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ON REFLECTIVE SUBCATEGORIES OF VARIETIES

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Abstract: Full reflective subcategories of varieties are characterized as the cocomplete categories with a regular generator, or as classes of algebras presented by "preequations". As a byproduct, a solution is presented to the problem of describing ω-orthogonality classes of locally finitely presentable categories in terms of closure properties.

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1. Introduction

By the Birkhoff Variety Theorem, equational classes of algebras (varieties) are exactly the classes closed under products, subalgebras and quotient algebras. Analogously, the quasivarieties, i.e., classes presented by *quasiequations* or implications of the following form

$$\forall (x_u)_{u\in U} \left[\bigwedge_{i\in I} \alpha_i(x_u) \to \bigwedge_{j\in J} \beta_j(x_u) \right]$$

where α_i and β_j are equations in the variables $\{x_u\}_{u\in U}$, are precisely the classes closed under products and subalgebras. That is, the full subcategories of $\mathrm{Alg}\,\Sigma$, where Σ is a (potentially infinitary, many-sorted) signature, which are reflective, and the reflections are regular epimorphisms. In the present paper we study full reflective subcategories of $\mathrm{Alg}\,\Sigma$ in general. We call them *prevarieties.*

Whereas quasivarieties (and varieties) have been characterized as the cocomplete categories with a regular generator formed by regular projectives (or exact projectives, respectively), see [2], we prove that prevarieties are just the cocomplete categories with a regular generator. All these results assume that the signature Σ is allowed to be large (a proper class of operations); in that case the definition of a prevariety $\mathcal{V}$ has to be supplemented by the requirement that free algebras exist.

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Prevarieties can be characterized syntactically as classes of algebras which can be presented by *preequations*, i.e., formulas of the following form

$$\forall (x_u)_{u\in U} \left[\bigwedge_{i\in I} \alpha_i(x_u) \to \exists! (y_v)_{v\in V} \bigwedge_{j\in J} \beta_j(x_u,\, y_v) \right] . \tag{1}$$

These are precisely the limit sentences in the logic $L_{\infty\infty}$ in the sense of [4].

Example: posets. The category $\mathcal{P}os$ of posets and order-preserving functions does not have a regularly projective regular generator, that is, this is not a quasivariety. But it is a prevariety, presented by two 2-sorted unary operations (source and target)

$$s,\, t : e \to v$$

where the set $S = \{e,\, v\}$ of sorts has two members: e for "edges" and v for "vertices". A natural presentation by preequations specifies that (1) an edge is determined by its domain and codomain:

$\forall (y, z)\,([(s\,y = s\,z) \wedge (t\,y = t\,z)] \to (y = z))$,

and that (2) the resulting relation is reflexive:

$\forall p\, \exists! z[(s\,z = p) \wedge (t\,z = p)]$,

antisymmetric:

$\forall (y, z)\,([(s\,y = t\,z) \wedge (s\,z = t\,y)] \to (y = z))$

and transitive:

$\forall (y, z)\,((t\,y = s\,z) \to \exists! x\,[(s\,x = s\,y) \wedge (t\,x = t\,z)])$

(Here p is a variable of sort v and x, y, z are variables of sort e.)

Prevarieties naturally generalize the locally presentable categories of Gabriel and Ulmer: if the given regular generator is assumed to consist of λ-presentable objects, then the prevariety is locally λ-presentable. And conversely, every locally λ-presentable category is equivalent to such a prevariety, see [3]. To mention examples outside of the realm of locally presentable categories: the category of compact T_2-spaces is a variety, thus, every reflective subcategory, e.g., the dual category of that of boolean algebras (zero-dimensional compact T_2-spaces) is a prevariety.

The most interesting special case of prevarieties are the *finitary prevarieties*, i.e., classes of finitary algebras presented by preequations of the finitary first-order logic (i.e., all the indexing sets I, J, U and V in (1) are finite) as the example $\mathcal{P}os$ above demonstrates.

We characterize finitary prevarieties as the classes $\mathcal{A}$ of finitary algebras closed in $\mathrm{Alg}\,\Sigma$ under

(i) limits

(ii) directed colimits

and

(iii) $\mathcal{A}$-pure subobjects.

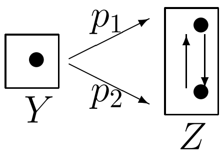
The last notion is a relativization of the concept of a pure subobject which is introduced in the present paper in order to solve the more general problem left open from previous work: a characterization of ω-orthogonality classes; see Section 5 for a short survey. Here we just recall that a homomorphism $m : B \hookrightarrow A$ in $\mathrm{Alg}\,\Sigma$ is called *pure* provided that every positive-primitive formula of the first-order logic valid in A is valid in B. Categorically, this means that in every commutative square

$$\begin{tikzcd} X \arrow[r, "f"] \arrow[d, "u"'] & Y \arrow[d, "v"] \\ B \arrow[r, "m"] & A \end{tikzcd}$$

where X and Y are finitely presentable Σ-algebras the homomorphism u factorizes through f. Unfortunately, it is not true in general that every class of Σ-algebras closed under limits, directed colimits and pure subojects is a finitary prevariety – a counterexample, essentially due to H. Volger ([11]), is given in 4.5 (see also Remark 5.5). We therefore introduce, for every full subcategory $\mathcal{A}$ of $\mathrm{Alg}\,\Sigma$, the following concept of an $\mathcal{A}$-pure subobject: it is precisely as above except that we request f to be an $\mathcal{A}$-epimorphism (i.e., given a parallel pair $p_1, p_2 : Y \to Z$ with $Z \in \mathcal{A}$ then $p_1 f = p_2 f$ implies $p_1 = p_2$). We prove that the above conditions (i)-(iii) characterize finitary prevarieties; the meaning of (iii) is, as expected, that for every algebra $A \in \mathcal{A}$ and every $\mathcal{A}$-pure $m : B \to A$ we have $B \in \mathcal{A}$. A surprising corollary is that if a class $\mathcal{A}$ of algebras is *cogenerating*, i.e., if for every pair of distinct homomorphisms $p_1, p_2 : Y \to Z$ in $\mathrm{Alg}\,\Sigma$ there exists $q : Z \to A$, $A \in \mathcal{A}$, with $qp_1 \neq qp_2$, then

$\mathcal{A}$ is a finitary prevariety $\Leftrightarrow$ $\mathcal{A}$ is a finitary quasivariety.

Thus, for example, every finitary prevariety of lattices containing the two-elements lattice is a quasivariety. In the above example $\mathcal{P}os$ cannot be cogenerating – in fact, consider the two graph homomorphism



as homomorphisms of Σ-algebras: we have $qp_1 = qp_2$ for every homomorphism q where the codomain is antisymmetric.

2. An abstract characterization

2.1. Definition. A category is called a *prevariety* if it is equivalent to a full reflective subcategory of a category monadic over a power of $\mathcal{S}et$.

2.2. Examples. (1) Every locally presentable category of Gabriel and Ulmer is a prevariety. In fact, let $\mathcal{K}$ be locally λ-presentable and let $\mathcal{A}$ be a small subcategory representing all λ-presentable objects. Then the canonical functor $E : \mathcal{K} \to \mathcal{S}et^{\mathcal{A}^{\mathrm{op}}}$, given by $K \mapsto \mathcal{K}(-, K)/\mathcal{A}^{\mathrm{op}}$, is a full and faithfull right adjoint, see [3]. The presheaf category $\mathcal{S}et^{\mathcal{A}^{\mathrm{op}}}$ is of course monadic over $\mathcal{S}et^S$, where $S = \mathrm{obj}(\mathcal{A})$, via the forgetful functor $F : \mathcal{S}et^{\mathcal{A}^{\mathrm{op}}} \to \mathcal{S}et^S$. And $\mathcal{K}$ is equivalent to the full reflective subcategory $E[\mathcal{K}]$.

(2) Every monadic category on $\mathcal{S}et^S$ is, of course, a prevariety. This includes examples such as compact Hausdorff topological spaces and complete semilattices.

(3) The dual of the category of boolean algebras, equivalently, the category of all zero-dimensional compact Hausdorff spaces, is a prevariety: the latter is a full reflective subcategory of the category of compact Hausdorff spaces. This shows that prevarieties are, in fact, a substantial extension of locally presentable categories (for which Gabriel and Ulmer showed that, with the exception of partially ordered classes, the dual category is never locally presentable).

2.3. Remark. Recall that a *regular generator* in a category $\mathcal{K}$ is a small collection $\mathbb{G}$ of objects such that for every K the canonical morphism

$$e_K : \coprod_{G \in \mathbb{G}} \mathcal{K}(G, K) \circ G \to K$$

is well-defined (i.e., the coproduct in the domain exists) and is a regular epimorphism. (Here $M \circ G$ denotes the copower of G indexed by M.)

Examples. (1) In an S-sorted quasivariety of algebras the collection $\{G_s\}_{s \in S}$, where G_s is a free algebra on one element of sort s, is a regular generator.

(2) In [1] a cocomplete category $\mathcal{B}$ is found which

(a) does not have a regular generator

but

(b) has an object B such that all objects of $\mathcal{B}$ are regular quotients of copowers of B

2.4. Theorem. *Prevarieties are precisely the cocomplete categories with a regular generator.*

Proof. Sufficiency follows from the well-known fact that, given an adjoint situation

$$F \dashv U : \mathcal{K} \to \mathcal{L} \qquad (\mathcal{L} \text{ cocomplete}),$$

if the counit $\varepsilon : FU \to Id$ has regular epimorphic components then the comparison functor $K : \mathcal{K} \to \mathcal{L}^{\mathbb{T}}$ of the corresponding monad $\mathbb{T}$ is full and faithful; and, if $\mathcal{L}$ has coequalizers, then K is a right adjoint. Thus, given a regular generator $\mathbb{G} = \{G_s\}_{s \in S}$ in $\mathcal{K}$, apply the above to the adjunction $F \dashv U$ where $U : \mathcal{K} \to \mathcal{S}et^S$ is the forgetful functor

$$UK = (\mathcal{K}(G_s, K))_{s \in S}$$

and F is its left adjoint

$$F(M_s)_{s \in S} = \coprod_{s \in S} M_s \circ G_s \,.$$

Since ε is formed by the canonical morphisms, which are regular epimorphisms by assumption on $\mathbb{G}$, we obtain a full and faithful right adjoint

$$K : \mathcal{K} \to \left(\mathcal{S}et^S\right)^{\mathbb{T}}$$

for the monad $\mathbb{T} = (U, F, \varepsilon, \eta)$. Consequently, $\mathcal{K}$ is equivalent to a full, reflective subcategory of the category $\left(\mathcal{S}et^S\right)^{\mathbb{T}}$.

For the necessity, let $\mathcal{K}$ be a full reflective subcategory of $\left(\mathcal{S}et^S\right)^{\mathbb{T}}$. Then $\mathcal{K}$ is cocomplete because $\left(\mathcal{S}et^S\right)^{\mathbb{T}}$ is: the latter follows from the fact that $\mathcal{S}et^S$ is cocomplete and has all epimorphisms split, see 7.9 in [8]. Moreover, $\left(\mathcal{S}et^S\right)^{\mathbb{T}}$ has a regular generator, e.g., $(F^{\mathbb{T}} X_s)_{s \in S}$ where X_s is the object of $\mathcal{S}et^S$ with all sorts empty except the sort s with a single element (and $F^{\mathbb{T}}$ is the left adjoint induced by the monad $\mathbb{T}$). It is obvious that for every full reflective subcategory $\mathcal{K}$ of $\left(\mathcal{S}et^S\right)^{\mathbb{T}}$ the reflections of the free algebras $F^{\mathbb{T}}X_s$ in $\mathcal{K}$ form a regular generator of $\mathcal{K}$. $\square$

2.5. Remark. (a) Analogously, quasivarieties are precisely the cocomplete categories with a regularly projective regular generator, see [2]. Observe that the concept of a regular generator is equivalent to $\mathcal{E}$-projective $\mathcal{E}$-generator for some class $\mathcal{E} \subseteq \mathcal{R}eg\mathcal{E}pi$. More precisely, a collection $\mathbb{G}$ of regular epimorphisms in a category $\mathcal{K}$ is a regular generator iff there is a class $\mathcal{E}$ of regular epimorphisms such that $\mathbb{G}$ is $\mathcal{E}$-projective (i.e., every hom-functor $\mathcal{K}(G, -), G \in \mathbb{G}$, maps $\mathcal{E}$-morphisms to epimorphisms) and an $\mathcal{E}$-generator (i.e., the above canonical morphisms e_X lie in $\mathcal{E}$ for all X). In fact, it is sufficient to denote by $\mathcal{E}$ the class of all regular epimorphisms w.r.t. which $\mathbb{G}$ is projective.

(b) Clearly a regular generator $\mathbb{G}$ in $\mathcal{K}$ is dense, in the sense that the closure under colimits of $\mathbb{G}$ is the whole category $\mathcal{K}$. Given a class of epimorphisms $\mathcal{F}$, let $\mathcal{F}'$ denote the largest pullback stable subclass of $\mathcal{F}$. In [10] it is shown that, under mild conditions on $\mathcal{F}$, a cocomplete category with pullbacks having an $\mathcal{F}'$-projective dense $\mathcal{F}$-generator $\mathbb{G}$ is a prevariety. In these circumstances, $\mathcal{F}'$ is just the class of all $\mathcal{F}$-morphisms to which $\mathbb{G}$ is projective.

2.6. Remark. In Lawvere's classical characterization of finitary varieties [7] the existence of colimits is weakened to that of (i) coproducts of objects from $\mathbb{G}$ and (ii) coequalizers of equivalence relations. In [2] the concept of *pseudoequivalences* was introduced; essentially, these are just equivalence relations precomposed with a regular epimorphism. Theorem 2.4 remains valid if cocompleteness is restricted to coproducts of objects of the generator and coequalizers of pseudoequivalences. This follows from the fact, proved in [2], that then all coequalizers exist.

2.7. Example. Recall that for locally presentable categories, we can work with strong generators rather than regular ones: a category is locally λ-presentable iff it is cocomplete and has a strong generator formed by λ-presentable objects. The analogous result does not hold for prevarieties: the category $\mathcal{B}$ of Example 2.3(2) is not a prevariety, although it is cocomplete and has a strong generator B.

3. A concrete characterization

3.1. Recall that, for every set S of sorts, monadic categories on $\mathcal{S}et^S$ are precisely those equivalent to S-sorted varieties. More detailed: consider any (possibly large) signature Σ of S-sorted operation symbols σ of arities

$$\sigma : (s_i)_{i<n} \to s$$

where n is a cardinal and s_i and s are sorts. We can form the quasicategory

$$\mathrm{Alg}\,\Sigma$$

of all S-sorted Σ-algebras and homomorphisms – this is, in general, not a legitimate category since, whenever Σ is a large signature, the collection of all Σ-algebras on the set $\{0, 1\}$ is as large as $\exp \mathcal{C}ard$ (the collection of all subclasses of the proper class $\mathcal{C}ard$). By a *variety* of Σ-algebras we mean a class $\mathcal{A}$ of Σ-algebras (considered as a full subcategory of $\mathrm{Alg}\,\Sigma$-algebras and equipped with the natural forgetful functor $U : \mathcal{A} \to \mathcal{S}et^S$) such that

(1) $\mathcal{A}$ has free algebras, i.e., U is a right adjoint
and
(2) $\mathcal{A}$ can be presented by equations.
For every variety $\mathcal{A}$ the forgetful functor $U : \mathcal{A} \to \mathcal{S}et^S$ is monadic.

Conversely, for every monadic functor $U_0 : \mathcal{A}_0 \to \mathcal{S}et^S$ there exists a variety $U : \mathcal{A} \to \mathcal{S}et^S$ of S-sorted algebras *concretely equivalent* to $\mathcal{A}_0$, i.e., such that there exists an equivalence functor $E : \mathcal{A}_0 \to \mathcal{A}$ for which the following triangle

$$\begin{tikzcd} \mathcal{A}_0 \arrow[rr, "E"] \arrow[dr, "U_0"'] & \arrow[d, phantom, "\cong"] & \mathcal{A} \arrow[dl, "U"] \\ & \mathcal{S}et^S & \end{tikzcd}$$

commutes up to natural isomorphism. This has been proved in [8], 5.45, in the one-sorted case. A generalization to $\mathcal{S}et^S$ is straightforward.

3.2. Remark. Let Σ be an S-sorted signature and X an S-sorted set, i.e., an object of $\mathcal{S}et^S$. We can form the *terms* over X in the usual manner, but we do not obtain an algebra (since all terms will typically form a proper class). That is, we define an S-sorted collection

$$T_\Sigma X = (T_{\Sigma,\,s} X)_{s \in S}$$

of terms over X to be the collection of the smallest classes such that

(1) every variable of sort s is a term of sort s: $X_s \subseteq T_{\Sigma,\,s} X$;
and

(2) given an operation symbol $\sigma \in \Sigma$ of arity $\sigma : (s_i)_{i\in I} \to s$ then for every collection of terms t_i of sort s_i $(i \in I)$ we have a term $\sigma(t_i)_{i\in I}$ of sort s.

For every Σ-algebra A and every S-sorted function $f : X \to UA$ we denote by

$$f^\sharp : T_\Sigma X \to A$$

the computation of terms, i.e., the S-sorted function $f^\sharp = (f^\sharp_s)_{s\in S}$ extending f and such that for the term $\sigma(t_i)$ above we always have $f^\sharp_s(\sigma(t_i)_{i\in I}) = \sigma_A(f^\sharp_{s_i}(t_i))_{i\in I}$.

3.3. Definition. By a *prequation* is meant a formula of the form

$$\forall (x_u)_{u\in U}\,(E \to \exists ! (y_v)_{v\in V}\, E')\ , \tag{2}$$

where E is a conjunction of equations (between terms of the same sort over the S-sorted set $X = \{x_u\}_{u\in U}$ of variables) and E' is a conjunction of equations (between terms of the same sort over $X + Y$ where Y is the S-sorted set $Y = \{y_v\}_{v\in V}$).

Remark. A Σ-algebra A is said to *satisfy the prequation* (2) provided that for every S-sorted function $f : X \to UA$ such that

$f^\sharp_s(t(x_u)) = f^\sharp_s(t'(x_u))$ for every equation $t(x_u) = t'(x_u)$ of sort s in E

there exists a unique S-sorted function $g : Y \to UA$ such that

$[f, g]^\sharp_s(u(x_u, y_v)) = [f, g]^\sharp_s(u'(x_u, y_v))$ for every $u(x_u, y_v) = u'(x_u, y_v)$ of sort s in E'.

3.4. Examples. (1) In the variety of monoids we have the full subcategory of groups. This is, obviously, not a subquasivariety. But it is a subprevariety because it can be presented by the following preequation

$$\forall x \exists ! y (xy = e)$$

(2) See Introduction for a preequational presentation of posets.

3.5. In the following definition, by an S-sorted set is simply meant an object $X = (X_s)_{s\in S}$ of $\mathcal{S}et^s$. If $\sum\limits_{s\in S} \mathrm{card} X_s = n$, we say that X has n elements.

Definition. Let $\mathcal{A}$ be a class of Σ-algebras. We say that an algebra $A \in \mathcal{A}$ is $\mathcal{A}$-*generated* by an S-sorted subset X of UA provided that A has no proper subalgebra in $\mathcal{A}$ containing X.

$\mathcal{A}$ is said to have *bounded generation* provided that for every cardinal n there is, up to isomorphism, only a set of objects in $\mathcal{A}$ which are $\mathcal{A}$-generated by a set of n elements.

Remark. Bounded generation of $\mathcal{A}$, jointly with closedness under intersection of subalgebras, implies that the forgetful functor $U : \mathcal{A} \to \mathcal{S}et^S$ satisfies the solution-set condition. Not conversely: in 3.8 we present an example of a category $\mathcal{A}$ of algebras on two unary operations which does not have bounded generation, although U is a right adjoint.

3.6. Theorem. *For a class $\mathcal{A}$ of Σ-algebras with bounded generation the following are equivalent:*

(i) *$\mathcal{A}$ is closed under limits in $Alg\,\Sigma$;*
(ii) *$\mathcal{A}$ is reflective in $Alg\,\Sigma$;*
(iii) *$\mathcal{A}$ can be presented by preequations.*

Proof. (i)→(ii): This follows from the Adjoint Functor Theorem: bounded generation yields the solution-set condition for the embedding $E : \mathcal{A} \to \mathrm{Alg}\,\Sigma$. In fact, for every Σ-algebra B on n elements, a solution set is obtained by considering all homomorphisms $h : B \to A$ such that $A \in \mathcal{A}$ and the set $h[B]$ (of at most n elements) $\mathcal{A}$-generates A. Every homomorphism $f : B \to C$ with $C \in \mathcal{A}$ factorizes through one of those: denote by A the intersection of all subalgebras of C lying in $\mathcal{A}$ and containing $f[B]$. The codomain restriction $h : B \to A$ fulfils $f = mh$ for the inclusion $m : A \to C$. There is only a set of such homomorphisms $h : B \to A$ because A is generated by at most n elements.

(ii)→(iii): Bounded generation and the fact that, being reflective in $\mathrm{Alg}\,\Sigma$, $\mathcal{A}$ is closed under intersections, provide the solution-set condition of the forgetful functor $U : \mathcal{A} \to \mathcal{S}et^S$, thus U has a left adjoint F with unit $\eta : Id \to UF$. For every S-sorted set X of variables let $\approx_X$ denote the kernel equivalence of

$$\eta_X^\sharp : T_\Sigma X \to FX\,.$$

Since FX is a set, $\approx_X$ has a set of representatives, and we choose one such set. For every term $t \in T_\Sigma X$ we denote by $[t] \in T_\Sigma X$ the representative of the class of t. Then every algebra $A \in \mathcal{A}$ fulfils the equation $t = [t]$: given any interpretation $f : X \to UA$ of the variables, then the unique homomorphism $\bar{f} : FX \to A$ extending f forms a commutative triangle

$$\begin{tikzcd} T_\Sigma X \arrow[rr, "\eta^\sharp_X"] \arrow[dr, "f^\sharp"'] & & FX \arrow[dl, "\bar{f}"] \\ & A & \end{tikzcd}$$

and thus from $\eta^\sharp_X(t) = \eta^\sharp_X([t])$ we conclude $f^\sharp(t) = f^\sharp([t])$.

For an arbitrary Σ-algebra B we form a conjunction of equations called the $\mathcal{A}$-*graph* of B as follows. These equations use the set $X = UB$ of variables. Consider an arbitrary operation symbol $\sigma : (s_i)_{i<n} \to s$ in Σ and arbitrary elements $x_i \in B_{s_i}$ and $x \in B_s$ such that

$$\sigma_B(x_i)_{i<n} = x\,. \tag{3}$$

Then $\sigma(x_i)_{i<n}$ and x are two terms in $T_\Sigma X$, and we can turn to their representatives $[\sigma(x_i)]$ and $[x]$, respectively. We define the $\mathcal{A}$-graph as the following conjunction

$$\mathrm{gr}_{\mathcal{A}} B = \bigwedge ([\sigma(x_i)] = [x])$$

ranging over all σ, x_i and x as in (3) above.

The $\mathcal{A}$-graph of B has the following property:

> Given a Σ-algebra A satisfying $[t] = t$ for all terms t and an S-sorted function $f : X \to UA$, then f is a homomorphism from B to A iff $\mathrm{gr}_{\mathcal{A}} B$ holds in $\mathcal{A}$ under the interpretation f (i.e., iff (3) implies $f^\sharp_s([\sigma(x_i)]) = f_s([x])$).

In fact, if f is a homomorphism, then (3) implies

$$\begin{aligned} f^\sharp_s([\sigma(x_i)]) &= f^\sharp_s(\sigma(x_i)) && (A \text{ fulfils } [t] = t) \\ &= \sigma_A(f_{s_i}(x_i)) && (\text{definition of } f^\sharp) \\ &= f_s(\sigma_A(x_i)) && (f \text{ is a homomorphism}) \\ &= f_s(x) && (\text{see (3)}) \\ &= f^\sharp_s([x]) && (A \text{ fulfils } [t] = t). \end{aligned}$$

Conversely, if $f^\sharp_s([\sigma(x_i)]) = f^\sharp_s([x])$ holds whenever (3) does, then we have $f^\sharp_s(\sigma(x_i)) = f^\sharp_s(x)$, due to $[t] = t$ in A, i.e., $\sigma_A(f_{s_i}(x_i)) = f_s(x)$ – this proves that f is a homomorphism.

We are prepared to define the preequation $\mathrm{pr}B$ which is satisfied by every algebra of $\mathcal{A}$ – we derive, then, that these preequations and the equations $[t] = t$ present the class $\mathcal{A}$.

Let $r : B \to B^*$ be the reflection of B into $\mathcal{A}$ and let $Y = UB^*$. We assume, without loss of generality, that X and Y are disjoint in every sort.

Observe that for every variable $x \in X$ we have an equation $x = r(x)$ in the variables $X \cup Y$. Put

$$\mathrm{pr}B \equiv (\forall \vec{x}) \left[\mathrm{gr}_{\mathcal{A}} B \to (\exists! \vec{y}) \left(\mathrm{gr}_{\mathcal{A}} B^* \wedge \bigwedge_{x \in X} (x = r(x)) \right) \right]$$

where $\vec{x}$ is a list of all elements of X and $\vec{y}$ is a list of all elements of Y. We claim that every algebra $A \in \mathcal{A}$ satisfies $\mathrm{pr}B$. In fact, let $f : X \to UA$ be an interpretation of variables from X under which $\mathrm{gr}_{\mathcal{A}} B$ holds. Equivalently, let $f : B \to A$ be a homomorphism. Then there exists a unique homomorphism $f^* : B^* \to A$ with $f = f^* \cdot r$ – that is, a unique interpretation $f^* : Y \to UA$ of the variables in Y such that $\mathrm{gr}B^*$ is satisfied and $x = r(x)$ are satisfied (by $[f, f^*]^\sharp : T_\Sigma(X + Y) \to A$), equivalently, $f(x) = f^*(r(x))$ holds for all $x \in X$.

The class $\mathcal{A}$ is presented by the collection of

(α) all preequations $\mathrm{pr}B$, where B ranges over all Σ-algebras

and

(β) all equations $t = [t]$, where t ranges over all terms.

In fact, every algebra in $\mathcal{A}$ satisfies (α) and (β). Conversely, if B satisfies (α) and (β), we show that the reflection $r : B \to B^*$ is a split subobject; $\mathcal{A}$, being closed under limits, is closed under split subobjects, thus $B \in \mathcal{A}$. Since B satisfies $\mathrm{pr}B$ and since the trivial interpretation id_X of variables has the property that all equations of $\mathrm{gr}_{\mathcal{A}} B$ hold in B, we conclude that there exists a unique interpretation $g : Y \to UB$ of the variables in Y such that (a) $\mathrm{gr}_{\mathcal{A}} B^*$ holds in B under the interpretation g and (b) $x = g(r(x))$ holds for all $x \in X$. Now (a) guarantees that $g : B^* \to B$ is a homomorphism and (b) yields $g \cdot r = id$, as desired.

(iii)$\to$(i): It is straightforward (see [3], 5.7). □

3.7. Corollary. *Prevarieties are precisely the categories equivalent to preequational classes of algebras with bounded generation.*

Proof. In fact, monadic categories $\mathcal{A}$ over $\mathcal{S}et^S$ are precisely the equational classes of S-sorted algebras (over large signatures) with bounded generation, or, equivalently, with free algebras; see e.g. [8]. Every reflective subcategory of $\mathcal{A}$ is preequational, as we have proved above. Conversely, a preequational class $\mathcal{A}$ with bounded generation is reflective in $\mathrm{Alg}\,\Sigma$. The closure $\bar{\mathcal{A}}$ of the class $\mathcal{A}$ under subalgebras and regular quotients (homomorphic images) has the same free algebras as $\mathcal{A}$, therefore, $\bar{\mathcal{A}}$ is a variety, i.e., a category monadic over $\mathcal{S}et^S$. And $\mathcal{A}$ is reflective in $\bar{\mathcal{A}}$. □

3.8. Example of a class $\mathcal{A}$ of unary algebras on two operations which
(i) is closed under limits,
(ii) has free algebras
and
(iii) is not a reflective subcategory of $\operatorname{Alg}\Sigma$.
This shows that the assumption of bounded generation cannot, in Theorem 3.6, be weakened to the existence of free algebras.

We use 2-sorted algebras with sorts $S = \{e, v\}$ and with two unary operations, s and t, of sort $e \to v$. Thus, $\operatorname{Alg}\Sigma = \mathcal{G}ra$ is the category of graphs and homomorphisms. For our example we need to assume that

$$(*) \qquad \text{a full embedding } E : \mathcal{O}rd^{\mathrm{op}} \to \mathcal{G}ra \text{ exists}$$

where $\mathcal{O}rd^{\mathrm{op}}$ is the linearly ordered class of all ordinals with the dual of the usual ordering. This assumption is fulfilled whenever our set theory does not have measurable cardinals, see A7 in [3].

Given the embedding E as above, we denote by $\mathcal{A}$ the class of all graphs G such that

either there exists an ordinal i such that

$$(1) \quad \hom(E_j, G) = \begin{cases} \emptyset & \text{if } j < i \\ \text{a singleton set} & \text{if } j \geq i \end{cases}$$

or G has no path of length 2, in other words,

$$(2) \quad \hom(P, G) = \emptyset.$$

Here P denotes the graph

$$0 \to 1 \to 2$$

with $P_v = \{0, 1, 2\}$ and $P_e = \{(0,1), (0,2)\}$ whose operations s and t are the two projections.

The class $\mathcal{A}$ clearly has all free algebras: a free algebra on a set A of arrows and a set X of elements is the graph having pairwise disjoint arrows indexed by A and nodes without arrows indexed by X – it has no path of length 2. The collection $\mathcal{A}_1$ of all graphs satisfying (1) above is obviously closed under limits. It follows that $\mathcal{A}$ is also closed under limits: condition (2) is namely equivalent to

$$\hom(G, Q) \neq \emptyset$$

where Q is the single arrow, i.e., $Q_e = \{q\}$ and $Q_v = \{0, 1\}$ with $s(q) = 0$ and $t(q) = 1$. (In fact, if G has no path of length 2, we have a homomorphism $h : G \to Q$ mapping an element x of G to 0 iff x lies in the image of s; the converse is also evident.) A limit of a diagram lying in $\mathcal{A}_1$ lies in $\mathcal{A}_1$, and for a diagram where some object has a homomorphism into Q the limit also has such a homomorphism.

Assuming that P has a reflection

$$r : P \to \bar{P}$$

in $\mathcal{A}$, we derive a contradiction. Since $\bar{P}$ does not satisfy (2) above, it lies in $\mathcal{A}_1$, thus, there exists a homomorphism

$$h : E_i \to \bar{P}$$

for some ordinal i. Observe that $E_{i+1} \in \mathcal{A}$ and conclude that

$$\hom(P, E_{i+1}) = \emptyset\,.$$

(In fact, every homomorphism $P \to E_{i+1}$ extends uniquely to a homomorphism $\bar{P} \to E_{i+1}$ which, composed with h above, yields a homomorphism $E_i \to E_{i+1}$ – a contradiction to the fullness of E.) In other words, we have proved that

$$\hom(E_{i+1}, Q) \neq \emptyset\,.$$

Since certainly

$$\hom(Q, E_{i+2}) \neq \emptyset$$

(the graph E_{i+2} has at least one arrow), this yields the desired contradiction: $\hom(E_{i+1},\, E_{i+2}) \neq \emptyset$.

4. λ-ary prevarieties

4.1. Remark. So far we have worked in the logic $L_{\infty\infty}$ in which conjunctions over any set (of equations) and quantifications over any set of variables are allowed. We want to restrict ourselves to the finitary logic $L_{\omega\omega}$ in which a *finitary preequation* is a formula

$$\forall(x_1, ..., x_n)\,(E \to \exists!(y_1, ..., y_t)\, E')$$

where E and E' are finite conjunctions of equations. Or, more generally, to the logic $L_{\lambda\lambda}$, where λ is an infinite regular cardinal (i.e., a cardinal equal to its cofinality). Here we speak about λ*-ary preequations* of the form 3.3 where U and V are sets of cardinality less than λ and also E and E' are conjunctions of less than λ equations.

4.2. Definition. By a λ*-ary prevariety of* Σ*-algebras*, where Σ is a (small) λ-ary signature, is meant a full subcategory of $\operatorname{Alg}\Sigma$ which can be presented by λ-ary preequations. If $\lambda = \omega$ we speak about *finitary prevarieties.*

Examples (1) The category of posets is a finitary prevariety, see Introduction.

(2) Every locally finitely presentable category $\mathcal{A}$ of Gabriel and Ulmer is equivalent to a finitary prevariety. In fact, $\mathcal{A}$ is equivalent to an ω-orthogonality class of $\mathcal{S}et^{\mathcal{B}}$ for some small subcategory $\mathcal{B}$, see [3], 1.46, i.e., there exists a set $\mathcal{M}$ of morphisms $m : X \to Y$ in $\mathcal{S}et^{\mathcal{B}}$ with X and Y finitely presentable such that the full subcategory $\mathcal{M}^{\perp}$ of all objects Z of $\mathcal{S}et^{\mathcal{B}}$ orthogonal to each m (i.e., for every morphism $X \to Z$ there exists a unique factorization through m) is equivalent to $\mathcal{A}$. Now $\mathcal{S}et^{\mathcal{B}}$ is a variety of unary algebras with $S = \mathcal{B}^{\mathrm{obj}}$ and $\Sigma = \mathcal{B}^{\mathrm{mor}}$ (and the sorting given by the domain and codomain). And the orthogonality to m can be expressed by a limit sentence in this signature, see [3], 5.6, which is another name for finitary preequation (in any signature without relational symbols).

More generally:

4.3. Proposition. *For every λ-ary preequation there exists a homomorphism $m : A \to \bar{A}$ between λ-presentable Σ-algebras A and $\bar{A}$ such that a Σ-algebra K satisfies the preequation iff K is orthogonal to m.*

Proof. We are given a preequation as follows

$$\forall (x_i)_{i \in I} \left(\left(\bigwedge_{u \in U} t_u(x_i) = t'_u(x_i) \right) \to \exists! (y_j)_{j \in J} \left(\bigwedge_{v \in V} s_v(x_i, y_j) = s'_v(x_i, y_j) \right) \right) \tag{4}$$

where I, U, J and V are sets of less than λ elements. We define a homomorphism

$$m : A \to \bar{A}$$

in $\mathrm{Alg}\,\Sigma$ with the following property: A and $\bar{A}$ are λ-presentable algebras and

$$\text{satisfaction of (4)} \iff \text{orthogonality to } m.$$

That is, a Σ-algebra K satisfies (4) iff for every homomorphism $f : A \to K$ there exists a unique $\bar{f} : \bar{A} \to K$ with $f = \bar{f} \cdot m$.

Let $F : \mathcal{S}et^S \to \mathrm{Alg}\,\Sigma$ and $\eta : Id \to UF$ denote the left adjoint and the unit of the forgetful functor U (i.e., FX is a free Σ-algebra on X). We denote by

$$e : FX \to A$$

the quotient of the free algebra on $X = \{x_i\}_{i \in I}$ modulo the congruence generated by $t_u(x_i) = t'(x_i)$ for all $u \in U$. Then an algebra K satisfies $\bigwedge_{u \in U} (t_u(x_i) = t'_u(x_i))$ under the interpretation $h_o : X \to UK$ of variables iff there is a homomorphism $h : A \to K$ with

$$h_o = U(he)\eta_X;$$

and h is uniquely determined by h_o. We also have a quotient, for $Y = \{y_j\}_{j\in J}$,

$$e^* : F(X+Y) \to A^*$$

of the free algebra on $X+Y$ modulo the congruence generated by $s_v(x_i, y_j) = s'_v(x_i, y_j)$ for all $v \in V$. Then homomorphisms from A^* to K correspond to the interpretations of variables in $X+Y$ satisfying the latter equations: The coproduct injection $m_1 : X \to X+Y$ yields a homomorphisms $Fm_1 : FX \to F(X+Y)$. Let us form a pushout

$$\begin{tikzcd} FX \arrow[r, "Fm_1"] \arrow[d, "e"'] & F(X+Y) \arrow[r, "e^*"] & A^* \arrow[d, "\bar{e}"] \\ A \arrow[rr, "m"'] & & \bar{A} \end{tikzcd}$$

in $\mathrm{Alg}\,\Sigma$. Since FX, A and A^* are λ-presentable algebras, so is $\bar{A}$.

I. If an algebra K satisfies (4), then it is orthogonal to m. In fact, given a homomorphism

$$h : A \to K$$

then the interpretation of variables

$$h_o = U(he)\eta_X : X \to UK$$

satisfies all equations $t_u(x_i) = t'_u(x_i)$, thus, there exists a unique interpretation of variables from $X+Y$ extending h_o and satisfying all the equations $s_v(x_i, y_j) = s'_v(x_i, y_j)$ – in other words, there exists a unique homomorphism

$$h^* : A^* \to K$$

such that

$$h_o = U(h^* e^* Fm_1)\eta_X\,.$$

We conclude

$$h^* e^* Fm_1 = he : FX \to K$$

since both sides are homomorphisms extending h_o We obtain a unique homomorphism $\bar{h}$ such that the following diagram

commutes. To prove that $\bar{h}$ is uniquely determined by $\bar{h}m = h$, recall that K satisfies (4), consequently, a homomorphism from A^* to K (which is an interpretation of the variables in $X+Y$ satisfying $s_v(x_i, y_j) = s'_v(x_i, y_j)$ for all $v \in V$) is uniquely determined by its values on $m_1 : X \to X + Y$. That is, given a homomorphism $k : \bar{A} \to K$ with

$$km = h\,,$$

we prove that $k = \bar{h}$ by verifying

$$k\bar{e} = \bar{h}\bar{e} : A^* \to K$$

which is equivalent to

$$U(k\bar{e}e^*)\eta_{X+Y}m_1 = U(\bar{h}\bar{e}e^*)\eta_{X+Y}m_1 : X \to UK\,.$$

The last equation follows easily:

$$\begin{aligned} U(k\bar{e}e^*)\eta_{X+Y}m_1 &= U(k\bar{e}e^*Fm_1)\eta_X \\ &= U(\bar{h}me)\eta_X \\ &= U(\bar{h}\bar{e}e^*)\eta_{X+Y}m_1\,. \end{aligned}$$

II. If an algebra K is orthogonal to m, then for every interpretation $h_o : X \to UK$ of variables satisfying $t_u(x_i) = t'_u(x_i)$ for all $u \in U$ we have the homomorphism $h : A \to K$ determined by $h_o = U(he)\eta_X$. Let $\bar{h} : \bar{A} \to K$ be the unique homomorphism with $h = \bar{h}m$. Then $\bar{h}\bar{e} : A^* \to K$ corresponds to an interpretation of the variables in $X + Y$ which satisfies all $s_v(x_i, y_j) = s'_v(x_i, y_j)$, and we conclude that $\bar{h}\bar{e}$ is uniquely determined by h_o, since it acts on X as h_o:

$$\begin{aligned} U(\bar{h}\bar{e}) \cdot Ue^* \cdot \eta_{X+Y} \cdot m_1 &= U(\bar{h}\bar{e}e^*Fm_1)\eta_X \\ &= U(he)\eta_X \\ &= h_o\,. \end{aligned}$$

In other words, for the interpretation h_o we obtain a unique extension to an interpretation $X+Y \to UK$ such that all the equations $s_v(x_i, y_j) = s_{v'}(x_i, y_j)$ for $v \in V$ hold. This proves that K satisfies (4). □

4.4. Corollary. *For every uncountable regular cardinal λ and every (small) λ-ary signature Σ a class of Σ-algebras is a λ-ary prevariety iff it is closed in $Alg\,\Sigma$ under limits and λ-filtered colimits.*

Proof. It is obvious that every λ-ary prevariety is closed under limits and λ-filtered colimits. The converse follows from the result of Hébert and Rosický [6] that full subcategories closed under limits and λ-filtered colimits are λ-orthogonality classes; see [3], 5.28, for a description of a λ-ary preequation

(π_h) characterizing orthogonality to a homomorphism $h : A \to A'$ having λ-presentable domain and codomain. □

4.5. Example. (see [5]) A class of unary algebras which is closed under limits and filtered colimits but is not a finitary prevariety. Let $\Sigma = \{\alpha, a\}$ with α unary and a nullary. Denote by $\mathcal{A}$ the class of all algebras which

(1) have a unique sequence $a = y_0, y_1, y_2, \ldots$ of elements with $\alpha y_{n+1} = y_n$ for

all $n = 1, 2, \ldots$

and

(2) fulfil $(\alpha^2 z = y_n) \Rightarrow (\alpha z = y_{n+1})$ for all elements z and all $n = 0, 1, 2, \ldots.$

This class is easily seen to be closed under limits – in fact it is an ω_1-ary prevariety presented by the preequation

$$\exists!(y_0, y_1, y_2, \ldots) \left[(a = y_0) \wedge \bigwedge_{n\in\omega} (\alpha y_{n+1} = y_n) \right]$$

and the following implications, one for every $k = 0, 1, 2, \ldots$

$$\forall(z, y_0, y_1, y_2 \ldots) \left(\left[(a = y_0) \wedge \bigwedge_{n\in\omega} (\alpha y_{n+1} = y_n) \wedge (\alpha^2 z = y_k) \right] \to (\alpha z = y_{k+1}) \right)$$

It has been proved in [5] that $\mathcal{A}$ is not an ω-orthogonality class, thus, by Proposition 4.3, $\mathcal{A}$ cannot be presented by finitary preequations.

5. Finitary prevarieties and ω-orthogonality classes in general

5.1. In the present section we characterize finitary prevarieties, i.e., ω-orthogonality classes of the category $\mathrm{Alg}\,\Sigma$, see Proposition 4.3. In fact, we present a new characterization of ω-orthogonality classes in any locally finitely presentable category $\mathcal{K}$. This solves an open problem in a realm where all "natural" related characterizations have been known for some time already. Let us mention these first.

Recall that for a class $\mathcal{M}$ of morphisms in $\mathcal{K}$ we have two natural full subcategories "presented" by $\mathcal{M}$:

$\mathcal{M}$-Inj,

the injectivity class of $\mathcal{M}$, consists of all objects K injective w.r.t. members of $\mathcal{M}$, i.e., such that $\hom(-, K)$ sends every member of $\mathcal{M}$ to an epimorphism in $\mathcal{S}et$; and

$\mathcal{M}^{\perp}$,
the orthogonality class of $\mathcal{M}$, contains just all objects K orthogonal to the members of $\mathcal{M}$, i.e., such that $\hom(-,K)$ sends every member of $\mathcal{M}$ to an isomorphism.

By an *ω-injectivity* or *ω-orthogonality class* in $\mathcal{K}$ is meant a full subcategory $\mathcal{A}$ for which there exists a set $\mathcal{M}$ of morphisms with finitely presentable domains and codomains such that

$$\mathcal{A} = \mathcal{M}\text{-Inj} \quad \text{or} \quad \mathcal{A} = \mathcal{M}^{\perp},$$

respectively. The former concept has been characterized in [9] using the following definition

5.2. Definition. A morphism $m : B \to A$ is said to be *pure* provided that for every commutative square

$$\begin{tikzcd} X \arrow[r, "f"] \arrow[d, "u"'] & Y \arrow[d, "v"] \\ B \arrow[r, "m"] & A \end{tikzcd}$$

with X and Y finitely presentable the morphism u factorizes through f (i.e., $u = u'f$ for some $u' : Y \to B$).

5.3. Remark. (a) Let $\mathcal{K}$ be a locally finitely presentable category. Then a morphism m is pure iff it is, as an object of the arrow category $\mathcal{K}^{\to}$, a filtered colimit of split monomorphisms. Consequently, every split monomorphism is pure, and every pure morphism is a strong monomorphism; see [3].

(b) More generally, m is called λ-pure if the above conditions holds whenever X and Y are λ-presentable.

5.4. Theorem. (see [9]) A full subcategory $\mathcal{A}$ of $\mathcal{K}$ is an ω-injectivity class iff it is closed in $\mathcal{K}$ under

(i) products,

(ii) filtered colimits,

and

(iii) pure subobjects – that is, given $m : B \to A$ pure with $A \in \mathcal{A}$, then $B \in \mathcal{A}$.

5.5. Remark. The "expected" characterization of ω-orthogonality classes as classes closed under limits, filtered colimits, and pure subobjects is not true, see Example 4.5. (The subcategory $\mathcal{A}$ is closed under pure subobjects. In fact, let $m : B \to A$, $A \in \mathcal{A}$, be pure. To prove $B \in \mathcal{A}$, it is sufficient to verify that $m[B]$ contains y_n for every n. This is clear for $n = 0$ since $y_0 = a$, the constant. For the induction step take the square in 5.2 with $X = \{y_0, ..., y_n\}$ and $Y = \{y_0, ..., y_{n+2}\}$ where u and v are the inclusion maps.) This is all the more surprising since ω is the only exception. That is, let λ be a cardinal with uncountable cofinality. Then the λ-orthogonality classes (i.e., $\mathcal{A} = \mathcal{M}^{\perp}$ where domains and codomains of morphisms of $\mathcal{M}$ are λ-presentable) are precisely the classes closed under limits, λ-filtered colimits and λ-pure subobjects; see [6].

To find a remedy for this lack of $\lambda = \omega$, we introduce the following new concept, where a morphism $f : X \to Y$ in $\mathcal{K}$ is called an $\mathcal{A}$-*epimorphism* provided that the implication

$$uf = vf \ \text{ implies } \ u = v$$

holds for all pairs $u, v : Y \to A$ with $A \in \mathcal{A}$.

5.6. Definition. Let $\mathcal{A}$ be a full subcategory of $\mathcal{K}$. A morphism $m : B \to A$ in $\mathcal{K}$ is called $\mathcal{A}$-*pure* provided that in every commutative square

$$\begin{tikzcd} X \arrow[r, "f"] \arrow[d, "u"'] & Y \arrow[d, "v"] \\ B \arrow[r, "m"] & A \end{tikzcd}$$

with X and Y finitely presentable and f an $\mathcal{A}$-epimorphism the morphism u factorizes through f.

5.7. Examples. (1) Every pure morphism is $\mathcal{A}$-pure.

(2) Let $\mathcal{K}$ have the property that every epimorphism is strong (e.g., $\mathcal{K} = \mathrm{Alg}\,\Sigma$ for any signature Σ, see [3], Exercise 3.b). Let $\mathcal{A}$ be *cogenerating*, i.e., given morphisms $u_1, u_2 : K \to L$ in $\mathcal{K}$ with $u_1 \neq u_2$ there exists $f : L \to A$, $A \in \mathcal{A}$, with $fu_1 \neq fu_2$. Then $\mathcal{A}$-epimorphisms are epimorphisms. Therefore

$$\mathcal{A}\text{-pure} \Leftrightarrow \text{monomorphism}.$$

In fact the implication $\Leftarrow$ follows from the diagonal fill-in property between strong epimorphisms ($=$ $\mathcal{A}$-epimorphisms) and monomorphisms. The reverse implication holds generally:

5.8. Lemma. *In every locally finitely presentable category all $\mathcal{A}$-pure morphisms are monomorphisms.*

Proof. Let $m : A \to B$ be $\mathcal{A}$-pure. It is sufficient to prove that for every finitely presentable object Y every pair $u_1, u_2 : Y \to A$ with $mu_1 = mu_2 = v$ fulfils $u_1 = u_2$. In fact, the following square

$$\begin{tikzcd} Y+Y \arrow[r, "\nabla"] \arrow[d, "{[u_1, u_2]}"'] & Y \arrow[d, "v"] \\ A \arrow[r, "m"] & B \end{tikzcd}$$

commutes. Since $Y + Y$ is finitely presentable and the codiagonal ∇ is an epimorphism, we conclude that $[u_1, u_2]$ factorizes through ∇ – thus, $u_1 = u_2$. □

5.9. Theorem. *In every locally finitely presentable category the ω-orthogonality classes are precisely the full subcategories $\mathcal{A}$ closed under*
(i) limits
(ii) filtered colimits
and
(iii) $\mathcal{A}$-pure subobjects.

Proof. I. Sufficiency: Let $\mathcal{A}$ be a full subcategory of $\mathcal{K}$ which fulfils (i)-(iii). Denote by $\mathcal{M}$ the set of all $\mathcal{K}$-morphisms $f : X \to Y$ such that X and Y are finitely presentable, and all objects of $\mathcal{A}$ are orthogonal to f. We prove

$$\mathcal{A} = \mathcal{M}^{\perp}.$$

Recall from [3] that (i) and (ii) imply that $\mathcal{A}$ is a reflective subcategory whose reflector $R : \mathcal{K} \to \mathcal{A}$ preserves filtered colimits; we denote by $r_K : K \to RK$ the reflection maps.

Given an object $B \in \mathcal{M}^{\perp}$ we prove $B \in \mathcal{A}$, thus establishing that $\mathcal{A} = \mathcal{M}^{\perp}$. It is sufficient to verify that the reflection r_B of B is $\mathcal{A}$-pure. Thus, let

$$\begin{tikzcd} X \arrow[r, "f"] \arrow[d, "u"'] & Y \arrow[d, "v"] \\ B \arrow[r, "r_B"] & RB \end{tikzcd}$$

be a commutative square where f is an $\mathcal{A}$-epimorphism and X and Y are finitely presentable. Express B as a filtered colimit $(b_i : B_i \to B)_{i\in I}$ of finitely presentable objects. The reflection arrows r_{B_i} form a filtered diagram in $\mathcal{K}^{\to}$ with the colimit $(b_i, Rb_i) : r_{B_i} \to r_B$ $(i \in I)$. This follows easily from (ii) and from R preserving filtered colimits. Since f is a finitely presentable object of $\mathcal{K}^{\to}$ (see 1.55 of [3]), it follows that the morphism $(u, v) : f \to r_B$ factorizes through one of the colimit morphisms $(b_i, Rb_i) : r_{B_i} \to r_B$. That is, there exist u', v' such that the following diagram

$$\begin{tikzcd}
X \arrow[r, "f"] \arrow[d, "u'"'] & Y \arrow[d, "v'"] \\
B_i \arrow[r, "r_{B_i}"] \arrow[d, "b_i"'] & RB_i \arrow[d, "Rb_i"] \\
B \arrow[r, "r_B"'] & RB
\end{tikzcd}$$

commutes. Let us form a pushout P of u' and f, and denote by t the obvious factorization morphism:

$$\begin{tikzcd}
X \arrow[rr, "f"] \arrow[dd, "u'"'] & & Y \arrow[dl, "\bar{u}"'] \arrow[dd, "v'"] \\
& P \arrow[dr, "t"] & \\
B_i \arrow[ur, "\bar{f}"] \arrow[rr, "r_{B_i}"'] & & RB_i
\end{tikzcd}$$

The morphism $\bar{f}$ lies in $\mathcal{M}$. (In fact, since B_i, X and Y are finitely presentable, so is P. Since f is an $\mathcal{A}$-epimorphism, so is $\bar{f}$. And for every morphism $p : B_i \to A$, where $A \in \mathcal{A}$, then there exists a factorization through $\bar{f}$: we have a unique $p' : RB_i \to A$ with $p = p' \cdot r_{B_i}$ thus, $p = (p't)\bar{f}$.) Since $B \in \mathcal{M}^{\perp}$, we conclude that b_i factorizes through $\bar{f}$, say,

$$b_i = q\bar{f} \quad \text{for} \quad q : P \to B\,.$$

Then u factorizes throug f, as requested:

$$u = b_i u' = q\bar{f}u' = q\bar{u}f\,.$$

This proves the $\mathcal{A}$-purity of r_B, thus, $B \in \mathcal{A}$.

II. Necessity. It is easy to see that every ω-orthogonality class $\mathcal{M}^{\perp}$ (where all morphisms in $\mathcal{M}$ have finitely presentable domains and codomains) is closed under limits and filtered colimits. Let us prove that for every $\mathcal{M}^{\perp}$-pure subobject $m : B \to A$ with $A \in \mathcal{M}^{\perp}$ we have $B \in \mathcal{M}^{\perp}$. Given $f : X \to Y$ in $\mathcal{M}$, for every $u : X \to B$ there exists $v : Y \to A$ with $mu = vf$. Now $f \in \mathcal{M}$ is clearly an $\mathcal{M}^{\perp}$-epimorphism, therefore, the last equality implies that u factorizes through f. To prove that the factorization is unique, use the fact that A is orthogonal to f, and m is a monomorphism (by Lemma 5.8). □

5.10. Corollary. *Finitary prevarieties are precisely the classes $\mathcal{A}$ of Σ-algebras closed in $Alg\,\Sigma$ under limits, filtered colimits and $\mathcal{A}$-pure subobjects.*

In fact, we know that finitary prevarieties are precisely the ω-orthogonality classes (see Examples 4.2 (2) and Proposition 4.3).

5.11. Corollary. *Every finitary prevariety $\mathcal{A}$ which is cogenerating in $Alg\,\Sigma$ is a finitary quasivariety.*

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