}_{F(L)})$. Obviously $z \leq \neg w \leq y$. The conclusion now follows from the fact that

$$y = \bigvee\{z \in \mathcal{L}_2(\mathcal{E}_{F(L)}) \mid z \overset{\mathcal{E}_{F(L)}}{\lhd_2} y\}.$$

This shows that $\mathcal{B}_F(L)$ is strictly zero-dimensional.

By the definition of T-section, $\mathcal{L}_1(\mathcal{E}_{F(L)}) \cong L \cong \nabla L$. Since Frm is an algebraic category, it has presentations by generators and relations. By what we have seen above, both $\mathcal{L}_2(\mathcal{E}_{F(L)})$ and ΔL are models for the presentation $\mathsf{Frm} < \mathcal{G} \mid \mathcal{R} >$, for generators

$$\mathcal{G} = \{\neg x \mid x \in \mathcal{L}_1(\mathcal{E}_{F(L)})\}$$

and relations

$$\neg(x \wedge y) = \neg x \vee \neg y \qquad (x, y \in \mathcal{L}_1(\mathcal{E}_{F(L)}))$$

$$\neg(\bigvee x_i) = \bigwedge(\neg x_i) \qquad (x_i \in \mathcal{L}_1(\mathcal{E}_{F(L)})).$$

Therefore $F_0(L) = \mathcal{L}_1(\mathcal{E}_{F(L)}) \vee \mathcal{L}_2(\mathcal{E}_{F(L)}) \cong \mathfrak{C}L$ and $\mathcal{B}_F(L) \cong Sk(L)$. ■

3.6. Properties of T-sections. Let F be a T-section. By Theorem 3.2, we may endow $F_0(L)$ with the Frith quasi-uniformity $\mathcal{F}_{F_0(L)}$, which is compatible with L_0. This transitive quasi-uniformity is coarser than the original quasi-uniformity $\mathcal{E}_{F(L)}$:

Lemma. $\mathcal{F}_{F_0(L)} \subseteq \mathcal{E}_{F(L)}$.

Proof: We need to show that, for each $a \in \mathcal{L}_1(\mathcal{E}_{F(L)})$, $E_a \in \mathcal{E}_{F(L)}$. For this, consider the frame homomorphism $f_a : \mathsf{3} \to L$ defined by $f_a(\alpha) = i_{F(L)}^{-1}(a)$. By Lemma 3.5, $\mathcal{B}_F(\mathsf{3}) \cong Sk(\mathsf{3})$ so $F(\mathsf{3})$ is necessarily isomorphic to the Sierpiński quasi-uniform frame. On the other hand, $F(f_a) : F(\mathsf{3}) \to F(L)$ is a quasi-uniform homomorphism thus $(F(f_a) \oplus F(f_a))(E_{\nabla_\alpha}) \in \mathcal{E}_{F(L)}$. By the commutativity of diagram (3.5.1),

$$F(f_a)(\nabla_\alpha) = TF(f_a)(\nabla_\alpha) = TF(f_a)(i_{F(\mathsf{3})}(\alpha)) = i_{F(L)}(f_a(\alpha)) = a.$$

Then $F(f_a)(\Delta_\alpha) = \neg a$ since $F(f_a)(\Delta_\alpha) \vee a = F(f_a)(\Delta_\alpha) \vee F(f_a)(\nabla_\alpha) = F(f_a)(1) = 1$ and $F(f_a)(\Delta_\alpha) \wedge a = F(f_a)(\Delta_\alpha) \wedge F(f_a)(\nabla_\alpha) = F(f_a)(0) = 0$. Hence

$$\begin{aligned}(F(f_a) \oplus F(f_a))(E_{\nabla_\alpha}) &= (F(f_a)(\nabla_\alpha) \oplus 1) \vee (1 \oplus F(f_a)(\Delta_\alpha)) \\ &= (a \oplus 1) \vee (1 \oplus \neg a) \\ &= E_a\end{aligned}$$

and $E_a \in \mathcal{E}_{F(L)}$, as required. ∎

By Lemma 3.2 and Proposition 3.5, for each frame L there exists a biframe isomorphism $\bar{i}_{F(L)} : Sk(L) \to \mathcal{B}_F(L)$ such that the diagram

$$\begin{tikzcd} L \arrow[r, "\nabla_L"] \arrow[d, "i_{F(L)}"'] & \mathfrak{C}L \arrow[ddl, dashed, "\bar{i}_{F(L)}"] \\ \mathcal{L}_1(\mathcal{E}_{F(L)}) \arrow[d, hook] & \\ F_0(L) & \end{tikzcd}$$

commutes.

Proposition. *Let F be a T-section. Then, for each frame L, we have:*

(1) $\bar{i}_{F(L)} : \mathcal{C}_1^*(L) \to (F_0(L), \mathcal{F}_{F_0(L)})$ *is a uniform isomorphism;*

(2) $\bar{i}_{F(L)} : \mathcal{C}_1^*(L) \to F(L)$ *is a uniform homomorphism.*

Proof: (1) We have

$$\begin{aligned}(\bar{i}_{F(L)} \oplus \bar{i}_{F(L)})(E_{\nabla_a}) &= (\bar{i}_{F(L)} \oplus \bar{i}_{F(L)})(\nabla_a \oplus 1) \vee (\bar{i}_{F(L)} \oplus \bar{i}_{F(L)})(1 \oplus \Delta_a) \\ &= (\bar{i}_{F(L)}(\nabla_a) \oplus 1) \vee (1 \oplus \bar{i}_{F(L)}(\Delta_a)) \\ &= (i_{F(L)}(a) \oplus 1) \vee (1 \oplus \neg i_{F(L)}(a))\end{aligned}$$

$$= \quad E_{i_{F(L)}(a)} \in \mathcal{F}_{F_0(L)}.$$

(2) It follows immediately from (1) and Lemma 3.6. ■

We end this section with the pointfree version of the classical result that the Pervin quasi-uniformity defines the coarsest T-section [4].

Theorem. *$\mathcal{C}_1^*$ is the coarsest section of T.*

Proof: Let F be a T-section. It suffices to verify that the maps $\bar{i}_{F(L)}$ of (2) in the Proposition define a natural transformation $\bar{i}_F : \mathcal{C}_1^* \Rightarrow F$ satisfying $T(\bar{i}_{F(L)}) \cdot \nabla_L = i_{F(L)}$ for each frame L.

Let $h : L \to M$ be a frame map. We need to show that the diagram

$$\begin{array}{ccc} \mathcal{C}_1^*(L) & \xrightarrow{\bar{i}_{F(L)}} & F(L) \\ {\scriptstyle \mathcal{C}_1^*(h)}\downarrow & & \downarrow{\scriptstyle F(h)} \\ \mathcal{C}_1^*(M) & \xrightarrow{\bar{i}_{F(M)}} & F(M) \end{array}$$

commutes. Let $\theta \in \mathfrak{C}L$. Then

$$\begin{aligned} F(h) \cdot \bar{i}_{F(L)}(\theta) &= F(h) \cdot \bar{i}_{F(L)}(\bigvee\{\Delta_a \wedge \nabla_b \mid (a,b) \in \theta, a \leq b\}) \\ &= F(h)(\bigvee\{\bar{i}_{F(L)}(\Delta_a) \wedge \bar{i}_{F(L)}(\nabla_b) \mid (a,b) \in \theta, a \leq b\}) \\ &= F(h)(\bigvee\{\neg i_{F(L)}(a) \wedge i_{FL}(b) \mid (a,b) \in \theta, a \leq b\}) \\ &= \bigvee\{\neg F(h)(i_{F(L)}(a)) \wedge F(h)(i_{F(L)}(b)) \mid (a,b) \in \theta, a \leq b\} \\ &= \bigvee\{\neg i_{F(M)}(h(a)) \wedge i_{F(M)}(h(b)) \mid (a,b) \in \theta, a \leq b\}. \end{aligned}$$

On the other hand,

$$\begin{aligned} \bar{i}_{F(M)} \cdot \mathcal{C}_1^*(h)(\theta) &= \bar{i}_{F(M)}(\overline{\nabla_M \cdot h}(\bigvee\{\Delta_a \wedge \nabla_b \mid (a,b) \in \theta, a \leq b\})) \\ &= \bar{i}_{F(M)}(\bigvee\{\neg\Delta_{h(a)} \wedge \nabla_{h(b)} \mid (a,b) \in \theta, a \leq b\}) \\ &= \bigvee\{\bar{i}_{F(M)}(\Delta_{h(a)}) \wedge \bar{i}_{F(M)}(\nabla_{h(b)}) \mid (a,b) \in \theta, a \leq b\}. \\ &= \bigvee\{\neg i_{F(M)}(h(a)) \wedge i_{F(M)}(h(b)) \mid (a,b) \in \theta, a \leq b\}. \end{aligned}$$

Thus $\bar{i}_{F(M)} \cdot \mathcal{C}_1^*(h)(\theta) = F(h) \cdot \bar{i}_{F(L)}(\theta)$.

Trivially, $(T(\bar{i}_{F(L)}) \cdot \nabla_L)(a) = \bar{i}_{F(L)}(\nabla_a) = i_{F(L)}(a)$, for every $a \in L$. ■

4. Functorial aspects of the Fletcher construction

4.1. Interior-preserving and Fletcher covers. We recall from [8] that a cover A of L is *interior-preserving* if, for each $B \subseteq A$,

$$\bigvee_{b \in B} \Delta_b = \Delta_{\bigwedge B}.$$

More generally, A is *weakly interior-preserving* if, for each $B \subseteq A$,

$$\bigwedge_{b \in B} \nabla_b = \nabla_{\bigwedge B}.$$

Further, a cover A is a *Fletcher cover* whenever

$$R_A := \bigcap_{a \in A} (\nabla_a \oplus 1) \cup (1 \oplus \Delta_a)$$

is a Weil entourage of $\mathfrak{C}L$ or, equivalently,

$$\bigvee \{ (\bigwedge_{a \in A_1} \nabla_a) \wedge (\bigwedge_{a \in A_2} \Delta_a) \mid A_1 \cup A_2 = A \} = 1 \quad \text{([8], Proposition 4.1).}$$

Examples of interior-preserving Fletcher covers are finite covers, locally finite covers, spectra and well-monotone covers (see [8] for the details).

It is also worth pointing out that, for any covers A, B of L,

$$R_A \cap R_B = R_{A \wedge B} \text{ ([8], Lemma 4.1)} \tag{4.1.1}$$

For the remainder of the paper we shall denote the entourage $E_{\nabla_a} = (\nabla_a \oplus 1) \cup (1 \oplus \Delta_a)$ simply by E_a and, for each frame homomorphism $h : L \to M$, we denote by $\overline{h} : \mathfrak{C}L \to \mathfrak{C}M$ the morphism given by (2.2.3). Note that $(\overline{h} \oplus \overline{h})(E_a) = E_{h(a)}$.

Interior-preserving covers and Fletcher covers behave well with respect to morphisms:

Proposition. *Let $h : L \to M$ be a frame homomorphism. Then:*

(1) *For every Fletcher cover A of L, $h[A]$ is a Fletcher cover of M;*
(2) *For every interior-preserving cover A of L, $h[A]$ is an interior-preserving cover of M.*

Proof: (1) Since R_A is a Weil entourage of L, $(\overline{h} \oplus \overline{h})(R_A)$ is a Weil entourage of M. But, clearly, $(\overline{h} \oplus \overline{h})(R_A) \subseteq \bigcap_{a \in A} (\overline{h} \oplus \overline{h})(E_a) = \bigcap_{a \in A} E_{h(a)} = R_{h[A]}$. Thus $R_{h[A]}$ is also a Weil entourage of M.

(2) For each $B \subseteq A$ we have, using the hypothesis,

$$\bigvee_{b\in B} \Delta_{h(b)} = \bigvee_{b\in B} \overline{h}(\Delta_b) = \overline{h}(\bigvee_{b\in B} \Delta_b) = \overline{h}(\Delta_{\bigwedge B}) = \Delta_{h(\bigwedge B)} \geq \Delta_{\bigwedge h[B]}.$$

The reverse inequality $\bigvee_{b\in B} \Delta_{h(b)} \leq \Delta_{\bigwedge h[B]}$ is always true. ■

In general $\overline{h}$ does not preserve arbitrary meets. But, clearly, $\overline{h}(\bigwedge_{b\in B} \Delta_b) = \bigwedge_{b\in B} \overline{h}(\Delta_b)$, for any $B \subseteq L$. Moreover:

Lemma. *Let A be an interior-preserving cover of L. Then:*
(1) $\overline{h}(\bigwedge_{b\in B} \nabla_b) = \bigwedge_{b\in B} \overline{h}(\nabla_b)$ *for every* $B \subseteq A$.
(2) $(\overline{h} \oplus \overline{h})(R_A) = R_{h[A]}$.
(3) *For each* $x \in L$, $st_1(\overline{h}(\nabla_x), (\overline{h} \oplus \overline{h})(R_A)) \leq \overline{h}(st_1(\nabla_x, R_A))$.
(4) *For each* $x \in L$, $st_2(\overline{h}(\Delta_x), (\overline{h} \oplus \overline{h})(R_A)) \leq \overline{h}(st_2(\Delta_x, R_A))$.

Proof: (1)

$$\begin{aligned}\overline{h}(\bigwedge_{b\in B} \nabla_b) &= \overline{h}(\nabla_{\bigwedge B}) = \overline{h}(\Delta_{\bigwedge B})^* = \overline{h}(\bigvee_{b\in B} \Delta_b)^* \\ &= (\bigvee_{b\in B} \overline{h}(\Delta_b))^* = \bigwedge_{b\in B} \overline{h}(\nabla_b).\end{aligned}$$

(2) The inclusion $(\overline{h} \oplus \overline{h})(R_A) \subseteq R_{h[A]}$ is trivial.

On the other hand, let $(\alpha, \beta) \in R_{h[A]}$. This means that, for every $a \in A$, $\alpha \leq \nabla_{h(a)}$ or $\beta \leq \Delta_{h(a)}$, that is, $\alpha \leq \bigwedge_{a\in A_1} \nabla_{h(a)}$ and $\beta \leq \bigwedge_{a\in A_2} \Delta_{h(a)}$ for some partition $A_1 \cup A_2$ of A. Consequently, by (1), $\alpha \leq \overline{h}(\bigwedge_{a\in A_1} \nabla_a)$, and, on the other hand, $\beta \leq \overline{h}(\bigwedge_{a\in A_2} \Delta_a)$. But $(\bigwedge_{a\in A_1} \nabla_a, \bigwedge_{a\in A_2} \Delta_a) \in R_A$ thus

$$(\alpha, \beta) \leq (\overline{h}(\bigwedge_{a\in A_1} \nabla_a), \overline{h}(\bigwedge_{a\in A_2} \Delta_a)) \in (\overline{h} \oplus \overline{h})(R_A).$$

(3) It suffices to check that $st_1(\nabla_{h(x)}, R_{h[A]}) \leq \overline{h}(st_1(\nabla_x, R_A))$. Let $(\alpha, \beta) \in R_{h[A]}$ with $\beta \wedge \nabla_{h(x)} \neq 0$. Then $\alpha \leq \bigwedge_{a\in A_1} \nabla_{h(a)}$ and $\beta \leq \bigwedge_{a\in A_2} \Delta_{h(a)}$ for some partition $A_1 \cup A_2$ of A. But, by (1), $\bigwedge_{a\in A_1} \nabla_{h(a)} = \bigwedge_{a\in A_1} \overline{h}(\nabla_a) = \overline{h}(\bigwedge_{a\in A_1} \nabla_a)$ so we only need to show that $\bigwedge_{a\in A_1} \nabla_a \leq st_1(\nabla_x, R_A)$ which is easy since

$$(\bigwedge_{a\in A_1} \nabla_a, \bigwedge_{a\in A_2} \Delta_a) \in R_A$$

and $\beta \wedge \nabla_{h(x)} \neq 0$ implies $\overline{h}(\bigwedge_{a \in A_2} \Delta_a \wedge \nabla_x) = \bigwedge_{a \in A_2} \Delta_{h(a)} \wedge \nabla_{h(x)} \neq 0$, that is, $\bigwedge_{a \in A_2} \Delta_a \wedge \nabla_x \neq 0$.

(4) Similar to (3). ■

4.2. The Fletcher construction is functorial. It is now our goal to study the functoriality of the pointfree version of Fletcher's construction presented by the authors in [8]. We begin by briefly recalling this method of constructing compatible quasi-uniformities for arbitrary frames.

For any frame L, let $\mathcal{A}_L$ be a collection of (weakly) interior-preserving Fletcher covers of L and let $\mathcal{E}_{\mathcal{A}_L}$ be the filter of $WEnt(\mathfrak{C}L)$ generated by $\{R_A \mid A \in \mathcal{A}_L\}$.

In general,

$$\{\bigwedge_{b \in B} \nabla_b \mid B \subseteq A, A \in \mathcal{A}_L\} \subseteq \mathcal{L}_1(\mathcal{E}_{\mathcal{A}_L}) \subseteq \nabla L \tag{4.2.1}$$

and

$$\{\bigwedge_{b \in B} \Delta_b \mid B \subseteq A, A \in \mathcal{A}_L\} \subseteq \mathcal{L}_2(\mathcal{E}_{\mathcal{A}_L}) \subseteq \Delta L. \tag{4.2.2}$$

If $\mathcal{L}_1(\mathcal{E}_{\mathcal{A}_L}) = \nabla L$ and $\mathcal{L}_2(\mathcal{E}_{\mathcal{A}_L}) = \Delta L$, $\mathcal{E}_{\mathcal{A}_L}$ is a quasi-uniformity on $\mathfrak{C}L$. Otherwise, it is not; there is, however, an easy way of obtaining a quasi-uniform frame by modifying $\mathcal{E}_{\mathcal{A}_L}$ (and $\mathfrak{C}L$): denoting by $\mathfrak{C}L'$ the subframe of $\mathfrak{C}L$ generated by $\mathcal{L}_1(\mathcal{E}_{\mathcal{A}_L}) \cup \mathcal{L}_2(\mathcal{E}_{\mathcal{A}_L})$, each $R'_A := R_A \cap (\mathfrak{C}L' \times \mathfrak{C}L')$ is a Weil entourage of $\mathfrak{C}L'$ and $\{R'_A \mid A \in \mathcal{A}_L\}$ generates a transitive quasi-uniformity $\mathcal{E}'_{\mathcal{A}_L}$ on $\mathfrak{C}L'$ such that $\mathcal{L}_1(\mathcal{E}'_{\mathcal{A}_L}) = \mathcal{L}_1(\mathcal{E}_{\mathcal{A}_L})$ and $\mathcal{L}_2(\mathcal{E}'_{\mathcal{A}_L}) = \mathcal{L}_2(\mathcal{E}_{\mathcal{A}_L})$ (see [8] for the details). The quasi-uniform frame $(\mathfrak{C}L', \mathcal{E}'_{\mathcal{A}_L})$ is not, in general, compatible with L. However, by Lemma 6.2 of [8], when $\bigcup \mathcal{A}_L$ is a subbase for L, we have $\mathcal{L}_1(\mathcal{E}'_{\mathcal{A}_L}) = \mathcal{L}_1(\mathcal{E}_{\mathcal{A}_L}) = \nabla L$ and the compatibility of $\mathcal{E}'_{\mathcal{A}_L}$ with the given L is ensured.

Following the classical terminology, we say that a *natural kind of covers* in Frm is an indexed class $\mathcal{A} = (\mathcal{A}_L)_{L \in \mathsf{Frm}}$ such that:

(1) Each $\mathcal{A}_L$ is a set of interior-preserving Fletcher covers of L;
(2) For every frame homomorphism $h : L \to M$ and every $A \in \mathcal{A}_L$, $h[A] \in \mathcal{A}_M$.

Lemma 1. *Let $\mathcal{A} = (\mathcal{A}_L)_{L \in \mathsf{Frm}}$ be a natural kind of covers and let $h : L \to M$ be a frame homomorphism. Then:*

(1) $\nabla_y \overset{\mathcal{E}_{\mathcal{A}_L}}{\triangleleft_1} \nabla_x$ *implies* $\overline{h}(\nabla_y) \overset{\mathcal{E}_{\mathcal{A}_M}}{\triangleleft_1} \overline{h}(\nabla_x)$.

(2) $\Delta_y \overset{\mathcal{E}_{\mathcal{A}_L}}{\lhd_2} \Delta_x$ *implies* $\overline{h}(\Delta_y) \overset{\mathcal{E}_{\mathcal{A}_M}}{\lhd_2} \overline{h}(\Delta_x)$.
(3) $\overline{h}(\mathcal{L}_i(\mathcal{E}'_{\mathcal{A}_L})) \subseteq \mathcal{L}_i(\mathcal{E}'_{\mathcal{A}_M})$ *($i = 1, 2$).*

Proof: (1) Consider $A_1, \ldots, A_n \in \mathcal{A}_L$ such that $st_1(\nabla_y, \bigcap_{i=1}^n R_{A_i}) \leq \nabla_x$. Then $\overline{h}(st_1(\nabla_y, \bigcap_{i=1}^n R_{A_i})) \leq \overline{h}(\nabla_x)$. But

$$\begin{aligned} \overline{h}(st_1(\nabla_y, \textstyle\bigcap_{i=1}^n R_{A_i})) &= \overline{h}(st_1(\nabla_y, R_{\bigwedge_{i=1}^n A_i})) && \text{by (4.1.1)} \\ &\geq st_1(\overline{h}(\nabla_y), (\overline{h} \oplus \overline{h})(R_{\bigwedge_{i=1}^n A_i})) && \text{by Lemma 4.1(3).} \end{aligned}$$

Clearly, each A_i being interior-preserving, $\bigwedge_{i=1}^n A_i$ is also interior-preserving. Thus, by Lemma 4.1(2), we get

$$\begin{aligned} \overline{h}(st_1(\nabla_y, \bigcap_{i=1}^n R_{A_i})) &\geq st_1(\overline{h}(\nabla_y), R_{h[\bigwedge_{i=1}^n A_i]}) \\ &= st_1(\overline{h}(\nabla_y), R_{\bigwedge_{i=1}^n h[A_i]}) \\ &= st_1(\overline{h}(\nabla_y), \bigcap_{i=1}^n R_{h[A_i]}). \end{aligned}$$

In conclusion, $st_1(\overline{h}(\nabla_y), \bigcap_{i=1}^n R_{h[A_i]}) \leq \overline{h}(\nabla_x)$, which shows that $\overline{h}(\nabla_y) \overset{\mathcal{E}_{\mathcal{A}_M}}{\lhd_1} \overline{h}(\nabla_x)$.

(2) Similar to (1).

(3) Let $\nabla_x \in \mathcal{L}_1(\mathcal{E}'_{\mathcal{A}_L}) = \mathcal{L}_1(\mathcal{E}_{\mathcal{A}_L}) \subseteq \nabla L$. Then $\nabla_x = \bigvee\{\nabla_y \mid \nabla_y \overset{\mathcal{E}_{\mathcal{A}_L}}{\lhd_1} \nabla_x\}$ and, by (1), it follows that

$$\begin{aligned} \overline{h}(\nabla_x) &= \bigvee\{\overline{h}(\nabla_y) \mid \nabla_y \overset{\mathcal{E}_{\mathcal{A}_L}}{\lhd_1} \nabla_x\} \\ &\leq \bigvee\{\theta \in \mathfrak{C}L \mid \theta \overset{\mathcal{E}_{\mathcal{A}_M}}{\lhd_1} \overline{h}(\nabla_x)\} \\ &\leq \overline{h}(\nabla_x). \end{aligned}$$

Hence $\overline{h}(\nabla_x) = \bigvee\{\theta \in \mathfrak{C}L \mid \theta \overset{\mathcal{E}_{\mathcal{A}_M}}{\lhd_1} \overline{h}(\nabla_x)\}$, which means that $\overline{h}(\nabla_x) \in \mathcal{L}_1(\mathcal{E}_{\mathcal{A}_M}) = \mathcal{L}_1(\mathcal{E}'_{\mathcal{A}_M})$. ■

It follows immediately from Lemma 1 that $\overline{h} : \mathfrak{C}L \to \mathfrak{C}M$ defines, by restriction, a biframe map

$$\overline{h} : (\mathfrak{C}L', \mathcal{L}_1(\mathcal{E}'_{\mathcal{A}_L}), \mathcal{L}_2(\mathcal{E}'_{\mathcal{A}_L})) \to (\mathfrak{C}M', \mathcal{L}_1(\mathcal{E}'_{\mathcal{A}_M}), \mathcal{L}_2(\mathcal{E}'_{\mathcal{A}_M})).$$

Statement (2) of Lemma 4.1 is also true for the restricted entourages $R'_A = R_A \cap (\mathfrak{C}L' \times \mathfrak{C}L')$:

Lemma 2. *Let $\mathcal{A} = (\mathcal{A}_L)_{L\in\mathsf{Frm}}$ be a natural kind of covers and let $h : L \to M$ be a frame homomorphism. Then, for each $A \in \mathcal{A}_L$, $(\overline{h} \oplus \overline{h})(R'_A) = R'_{h[A]}$.*

Proof: The inclusion $(\overline{h} \oplus \overline{h})(R'_A) \subseteq R'_{h[A]}$ is trivial.

Let $(\alpha, \beta) \in R'_{h[A]}$. This means that, for every $a \in A$, $\alpha \leq \nabla_{h(a)}$ or $\beta \leq \Delta_{h(a)}$, that is, $\alpha \leq \bigwedge_{a\in A_1} \nabla_{h(a)}$ and $\beta \leq \bigwedge_{a\in A_2} \Delta_{h(a)}$ for some partition $A_1 \cup A_2$ of A. Consequently, by Lemma 4.1(1), $\alpha \leq \overline{h}(\bigwedge_{a\in A_1} \nabla_a)$, and, on the other hand, $\beta \leq \overline{h}(\bigwedge_{a\in A_2} \Delta_a)$. But $(\bigwedge_{a\in A_1} \nabla_a, \bigwedge_{a\in A_2} \Delta_a) \in R'_A$ since it belongs to R_A and, by (4.2.1) and (4.2.2), $\bigwedge_{a\in A_1} \nabla_a \in \mathcal{L}_1(\mathcal{E}'_{A_L})$ and $\bigwedge_{a\in A_2} \Delta_a \in \mathcal{L}_2(\mathcal{E}'_{A_L})$. Thus

$$(\alpha, \beta) \leq (\overline{h}(\bigwedge_{a\in A_1} \nabla_a), \overline{h}(\bigwedge_{a\in A_2} \Delta_a)) \in (\overline{h} \oplus \overline{h})(R'_A). \qquad \blacksquare$$

Proposition. *$\overline{h}$ is a uniform homomorphism from $(\mathfrak{C}L', \mathcal{E}'_{\mathcal{A}_L})$ to $(\mathfrak{C}M', \mathcal{E}'_{\mathcal{A}_M})$.*

Proof: Let $E \in \mathcal{E}'_{\mathcal{A}_L}$. Then $\bigcap_{i=1}^n R'_{A_i} \subseteq E$ for some $A_1, \ldots, A_n \in \mathcal{A}_L$, from which it follows that $(\overline{h} \oplus \overline{h})(\bigcap_{i=1}^n R'_{A_i}) \subseteq (\overline{h} \oplus \overline{h})(E)$. On the other hand, by Lemma 2,

$$(\overline{h} \oplus \overline{h})(\bigcap_{i=1}^{n} R'_{A_i}) = \bigcap_{i=1}^{n} (\overline{h} \oplus \overline{h})(R'_{A_i}) = \bigcap_{i=1}^{n} R'_{h[A_i]} \in \mathcal{E}'_{\mathcal{A}_M}.$$

Hence $(\overline{h} \oplus \overline{h})(E) \in \mathcal{E}'_{\mathcal{A}_M}$. $\blacksquare$

This defines a (transitive) functor $Q_\mathcal{A} : \mathsf{Frm} \to \mathsf{QUFrm}$.

4.3. When does the Fletcher construction induce a T-section? Of course, we are interested in the case when, for every L, $Q_\mathcal{A}(L)$ is a quasi-uniform frame compatible with L, that is, when $Q_\mathcal{A}$ is a T-section. First, we need to recall the following from [8]:

Let $\mathcal{E}$ be a transitive quasi-uniformity on a subframe $\mathfrak{C}L'$ of $\mathfrak{C}L$, compatible with L, and consider a transitive subbase $\mathcal{S}$ of $\mathcal{E}$. Since each $E \in \mathcal{S}$ is transitive,

$$st_i(\theta, E) \overset{\mathcal{E}}{\triangleleft_i} st_i(\theta, E) \text{ for every } \theta \in \mathfrak{C}L' \quad (i = 1, 2).$$

Therefore, $st_1(\theta, E) \in \mathcal{L}_1(\mathcal{E})$ and $st_2(\theta, E) \in \mathcal{L}_2(\mathcal{E})$. So, by the isomorphism $\mathcal{L}_1(\mathcal{E}) \cong \nabla L$, each $st_1(\theta, E)$ corresponds to $\nabla_{E[\theta]}$ for some element $E[\theta] \in L$. Set $Cov E = \{E[\theta] \mid (\theta, \theta) \in E\}$.

Proposition. *Let $\mathcal{E}$ be a transitive quasi-uniformity on a subframe $\mathfrak{C}L'$ of $\mathfrak{C}L$, compatible with L, and consider a transitive subbase $\mathcal{S}$ of $\mathcal{E}$. Then:*

(1) *Each $CovE$ is an interior-preserving cover of L.*

(2) *$\bigcup_{E\in\mathcal{S}} CovE$ is a subbase for L.*

Proof: (1) Proposition 7.3 of [8].

(2) Proposition 7.2 of [8]. ∎

When $\mathcal{E}$ is the quasi-uniformity $\mathcal{E}'_{\mathcal{A}_L}$ generated by a family $\mathcal{A}_L$ of (weakly) interior-preserving Fletcher covers of L, constructed in 4.2, we have:

Lemma. *Let $\mathcal{A}_L$ be a family of (weakly) interior-preserving Fletcher covers of L. If $\bigcup\{CovR'_A \mid A\in\mathcal{A}_L\}$ is a subbase for L then $\mathcal{L}_1(\mathcal{E}'_{\mathcal{A}_L}) = \nabla L$.*

Proof: Let $x\in L$. By hypothesis, we may write

$$x = \bigvee_{i\in I}(R'_{A^i_1}[\theta_1]\wedge\ldots\wedge R'_{A^i_{n_i}}[\theta_{n_i}])$$

for some $A^i_j\in\mathcal{A}$ and $(\theta_j,\theta_j)\in R'_{A^i_j}$ $(i\in I, j\in\{1,\ldots,n_i\})$. Then

$$\nabla_x = \bigvee_{i\in I}(\nabla_{R'_{A^i_1}[\theta_1]}\wedge\ldots\wedge\nabla_{R'_{A^i_{n_i}}[\theta_{n_i}]}) = \bigvee_{i\in I}(st_1(\theta_1,R'_{A^i_1})\wedge\ldots\wedge st_1(\theta_{n_i},R'_{A^i_{n_i}})).$$

So, in order to show that $\nabla_x\in\mathcal{L}_1(\mathcal{E}'_{\mathcal{A}_L})$ it suffices to check that, for each i,

$$st_1(\theta_1,R'_{A^i_1})\wedge\ldots\wedge st_1(\theta_{n_i},R'_{A^i_{n_i}}) \overset{\mathcal{E}'_{\mathcal{A}_L}}{\lhd_1} \nabla_x.$$

For each i, take $\bigcap_{j=1}^{n_i} R'_{A^i_j}\in\mathcal{E}'_{\mathcal{A}_L}$. Then, by properties (S3) and (S4),

$$\begin{aligned}
& st_1(st_1(\theta_1,R'_{A^i_1})\wedge\ldots\wedge st_1(\theta_{n_i},R'_{A^i_{n_i}}),\bigcap_{j=1}^{n_i} R'_{A^i_j}) \\
\leq\ & \bigwedge_{j=1}^{n_i} st_1(st_1(\theta_1,R'_{A^i_1})\wedge\ldots\wedge st_1(\theta_{n_i},R'_{A^i_{n_i}}),R'_{A^i_j}) \\
\leq\ & \bigwedge_{j=1}^{n_i} st_1(st_1(\theta_j,R'_{A^i_j}),R'_{A^i_j}) \leq \bigwedge_{j=1}^{n_i} st_1(\theta_j,R'_{A^i_j}\circ R'_{A^i_j}) \\
=\ & \bigwedge_{j=1}^{n_i} st_1(\theta_j,R'_{A^i_j}) \leq \nabla_x.
\end{aligned}$$

∎

The following statements are a reformulation of results in [8].

Theorem. *Let $\mathcal{A}_L$ be a set of covers of a frame L. Then $\{R'_A \mid A \in \mathcal{A}_L\}$ is a subbase for a transitive quasi-uniformity on the subframe $\mathfrak{C}L'$ of $\mathfrak{C}L$, compatible with L, if and only if $\mathcal{A}_L$ is a set of weakly interior-preserving Fletcher covers of L such that $\bigcup\{CovR'_A \mid A \in \mathcal{A}_L\}$ is a subbase for L.*

Proof: Let $\mathcal{E}'_{\mathcal{A}_L}$ denote the quasi-uniformity generated by $\mathcal{S} = \{R'_A \mid A \in \mathcal{A}_L\}$. Since each R'_A is an entourage, each $A \in \mathcal{A}_L$ is a Fletcher cover. By (2) in the Proposition, $\bigcup\{CovR'_A \mid A \in \mathcal{A}_L\}$ is a subbase of L. Finally, each $A \in \mathcal{A}_L$ is weakly interior-preserving. Indeed, by Lemma 6.1.1 (a) of [8],

$$st_1(\bigwedge_{b\in B} \nabla_b, R'_A) \leq st_1(\bigwedge_{b\in B} \nabla_b, R_A) = \bigwedge_{b\in B} \nabla_b$$

for every $B \subseteq A$, thus $\bigwedge_{b\in B} \nabla_b \in \mathcal{L}_1(\mathcal{E}'_{\mathcal{A}_L}) \cong \nabla L$, by the compatibility of the quasi-uniformity.

The converse is obvious: the quasi-uniformity $\mathcal{E}'_{\mathcal{A}_L}$ of 4.2, which as $\{R'_A \mid A \in \mathcal{A}_L\}$ as a subbase, is compatible with L, by the Lemma. ■

Corollary. *Let $\mathcal{A}$ be a natural kind of covers. The induced transitive functor $Q_{\mathcal{A}}$ is a T-section if and only if, for each frame L, $\bigcup\{CovR'_A \mid A \in \mathcal{A}_L\}$ is a subbase for L.* ■

5. The construction of all transitive T-sections

Finally, with the help of results from [8], we may conclude that the functor $Q_{\mathcal{A}}$ induced by Fletcher's construction describes all transitive T-sections.

We say that a natural kind of covers $\mathcal{A} = (\mathcal{A}_L)_{L\in\mathsf{Frm}}$ is an *adequate kind of covers* if, for each frame L, $\bigcup \mathcal{A}_L$ is a subbase for L. Then we have:

Theorem 1. *For each adequate kind of covers $\mathcal{A}$, the induced transitive functor $Q_{\mathcal{A}}$ is a transitive T-section.*

Proof: By Proposition 4.2, $Q_{\mathcal{A}}$ is a transitive functor. The conclusion that it is a T-section follows immediately from Theorem 6.3 of [8], which asserts that, for every nonempty family $\mathcal{A}_L$ of weakly interior-preserving Fletcher covers of L such that $\bigcup \mathcal{A}_L$ is a subbase for L, $\mathcal{E}'_{\mathcal{A}_L}$ is a transitive quasi-uniformity on $\mathfrak{C}L'$, compatible with L. ■

Then, by Proposition 3.5, when $Q_{\mathcal{A}}$ is a T-section, each $Q_{\mathcal{A}}(L)$ is isomorphic to the Skula biframe so $\mathfrak{C}L' = \mathfrak{C}L$ and $\mathcal{E}'_{\mathcal{A}_L} = \mathcal{E}_{\mathcal{A}_L}$, that is, $Q_{\mathcal{A}}(L) = (\mathfrak{C}L, \mathcal{E}_{\mathcal{A}_L})$. More generally, for any T-section F, also by Proposition 3.5,

$\mathcal{B}_F(L) \cong (\mathfrak{C}L, \nabla L, \Delta L)$, and we may assume, to simplify notation, that $F(L) = (\mathfrak{C}L, \mathcal{E}_{F(L)})$.

Theorem 2. *Let F be a transitive T-section. For each frame L, let*

$$\mathcal{A}_L = \{A \mid A \textit{ interior-preserving cover of } L, R_A \in \mathcal{E}_{F(L)}\}.$$

Then $\mathcal{A} = (\mathcal{A}_L)_{L \in \mathsf{Frm}}$ is an adequate kind of covers such that $Q_{\mathcal{A}} = F$. Moreover, $\mathcal{A}$ is the largest adequate kind of covers whose induced functor is the given F.

Proof: We prove that $\mathcal{A}$ is adequate. Trivially each $A \in \mathcal{A}_L$ is an interior-preserving Fletcher cover of L. Let $h : L \to M$ be a frame homomorphism. Then, for each $A \in \mathcal{A}_L$, $R_A \in \mathcal{E}_{F(L)}$ thus $(\overline{h} \oplus \overline{h})(R_A) \in \mathcal{E}_{F(M)}$. By Lemma 4.1(2), this means that $R_{h[A]} \in \mathcal{E}_{F(M)}$. Consequently, $h[A] \in \mathcal{A}_M$.

Since $\{CovR_A \mid A \in \mathcal{A}_L\} \subseteq \mathcal{A}_L$, it follows from Proposition 4.3(2) that $\bigcup \mathcal{A}_L$ is a subbase for L. The remaining claim follows from Theorem 7.4.2(a) of [8] that asserts that for any compatible transitive quasi-uniformity $\mathcal{E}$ on $\mathfrak{C}L$, $\mathcal{A}_L = \{A \mid A \in CovL, R_A \in \mathcal{E}\}$ is the largest set of covers of L that induces $\mathcal{E}$. ■

Examples. Many kinds of interior-preserving Fletcher covers induce transitive T-sections. The following are examples of adequate kinds of covers and of their induced transitive T-sections.

kind $\mathcal{A}$ of covers	Transitive T-section $Q_{\mathcal{A}}$
Interior-preserving Fletcher covers	$\mathcal{FT}$: "Fine transitive section"
Finite	$\mathcal{F}$: "Frith section"
Locally finite	$\mathcal{LF}$: "Locally finite section"
Well-monotone	$\mathcal{W}$: "Well-monotone section"
Spectra	$\mathcal{SC}$: "Semi-continuous section"

Indeed, they are examples of collections $\mathcal{A}_L$ of interior-preserving Fletcher covers such that $\bigcup \mathcal{A}_L$ is a subbase of L, as we proved in the last section of [8], thus adequateness follows from the following result.

Proposition. *Let $h : L \to M$ be a frame homomorphism. For every locally finite (resp. spectrum, well-monotone) cover A of L, $h[A]$ is a locally finite (resp. spectrum, well-monotone) cover of M.*

Proof: (1) Let A be a locally finite cover, that is, a cover for which there exists a cover C such that $A_c := \{a \in A \mid a \wedge c \neq 0\}$ is finite for every $c \in C$. Then $h[C]$ is a cover of M and, for every $c \in C$, $h[A]_{h(c)} \subseteq \{h(a) \mid a \in A_c\}$, since $h(a) \wedge h(c) \neq 0$ implies $a \wedge c \neq 0$. Thus $h[A]$ is locally finite.

(2) In case $A = \{a_n \mid n \in \mathbb{Z}\}$ is a spectrum cover of L, that is, a cover of L satisfying $a_n \leq a_{n+1}$, for each $n \in \mathbb{Z}$, and $\bigvee_{n\in\mathbb{Z}} \Delta_{a_n} = 1$, then, immediately, $h[A]$ is a cover of M, $h(a_n) \leq h(a_{n+1})$, for each $n \in \mathbb{Z}$, and $\bigvee_{n\in\mathbb{Z}} \Delta_{h(a_n)} = \bigvee_{n\in\mathbb{Z}} \overline{h}(\Delta_{a_n}) = \overline{h}(\bigvee_{n\in\mathbb{Z}} \Delta_{a_n}) = \overline{h}(1) = 1$.

(3) Finally, the case when A is well-monotone, that is, well-ordered by the partial order of L, is obvious. ■

References

[1] B. Banaschewski and G.C.L. Brümmer, Strong zero-dimensionality of biframes and bispaces, *Quaest. Math.* 13 (1990) 273–290.
[2] B. Banaschewski and G.C.L. Brümmer, Functorial uniformities on strongly zero-dimensional frames, *Kyungpook Math. J.* 41 (2001) 179–190.
[3] B. Banaschewski, G.C.L. Brümmer and K.A. Hardie, Biframes and bispaces, *Quaestiones Math.* 6 (1983) 13–25.
[4] G.C.L. Brümmer, Initial quasi-uniformities, *Indag. Math.* 31 (1969) 403–409.
[5] G.C.L. Brümmer, *A Categorical Study of Initiality in Uniform Topology*, Ph.D. Thesis, University of Cape Town, 1971.
[6] G.C.L. Brümmer, Functorial transitive quasi-uniformities, in: *Categorical Topology* (Proceedings Conference Toledo, 1983), Heldermann Verlag, Berlin, 1984, pp. 163–184.
[7] G.C.L. Brümmer, *Some problems in categorical and asymmetric topology*, ICAGT (Ankara, Turkey), August 2001.
[8] M.J. Ferreira and J. Picado, *A method of constructing compatible quasi-uniformities for an arbitrary frame*, preprint 03-06 (University of Coimbra, 2003).
[9] P. Fletcher, On totally bounded quasi-uniform spaces, *Arch. Math.* 21 (1970) 396–401.
[10] P. Fletcher and W.F. Lindgren, *Quasi-uniform spaces*, Marcel Dekker, New York, 1982.
[11] J. Frith, *Structured Frames*, Ph.D. Thesis, University of Cape Town, 1987.
[12] P.T. Johnstone, *Stone Spaces*, Cambridge Studies in Advanced Mathematics, Vol. 3, Cambridge University Press, Cambridge, 1982
[13] W. Hunsaker and J. Picado, Frames with transitive structures, *Appl. Categ. Structures* 10 (2002) 63–79.
[14] W.J. Pervin, Quasi-uniformization of topological spaces, *Math. Ann.* 147 (1962) 316–317.
[15] J. Picado, *Weil Entourages in Pointfree Topology*, Ph.D. Thesis, University of Coimbra, 1995.
[16] J. Picado, Frame quasi-uniformities by entourages, in: *Symposium on Categorical Topology* (University of Cape Town 1994), Department of Mathematics, University of Cape Town, 1999, pp. 161–175.
[17] J. Picado, Structured frames by Weil entourages, *Appl. Categ. Structures* 8 (2000) 351-366.

[18] S. Salbany, *Bitopological spaces, compactifications and completions*, Mathematical Monographs of the University of Cape Town, N. 1, Department of Mathematics, University of Cape Town, 1974.
[19] S. Vickers, *Topology via Logic*, Cambridge Tracts in Theoretical Computer Science, Vol. 5, Cambridge University Press, Cambridge, 1985.

Maria João Ferreira
Departamento de Matemática, Universidade de Coimbra, 3001-454 Coimbra, PORTUGAL
E-mail address: mjrf@mat.uc.pt

Jorge Picado
Departamento de Matemática, Universidade de Coimbra, 3001-454 Coimbra, PORTUGAL
E-mail address: picado@mat.uc.pt