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# ELECTORAL CELLS OF LARGEST REMAINDERS METHOD

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ABSTRACT: In an election process, $p$ parties compete for $\Sigma$ seats in a parliament. After votes are cast, the electoral result may be thought of as an element $x \in \mathbb{R}^p$. Given $x$, the so-called *largest remainders method* determines the number $a_i$ of seats party $i$ gets in the parliament. The *electoral cell* determined by $(a_1, \dots, a_p)$ is the closure of the set of all results $x$ that determine $a_i$ seats for party $i$, $1 \leqslant i \leqslant p$. The electoral cells are convex polytopes and tile a hyperplane of $\mathbb{R}^p$.
In this paper we give a description of the electoral cells. For a single cell we identify and classify the cell's faces, completely describe its face lattice, and determine its group of automorphisms. It turns out that each face of dimension $d$ arises from a $d$-unit-cube by a compression along a diagonal.

KEYWORDS: polytopes, convexity, faces, tilings.
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## 1. Introduction

The results in this paper came as an aftermath of two of the problems in H. Steinhaus' famous book *Mathematical Snapshots* [6]. We have in mind the "elections problems" contained in pages 72-73 and 210 of the 1999 edition of [6]. In the first of those problems (page 72-73 of [6]) Steinhaus considers the outcome of an electoral process where 3 parties compete for a given number of seats in a parliament, the seats being assigned to the parties by the so-called proportional largest remainders method (to be explained below); the 4-party conterpart is considered later in the book [6, p. 210].

The natural generalization of the *Snapshots* elections problems goes as follows: $p$ parties compete for $\Sigma$ seats in a parliament; after the electors cast their votes, the total number $N$ of votes (abstentions not counted) is decomposed into $p$ parts, say $N = N_1 + \cdots + N_p$, where $N_i$ is the number of votes in party $i$. The *quota* of party $i$ is the number $x_i := (N_i/N)\Sigma$. We assume that this election is based on a *proportional method* of seat assignment (cf. [2, p. 25]); this means $x_i$ represents the (approximate) fraction of the $\Sigma$

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seats obtained by party $i$. This party will directly obtain $\lfloor x_i \rfloor$ (the integer part of $x_i$) seats in the parliament, and its partisans expect to obtain one more seat if its *remainder*, $r_i := x_i - \lfloor x_i \rfloor$, is high enough when compared with the other parties remainders. Of course, the sum $s$ of all remainders,

$$s := r_1 + \cdots + r_p,$$

is the number of seats not directly assigned to the parties; it is an integer such that $0 \leqslant s < p$.

Among the many algorithms in use (cf. [2, p. 25 ff]) to assign the remaining $s$ seats to the $p$ parties, the *Snapshots* adopt the most 'natural' one of them, which is known by many names, e.g., *Hamilton's method*, or the *natural quota*, or *largest remainders* method: the parties are ordered according to their remainders, in nonincreasing order, and one more seat is given to each one of the best $s$ parties on that arrangement.

In the sequel, the $p$-tuple $x = (x_1, \ldots, x_p)$ is called the *result* of the electoral process. Eventual ties on the remainders may imply a stalemate in this process; in such cases, the result $x$ is said to be *undetermined*; in all other cases, we say the result is *determined*. It is obvious that $x$ is determined iff the $s$-th largest remainder is strictly greater than the $(s+1)$-th greatest remainder.

Clearly $x$ is an element of $\mathbb{R}^p$, with rational coordinates of sum $\Sigma$. Moreover, in the real world, for a fixed number $N$ of votes there is only a finite number of possible electoral results. We eliminate this inconvenience by accepting as ideally feasible electoral results *all* the real $p$-tuples, of sum $\Sigma$, even those with irrational coordinates. Everything will take place on the hyperplane of the $x$'s with sum $\Sigma$, that we denote by $\mathfrak{H}_\Sigma$:

$$\mathfrak{H}_\Sigma = \{x \in \mathbb{R}^p : \ x_1 + \cdots + x_p = \Sigma\}.$$

So, Hamilton's method assigns to each result $x \in \mathfrak{H}_\Sigma$ a $p$-tuple of integers, say $a = (a_1, \ldots, a_p) \in \mathfrak{H}_\Sigma$, where $a_i$ is the number of seats obtained by party $i$. In this paper we are mainly concerned with the

> **Electoral Cell Problem.** *Given the $p$-tuple of nonegative integers $a = (a_1, \ldots, a_p) \in \mathfrak{H}_\Sigma$, describe the* 'electoral cell' *corresponding to $a$, i.e., the set of all electoral results $x$ that assign $a_1, \ldots, a_p$ seats in the parliament to the parties $1, \ldots, p$, respectively.*

In the case of 3 parties, the *Snapshots* [6, p. 73] represent each electoral result $(x_1, x_2, x_3)$ by a point $R$ inside an equilateral triangle of height $\Sigma$, where the Cartesian coordinates now play the role of, say, 'triangular coordinates' (see figure 1).

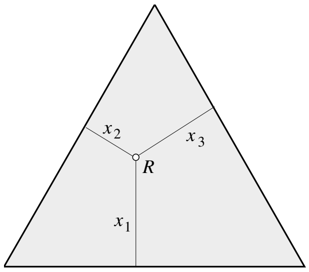


*figure* 1

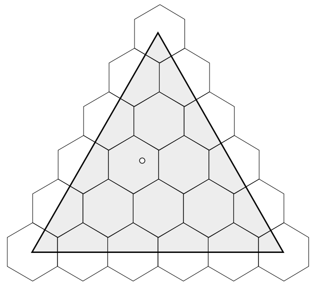

*figure* 2

In this triangular representation, Steinhaus says, each 'electoral cell' is represented by a regular hexagon, and these hexagons determine the honeycomb tiling of the triangle, as the second figure shows for a 5-seat parliament: the triangle has height 5, and $R$ lies inside the cell corresponding to $a = (2, 1, 2)$.

The *Snapshots* briefly discuss, later on, on page 210, the 4-party case; the problem is modeled in 'tetrahedral coordinates': on a reference regular tetrahedron of hight $\Sigma$, we choose one face for each party; the electoral result $(x_1, x_2, x_3, x_4)$ is then identified with the point $R$ located at distance $x_i$ of party $i$'s face, for each $i$. The *Snapshots* then refer the electoral cells as *"a tiling composed of regular tetrahedra and regular octahedra"*. As a matter of fact, in the 4-party case, the cells are regular rhombic dodecahedra, tiling the tridimensional space in a simple, well-known manner (the Voronoi tiling of the face-centered cubic lattice $A_3$ [1, pp. 112, 459]), as figures 3 and 4 suggest for a 5-seat parliament. In figure 3 we represent the reference tetrahedron partly covered by rhombic dodecahedra; in figure 4 the tetahedron is completely covered; if we cut the complex of figure 4 by the plane of one of the faces of the tetrahedron, we obtain the honeycomb covering of that face, as shown in figure 2.

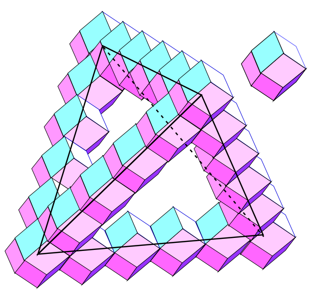

*figure* 3

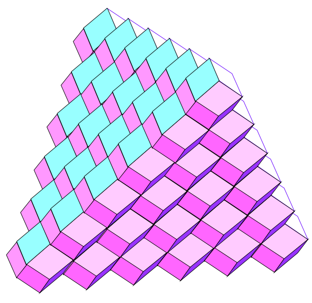

*figure* 4

For $p$ parties, the electoral cells turn out to be the Dirichlet-Voronoi cells for the lattice $A_{p-1}$ (see, *e.g.*, [1, 3]), so each cell is a convex polytope of dimension $p-1$. For a single cell we identify the cell's faces and completely describe its face lattice. Among other things, we determine the baricenter, diagonals and axes of each face, and its group of automorphisms. All faces of a given dimension are geometrically equal; in fact, each face of dimension $d$ arises from a $d$-unit-cube by a compression along a diagonal.

## 2. Electoral Cells and Voronoi Tilings

For concepts related to the theory of polytopes we send the reader to [4]. The following notation will be used throughout: $p$ is a fixed positive integer; $\langle\cdot|\cdot\rangle$ is the usual inner product in $\mathbb{R}^p$; for $S \subseteq \{1,\ldots,p\}$, $|S|$ denotes the cardinality of $S$, and $S^c$ is the complementary set $\{1,\ldots,p\}\backslash S$. In the sequel, as in the 'Electoral Cell Problem', $a = (a_1,\ldots,a_p)$ denotes a $p$-tuple of integers.

The set of all $x$ in $\mathfrak{H}_\Sigma$ with nonnegative entries is a regular simplex $\mathcal{S}$ of dimension $p-1$. The facets of $\mathcal{S}$ are the simplices $\Phi_k := \mathcal{S} \cap \{x : x_k = 0\}$. The distance of $x \in \mathcal{S}$ to the facet $\Phi_k$ is called the *$k$-th simplicial coordinate of $x$ in $\mathcal{S}$*. It is an easy exercise to show that this distance is $\alpha^{-1}x_k$, where $\alpha := \sqrt{1-1/p}$. This shows that we may represent each such $x$ by a point $R$ in the regular simplex $\alpha\mathcal{S}$ such that $x_k$ is the $k$-th simplicial coordinate of $R$ in $\alpha\mathcal{S}$. This generalizes the *Snapshots*' triangular and tetrahedral coordinates.

Moreover, this shows that the electoral cells as defined in the 'Electoral Cell Problem', and the corresponding tiling of $\mathcal{S}$, are homothetic (with factor $\alpha$) to their representations in simplicial coordinates in the simplex $\alpha\mathcal{S}$. So the Cartesian and the simplicial representations are essentially the same. In the sequel, we shall consider only the Cartesian setting.

**Theorem 2.1.** *Let $x$ be a real $p$-tuple of sum $\Sigma$. Then $x$ belongs to the electoral cell corresponding to $a$ if and only if, for all $i \neq j$:*

$$(a_i - x_i) + (x_j - a_j) < 1\,. \tag{1}$$

*Proof.* Assume $x$ determines $a$. Clearly $|x_i - a_i| < 1$, for all $i$; this implies (1) in case $x_i \geqslant a_i$ or $x_j \leqslant a_j$. If $x_i < a_i$ and $x_j > a_j$, the remainders are $r_i = 1 - a_i + x_i$ and $r_j = x_j - a_j$; in this case we have $r_i > r_j$, i.e., (1) holds, because party $i$ was assigned an extra seat, and party $j$ wasn't.

Conversely, assume (1) holds. As $x$ and $a$ have the same sum, there exist $i$ and $j$ such that $x_i \leqslant a_i$ and $x_j \geqslant a_j$; therefore $|x_k - a_k| < 1$, for all $k$.

Define: $P^+ := \{i : x_i < a_i\}$ and $P^- := \{j : x_j \geqslant a_j\}$. The remainders are $r_i = 1 + x_i - a_i$ for $i \in P^+$, and $r_j = x_j - a_j$ for $j \in P^-$. So the sum of all remainders is the cardinality of $P^+$. As (1) implies $r_i > r_j$ for all $i \in P^+$ and $j \in P^-$, the largest remainders method assigns one extra seat to each party in $P^+$, and no extra seat to the parties in $P^-$. So $x$ determines $a$. □

We shall work with the closure of the electoral cells. We get rid of parties and deputies and define, for any integer $p$-tuple $a$, the *a-cell*, denoted $\mathfrak{C}_a$, as the set of all $x$ such that:

$$\begin{cases} x_1 + \cdots + x_p = a_1 + \cdots + a_p \\ (a_i - x_i) + (x_j - a_j) \leqslant 1, \qquad \text{for all } i \neq j\,. \end{cases} \tag{2}$$

As $\mathfrak{C}_a$ is bounded and $x = a$ satisfies (2) holds with strict inequalities, $\mathfrak{C}_a$ is a convex polytope of dimension $p-1$. Clearly, $\mathfrak{C}_a = a + \mathfrak{C}_0$, and the family of $a$-cells contained in the hyperplane $\mathfrak{H}_0$ is a lattice tiling of the hyperplane $\mathfrak{H}_0$, with prototile $\mathfrak{C}_0$ [3, p. 756]. It is easy to check, using definition (2), that this is the Dirichlet-Voronoi tiling determined by the lattice $A_{p-1}$ of all integral points of $\mathfrak{H}_0$ ([1, p. 459], [3, p.756]). So Hamilton's method assigns, to each result $x$ of sum $\Sigma$, the integral point of sum $\Sigma$ which is closest to $x$ in the Euclidean metric (see [5, p.248]).

## 3. Remainders and Vertices of $\mathfrak{C}_0$

According to (2), $x \in \mathfrak{C}_0$ iff $x_1 + \cdots + x_p = 0$, and $x_j \leqslant 1 + x_i$, for $i \neq j$. We may characterize $\mathfrak{C}_0$, its relative interior and boundary, and its vertices in a simple way, by means of remainders:

**Theorem 3.1.**

(a) *Let $x \in \mathbb{R}^p$ have zero sum, and let $s$ be the number of negative coordinates of $x$. Then $x \in \mathfrak{C}_0$ iff there exist real numbers, $r_1, \ldots, r_p$, and a permutation matrix $P$, such that $1 > r_1 \geqslant \cdots \geqslant r_p \geqslant 0$ and*

$$Px = (r_1, r_2, \ldots, r_p) - (\underbrace{1, \ldots, 1}_{s}, 0, \ldots, 0)\,. \tag{3}$$

*This representation is uniquely determined by $x$.*

(b) *A point $x$ of $\mathfrak{C}_0$ belongs to the relative boundary of $\mathfrak{C}_0$ iff $r_s = r_{s+1}$.*

(c) *A point $x$ of $\mathfrak{C}_0$ belongs to the relative interior of $\mathfrak{C}_0$ iff $r_s > r_{s+1}$;*

(d) *A point $x$ of $\mathfrak{C}_0$ satisfies $|x_k| \leqslant 1 - 1/p$, for all $k$;*

(e) *A point $x$ of $\mathfrak{C}_0$ is a vertex of $\mathfrak{C}_0$ iff $x \neq 0$, and $r_1 = \cdots = r_p$.*

*Proof.* (a) is a reformulation of the last remainders method. We just sketch the 'only if' part of the proof. For $x \in \mathfrak{C}_0$, define $r_i := x_i$ if $x_i \geqslant 0$, and $r_j := 1 + x_j$ if $x_j < 0$. To get (3), reorder the $x_k$'s in such a way that the respective $r_k$'s come out in nonincreasing order. The other details follow from the inequalities $x_i \leqslant 1 + x_j$, and the uniqueness of (3) is obvious.

(b) We may assume that $x$ is already reordered to match the right hand side of (3). Let $x$ lie in the relative boundary of $\mathfrak{C}_0$. This is equivalent to $x_u = 1 + x_v$ for some $u, v$. We then have $x_u > 0$ and $x_v < 0$; therefore $v \leqslant s < u$, and $r_u = r_v$. This proves $r_s = r_{s+1}$. The converse follows by reversing the argument.

(c) is a simple consequence of (b).

(d) Let us fix $k$. As $-x \in \mathfrak{C}_0$, we may assume, without loss of generality, that $x_k \geqslant 0$. From $x_k > 1 - 1/p$ we get the following contradiction: from $x_k \leqslant 1 + x_i$ we have $x_i > -1/p$, for $i \neq k$, and so $x$ has a positive sum.

(e) We assume that $x$ coincides with the right hand side of (3). To prove the 'only if' part, we show that $x$ is not a vertex of $\mathfrak{C}_0$, whenever (i) $x = 0$, or (ii) two of the $r_i$ are distinct. Case (i) is obvious. In case (ii), there exists $t$, $0 < t < p$, such that $r_t > r_{t+1}$. Define

$$w := (\underbrace{-\epsilon, \ldots, -\epsilon}_{t}, \eta, \ldots, \eta),$$

where $\epsilon$ and $\eta$ are positive and $\eta = t\epsilon/(p - t)$. Then $w$ has zero sum. Now define two perturbations of $x$, say: $x^+ := x + w$, and $x^- := x - w$. Let $\epsilon$ be small enough so that, for all $i \neq j$:

$$x_j < 1 + x_i \;\Rightarrow\; \left[x_j^+ < 1 + x_i^+ \quad \text{and} \quad x_j^- < 1 + x_i^-\right].$$

Note that, in case $x_j = 1 + x_i$, we have $r_i = r_j$; therefore: $i, j \leqslant t$, or $i, j > t$; this implies $x_j^+ = 1 + x_i^+$ and $x_j^- = 1 + x_i^-$. Conclusion: for such small positive $\epsilon$, $x^+$ and $x^-$ belong to $\mathfrak{C}_0$. Thus $x$ is not a vertex of $\mathfrak{C}_0$.

Conversely, we show that the vectors

$$v_s := \left(\tfrac{s}{p}, \tfrac{s}{p}, \ldots, \tfrac{s}{p}\right) - (\underbrace{1, \ldots, 1}_{s}, 0, \ldots, 0), \tag{4}$$

for $0 < s < p$, are vertices of $\mathfrak{C}_0$. Assume that, for a given $w \in \mathbb{R}^p$, the two vectors $v_s \pm w$ are in $\mathfrak{C}_0$. We have $[v_s \pm w]_j - [v_s \pm w]_i \leqslant 1$, for all $i, j$. Looking at these inequalities in the cases $i \leqslant s < j$, we get $\pm w_j \leqslant \pm w_i$, that is, $w_i = w_j$. So the coordinates of $w$ are pairwise equal; as $w$ has zero sum, we must have $w = 0$. So $w$ is a vertex. $\square$

The following lemma will be useful in computing dimensions.

**Lemma 3.2.** *The $p-1$ vertices defined in* (4) *are linearly independent.*

*Proof.* Consider the $(p-1)\times p$ matrix

$$\begin{bmatrix} v_1 \\ \vdots \\ v_{p-1} \end{bmatrix} = \frac{1}{p}\begin{bmatrix} 1 & 1 & 1 & \dots & 1 \\ 2 & 2 & 2 & \dots & 2 \\ 3 & 3 & 3 & \dots & 3 \\ \vdots & \vdots & \vdots & & \vdots \\ {\scriptstyle p-1} & {\scriptstyle p-1} & {\scriptstyle p-1} & \dots & {\scriptstyle p-1} \end{bmatrix} - \begin{bmatrix} 1 & 1 & & & & \\ 1 & 1 & 1 & & & \\ 1 & 1 & 1 & 1 & & \\ \vdots & & & & \ddots & \\ 1 & 1 & 1 & 1 & \dots & 1 \end{bmatrix},$$

Let $M$ be the submatrix of the last $p-1$ columns:

$$M = \frac{1}{p}\begin{bmatrix} 1 & 1 & 1 & \dots & 1 \\ 2 & 2 & 2 & \dots & 2 \\ 3 & 3 & 3 & \dots & 3 \\ \vdots & \vdots & \vdots & & \vdots \\ {\scriptstyle p-1} & {\scriptstyle p-1} & {\scriptstyle p-1} & \dots & {\scriptstyle p-1} \end{bmatrix} - \underbrace{\begin{bmatrix} 1 & & & & \\ 1 & 1 & & & \\ 1 & 1 & 1 & & \\ & \vdots & & \ddots & \\ 1 & 1 & 1 & \dots & 1 \end{bmatrix}}_{L}.$$

If we multiply $M$, on the left, by the inverse of $L$, we obtain $\Omega_p - I_{p-1}$, where $\Omega_p$ denotes the $(p-1)\times(p-1)$ matrix with all entries equal to $1/p$. As $\Omega_p$ has eigenvalues $1-1/p$ and $0$, $\Omega_p - I_\rho$ is nonsingular; therefore $M$ is nonsingular. This proves the lemma. □

We have seen that the vertices of $\mathfrak{C}_0$ are the vectors obtained by permutations on the entries of the $v_s$ defined in (4). In other words, each proper, nonempty subset $S$ of $\{1,\dots,p\}$ determines a vertex, denoted by $\mathfrak{V}(S)$, given by

$$\big[\mathfrak{V}(S)\big]_i := \begin{cases} s/p-1, & \text{if } i\in S \\ s/p, & \text{if } i\notin S\,, \end{cases} \tag{5}$$

where $s$ denotes the cardinality of $S$. All vertices of $\mathfrak{C}_0$ are of this kind. To this, we add the conventions $\mathfrak{V}(\varnothing) = \mathfrak{V}(\{1,\dots,p\}) := 0$. We often simplify the notation, using $\mathfrak{V}_S$ for $\mathfrak{V}(S)$, and $\mathfrak{V}_t$ instead of $\mathfrak{V}(\{t\})$ (the context will make these notations clear). A straightforward calculation shows that $\mathfrak{V}(S\cup\{t\}) = \mathfrak{V}(S)+\mathfrak{V}_t$, in case $t$ is not an element of $S$. From this, and induction, we get the useful formula

$$\mathfrak{V}_{\{u_1,\dots,u_k\}} = \mathfrak{V}_{u_1}+\cdots+\mathfrak{V}_{u_k}\,, \tag{6}$$

where $u_1, \dots, u_k$ are distinct indices. This implies $\mathfrak{V}(S^c) = -\mathfrak{V}(S)$, and for any subsets $S$ and $T$ of $\{1, \dots, p\}$,

$$\mathfrak{V}(S) + \mathfrak{V}(T) = \mathfrak{V}(S \cup T) + \mathfrak{V}(S \cap T)\,. \tag{7}$$

**Theorem 3.3.** *$\mathfrak{C}_0$ is the orthogonal projection of the unit $p$-cube $[0,1]^p$ into the subspace $\mathfrak{H}_0$ of all $x$ of zero sum.*

*Proof.* The projection onto $\mathfrak{H}_0$ is given by

$$\operatorname{proj}(x) = x - \frac{x_1 + \cdots + x_p}{p}\,(1, \dots, 1)\,,$$

The vertices of $[0,1]^p$ are the $2^p$ $p$-tuples of 0's and 1's. If $w$ is such a vertex, it is easy to see that $\operatorname{proj}(w) = \mathfrak{V}_T$, where $T$ is the set of indices $t$ such that $w_t = 0$. As the projection of any polytope is the convex hull of the projections of its vertices, and any vertex of $\mathfrak{C}_0$ is the image of a vertex of $[0,1]^p$, the proposition is true. □

## 4. The Faces of $\mathfrak{C}_0$

In this section, we completely determine the lattice of faces of $\mathfrak{C}_0$, and the *scheme* of $\mathfrak{C}_0$ in the sense of [4, p.90]. For $i \neq j$, let $\mathcal{H}_{ij}$ be the hyperplane of equation $x_j = 1 + x_i$. $\mathfrak{C}_0$ has dimension $p-1$ and is bounded by the $p^2 - p$ hyperplanes $\mathcal{H}_{ij}$. Any set of the form

$$F := \mathcal{H}_{i_1 j_1} \cap \cdots \cap \mathcal{H}_{i_t j_t} \cap \mathfrak{C}_0 \tag{8}$$

is, therefore, a face of $\mathfrak{C}_0$, and any face of $\mathfrak{C}_0$ may be represented in this way. Clearly, $F$ is a proper face iff $t \geqslant 1$. Given a proper face (8) we define the sets:

$$I := \{i_1, \dots, i_t\} \quad \text{e} \quad J := \{j_1, \dots, j_t\}\,.$$

**Theorem 4.1.** *The vertex $\mathfrak{V}_S$ lies in $F$ iff $S \supseteq I$ and $S \cap J = \varnothing$.*

*Proof.* Let us be given $\mathfrak{V}_S \in F$, $i \in I$ and $j \in J$; then $\mathfrak{V}_S \in \mathcal{H}_{ij}$, that is, $\left[\mathfrak{V}_S\right]_j = 1 + \left[\mathfrak{V}_S\right]_i$; so we have $\left[\mathfrak{V}_S\right]_j > 0$ and $\left[\mathfrak{V}_S\right]_i < 0$; by definition (5) we must have $i \in S$ and $j \notin S$. The proof that $i \in S, j \notin S$ implies $\mathfrak{V}_S \in F$ is also easy. □

So the face (8) depends only on the *sets* $I$ and $J$ and not on the manner the $i_w, j_w$ are arranged in pairs. Accordingly, from now on we shall denote the face (8) by $\mathfrak{F}(I,J)$, or $\mathfrak{F}_{IJ}$, and the symbols $\mathfrak{F}, I$ and $J$ will consistently have the above meanings.

**Theorem 4.2.** *Let $I$ and $J$ be nonempty subsets of $\{1, \ldots, p\}$.*

(a) *The face $\mathfrak{F}_{IJ}$ is nonempty iff $I$ and $J$ are disjoint;*

(b) *The face $\mathfrak{F}_{IJ}$ has dimension $d := p - |I| - |J|$;*

(c) *The face $\mathfrak{F}_{IJ}$ has $2^d$ vertices;*

(d) *Given nonempty subsets, $I', J' \subseteq \{1, \ldots, p\}$ we have $\mathfrak{F}_{I'J'} \subseteq \mathfrak{F}_{IJ}$ if and only if: $I' \supseteq I$ and $J' \supseteq J$;*

(e) *$(I, J) \mapsto \mathfrak{F}_{IJ}$ is a one-to-one mapping from the set of of pairs $(I, J)$, with $I, J$ nonempty disjoint subsets of $\{1, \ldots, p\}$, onto the set of proper, nonempty faces of $\mathfrak{C}_0$;*

(f) *The number of faces of $\mathfrak{C}_0$ of dimension $d$, with $0 \leqslant d < p - 1$, is*

$$f_d = \binom{p}{d} \left[2^{p-d} - 2\right];$$

*$\mathfrak{C}_0$ has a total of $3^p - 2^{p+1} + 1$ of proper, nonempty faces.*

*Proof.* (a) is an immediate consequence of Theorem 4.1.

(b) $x$ lies in the face (8) iff $x$ satisfies the identities $x_{i_u} = 1 + x_{j_u}$, for $u = 1, \ldots, t$, and the inequalities $x_i \leqslant 1 + x_j$ for all $i, j$. Therefore, all $x_{i_u}$'s [all $x_{j_u}$'s] are pairwise equal. This means that $\mathfrak{F}_{IJ}$ is contained in the flat determined by the following $|I| + |J|$ equations:

$$\begin{cases} x_i = x_w & i, w \in I, \quad i \neq w \\ x_j = x_v & j, v \in J, \quad j \neq v \\ x_{i_1} = 1 + x_{j_1} & \\ \sum_i x_i = 0\,. & \end{cases} \tag{9}$$

It is easy to check that the equations (9) are linearly independent. Therefore, the flat (9) has dimension $d := p - |I| - |J|$, and so the dimension of $\mathfrak{F}_{IJ}$ is $\leqslant d$. To prove that $\mathfrak{F}_{IJ}$ has dimension $\geqslant d$, it is enough to find $d + 1$ affinely independent vertices of $\mathfrak{F}_{IJ}$. To do so we may assume, without loss of generality, that $I = \{1, \ldots, s\}$ e $J = \{s + d + 1, s + d + 2, \ldots, p\}$; with this reduction, the $d + 1$ vertices $v_s, v_{s+1}, \ldots, v_{s+d}$, as given in (4), all lie in $\mathfrak{F}_{IJ}$. Finally, by Lemma 3.2, these vertices are linearly independent.

(c) is obvious from the previous results.

(d) may be obtained from the formula $\mathfrak{F}_{IJ} \cap \mathfrak{F}_{KL} = \mathfrak{F}_{I\cup K, J\cup L}$, that the reader may easily prove by checking that both sides of the identity have the same vertices. The formula implies that $\mathfrak{F}_{I'J'} \subseteq \mathfrak{F}_{IJ}$ is equivalent to $\mathfrak{F}_{I'J'} = \mathfrak{F}_{I\cup I', J\cup J'}$; and the latter identity is equivalent to $(I', J') = (I \cup I', J \cup J')$, as you may prove using the dimension result (b).

The rest of the proof is left to the reader. □

## 5. The form of a proper face

In this section we give some geometrical features of the proper faces of $\mathfrak{C}_0$. In particular we determine the 'inner diagonals' of a face, and then show that any face may be obtained by a linear 'compression' applied to a unit cube of appropriate dimension.

We have seen that the vertices of the face $\mathfrak{F}_{IJ}$ are the $\mathfrak{V}(I \cup K)$, where $K$ is any subset of $(I \cup J)^c$.

**Theorem 5.1.** *The face $\mathfrak{F}_{IJ}$ is centrally symmetric, and its center of symmetry is $c_{IJ} := (\mathfrak{V}_I - \mathfrak{V}_J)/2$. For each $K \subseteq (I \cup J)^c$, define $K^* := (I \cup J)^c \backslash K$. Then $\mathfrak{V}_K$ and $\mathfrak{V}_{K^*}$ are opposite vertices of $\mathfrak{F}_{IJ}$.*

*Proof.* The unordered pairs $\{\mathfrak{V}(I \cup K)\,,\, \mathfrak{V}(I \cup K^*)\}$ (for all $K \subseteq (I \cup J)^c$) form a disjoint partition of the set of vertices of $\mathfrak{F}_{IJ}$. Using formulas (6)-(7) we obtain

$$\begin{aligned}\mathfrak{V}(I \cup K) + \mathfrak{V}(I \cup K^*) &= 2\mathfrak{V}(I) + \mathfrak{V}(K \cup K^*)\\ &= 2\mathfrak{V}(I) - \mathfrak{V}(I \cup J) = \mathfrak{V}_I - \mathfrak{V}_J.\end{aligned}$$

This shows that the segment

$$\big[\mathfrak{V}(I \cup K)\,,\, \mathfrak{V}(I \cup K^*)\big] \tag{10}$$

has $c_{IJ}$ as mid point. So the theorem follows. □

The segment (10) is said to be an *inner diagonal* of the face $\mathfrak{F}_{IJ}$. With some boring, straightforward calculations we may get an explicit value for the length of an inner diagonal:

$$\begin{aligned}\text{legth of (10)} &= \big\|\mathfrak{V}(I \cup K) - \mathfrak{V}(I \cup K^*)\big\|\\ &= \big\|\mathfrak{V}(K) - \mathfrak{V}(K^*)\big\| = \cdots = \sqrt{d - (d-2k)^2/p}\,,\end{aligned}$$

where $k$ is the cardinality of $K$, and $d$ is the dimension of $\mathfrak{F}_{IJ}$. Therefore, $[\mathfrak{V}(I)\,,\, \mathfrak{V}(J^c)]$ is the shortest inner diagonal and its length is $\sqrt{d(1-d/p)}\,$.

**Theorem 5.2.** *A proper face $\mathfrak{F}_{IJ}$ is a $d$-parallelotope. More explicitly, for any $K \subseteq (I \cup J)^c$, $\mathfrak{F}_{IJ} - \mathfrak{V}(I \cup K)$ is the following $d$-parallelotope (recall $K^* := (I \cup J \cup K)^c$):*

$$\mathfrak{F}_{IJ} - \mathfrak{V}(I \cup K) = \sum_{t \in K^*} [0, \mathfrak{V}_t] - \sum_{k \in K} [0, \mathfrak{V}_k]. \tag{11}$$

*Proof.* The proper face $\mathfrak{F}_{IJ}$ has $2d$ facets, namely the following

$$\mathfrak{F}(I\cup\{z\},J) \quad \text{e} \quad \mathfrak{F}(I,J\cup\{z\})\,, \tag{12}$$

where $z$ runs over $(I\cup J)^c$. Combining (7) with Theorem 4.1, if $\mathfrak{V}_S$ is a vertex of the latter facet in (12), then $\mathfrak{V}_S+\mathfrak{V}_z$ is a vertex of the former facet, and all vertices of the latter are translates, by $\mathfrak{V}_z$, of vertices of the former. Moreover, each vertex of $\mathfrak{F}_{IJ}$ is a vertex of one of the facets (12). So, each facet in (12) is a translate of the other (by $\pm\mathfrak{V}_z$), and $\mathfrak{F}_{IJ}$ is the convex hull of the union of these two facets. Therefore

$$\mathfrak{F}_{IJ} = \mathfrak{F}(I,J\cup\{z\})+[0,v(\{z\})] \tag{13}$$

$$= \mathfrak{F}(I\cup\{z\},J)-[0,v(\{z\})]. \tag{14}$$

We now go by induction on $d$. For $d=0$ the theorem is trivial. For a positive $d$, one of the sets $K,K^*$ is nonempty. Assume $K$ is nonempty, and let $z\in K$. By induction, we apply formula (11) with sets $I,J,K$ replaced by $I\cup\{z\}, J, K\backslash\{z\}$, respectively:

$$\mathfrak{F}(I\cup\{z\},J)-\mathfrak{V}(I\cup K)=\sum_{t\in K^*}[0,\mathfrak{V}_t]-\sum_{k\in K\backslash\{z\}}[0,\mathfrak{V}_k].$$

Combining this identity with (14) gives (11). The case of nonempty $K^*$ is similarly treated. □

We now single out with no proofs some simple consequences of the above results. If a segment $[a,b]$ is an edge of a polytope, we say that the two $p$-tuples $\pm(a-b)$ are the *vectors along* the edge $[a,b]$.

**Corollary 5.3.** *Let $S$ and $T$ be nonempty proper subsets of $\{1,\dots,p\}$.*

(i) *The segment $[\mathfrak{V}_S,\mathfrak{V}_T]$ is an edge of $\mathfrak{C}_0$ iff one of the sets $S,T$ is obtained from the other by removing one element.*

(ii) *The* degree *of a vertex $\mathfrak{V}_S$, i.e., the number of edges of $\mathfrak{C}_0$ emerging from $\mathfrak{V}_S$, is either $p$ or $p-1$. It is $p-1$ iff $S$ is a singleton or $S^c$ is a singleton.*

(iii) *The vectors along the edges of a given face $\mathfrak{F}_{IJ}$, are the vectors $\pm\mathfrak{V}_z$, for $z\in(I\cup J)^c$.* □

Given a nonzero $D\in\mathbb{R}^p$, and $\theta\in[0,1]$, the *$\theta$-compression along $D$* is the linear operator in $\mathbb{R}^p$, $x\mapsto x'$, that transforms $D$ in $D':=\theta D$, and fixes all elements of the subspace orthogonal to $D$. In case $\theta > 1$ we call such operator the *$\theta$-elongation along $D$*. Clearly

$$x' = x + (\theta - 1)\frac{\langle x|D\rangle}{\|D\|^2}\, D\,.$$

Now we fix a proper face $\mathfrak{F}_{IJ}$, and consider the elongation with the following parameters:

$$\theta_0 := 1/\sqrt{1 - d/p} \quad \text{and} \quad D_0 := \mathfrak{V}_{J^c} - \mathfrak{V}_I\,.$$

This is an elongation along the shortest inner diagonal of $\mathfrak{F}_{IJ}$, $[\mathfrak{V}_I\,,\,\mathfrak{V}_{J^c}]$.

**Theorem 5.4.** *The $\theta_0$-elongation along the shortest inner diagonal of $\mathfrak{F}_{IJ}$ transforms $\mathfrak{F}_{IJ}$ into a unit $d$-cube.*

*Proof.* We have $\langle \mathfrak{V}_w|\mathfrak{V}_z\rangle = \delta_{wz} - 1/p$, (the Kronecker 'delta'). Therefore

$$\langle \mathfrak{V}(S)|\mathfrak{V}(T)\rangle = |S \cap T| - |S| \cdot |T|/p\,.$$

From this general formula we get the image of $\mathfrak{V}_w$ by the $\theta$-elongation:

$$\mathfrak{V}_w{}' = \mathfrak{V}_w + \begin{cases} \big[(\theta_0 - 1)/d\big]\, D_0\,, & \text{if } w \notin I \cup J \\ \big[(\theta_0 - 1)/(d - p)\big]\, D_0\,, & \text{if } w \in I \cup J. \end{cases}$$

More boring computation then yields, for $w, z \notin I \cup J$: $\langle \mathfrak{V}_w{}'|\mathfrak{V}_z{}'\rangle = \delta_{wz}$. Now, apply our elongation to both members of (11) with, say, $K = \varnothing$:

$$\mathfrak{F}_{IJ}{}' - \mathfrak{V}(I)' = \sum_{t \notin I \cup J} [0, \mathfrak{V}_t{}'].$$

As we have seen, the $d$ vectors $\mathfrak{V}_t{}'$ form an orthonormal set. So the right hand side of the last identity is a unit $d$-cube. □

Note that each face $\mathfrak{F}_{IJ}$ may be elongated in many ways to be transformed into a unit cube. In general, we may choose any face $\mathfrak{F}_{RS}$ containing $\mathfrak{F}_{IJ}$ and then elongate $\mathfrak{F}_{RS}$ along the shortest inner diagonal of $\mathfrak{F}_{RS}$, with parameter $\theta = 1/\sqrt{1 - \epsilon/p}$, where $\epsilon$ is the dimension of $\mathfrak{F}_{RS}$. Then $\mathfrak{F}_{RS}$ transforms into a unit $\epsilon$-cube; so all its faces, including $\mathfrak{F}_{IJ}$, are transformed into unit cubes of appropriate dimensions.

# 6. The Automorphisms of $\mathfrak{C}_0$

An *automorphism of* $\mathfrak{C}_0$ is a linear operator of the space $\mathfrak{H}_0$ that transforms $\mathfrak{C}_0$ onto itself. It is easy to determine the group $\operatorname{Aut}\mathfrak{C}_0$ of automorphisms of $\mathfrak{C}_0$. For that purpose, we let $\mathbb{S}$ be the convex hull of the $p$ vertices $\mathfrak{V}_1,\ldots,\mathfrak{V}_p$ of $\mathfrak{C}_0$ [cf. notation (5)-(6)], and denote by $\mathbb{H}$ the convex hull of $\mathbb{S}\cup(-\mathbb{S})$. Note that, by Corollary 5.3, $\mathbb{H}$ is the convex hull of the vertices of $\mathfrak{C}_0$ of degree $p-1$. As an automorphism of a polytope is a degree preserving mapping, we have $\operatorname{Aut}\mathfrak{C}_0\subseteq\operatorname{Aut}\mathbb{H}$ . $\mathbb{S}$ is a regular simplex centered at the origin of $\mathfrak{H}_0$. So the automorphisms of $\mathbb{S}$ are the mappings of the kind $\mathfrak{H}_0\to\mathfrak{H}_0$, $x\mapsto xM$, where $M$ is a $p\times p$ permutation matrix.

**Lemma 6.1.** *The image of* $\mathbb{S}$ *under an automorphism of* $\mathbb{H}$ *is either* $\mathbb{S}$ *or* $-\mathbb{S}$.

*Proof.* Let $\varphi$ be an automorphism of $\mathbb{H}$. For each vertex $w$ of $\mathbb{S}$, $\varphi(w)$ is one of the $2p$ vertices $\pm\mathfrak{V}_1,\ldots,\pm\mathfrak{V}_p$. Let $K$ be the set of the $k$ such that $\mathfrak{V}_k$ is a vertex of $\varphi[\mathbb{S}]$. Then the other vertices of $\varphi[\mathbb{S}]$ are the $-\mathfrak{V}_t$ for $t\notin K$. The sum of the vertices of $\mathbb{S}$, and the sum of the vertices of $\varphi[\mathbb{S}]$ are both 0; therefore, $\sum_{k\in K}\mathfrak{V}_k=0$. So the face $\operatorname{conv}\{\mathfrak{V}_k : k\in K\}$ of $\mathbb{S}$ is either empty or it is $\mathbb{S}$ itself. The lemma follows. $\square$

This lemma shows that $\operatorname{Aut}\mathbb{H}=\operatorname{Aut}\mathbb{S}\cup[-\operatorname{Aut}\mathbb{S}]$. As $\mathfrak{C}_0$ is invariant under permutations of the $\mathbb{R}^p$'s coordinates and $-\mathfrak{C}_0=\mathfrak{C}_0$, we obtain

**Theorem 6.2.** $\operatorname{Aut}\mathfrak{C}_0=\operatorname{Aut}\mathbb{H}=\operatorname{Aut}\mathbb{S}\cup[-\operatorname{Aut}\mathbb{S}]$. $\square$

# 7. Examples in Low Dimensions

*Case* $p=3$.

In this case, $\mathfrak{C}_0$ is a regular hexagon in the plane $x_1+x_2+x_3=0$. $\mathbb{S}$ is an equilateral triangle, and $\mathfrak{C}_0=\mathbb{H}$. Check figures 2 and 4.

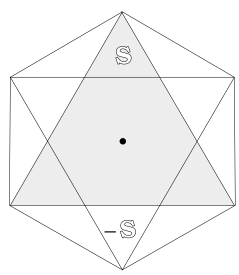


*figure* 5

*Case* $p = 4$.

The cell $\mathfrak{C}_0$ is a regular rhombic dodecahedron. $\mathbb{S}$ and $-\mathbb{S}$ are regular tetrahedra, and $\mathbb{H}$ is a cube; $\mathbb{S}$, $-\mathbb{S}$ and $\mathbb{H}$ are depicted in figure 5. The vertices of the cube are those of the dodecahedron of degree 3. The convex hull of the vertices of degree 4 is an octahedron (see figure 6). Obviously, the cube and the dodecahedron have the same automorphisms (Theorem 6.2).

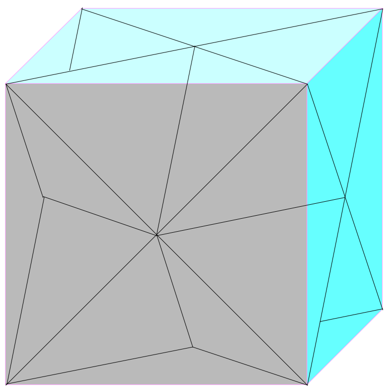

*figure* 6

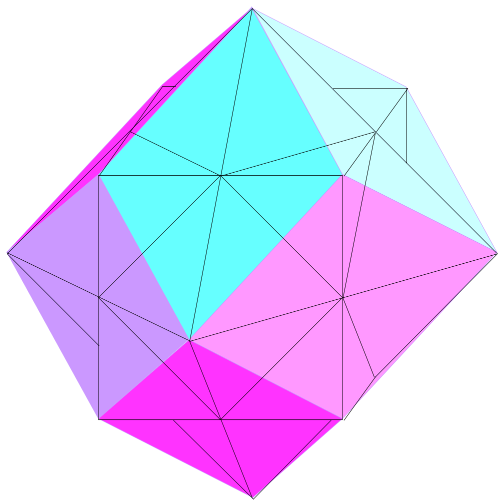

*figure* 7

## 8. Neighbouring Cells

In this section we briefly present some simple, expected facts on the relationship between $\mathfrak{C}_0$ and those cells $\mathfrak{C}_a$ that intersect $\mathfrak{C}_0$. We just mention the facts and leave the proofs as exercises. Here, $a = (a_1, \dots, a_p)$ denotes a $p$-tuple of integer coordinates of sum 0.

(1) *$\mathfrak{C}_a$ intersects $\mathfrak{C}_0$ iff any coordinate of $a$ is* 0, 1 *or* $-1$.
(2) *In that case, the intersection of $\mathfrak{C}_a$ with $\mathfrak{C}_0$ is the face $\mathfrak{F}_{IJ}$ of $\mathfrak{C}_0$, where $I := \{i : a_i = 1\}$ and $J := \{j : a_j = -1\}$.*
(3) *So, the dimension of $\mathfrak{C}_a \cap \mathfrak{C}_0$ is the number of zero entries of $a$.*
(4) *$\mathfrak{F}_{KL}$ is the intersection of $\mathfrak{C}_0$ with a neighbouring cell iff $|K| = |L|$.*
(5) *The dimension of the intersection of two neighbour cells has the same parity as $p$.*
(6) *The set $\mathfrak{F}_{IJ}$ is a face of $\binom{|I|+|J|}{|I|}$ distinct cells $\mathfrak{C}_a$.*
(7) *The total number of cells $\mathfrak{C}_a$ whose intersection with $\mathfrak{C}_0$ has dimension $d$ is $\binom{p}{k}\binom{p-k}{k}$, where $k$ denotes $(p-d)/2$.*

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