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# LOCAL HOMEOMORPHISMS VIA ULTRAFILTER CONVERGENCE

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ABSTRACT: Using the ultrafilter-convergence description of topological spaces, we generalize Janelidze-Sobral characterization of local homeomorphisms between finite topological spaces, showing that local homeomorphisms are the pullback-stable discrete fibrations.

KEYWORDS: local homeomorphism, ultrafilter.
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## Introduction

In this paper we complete the study of some special classes of continuous maps using convergence (see [7, 6, 2, 3, 4]). The recent interest on this had as starting point Janelidze-Sobral paper [6], where the authors presented characterizations of proper, perfect, open, triquotient maps and local homeomorphisms between finite topological spaces using point convergence. This motivated the study of the infinite version of these characterizations, successfully established in [3], with the exception of local homeomorphisms. This paper fills the remaining gap.

In [6], interpreting a finite topological space as a category via its point-convergence description, it is proved that:

> **Theorem I.** *A continuous map between finite topological spaces is a local homeomorphism if and only if it is a discrete fibration.*

For infinite spaces, replacing point convergence by ultrafilter convergence, local homeomorphisms turn out to be the pullback-stable discrete fibrations. Indeed, in this paper we present an example of a discrete fibration which is not a local homeomorphism, and we prove:

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**Theorem II.** *A continuous map between topological spaces is a local homeomorphism if and only if it is a pullback-stable discrete fibration.*

Note that since general (infinite) topological spaces cannot be viewed as categories, what we call the discrete fibrations of topological spaces has to be *defined*; however our definition (see Section 2) is a straightforward imitation of the categorical one: *just interpret arrows as convergence.*

We also introduce the notion of a $\lambda$-*space* (where $\lambda$ is a cardinal number), and show that pullback stability comes for free whenever the domain of the given map is a $\lambda$-space. This is important because amongst the $\lambda$-spaces we have both *Alexandrov* and *first countable T1-spaces*, that is, we have the "strange spaces" we generalized and the spaces that occur in classical geometry at the same time.

## 1. Local homeomorphisms and open maps

We recall that a *local homeomorphism* is a continuous map $f : X \to Y$ which is locally a homeomorphism; that is, each $x \in X$ has an open neighbourhood $U$ such that $f(U)$ is open and the map $f|_U : U \to f(U)$ is a homeomorphism.

It is well-known that:

**Proposition 1.** (1) *For a continuous map $f : X \to Y$, consider the following commutative diagram, where the square is a pullback:*

$$\begin{tikzcd}
X \arrow[dr, "\delta_f"] \arrow[drr, bend left, "1_X"] \arrow[ddr, bend right, "1_X"'] & & \\
& X \times_Y X \arrow[r, "\pi_2"] \arrow[d, "\pi_1"'] & X \arrow[d, "f"] \\
& X \arrow[r, "f"] & Y.
\end{tikzcd} \tag{A}$$

*The following assertions are equivalent:*

(a) *$f$ is a local homeomorphism;*

(b) *$f$ is open and locally injective;*

(c) *both $f$ and $\delta_f$ are open maps.*

(2) *Local homeomorphisms are pullback-stable.*

The following results can be found in [3]:

**Theorem 1.** (1) *A continuous map $f : X \to Y$ is open if and only if, for each $x \in X$ and each ultrafilter $\mathfrak{y}$ with $\mathfrak{y} \to f(x)$ in $Y$, there exists an ultrafilter $\mathfrak{x}$ such that $\mathfrak{x} \to x$ in $X$ and $f(\mathfrak{x}) = \mathfrak{y}$; we display this as*

$$\begin{tikzcd} X \arrow[d, "f"'] & \mathfrak{x} \arrow[r, dashed] \arrow[d, mapsto] & x \arrow[d, mapsto] \\ Y & \mathfrak{y} \arrow[r] & f(x). \end{tikzcd} \tag{B}$$

(2) *If $f : X \to Y$ is a local homeomorphism, then the ultrafilter $\mathfrak{x}$ in* (A) *is unique.*

## 2. Discrete fibrations

One can obviously imitate many categorical notions in topology simply by replacing arrows=morphisms with arrows representing ultrafilter convergence. In particular the continuous maps of topological spaces will then come up as an imitation of functors, and then the existence and uniqueness of liftings displayed in (B) will come up as an imitation of the discrete-fibration condition. Hence we define a discrete fibration of topological spaces as a continuous map $f : X \to Y$ with

$$\begin{tikzcd} X \arrow[d, "f"'] & (\exists^1)\ \mathfrak{x} \arrow[r, dashed] \arrow[d, mapsto] & x \arrow[d, mapsto] \\ Y & \mathfrak{y} \arrow[r] & f(x), \end{tikzcd}$$

which means that, for each $x \in X$ and each ultrafilter $\mathfrak{y}$ with $\mathfrak{y} \to f(x)$ in $Y$, there exists a unique ultrafilter $\mathfrak{x}$ such that $\mathfrak{x} \to x$ in $X$ and $f(\mathfrak{x}) = \mathfrak{y}$.

The class of discrete fibrations of topological spaces will be denoted by $\mathcal{H}$.

For each topological space $X$, we denote by $\mathrm{Conv}(X)$ the set of pairs $(\mathfrak{x}, x)$, where $x$ is a point, and $\mathfrak{x}$ an ultrafilter converging to $x$ in $X$. The set $\mathrm{Conv}(X)$ has a canonical, but not necessarily topological, convergence structure, giving rise to a functor Ult studied in [3]; however we will not use that structure here. Each continuous map $f : X \to Y$ induces a map $\mathrm{Conv}(f) : \mathrm{Conv}(X) \to \mathrm{Conv}(Y)$ with $(\mathfrak{x}, x) \mapsto (f(\mathfrak{x}), f(x))$. Clearly, a continuous map $f : X \to Y$

belongs to $\mathcal{H}$ if and only if the diagram

$$\begin{tikzcd}
\mathrm{Conv}(X) \arrow[r, "\mathrm{Conv}(f)"] \arrow[d, "\pi_X"'] & \mathrm{Conv}(Y) \arrow[d, "\pi_Y"] \\
X \arrow[r, "f"] & Y
\end{tikzcd}$$

is a pullback in **Set**; here and below $\pi_X$ and $\pi_Y$ are the projection maps.

Using this formulation it is easy to prove that:

**Proposition 2.** (1) *Let $f : X \to Y$ and $g : Y \to Z$ be continuous maps. If $g \cdot f$ and $g$ are in $\mathcal{H}$, then so is $f$.*

(2) *Given, in* **Top**, *a pullback diagram*

$$\begin{tikzcd}
W \arrow[r, "k"] \arrow[d, "h"'] & X \arrow[d, "f"] \\
Z \arrow[r, "g"] & Y
\end{tikzcd}$$

*such that $k$ is an injective map, if $f$ is in $\mathcal{H}$ then so is $h$. In particular, $\mathcal{H}$ is stable under pullbacks along injective continuous maps.*

*Proof*: To prove (1), consider the following commutative diagram in **Set**

$$\begin{tikzcd}
\mathrm{Conv}(X) \arrow[r, "\mathrm{Conv}(f)"] \arrow[d, "\pi_X"'] & \mathrm{Conv}(Y) \arrow[r, "\mathrm{Conv}(g)"] \arrow[d, "\pi_Y"'] & \mathrm{Conv}(Z) \arrow[d, "\pi_Z"] \\
X \arrow[r, "f"] & Y \arrow[r, "g"] & Z.
\end{tikzcd}$$

Since, by hypothesis, the outer diagram and the right-hand square are pullback diagrams, also the left-hand square is a pullback, and the result follows.

(2): We only need to show that, in the commutative diagram

$$\begin{tikzcd}
& W \arrow[rr, "k"] \arrow[dd, "h" near start] & & X \arrow[dd, "f"] \\
\mathrm{Conv}(W) \arrow[ur, "\pi_W"] \arrow[rr, "\mathrm{Conv}(k)" near end] \arrow[dd, "\mathrm{Conv}(h)"'] & & \mathrm{Conv}(X) \arrow[ur, "\pi_X"] \arrow[dd, "\mathrm{Conv}(f)" near start] & \\
& Z \arrow[rr, "g" near start] & & Y \\
\mathrm{Conv}(Z) \arrow[ur, "\pi_Z"] \arrow[rr, "\mathrm{Conv}(g)"'] & & \mathrm{Conv}(Y) \arrow[ur, "\pi_Y"'] &
\end{tikzcd}$$

the maps $\pi_W : \mathrm{Conv}(W) \to W$ and $\mathrm{Conv}(h) : \mathrm{Conv}(W) \to \mathrm{Conv}(Z)$ are jointly monic. This is immediate since $\pi_X$ and $\mathrm{Conv}(f)$ are jointly monic, because $f$ is in $\mathcal{H}$ and $\mathrm{Conv}(k)$ is injective (since so is $k$) by hypothesis. ∎

Discrete fibrations do not need to be local homeomorphisms, as the following example shows:

**Example.** Let $\mathfrak{x}$ be a non-principal ultrafilter on the set $\mathbb{N}$ of natural numbers and consider $\mathbb{N}$ endowed with the topology $\{A \subseteq \mathbb{N} \,|\, 0 \in A \;\Rightarrow\; A \in \mathfrak{x}\}$. Consider the Sierpinski space $\{0,1\}$, with the non-trivial open subset $\{1\}$. Let

$$\begin{aligned} f : \mathbb{N} &\to \{0,1\} \\ n &\mapsto \begin{cases} 0 & \text{if } n = 0, \\ 1 & \text{otherwise.} \end{cases} \end{aligned}$$

Then $f$ is not a local homeomorphism since it is not injective on any neighbourhood of 0. However it belongs clearly to $\mathcal{H}$: if $n \in \mathbb{N} \setminus \{0\}$ and $\mathfrak{y} \to f(n) = 1$ in $\{0,1\}$, then $\mathfrak{y} = \dot{1}$ (=the ultrafilter generated by $\{1\}$) and so $\dot{n}$ is the unique ultrafilter in $\mathbb{N}$ with image $\mathfrak{y}$ converging to $n$; if $n = 0$, we have two possibilities: $\dot{0} \to 0$, whose unique lifting is $\dot{0}$, and $\dot{1} \to 0$, whose unique lifting is $\mathfrak{x}$.

## 3. The characterization theorem

We consider again Diagram (A).

**Theorem 2.** *For a continuous map $f : X \to Y$, the following conditions are equivalent:*

(i) *$f$ is a local homeomorphism;*
(ii) *$f$ is stably in $\mathcal{H}$;*
(iii) *both $f$ and $\pi_1 : X \times_Y X \to X$ belong to $\mathcal{H}$;*
(iv) *both $f$ and $\delta_f$ belong to $\mathcal{H}$;*
(v) *for each indiscrete space $Z$, $f \times 1_Z : X \times Z \to Y \times Z$ belongs to $\mathcal{H}$;*
(vi) *both $f$ and $f \times 1_{X'} : X \times X' \to Y \times X'$ belong to $\mathcal{H}$, whenever $X'$ is the indiscrete space with the underlying set of $X$.*

*Proof*: (i) $\Rightarrow$ (ii) follows from Proposition 1. (ii) $\Rightarrow$ (iii) is trivial, while to prove (iii) $\Rightarrow$ (iv) we use Proposition 2 and the equality $\pi_1 \cdot \delta_f = 1_X$.

From (iv) it follows that both $f$ and $\delta_f$ are open maps, hence $f$ is a local homeomorphism, by Proposition 1. Therefore the first four conditions are equivalent. It is also clear that (ii) $\Rightarrow$ (v) $\Rightarrow$ (vi), and so to complete the proof it suffices to show that (vi) implies (iii). This follows from Proposition 2, observing that the left-hand square in the diagram below is a pullback and $\langle f, 1\rangle$ is monic:

$$\begin{tikzcd}
X \times_Y X \arrow[r] \arrow[d, "\pi_1"'] \arrow[rr, bend left, "\pi_2"] & X \times X' \arrow[r, "p_X"] \arrow[d, "f\times 1_{X'}"] & X \arrow[d, "f"] \\
X \arrow[r, "\langle f,1\rangle"'] \arrow[rr, bend right, "f"'] & Y \times X' \arrow[r, "p_Y"'] & Y.
\end{tikzcd}$$

■

## 4. When local homeomorphisms and discrete fibrations coincide

The equivalence between local homeomorphisms and discrete fibrations obtained by Janelidze and Sobral in [6] for finite topological spaces can be obtained in a more general setting that includes Alexandroff spaces and first countable $T_1$-spaces. To present it, we introduce $\lambda$-*spaces*, for $\lambda$ a cardinal number: these are the topological spaces $X$ such that:

(a) every neighbourhood filter has a base with cardinality at most $\lambda$;
(b) each subset of $X$ with cardinality less than $\lambda$ is closed.

The 0-*spaces* are the indiscrete spaces, the 1-*spaces* ($n$-*spaces* for each finite cardinal $n$, respect.) are the Alexandroff ($T_1$-)spaces, while the $\aleph_0$-*spaces* are the first countable $T_1$-spaces. Assuming CH, an example of an $\aleph_1$-*space* is the line $\mathbb{R}$ with the cocountable topology.

**Theorem 3.** *If $X$ is a $\lambda$-space, then a continuous map $f : X \to Y$ is a local homeomorphism if and only if it is a discrete fibration.*

*Proof*: We only have to check that $f : X \to Y$ is locally injective whenever it is a discrete fibration. In order to do that, we assume that $f$ is not locally injective, that is

$$\exists x \in X \ : \ \forall U \in \mathcal{O}(x)\ \exists a, b \in U \ : \ f(a) = f(b),$$

where $\mathcal{O}(x)$ denotes the set of open neighbourhoods of $x$. Then there are two possibilities, which we will consider separately:

(a) We have

$$\forall U \in \mathcal{O}(x) \ \ \exists a_U \in U \setminus \{x\} \ : \ f(a_U) = f(x).$$

Consider an ultrafilter $\mathfrak{a}$ in $X$ such that $\mathfrak{a} \supseteq \mathcal{O}(x)$ and $\{a_U\,;\, U \in \mathcal{O}(x)\} \in \mathfrak{a}$. By construction it converges to $x$ and its image by $f$ is $\dot{f(x)}$ . Since $\dot{f(x)}$ may also be lifted by the ultrafilter $\mathfrak{b} = \dot{x} \neq \mathfrak{a}$, the assumption of the Theorem fails.

(b) If there is a neighbourhood $V$ of $x$ such that $f(x) \notin f(V \setminus \{x\})$, we have

$$\forall U \in \mathcal{O}(x) \ \ \exists a, b \in U \ : \ a \neq b \ \text{ and } \ f(a) = f(b) \neq f(x).$$

The result is obvious for $\lambda = 0$, and so we consider now $\lambda \neq 0$. Since each subset of $X$ with cardinality less than $\lambda$ is closed, for every neighbourhood $U$ the cardinality of the set $A := \{\{a, b\} \subseteq U \,|\, b \neq a \text{ and } f(a) = f(b) \neq f(x)\}$ is at least $\lambda$. Otherwise the set $U \setminus \bigcup A$, which does not satisfy the condition above, would be a neighbourhood of $x$.

Let $\{U_\alpha\,;\,\alpha \in \lambda\}$ be a neighbourhood base of $x$. We may choose two disjoint injective families $(a_\alpha)_{\alpha\in\lambda}$ and $(b_\alpha)_{\alpha\in\lambda}$ (that is, two injective maps $a, b : \lambda \to X$ with disjoint images) such that both $a_\alpha$ and $b_\alpha$ belong to $U_\alpha$ and $f(a_\alpha) = f(b_\alpha) \neq f(a)$ for each $\alpha \in \lambda$. Let $\mathfrak{a}$ be an ultrafilter converging to $x$ and to which $\{a_\alpha\,;\,\alpha \in \lambda\}$ belongs. The ultrafilter $\mathfrak{b}$ obtained by replacing in $\mathfrak{a}$ each $a_\alpha$ with $b_\alpha$ is different from $\mathfrak{a}$, although $f(\mathfrak{b}) = f(\mathfrak{a})$ and both converge to $x$. ∎

**Remarks.** (1) The part (a) of our proof does not use $\lambda$ and is obviously valid for arbitrary spaces. In fact this shows that every discrete fibration is an open map with discrete fibres. Therefore the class of discrete fibrations, which itself is not pullback stable, is however strictly between two pullback stable classes, that are not too far from each other – namely of local homeomorphisms and of open maps with discrete fibres.

(2) Is there a reasonable simultaneous generalization of the notions of a discrete fibration for categories and for topological spaces? Most probably this question has several good answers; a straightforward one would be just to copy both (=any of the two) definitions to $T$-categories in the sense of Burroni [1]. One could also try the far more general context of Clementino-Tholen [5].

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