Pré-Publicações do Departamento de Matemática
Universidade de Coimbra
Preprint Number 03–19

ON SOME ALGEBRAS RELATED WITH THE SIMPLE 8–DIMENSIONAL TERNARY MALCEV ALGEBRA M_8

ALEXANDRE P. POJIDAEV AND PAULO SARAIVA

ABSTRACT: Some results on derivations and representations of the simple Malcev algebra M_8 over a field of characteristic zero and its algebra of multiplications are obtained.

KEYWORDS: Ternary Malcev algebras, Multiplication algebras, Weight spaces.
AMS SUBJECT CLASSIFICATION (2000): 17A40, 17A42, 17D10.

1. Introduction

Ternary Malcev algebras are a particular case of $n-$ary Malcev algebras, first defined in [7], and these naturally arise from the classification of $n-$ary vector cross product algebras [2]. Indeed, the classification theorem for these asserts that, in the case $n = 2$, the only possible algebras are the simple $3-$dimensional Lie algebra $sl(2)$ and the simple $7-$dimensional Malcev algebra C_7; in the case $n \geq 3$, those are the simple $(n + 1)-$dimensional $n-$Lie algebras (which are a natural generalization of Lie algebras to the case of an $n-$ary multiplication [4], and nowadays these algebras are called *Filippov algebras*) with vector cross product, being analogues of $sl(2)$, and also some exclusive ternary algebras arising on composition algebras.

It has been proved [7] that the latter ones are ternary central simple Malcev algebras, which are not $3-$Lie algebras if the characteristic of the ground field is different from 2 and 3 (more generally, the result states that every $n-$ary vector cross product algebra is an $n-$ary central simple Malcev algebra).

The class of $n-$ary Malcev algebras has also the following interesting properties:
1. It is an extension of the class of $n-$Lie algebras, *i.e.*, every $n-$Lie algebra is an $n-$ary Malcev algebra (generalizing the fact that every

The work of the second author was supported by ISR.

Lie algebra is a Malcev algebra);
2. Fixing an arbitrary component in the multiplication (*i.e.*, defining a new reduced operation on the vector space A of the $n-$ary Malcev algebra by the rule $[x_1, \ldots, x_{n-1}]_a = [a, x_1, \ldots, x_{n-1}]$, (*reduced algebra*)), we obtain an $(n-1)$-ary Malcev algebra.

Really, at the moment, the only known example of a simple $n-$ary Malcev algebra which is not an $n-$Lie algebra is the above mentioned ternary central simple Malcev algebra arising on an 8-dimensional composition algebra.

In this article we continue investigating the properties of this ternary simple Malcev algebra. We obtain some results on its derivation and innerderivation algebras over a field of characteristic 0 (namely, concluding that these coincide) and its associative and Lie algebras of multiplications. These results are necessary to classify the faithful irreducible finite-dimensional representations of this ternary algebra. Further, we describe the algebra of quasi-derivations of the ternary Malcev algebra mentioned above.

We start recalling some definitions. Let Φ be an associative, commutative ring with unity. An $\Omega-$*algebra* over Φ is a unital module over Φ, on which we define a system of multilinear algebraic operations $\Omega = \{\omega_i : |\omega_i| = n_i \in N,\ i \in I\}$, where $|\omega_i|$ denotes the arity of ω_i. Henceforth, an $\Omega-$algebra is sometimes briefly called an algebra.

An $n-$*Lie algebra* $(n \geq 3)$ is an $\Omega-$algebra L with one $n-$ary operation $[x_1, \ldots, x_n]$ satisfying the identities

$$[x_1, \ldots, x_n] = sgn(\sigma)[x_{\sigma(1)}, \ldots, x_{\sigma(n)}], \tag{1.1}$$

$$[[x_1, \ldots, x_n], y_2, \ldots, y_n] = \sum_{i=1}^{n} [x_1, \ldots, [\, x_i, y_2, \ldots, y_n], \ldots, x_n], \tag{1.2}$$

where σ is a permutation in the symmetric group S_n, with sign denoted by $sgn(\sigma)$. The relation (1.1) is called the anticommutativity identity and (1.2) is the generalized Jacobi identity (or Filippov identity).

By an $n-$ary Jacobian, we mean the following function defined on an $n-$ary algebra:

$$J(x_1,\ldots,x_n;y_2,\ldots,y_n)=$$
$$[\,[x_1,\ldots,x_n],y_2,\ldots,y_n]-\sum_{i=1}^{n}[\,x_1,\ldots,[\,x_i,y_2,\ldots,y_n],\ldots,x_n].$$

Note that, in an $n-$Lie algebra, $J(x_1,\ldots,x_n;y_2,\ldots,y_n)$ is skew-symmetric on each of the sets $x_1,\ldots,x_n$ and $y_2,\ldots,y_n$, but not on their totality. It follows from the definition that if A is an $n-$Lie algebra then

$$J(x_1,\ldots,x_n;y_2,\ldots,y_n)=0$$

for all $x_1,\ldots,x_n,y_2,\ldots,y_n\in A$.

An $n-$ary *Malcev algebra* ($n\geq 3$) is an $\Omega-$algebra L with one anticommutative $n-$ary operation $[x_1,\ldots,x_n]$ satisfying the identity

$$\sum_{i=2}^{n}[[z,x_2,\ldots,x_n],x_2,\ldots,[x_i,y_2,\ldots,y_n],\ldots,x_n] \tag{1.3}$$
$$+\sum_{i=2}^{n}[[z,x_2,\ldots,[x_i,y_2,\ldots,y_n],\ldots,x_n],x_2,\ldots,x_n]$$
$$=[[[z,x_2,\ldots,x_n],x_2,\ldots,x_n],y_2,\ldots,y_n]$$
$$-[[[z,y_2,\ldots,y_n],x_2,\ldots,x_n],x_2,\ldots,x_n]\ .$$

In terms of right multiplications, this identity is equivalent to:

$$R_x(\sum_{i=2}^{n}R_{x_2,\ldots,x_iR_y,\ldots,x_n})+(\sum_{i=2}^{n}R_{x_2,\ldots,x_iR_y,\ldots,x_n})R_x=R_x^2R_y-R_yR_x^2,$$

where $R_x=R_{x_2,\ldots,x_n}$ and $R_y=R_{y_2,\ldots,y_n}$ are right multiplication operators: $zR_x=[z,x_2,\ldots,x_n]$. Note also that we can rewrite (1.3) as

$$-J(zR_x,x_2,\ldots,x_n;y_2,\ldots,y_n)=J(z,x_2,\ldots,x_n;y_2,\ldots,y_n)R_x$$

A version of the ternary case of (1.3) can be written as follows:

$$[[x,y,z],[y,u,v],z]+[[x,y,z],y,[z,u,v]]$$
$$+[[x,[y,u,v],z],y,z]+[[x,y,[z,u,v]],y,z]$$
$$=[[[x,y,z],y,z],u,v]-[[[x,u,v],y,z],y,z].$$

Henceforth, we assume that Φ is a field of characteristic 0 and denote by A a composition algebra over Φ with an involution $^- : a \mapsto \bar{a}$ and unity 1. The symmetric, bilinear form $\langle x, y\rangle = \frac{1}{2}(x\bar{y} + y\bar{x})$ defined on A is supposed to be nonsingular and we can define the norm of each $a \in A$ by the rule $n(a) = \langle a, a\rangle$. Equip A with a ternary multiplication $[\cdot,\cdot,\cdot]$ by the rule

$$[x, y, z] = x\bar{y}z - \langle y, z\rangle\, x + \langle x, z\rangle\, y - \langle x, y\rangle\, z.$$

Then A becomes a ternary Malcev algebra with respect to this operation which will be denoted by $M(A)$. If $\dim A = 8$ then $M(A)$ is not a 3$-$Lie algebra and we denote it by M_8 or simply by M.

2. Algebras of multiplications of M

Let M be the simple 8-dimensional ternary Malcev algebra over a field Φ of characteristic 0 and $\mathcal{R}$ the vector space generated by the right multiplications of M. Let $Ass(\mathcal{R})$ and $Lie(\mathcal{R})$ be, respectively, the associative and Lie algebras generated by $\mathcal{R}$. Let $Der(M)$ and $Innder(M)$ be, respectively, the derivation and inner derivation algebras of M. Remind that a derivation is called *inner* if it belongs to the Lie algebra $Lie(\mathcal{R})$ of transformations.

Lemma 2.1. *We have:*
1. $Ass(\mathcal{R}) = M_{8,8}(\Phi)$;
2. $Ass(\mathcal{R}) =< \mathcal{R}^2 >$;
3. $Lie(\mathcal{R}) \cong D_4$;
4. $Lie(\mathcal{R}) = \mathcal{R}$ *as vector spaces;*
5. $Der(M) \cong B_3$;
6. $Der(M) = Innder(M)$.

Proof: Being $1, a, b, c$ orthonormal vectors of A, let us choose the following basis

$$\{e_1 = 1, e_2 = a, e_3 = b, e_4 = ab, e_5 = c, e_6 = ac, e_7 = bc, e_8 = abc\},$$

denoted by ε and called the *canonical basis* of M. For each $i \in \{2, \ldots, 8\}$, it is possible to choose j, k, l, m, s, t, all depending on i such that

$$e_i = e_1e_i = e_je_k = e_le_m = e_se_t$$

Let $R_{ij} = R_{e_i,e_j}$ and

$$\mathcal{S} = \{R_{ij} \ : \ i,j = 1,\dots,8,\ i<j\}.$$

We claim that $\mathcal{S}$ is linearly independent, that is,

$$\sum_{i<j} \alpha_{ij} R_{ij} = 0 \tag{2.1}$$

implies $\alpha_{ij} = 0$ for all $i,j = 1,\dots,8,\ i<j$. Consider the following partition of $\mathcal{S}$:

$$\mathcal{S} = \bigcup_{i=2}^{8} \mathcal{S}_i,$$

where for each $i \in \{2,\dots,8\}$, $\mathcal{S}_i = \{R_{1i}, R_{jk}, R_{lm}, R_{st}\}$. Fixing i in $\{2,\dots,8\}$ and applying the left part of (2.1) to e_1, we have:

$$-\alpha_{jk} - \alpha_{lm} - \alpha_{st} = 0. \tag{2.2}$$

Indeed, it is easy to see that

$$e_1 R_{1i} = 0 \quad \text{and} \quad e_1 R_{jk} = -e_i = e_1 R_{lm} = e_1 R_{st}.$$

Further, if $i' \neq i$ and we apply the right multiplications of $\mathcal{S}_{i'}$ to e_1, we never obtain a element of $\langle e_i \rangle$ (except 0, of course) as a consequence of the above partition. Therefore, from

$$e_1 \sum_{i<j} \alpha_{ij} R_{ij} = 0$$

we obtain (2.2). Analogously, applying the left side (2.1) to e_j, we obtain

$$-\alpha_{1i} - \alpha_{lm} - \alpha_{st} = 0, \tag{2.3}$$

since

$$e_j R_{jk} = 0 \quad \text{and} \quad e_j R_{1i} = -e_k = e_j R_{lm} = e_j R_{st}$$

and because no other right multiplication of S produces a vector of $\langle e_k \rangle$ when applied to e_j. Analogously proceeding, when we apply the left side of (2.1), respectively, to e_l and to e_s we obtain

$$-\alpha_{1i} - \alpha_{jk} - \alpha_{st} = 0 \tag{2.4}$$

and

$$-\alpha_{1i} - \alpha_{jk} - \alpha_{lm} = 0, \tag{2.5}$$

respectively. So, from (2.2)-(2.5) we conclude that

$$\alpha_{1i} = \alpha_{jk} = \alpha_{lm} = \alpha_{st} = 0$$

and thus $\mathcal{S}_i$ is linearly independent. Since the same can be applied for all $i \in \{2, \ldots, 8\}$, we conclude that $\mathcal{S}$ is linearly independent.

Consider R_{jk} as a linear transformation of the space M with basis ε. It is easy to see that

$$R_{jk} = \Delta_{i1} + \Delta_{ml} + \Delta_{ts}, \tag{2.6}$$

where $\Delta_{ij} = e_{ij} - e_{ji}$ and e_{ij} are the usual matrix units. Note that $\dim \Delta = 28$, where

$$\Delta = \langle \Delta_{ij} : i, j = 1, \ldots, 8 \rangle_{\Phi} .$$

By (2.6), $\langle S \rangle = \mathcal{R}$ is a subspace of Δ, with $\dim \mathcal{R} = 28$. Hence, $\mathcal{R} = \Delta$. The items 1. to 4. of the lemma easily follow from here.

To prove that $Der(M) \cong B_3$, let D be a derivation of M. Then D is an operator of M such that:

$$[x, y, z]\, D = [xD, y, z] + [x, yD, z] + [x, y, zD]\,, \tag{2.7}$$

for all $x, y, z \in M$. Applying this identity to every $x, y, z \in \varepsilon$ (it is sufficient to do it for every e_i, e_j, e_k, with $i < j < k$), and being $D = [a_{ij}]_{i,j=1,\ldots,8}$, we may conclude that:

$$\begin{cases} a_{ij} = -a_{ji},\ i < j; \\ a_{ii} = 0,\ i = 1, ..., 8; \\ a_{18} - a_{27} + a_{36} + a_{45} = 0; \\ a_{28} + a_{17} + a_{35} - a_{46} = 0; \\ a_{38} - a_{16} - a_{25} - a_{47} = 0; \\ a_{48} - a_{15} + a_{26} + a_{37} = 0; \\ a_{58} + a_{14} + a_{23} - a_{67} = 0; \\ a_{68} + a_{13} - a_{24} + a_{57} = 0; \\ a_{78} - a_{12} - a_{34} - a_{56} = 0. \end{cases} \tag{2.8}$$

Replacing, for instance, the elements a_{1i} and a_{i1} for $i = 2, ..., 8$, and using the operators Δ_{ij}, we may deduce, after (2.8) that

$$
\begin{aligned}
D = {} & a_{23}(\Delta_{23} - \Delta_{14}) + a_{24}(\Delta_{24} + \Delta_{13}) + a_{25}(\Delta_{25} - \Delta_{16}) \\
& + a_{26}(\Delta_{26} + \Delta_{15}) + a_{27}(\Delta_{27} + \Delta_{18}) + a_{28}(\Delta_{28} - \Delta_{17}) \\
& + a_{34}(\Delta_{34} - \Delta_{12}) + a_{35}(\Delta_{35} - \Delta_{17}) + a_{36}(\Delta_{36} - \Delta_{18}) \\
& + a_{37}(\Delta_{37} + \Delta_{15}) + a_{38}(\Delta_{38} + \Delta_{16}) + a_{45}(\Delta_{45} - \Delta_{18}) \\
& + a_{46}(\Delta_{46} + \Delta_{17}) + a_{47}(\Delta_{47} - \Delta_{16}) + a_{48}(\Delta_{48} + \Delta_{15}) \\
& + a_{56}(\Delta_{56} - \Delta_{12}) + a_{57}(\Delta_{57} - \Delta_{13}) + a_{58}(\Delta_{58} - \Delta_{14}) \\
& + a_{67}(\Delta_{67} + \Delta_{14}) + a_{68}(\Delta_{68} - \Delta_{13}) + a_{78}(\Delta_{78} + \Delta_{12}) .
\end{aligned}
$$

It is easy to conclude now that

$$Der(M) = \left\langle\ \Delta_{23} - \Delta_{14}; ...; \Delta_{78} + \Delta_{12}\ \right\rangle_{\Phi}$$

and $Der(M)$ is a simple 21-dimensional Lie algebra. Therefore,

$$Der(M) \cong B_3 ,$$

and the item 5. holds.

By (2.6) and the item 4 of the lemma, it is easy to compute that every $D \in Der(M)$ is such that $D \in Lie(\mathcal{R})$ and thus D is an inner derivation of M. ■

Lemma 2.2. *Let L be a ternary Malcev algebra and $V =< R_{x,y}R_{x,y} : x, y \in L >$. Then V is a module over the Lie algebra $Lie(L)$ of transformations under the action $v \circ R = vR - Rv$, where $v \in V,\ R \in Lie(L)$. In the case $L \cong M$ the module V is of dimension 36 and it contains an irreducible submodule of codimension 1. Moreover, $Ass(\mathcal{R})^{(-)} = Lie(\mathcal{R}) \oplus V$ is a Z_2-graded Lie algebra.*

Proof: Define the action of $Lie(L)$ on $End(M)$ by the rule: $v \circ R = vR - Rv$, where $v \in End(M),\ R \in Lie(L)$. Then

$$(v \circ R_1) \circ R_2 - (v \circ R_2) \circ R_1 = v(R_1R_2 - R_2R_1) - (R_1R_2 - R_2R_1)v =$$

$$v[R_1, R_2] - [R_1, R_2]v = v \circ [R_1, R_2].$$

Thus, $End(M)$ is a module over the Lie algebra $Lie(L)$.

Since the right side of the generalized Malcev identity is equal to $R^2_{x,y} \circ R_{u,v}$ and the left side is a linear combination of elements of V, then V is a Lie submodule over $Lie(L)$.

The knowledge of $\mathcal{R}$ and $Lie(\mathcal{R})$ allows us to observe that we can choose the following basis of V: $\{e_{ii}, e_{ij} + e_{ji} : i, j = 1, \ldots, 8\}$. The subspace

$$W = \langle e_{ii} - e_{jj}, e_{ij} + e_{ji} : i, j = 1, \ldots, 8\rangle$$

is an irreducible submodule in V. ■

Let us remind that by a *ternary Jacobian* we mean the following function defined on a ternary algebra L:

$$\begin{aligned}
&J(x_1, x_2, x_3; y_2, y_3) = \\
&[[x_1, x_2, x_3], y_2, y_3] - [[x_1, y_2, y_3], x_2, x_3] - [x_1, [x_2, y_2, y_3], x_3] \\
&-[x_1, x_2, [x_3, y_2, y_3]].
\end{aligned}$$

Let J_L denote the vector space generated by $J(x_1, x_2, x_3; y_2, y_3)$, $x_i, y_i \in L$. As we know, in the case of a simple Malcev algebra L we have $J_L = J(L, L, L) = L$. In the case of the ternary Malcev algebra M, we too have:

Lemma 2.3. $J_M = M$.

Proof: Consider the orthonormal basis ε of M. Then, by direct computations, we have

$$\begin{aligned}
&J\,(e_4, e_1, e_2; e_3, e_5) = [[e_4, e_1, e_2], e_3, e_5] - [[e_4, e_3, e_5], e_1, e_2] \\
&-[e_4, [e_1, e_3, e_5], e_2] - [e_4, e_1, [e_2, e_3, e_5]] \\
&= [e_3, e_3, e_5] - [e_6, e_1, e_2] + [e_4, e_7, e_2] + [e_4, e_1, e_8] = -3e_5,
\end{aligned}$$

and, analogously,

$$\begin{aligned}
&J\,(e_6, e_1, e_2; e_3, e_5) = 3e_3,\ J\,(e_4, e_1, e_2; e_3, e_6) = -3e_6, \\
&J\,(e_4, e_1, e_2; e_3, e_7) = -3e_7, J\,(e_4, e_1, e_2; e_3, e_8) = -3e_8, \\
&J\,(e_6, e_1, e_2; e_4, e_5) = 3e_4, J\,(e_7, e_4, e_5; e_1, e_2) = -3e_1, \\
&J\,(e_8, e_4, e_5; e_1, e_2) = -3e_2.
\end{aligned}$$

Therefore, $J_M = J\,(M, M, M, M, M)$ and the result is proved. ■

Lemma 2.4. *Let ε be the standard basis of M. Then, for any $x, y, z \in \varepsilon$,*

$$[[R_{x,y}, R_{x,z}], R_{y,z}] = 0. \tag{2.9}$$

Proof: In order to prove that (2.9) is true it is enough to show that

$$t[[R_{x,y}, R_{x,z}], R_{y,z}] = 0$$

for all $x, y, z, t \in \varepsilon$. This identity is equivalent to

$$\begin{aligned} &[[[t,x,y],x,z],y,z] - [[[t,x,z],x,y],y,z] \\ &- [[[t,y,z],x,y],x,z] + [[[t,y,z],x,z],x,y] = 0. \end{aligned} \tag{2.10}$$

Denoting the left hand side of (2.10) by $f(t,x,y,z)$, it is clear that f is symmetric on y, z. On the other hand, it is simple to observe that this identity is verified whenever three of the arguments are equal. If just two of the arguments are equal, then (2.10) is too satisfied. Indeed, the cases $x = y$, $x = z$ and $y = z$ are trivial. Suppose now that $t = x$. If we also have $y = z$ it is trivial that $f(t,t,y,y) = 0$. Otherwise,

$$\begin{aligned} f(t,t,y,z) &= -[[[t,y,z],t,y],t,z] + [[[t,y,z],t,z],t,y] \\ &= [z,t,z] + [y,t,y] = 0. \end{aligned}$$

If $t = y$, then

$$f(t,x,t,z) = -[[[t,x,z],x,t],t,z].$$

It is clear that, if we also have $x = z$, then $f(t,x,t,x) = 0$. If $x \neq z$, since $[[t,x,z],x,t] = t$, we conclude that

$$f(t,x,t,z) = -[z,y,z] = 0.$$

By the symmetry of f on y, z, the case $t = z$ follows from this.

Finally, admit that all arguments of f are pairwise different in ε. Each summand of $f(t,x,y,z)$ is easily computable, and we obtain:

$$\begin{aligned} [[[t,x,y],x,z],y,z] &= t - \langle t,[x,y,z]\rangle [x,y,z] = [[[t,y,z],x,y],x,z]; \\ [[[t,x,z],x,y],y,z] &= -t + \langle t,[x,y,z]\rangle [x,y,z] = [[[t,y,z],x,z],x,y]. \end{aligned}$$

Replacing in (2.10), we conclude that $f(t,x,y,z) = 0$, which concludes the proof. ∎

Let L be a ternary algebra, $a \in L$ and $D \in Der(L)$ such that $D(a) = 0$. It is easy to see that D is a derivation of the reduced algebra L_a.

It is also easy to observe that, even in the case when $L = M$, there exists $D \in Der(M)$ such that $D(a) \neq 0$ for all $a \in M,\ a \neq 0$. Take for instance $D = (\Delta_{14} - \Delta_{23}) + (\Delta_{56} + \Delta_{78})$.

Note also that, if V is a module over a ternary Malcev algebra L and ρ is a corresponding representation, then V is a module over the Malcev algebra L_a for any $a \in L$, simply by putting $x \cdot v = \rho(a, x)(v)$.

Let $L = M$ and $a \in M$ such that $M_a / < a >$ is a simple Malcev algebra (we know [8] that this happens when a is an element of the canonical basis, for example). Let $D \in Der(M)$ and $D(a) \neq 0$. Since D is a derivation of reduced simple Malcev algebra, and every derivation of such algebra is inner (*i.e.* it belongs to $\langle [R_x, R_y] + R_{xy} \rangle$), we have

$$D = \sum_i ([R_{a,x_1^i}, R_{a,x_2^i}] + R_{a,[x_1^i,a,x_2^i]}).$$

We know [8], that in the general case a derivation of the type

$$\left\langle [R_x, R_y] + R_{x \circ R_y} \right\rangle,$$

where $x = (x_2, \ldots, x_n) \in L^{\times(n-1)}$, $y = (y_2, \ldots, y_n) \in L^{\times(n-1)}$ and

$$x \circ R_y = \sum_{i=1}^{n-1} (x_2, \ldots, x_i R_y, \ldots, x_n) \in L^{\times(n-1)},$$

is not a derivation of an $n-$ary Malcev algebra L. Therefore, a natural question arises: is the operator $[R_{z,x}, R_{z,y}] + R_{z,[x,z,y]}$ a derivation of the ternary Malcev algebra M?

The answer is given by the following result.

Theorem 2.5. *Let $M(A)$ be a ternary Malcev algebra. For any $z, x, y \in A$*

$$[R_{z,x}, R_{z,y}] + R_{z,[x,z,y]} \in Der(M(A)). \tag{2.11}$$

Proof: The proof for this assertion, based on exhaustive but easy computations, is rather long. By this reason, we shall only give a sketch of it.

In the first part of the proof, we consider any e_i, e_j, e_k arbitrary elements on ε, the standard basis of $M(A)$, and prove that (2.11) holds for these, that is,

$$\left[R_{e_i,e_j}, R_{e_i,e_k}\right] + R_{e_i,[e_j,e_i,e_k]} \in Der(M(A)). \tag{2.12}$$

Let us denote the operator $\left[R_{e_i,e_j}, R_{e_i,e_k}\right] + R_{e_i,[e_j,e_i,e_k]}$ by $D\left(e_i, e_j, e_k\right)$ or simply by $D\left(i, j, k\right)$. It is an easy task to see that D is skew-symmetric on j, k and that

$$e_i D\left(i, j, k\right) = 0,\ e_j D\left(i, j, k\right) = -2e_k,\ e_k D\left(i, j, k\right) = 2e_j, \tag{2.13}$$

whenever e_i, e_j, e_k are different. Clearly, to show that $D\left(i, j, k\right)$ is a derivation of $M(A)$ for each $e_i, e_j, e_k \in \varepsilon$, it is sufficient to prove that

$$\begin{aligned} [x, y, z]\, D\left(i, j, k\right) &= \left[xD\left(i, j, k\right), y, z\right] \\ &\quad + \left[x, yD\left(i, j, k\right), z\right] + \left[x, y, zD\left(i, j, k\right)\right] \end{aligned} \tag{2.14}$$

for all $x, y, z \in \varepsilon$.

Hereinafter we admit that x, y, z are pairwise different in ε (due to the anticommutativity of $[.,.,.]$, it is clear that (2.14) is satisfied whenever two or even three elements among x, y, z are equal). Further, we will divide the proof in four main cases:

Case 1: $x, y, z \in \{e_i, e_j, e_k\}$.

Case 2: One of the elements x, y, z is not in $\{e_i, e_j, e_k\}$. Without loss of generality, we will admit that $x \notin \{e_i, e_j, e_k\}$, being $y, z \in \{e_i, e_j, e_k\}$. Three subcases must be considered:

2.1) $y = e_j, z = e_k$; **2.2)** $y = e_i, z = e_j$; **2.3)** $y = e_i, z = e_k$.

It is clear, by the skew-symmetry of $D\left(i, j, k\right)$ on j, k, that **2.3)** is a consequence of **2.2)**, so only the first two subcases must be analyzed.

Case 3: Just one of the elements x, y, z is in $\{e_i, e_j, e_k\}$. We can admit, without loss of generality, that $z \in \{e_i, e_j, e_k\}$ and $x, y \notin \{e_i, e_j, e_k\}$. Again, since $D\left(i, j, k\right)$ is skew-symmetric on j, k, among the following three subcases,

3.1) $z = e_i$; **3.2)** $z = e_j$; **3.3)** $z = e_k$,

the first two will be sufficient.

Case 4: x, y, z, e_i, e_j, e_k are pairwise different.

Now, the only cases which appeal to a deeper analysis are **3.2)** and **4**. since these lead, after the computation of each side of (2.14), to expressions which have not exactly the same shape (although we prove that those are identical for all elements satisfying the respective hypothesis). Moreover, we can add that this analysis can be done recalling that x, y, z, e_i, e_j, e_k can be considered as points in the Fano's plane (of course, the possibility of any of these being equal to the identity 1 must also be studied).

At this point, for each $z, x, y \in A$, let

$$D(z,x,y) = [R_{z,x}; R_{z,y}] + R_{z,[x,z,y]}. \tag{2.15}$$

Observe that the operator D is linear and skew-symmetric on x, y. Thus, in order to show that $D(z,x,y)$ is a derivation for all $z, x, y \in A$, it suffices to prove that $D(z,x,y)$ is a derivation, for any $z = \sum_{i=1}^{8} \alpha_i e_i \in A, \alpha_i \in \Phi$ and $x, y \in \varepsilon$. Now, using (2.15), it is easy to see that

$$D(z,x,y) \;=\; D(\sum_{i=1}^{8} \alpha_i e_i, x, y) = \sum_{i=1}^{8} \alpha_i^2 D(e_i, x, y)$$

$$+ \sum_{i,j=1, i<j}^{8} \alpha_i\alpha_j \left(\left[R_{e_i,x}; R_{e_j,y}\right] + \left[R_{e_j,x}; R_{e_i,y}\right] + R_{e_i,[x,e_j,y]} + R_{e_j,[x,e_i,y]} \right).$$

Observing that $D(e_i,x,y)$ is, by (2.12), a derivation (and thus the same happens with $\sum_{i=1}^{8} \alpha_i^2 D(e_i,x,y)$), we deduce that $D(z,x,y)$ is a derivation if

$$\left[R_{e_i,x}; R_{e_j,y}\right] + \left[R_{e_j,x}; R_{e_i,y}\right] + R_{e_i,[x,e_j,y]} + R_{e_j,[x,e_i,y]}$$

is a derivation, *i.e.*, if

$$\begin{aligned}
& [w_1,w_2,w_3]\left[R_{e_i,x}; R_{e_j,y}\right] + [w_1,w_2,w_3]\left[R_{e_j,x}; R_{e_i,y}\right] \\
& + [w_1,w_2,w_3] R_{e_i,[x,e_j,y]} + [w_1,w_2,w_3] R_{e_j,[x,e_i,y]} \\
= \;& \left[w_1\left[R_{e_i,x}; R_{e_j,y}\right], w_2, w_3\right] + \left[w_1\left[R_{e_j,x}; R_{e_i,y}\right], w_2, w_3\right] \\
& + \left[w_1 R_{e_i,[x,e_j,y]}, w_2, w_3\right] + \left[w_1 R_{e_j,[x,e_i,y]}, w_2, w_3\right] \\
& + \left[w_1, w_2\left[R_{e_i,x}; R_{e_j,y}\right], w_3\right] + \left[w_1, w_2\left[R_{e_j,x}; R_{e_i,y}\right], w_3\right] \\
& + \left[w_1, w_2 R_{e_i,[x,e_j,y]}, w_3\right] + \left[w_1, w_2 R_{e_j,[x,e_i,y]}, w_3\right] \\
& + \left[w_1, w_2, w_3\left[R_{e_i,x}; R_{e_j,y}\right]\right] + \left[w_1, w_2, w_3\left[R_{e_j,x}; R_{e_i,y}\right]\right] \\
& + \left[w_1, w_2, w_3 R_{e_i,[x,e_j,y]}\right] + \left[w_1, w_2, w_3 R_{e_j,[x,e_i,y]}\right]
\end{aligned} \tag{2.16}$$

for every $w_1, w_2, w_3, e_i, e_j, x, y \in \varepsilon$.

Analogously to the proof of (2.12), the proof of (2.16) is too long to include in this paper. Thus, we briefly give a sketch of it.

To simplify, $g_1(w_1, w_2, w_3, e_i, e_j, x, y)$ and $g_2(w_1, w_2, w_3, e_i, e_j, x, y)$ stand, respectively, for the left and right hand sides of the above identity. Further, each summand will be denoted by $g_{k,l}$ where $k = 1, 2$ and stands for the left or right hand side, respectively, and l stands for the order of the summand which is being considered.

Observe that the operator

$$\left[R_{e_i,x}; R_{e_j,y}\right] + \left[R_{e_j,x}; R_{e_i,y}\right] + R_{e_i,[x,e_j,y]} + R_{e_j,[x,e_i,y]},$$

which we want to prove that is a derivation, is skew-symmetric on the pairs x, y and symmetric on e_i, e_j. Thus, this properties still hold concerning g_1 and g_2. Further, these are skew-symmetric on the pairs $w_1, w_2; w_2, w_3$ and w_1, w_3. We will consider that w_1, w_2, w_3, are pairwise different and also that $e_i \neq e_j$ and $x \neq y$, because otherwise we obtain trivial cases. Further, denote $\{w_1, w_2, w_3\}$ by ε_1 and $\{e_i, e_j, x, y\}$ by ε_2. By the properties of g_1 and g_2, in order to prove that

$$g_1(w_1, w_2, w_3, e_i, e_j, x, y) = g_2(w_1, w_2, w_3, e_i, e_j, x, y),$$

we have to consider two main cases concerning the elements e_i, e_j, x, y:

I: $x = e_i$; **II:** e_i, e_j, x, y pairwise different.

CASE I: $x = e_i$ This means that ε_2 reduces to $\{e_i, x, y\}$.

Case 1: $\varepsilon_1 \cap \varepsilon_2 = \emptyset$.

1.1. $1 \in \varepsilon_1 \cup \varepsilon_2$. Since it is possible to show that

$$g_k(w_1, w_2, w_3, x, e_j, x, y) = -g_k(w_1, w_2, w_3, x, y, x, e_j), k = 1, 2,$$

we just have to analyze three subcases:

1.1.1. $x = 1$; **1.1.2.** $y = 1$; **1.1.3.** $w_1 = 1$.

1.2. $1 \notin \{w_1, w_2, w_3, x, e_j, y\}$.

Case 2: $\varepsilon_1 \cap \varepsilon_2 \neq \emptyset$.

2.1. ε_1 and ε_2 have three elements in common.

2.2. ε_1 and ε_2 have two elements in common. By the properties of g_1 and g_2, the following subcases are enough:

2.2.1. $w_1 = x, w_2 = y$; **2.2.2.** $w_1 = x, w_2 = e_j$; **2.2.3.** $w_1 = y, w_2 = e_j$.

2.3. The sets ε_1 and ε_2 have only one element in common. Again taking in consideration the properties of g_1 and g_2, we just have to analyze the following subcases:

$$\textbf{2.3.1.}\ w_1 = x; \quad \textbf{2.3.2.}\ w_1 = y; \quad \textbf{2.3.3.}\quad w_1 = e_j.$$

CASE II: x, y, e_i, e_j are pairwise different.

1. $\varepsilon_1 \cap \varepsilon_2 = \emptyset$.

1.1. $1 \in \varepsilon_1 \cup \varepsilon_2$.

According to the properties of g_1 and g_2, we have to analyze three main cases:

$$\textbf{1.1.1.}\quad x = 1; \quad \textbf{1.1.2.}\quad e_i = 1; \quad \textbf{1.1.3.}\quad w_1 = 1.$$

1.2. $1 \notin \{x, y, e_i, e_j, w_1, w_2, w_3\}$.

2. $\varepsilon_1 \cap \varepsilon_2 \neq \emptyset$.

Admit that there are three common elements. Due to the properties of (2.16), we just have to analyze two cases:

$$\textbf{2.1.}\ w_1 = x; w_2 = y; w_3 = e_i \quad \text{and} \quad \textbf{2.2.}\ w_1 = x; w_2 = e_i; w_3 = e_j.$$

Next, we analyze what happens when ε_1 and ε_2 have only two elements in common. Again recalling the properties of (2.16), we just have to observe the cases

$$\textbf{2.3.}\ w_1 = x, w_2 = y\ ; \quad \textbf{2.4.}\ w_1 = x, w_2 = e_i\ ; \quad \textbf{2.5.}\ w_1 = e_i, w_2 = e_j\ .$$

Finally, we analyze what happens when ε_1 and ε_2 have only one element in common. It is easy to see that we just have to check the cases

$$\textbf{2.6.}\ w_1 = x; \quad \textbf{2.7.}\ w_1 = e_i.$$

We recall again that a deeper analysis of all of these cases and subcases is made by considering $w_1, w_2, w_3, x, y, e_i, e_j$ as elements on the Fano's plane. ■

Another family of multiplication algebras that might be interesting to study is the algebra of quasi-derivations. According to R. E. Block [1], a linear operator $D : A \longrightarrow A$ is called *quasi-derivation* of a ring A, if it satisfies

$$[D, T] \in T(A), \text{ for all } T \in T(A),$$

where $T(A)$ stands for the Lie ring generated by the right and left multiplications by elements of A.

In the case of $n-$ary algebras we have the following.

Let A be an $n-$ary anticommutative algebra with multiplication $[\cdot, ..., \cdot]$. Consider the vector space $\mathcal{R}$ generated by the right multiplications $R_a = R_{a_2,...,a_n}$, $a_2, ..., a_n \in A$, $Ass\,(\mathcal{R})$ and $Lie\,(\mathcal{R})$, respectively, the associative and Lie algebra generated by $\mathcal{R}$. Every operator $D : A \longrightarrow A$ such that

$$[D, R_a] \in Lie\,(\mathcal{R})\,, \text{ for all } R_a \in Lie\,(\mathcal{R})\,, \tag{2.17}$$

is said to be a *quasi-derivation* of A.

Consider M the simple 8-dimensional ternary Malcev algebra over a field Φ of characteristic zero and let $\mathcal{R}$, $Ass\,(\mathcal{R})$ and $Lie\,(\mathcal{R})$ stand with the above described meaning (with M instead of A). As it has been proved, on lemma 1.1.,

$$Lie\,(\mathcal{R}) \cong D_4 = \{X \in M_{8\times 8}\,(\Phi) : X' = -X\}\,,$$

and

$$Lie\,(\mathcal{R}) = \langle \Delta_{ij} = e_{ij} - e_{ji}, \quad i, j = 1, ..., 8,\, i < j\rangle_{\Phi}\,.$$

Thus, by (2.17), in order to find the quasi-derivations of M, we have to find

$$D = (d_{kl}) \in M_{8\times 8}\,(\Phi) : [D, \Delta_{ij}] \in Lie\,(\mathcal{R})\,.$$

It is possible to observe, by direct computations, that

$$[D, \Delta_{ij}] = \begin{array}{cc}
\begin{array}{ccccccc} & & \downarrow i & & \downarrow j & & \end{array} & \\
\begin{bmatrix}
0 & \cdots & -d_{1j} & 0 & d_{1i} & \cdots & 0 \\
\vdots & \ddots & \vdots & & \vdots & & \vdots \\
-d_{j1} & \cdots & -d_{ij} - d_{ji} & \cdots & d_{ii} - d_{jj} & \cdots & -d_{j8} \\
0 & & \vdots & \ddots & \vdots & & 0 \\
d_{i1} & \cdots & -d_{jj} + d_{ii} & \cdots & d_{ji} + d_{ij} & \cdots & d_{i8} \\
\vdots & & \vdots & & \vdots & \ddots & \vdots \\
0 & \cdots & -d_{8j} & 0 & d_{8i} & \cdots & 0
\end{bmatrix} & \begin{array}{c} \\ \\ \\ \leftarrow i \\ \\ \leftarrow j \\ \\ \end{array}
\end{array}.$$

Therefore,

$$[D, \Delta_{ij}] \in Lie\,(\mathcal{R}) \Leftrightarrow \begin{cases} d_{kj} = -d_{jk}\,,\ k = 1, \ldots, \hat{\imath}, \ldots, \hat{\jmath}, \ldots, 8 \\ d_{ki} = -d_{ik}\,,\ k = 1, \ldots, \hat{\imath}, \ldots, \hat{\jmath}, \ldots, 8 \\ d_{ji} + d_{ij} = 0 \\ d_{ii} - d_{jj} = -\left(d_{ii} - d_{jj}\right) \end{cases}.$$

Since *char* $\Phi = 0$, the last identity implies $d_{ii} = d_{jj}$. Finally, recalling that $i, j = 1, ..., 8$, $i < j$, we may state the following result.

Theorem 2.6. *If D is a quasi-derivation of the ternary Malcev algebra M over a field of characteristic zero, then*

$$D = \alpha Id_8 + X, \text{ with } X' = -X \ .$$

3. On some nonassociative algebras arising on a ternary Malcev algebra

Following to the general ideas of S. Eilenberg [3], we may introduce the notion of an n-ary Malcev module. Admit that A is an n-ary Malcev algebra over a field Φ. A vector space V over Φ is said to be an *n-Malcev module* if the direct sum of vector spaces

$$B = V \oplus A$$

has the structure of n-ary Malcev algebra, such that A is an n-ary Malcev subalgebra of B and V is an abelian ideal (*i.e.*, $[V, V, B, \ldots, B] = 0$). In this case, to any set of elements $a_1, \ldots, a_{n-1} \in A$ it corresponds a linear transformation $\rho_{(a_1,\ldots,a_{n-1})}$ of V, acting by the rule

$$v\rho_{(a_1,\ldots,a_{n-1})} = [v, a_1, \ldots, a_{n-1}] \, ,$$

and we say that ρ is a *representation* of the n-ary Malcev algebra A in the space V.

We must point that, an analogous definition was earlier given by Sh. Kasymov and E. Kuz'min concerning Filippov algebras [6].

Let's define on $A = A_M = \wedge^2 M$ an operation

$$(y \wedge z)(u \wedge v) = [y, u, v] \wedge z + y \wedge [z, u, v]. \tag{3.1}$$

Let V be a module over M and ρ_0 a corresponding representation. We can define an action ρ of the algebra A on V by means of

$$\rho(y \wedge z)(v) = \rho_0(y, z)(v).$$

Then, for any $x = u_1 \wedge u_2, \ y = u_3 \wedge u_4$, where $u_i \in M$, we have

$$[\rho_x^2, \rho_y] = \rho_{xy}\rho_x + \rho_x\rho_{xy},$$

where $\rho_x = \rho(x)$. This equality is very near to one of the equalities concerning the representations of Malcev algebras.

Note that if the action ρ_0 is irreducible, then the action ρ of the algebra A on V is also irreducible.

At this point, we arrive to the question of studying the algebra A, which is nor commutative neither anticommutative. By these reasons, it will be also interesting to study its commutator algebra, $A^{(-)}$, *i.e.*, A with the operation:

$$[y \wedge z, u \wedge v] = [y, u, v] \wedge z + y \wedge [z, u, v] - [u, y, z] \wedge v - u \wedge [v, y, z].$$

Note also that if ϕ is an isomorphism of ternary Malcev algebras L_1 and L_2 then $\psi : x \wedge y \mapsto \phi(x) \wedge \phi(y)$ is an isomorphism of A_{L_1} and A_{L_2}.

It is possible to observe that there exists a basis of A whose elements have zero squares. It is easy to construct an example showing that A is not a power associative algebra. *E.g.*, taking $x = e_2 \wedge e_3 + e_4 \wedge e_5$, it is enough to consider the coefficient at $e_6 \wedge e_3$ on the elements $x^2 \cdot x$ and $x \cdot x^2$ to conclude that $x^2 \cdot x \neq x \cdot x^2$.

Prior to the question of knowing if $A^{(-)}$ is a Lie or a Malcev algebra, we put a more general question. Let F be a free anticommutative ternary algebra. Define on $A = \wedge^2 F$ an operation by the rule (3.1) and consider its commutator algebra. The problem is to find a minimal ideal I such that $(A/I, [.,.])$ is a Lie (or Malcev) algebra.

Theorem 3.1. *Let I be a nonzero minimal ideal of A with the property that $(A/I, [.,.])$ is a Lie algebra. Then*

$$I = id < [[x_1, x_2, x_3], x_4, x_5] - [[x_1, x_4, x_5], x_2, x_3] > .$$

Further, let I_1 be a nonzero minimal ideal of A such that $(A/I_1, [.,.])$ is a Malcev algebra. Then $I = I_1$ and $(A/I_1, [.,.])$ is a Lie algebra.

Proof: Using direct computations it is possible to see that the elements of the type

$$\begin{aligned} &-[[x_1, x_2, x_3], x_4, x_5] - [[x_5, x_1, x_2], x_3, x_4] \\ &+[[x_4, x_5, x_1], x_2, x_3] + [[x_3, x_4, x_5], x_1, x_2] \\ &+[[x_2, x_4, x_5], x_3, x_1] - [[x_1, x_2, x_4], x_5, x_3] \end{aligned} \tag{3.2}$$

belong to the ideal. Denote an element of the type (3.2) by $g(x_1, x_2, x_3, x_4, x_5)$. Then we have

$$g(x_1, x_2, x_3, x_4, x_5) + g(x_4, x_5, x_1, x_2, x_3)$$

$$= -[[x_5, x_1, x_2], x_3, x_4] - [[x_1, x_2, x_4], x_5, x_3]$$
$$+[[x_2, x_3, x_4], x_5, x_1] + [[x_5, x_2, x_3], x_1, x_4] = 0.$$

We can rewrite it as

$$[[x_2, x_1, x_5], x_3, x_4] - [[x_2, x_4, x_1], x_5, x_3] \tag{3.3}$$
$$[[x_2, x_3, x_4], x_1, x_5] + [[x_2, x_5, x_3], x_4, x_1] = 0.$$

Put $x_2 = x_5$ in (3.3). We have

$$-[[x_2, x_4, x_1], x_2, x_3] - [[x_2, x_3, x_4], x_1, x_2] = 0.$$

Linearizing and interchanging variables, we will obtain the identity which is analogous to (3.3), but having all coefficients equal to 1. Adding this identity with (3.3), we obtain what's required. ∎

Corollary 3.2. *Let L be a solvable anticommutative ternary algebra of derived length* 2*. Then* $(\wedge^2 L, [,])$ *is a Lie algebra.*

Corollary 3.3. *Let L be an anticommutative ternary algebra with an identity*

$$[[x_1, x_2, x_3], x_4, x_5] = [[x_1, x_4, x_5], x_2, x_3].$$

Then $(\wedge^2 L, [,])$ *is a Lie algebra.*

Let D be an endomorphism of a ternary algebra A over a field Φ. Then, following to V.T.Filippov [5], we call D a *δ-derivation*, where $\delta = (\alpha, \beta) \in \Phi^2$, if, for any $x, y, z \in A$, we have

$$[x, y, z]D = [xD, y, z] + \alpha[x, yD, z] + \beta[x, y, zD]. \tag{3.4}$$

Corollary 3.4. *Let L be an anticommutative ternary algebra whose right multiplication operators are* $(2, 2)$*-derivations. Then* $(\wedge^2 L, [,])$ *is a Lie algebra.*

Lemma 3.5. *There are no identities of degrees* 2 *for A and degree* 3 *for $A^{(-)}$. All identities of degree* 2 *for $A^{(-)}$ follow from anticommutativity. Further, there are no identities of degree* 3 *in A, neither of degree* 4 *in $A^{(-)}$.*

Proof: First, since the field Φ is a field of characteristic 0, we know (see [9], for example) that all identities of the algebra A follow from multilinear identities. It is easy to see that there are no identities of degree 2 for A and that all identities of degree 2 for $A^{(-)}$ follow from anticommutativity.

Let f be an identity of degree 3 for $A^{(-)}$. Then

$$f(x,y,z) = [[x,y],z] + \alpha[[z,x],y] + \beta[[y,z],x] = 0,$$

for all $x, y, z \in A^{(-)}$. It is easy to see that $f(x,y,y) = (1-\alpha)[[x,y],y]$. Taking $x = e_1 \wedge e_2,\ y = e_2 \wedge e_5$, we have

$$[x,y] = 2e_2 \wedge e_6, \quad \text{and} \quad [[x,y],y] = 4e_2 \wedge e_1,$$

whence $\alpha = 1$. Analogously, from $f(x,y,x) = (1-\beta)[[x,y],x]$ we obtain $\beta = 1$. Putting $z = e_2 \wedge e_7$, we have

$$[[x,y],z] = -4e_2 \wedge e_3 = [[z,x],y] = [[y,z],x],$$

from where everything follows.

An identity of degree 3 for A has the following expression: $f(x,y,z) = 0$, where

$$\begin{aligned} f(x,y,z) &= (xy)z + \alpha_2(yz)x + \alpha_3(zx)y + \alpha_4(yx)z \\ &\quad +\alpha_5(zy)x + \alpha_6(xz)y + \alpha_7 z(xy) + \alpha_8 x(yz) \\ &\quad +\alpha_9 y(zx) + \alpha_{10} z(yx) + \alpha_{11} x(zy) + \alpha_{12} y(xz), \end{aligned} \tag{3.5}$$

with $\alpha_i \in \Phi$.

Considering $x = y = z$ in (3.5), we have

$$\begin{aligned} f(x,x,x) &= (1 + \alpha_2 + \alpha_3 + \alpha_4 + \alpha_5 + \alpha_6)\, x^2 x \\ &\quad + (\alpha_7 + \alpha_8 + \alpha_9 + \alpha_{10} + \alpha_{11} + \alpha_{12})\, x x^2. \end{aligned} \tag{3.6}$$

Thus, taking for instance $x = e_2 \wedge e_3 + e_4 \wedge e_5$, it is possible to see that $x^2 x$ and $x x^2$ are linearly independents. Therefore, if $f(x,x,x) = 0$, then

$$\left\{ \begin{array}{c} 1 + \alpha_2 + \alpha_3 + \alpha_4 + \alpha_5 + \alpha_6 = 0 \\ \alpha_7 + \alpha_8 + \alpha_9 + \alpha_{10} + \alpha_{11} + \alpha_{12} = 0 \end{array} \right. . \tag{3.7}$$

Admit that $x = y$ in (3.5). Then

$$\begin{aligned} f(x,x,z) &= (1 + \alpha_4)\, x^2 z + (\alpha_2 + \alpha_6)\,(xz)x + (\alpha_3 + \alpha_5)\,(zx)x \\ &\quad + (\alpha_7 + \alpha_{10})\, z x^2 + (\alpha_8 + \alpha_{12})\, x(xz) + (\alpha_9 + \alpha_{11})\, x(zx). \end{aligned} \tag{3.8}$$

Taking $x = e_1 \wedge e_2$, $z = e_1 \wedge e_3$ and replacing in (3.8), the equality $f(x,x,z) = 0$ implies that

$$(\alpha_2 + \alpha_6) + (\alpha_9 + \alpha_{11}) = (\alpha_3 + \alpha_5) + (\alpha_8 + \alpha_{12}). \tag{3.9}$$

On the other hand, doing the same but now with $x = e_2 \wedge e_3 + e_4 \wedge e_5$, $z = e_1 \wedge e_2$, we obtain the set of equalities:

$$\left\{ \begin{array}{c} (\alpha_2 + \alpha_6) - 3(\alpha_3 + \alpha_5) - (\alpha_8 + \alpha_{12}) + (\alpha_9 + \alpha_{11}) = 0 \\ -(1 + \alpha_4) - (\alpha_2 + \alpha_6) - (\alpha_3 + \alpha_5) + 3(\alpha_7 + \alpha_{10}) = 0 \\ -(1 + \alpha_4) + (\alpha_3 + \alpha_5) + 3(\alpha_7 + \alpha_{10}) - (\alpha_8 + \alpha_{12}) - 2(\alpha_9 + \alpha_{11}) = 0 \\ (\alpha_2 + \alpha_6) - (\alpha_8 + \alpha_{12}) - 2(\alpha_9 + \alpha_{11}) = 0 \\ -2(1 + \alpha_4) - (\alpha_2 + \alpha_6) + 2(\alpha_8 + \alpha_{12}) + (\alpha_9 + \alpha_{11}) = 0 \\ 2(1 + \alpha_4) - 2(\alpha_3 + \alpha_5) - (\alpha_8 + \alpha_{12}) + (\alpha_9 + \alpha_{11}) = 0 \\ 2(\alpha_3 + \alpha_5) = 0. \end{array} \right. \tag{3.10}$$

The conjugation of (3.7), (3.9) and (3.10), leads to:

$$\alpha_4 = -1,\ \alpha_5 = -\alpha_3,\ \alpha_6 = -\alpha_2,\ \alpha_{10} = -\alpha_7,\ \alpha_{11} = -\alpha_9,\ \alpha_{12} = -\alpha_8.$$

Thus,

$$\begin{aligned} f(x,y,z) = {} & (xy)z + \alpha_2(yz)x + \alpha_3(zx)y - (yx)z - \alpha_3(zy)x \\ & -\alpha_2(xz)y + \alpha_7 z(xy) + \alpha_8 x(yz) + \alpha_9 y(zx) \\ & -\alpha_7 z(yx) - \alpha_9 x(zy) - \alpha_8 y(xz). \end{aligned} \tag{3.11}$$

Consider the case $x = z$ in (3.5). Then, from (3.11), we get

$$\begin{aligned} f(x,y,x) = {} & (\alpha_3 - \alpha_2) x^2 y + (1 - \alpha_3)(xy)x \\ & + (\alpha_2 - 1)(yx)x + (\alpha_9 - \alpha_8) yx^2 \\ & + (\alpha_7 - \alpha_9) x(xy) + (\alpha_8 - \alpha_7) x(yx). \end{aligned} \tag{3.12}$$

Proceeding as above, considering $x = e_1 \wedge e_2$, $y = e_1 \wedge e_3$ in (3.12), we obtain from $f(x,y,x) = 0$ the equality:

$$2 - \alpha_2 - \alpha_3 - 2\alpha_7 + \alpha_8 + \alpha_9 = 0.$$

Further, the consideration of $x = e_2 \wedge e_3 + e_4 \wedge e_5$, $y = e_1 \wedge e_2$ again in (3.12) leads to

$$\left\{ \begin{array}{c} \alpha_7 = \alpha_8 = \alpha_9, \\ \alpha_2 = \alpha_3 = 1. \end{array} \right.$$

Thus,

$$\begin{aligned} f(x,y,z) &= (xy)z + (yz)x + (zx)y - (yx)z - (zy)x - (xz)y \\ &\quad + \alpha_7 \left(z(xy) + x(yz) + y(zx) - z(yx) - x(zy) - y(xz) \right). \end{aligned} \tag{3.13}$$

Consider $x = e_1 \wedge e_2$, $y = e_2 \wedge e_3 + e_4 \wedge e_5$ and $z = e_1 \wedge e_6$ in (3.13). Then,

$$f(x,y,z) = (1-\alpha_7)\, w,$$

where $w = 2(e_1 \wedge e_4) - (e_2 \wedge e_3) - 3(e_2 \wedge e_7) + 2(e_5 \wedge e_8) + (e_6 \wedge e_7) - 3(e_1 \wedge e_8)$. Thus, $f(x,y,z) = 0$, leads to $\alpha_7 = 1$. Whence,

$$\begin{aligned} f(x,y,z) = {} & (xy)z + (yz)x + (zx)y - (yx)z - (zy)x - (xz)y \\ & + z(xy) + x(yz) + y(zx) - z(yx) - x(zy) - y(xz). \end{aligned}$$

Finally, if we now take

$$x = e_1 \wedge e_2, y = e_1 \wedge e_3 + e_4 \wedge e_5 \text{ and } z = e_6 \wedge e_2 + e_7 \wedge e_8,$$

we obtain

$$f(x,y,z) = -e_1 \wedge e_3 + e_1 \wedge e_4 + e_2 \wedge e_3 - e_2 \wedge e_4 + e_5 \wedge e_8 - e_6 \wedge e_7 \neq 0.$$

This way, there are no identities of degree 3 for A.

The identities of fourth degree in $A^{(-)}$ are of the type $f(x,y,z,w) = 0$, where

$$\begin{aligned} f(x,y,z,w) &= [[[x,y],z],w] \\ &\quad + \alpha_1 [[[x,y],w],z] + \alpha_2 [[[x,z],y],w] \\ &\quad + \alpha_3 [[[x,z],w],y] + \alpha_4 [[[x,w],y],z] + \alpha_5 [[[x,w],z],y] \\ &\quad + \alpha_6 [[[y,z],x],w] + \alpha_7 [[[y,z],w],x] + \alpha_8 [[[y,w],x],z] \\ &\quad + \alpha_9 [[[y,w],z],x] + \alpha_{10} [[[z,w],x],y] + \alpha_{11} [[[z,w],y],x] \\ &\quad + \alpha_{12} [[x,y],[z,w]] + \alpha_{13} [[x,z],[y,w]] + \alpha_{14} [[x,w],[y,z]], \end{aligned} \tag{3.14}$$

with $x,y,z,w \in A^{(-)}$.

Considering $x = y = z$ in the above formula, we have:

$$f(x,y,z,w) = (\alpha_4 + \alpha_5 + \alpha_8 + \alpha_9 + \alpha_{10} + \alpha_{11}) [[[x,w],x],x].$$

Taking $x = e_1 \wedge e_2$, $w = e_1 \wedge e_3$, we have $[[[x,w],x],x] = -8(e_1 \wedge e_4)$. Therefore, $f(x,y,z,w) = 0$ leads to

$$\alpha_4 + \alpha_5 + \alpha_8 + \alpha_9 + \alpha_{10} + \alpha_{11} = 0. \tag{3.15}$$

The consideration of the analogous cases $x=y=w$, $x=z=w$ and $y=z=w$ allows us to obtain, respectively, the identities:

$$\begin{aligned} \alpha_2+\alpha_3+\alpha_6+\alpha_7-\alpha_{10}-\alpha_{11} &= 0; \\ 1+\alpha_1-\alpha_6-\alpha_7-\alpha_8-\alpha_9 &= 0; \\ 1+\alpha_1+\alpha_2+\alpha_3+\alpha_4+\alpha_5 &= 0. \end{aligned} \tag{3.16}$$

By the conjugation of (3.15) and (3.16), we have:

$$\begin{cases} \alpha_1=-1+\alpha_6+\alpha_7+\alpha_8+\alpha_9 \\ \alpha_2=-\alpha_3-\alpha_6-\alpha_7+\alpha_{10}+\alpha_{11} \\ \alpha_4=-\alpha_5-\alpha_8-\alpha_9-\alpha_{10}-\alpha_{11} \end{cases}.$$

After replacing in (3.14), we have:

$$\begin{aligned} f(x,y,z,w) = {} & [[[x,y],z],w] \\ & +(-1+\alpha_6+\alpha_7+\alpha_8+\alpha_9)[[[x,y],w],z] \\ & +(-\alpha_3-\alpha_6-\alpha_7+\alpha_{10}+\alpha_{11})[[[x,z],y],w] \\ & +(-\alpha_5-\alpha_8-\alpha_9-\alpha_{10}-\alpha_{11})[[[x,w],y],z] \\ & +\alpha_3[[[x,z],w],y]+\alpha_5[[[x,w],z],y] \\ & +\alpha_6[[[y,z],x],w]+\alpha_7[[[y,z],w],x]+\alpha_8[[[y,w],x],z] \\ & +\alpha_9[[[y,w],z],x]+\alpha_{10}[[[z,w],x],y]+\alpha_{11}[[[z,w],y],x] \\ & +\alpha_{12}[[x,y],[z,w]]+\alpha_{13}[[x,z],[y,w]]+\alpha_{14}[[x,w],[y,z]]. \end{aligned} \tag{3.17}$$

Consider now, respectively, the cases $x=y$, $x=z$, $x=w$, $y=z$, $y=w$, $z=w$. From (3.17) we obtain, respectively:

$$\begin{aligned} f(x,x,z,w) = {} & (-\alpha_3-\alpha_7+\alpha_{10}+\alpha_{11})[[[x,z],x],w] \\ & +(\alpha_3+\alpha_7)[[[x,z],w],x] \\ & +(-\alpha_5-\alpha_9-\alpha_{10}-\alpha_{11})[[[x,w],x],z] \\ & +(\alpha_5+\alpha_9)[[[x,w],z],x] \\ & +(\alpha_{10}+\alpha_{11})[[[z,w],x],x]+(\alpha_{13}-\alpha_{14})[[x,z],[x,w]]; \end{aligned} \tag{3.18}$$

$$\begin{aligned} f(x,y,x,w) &= (1-\alpha_6)\,[[[x,y],x],w] \\ &\quad + (-1+\alpha_6+\alpha_8+\alpha_9)\,[[[x,y],w],x] \\ &\quad + (\alpha_5+\alpha_{10})\,[[[x,w],x],y] \\ &\quad + (-\alpha_5-\alpha_8-\alpha_9-\alpha_{10})\,[[[x,w],y],x] \\ &\quad + (\alpha_8+\alpha_9)\,[[[y,w],x],x] + (\alpha_{12}+\alpha_{14})\,[[x,y],[x,w]]; \end{aligned} \tag{3.19}$$

$$\begin{aligned} f(x,y,z,x) &= (-1+\alpha_6+\alpha_7+\alpha_9)\,[[[x,y],x],z] \\ &\quad + (1-\alpha_9)\,[[[x,y],z],x] + (\alpha_3-\alpha_{10})\,[[[x,z],x],y] \\ &\quad + (-\alpha_3-\alpha_6-\alpha_7+\alpha_{10})\,[[[x,z],y],x] \\ &\quad + (\alpha_6+\alpha_7)\,[[[y,z],x],x] + (-\alpha_{12}+\alpha_{13})\,[[x,y],[x,z]]; \end{aligned} \tag{3.20}$$

$$\begin{aligned} f(x,y,y,w) &= (1-\alpha_3-\alpha_6-\alpha_7+\alpha_{10}+\alpha_{11})\,[[[x,y],y],w] \\ &\quad + (-1+\alpha_3+\alpha_6+\alpha_7+\alpha_8+\alpha_9)\,[[[x,y],w],y] \\ &\quad + (-\alpha_8-\alpha_9-\alpha_{10}-\alpha_{11})\,[[[x,w],y],y] \\ &\quad + (\alpha_8+\alpha_{10})\,[[[y,w],x],y] \\ &\quad + (\alpha_9+\alpha_{11})\,[[[y,w],y],x] + (\alpha_{12}+\alpha_{13})\,[[x,y],[y,w]]; \end{aligned} \tag{3.21}$$

$$\begin{aligned} f(x,y,z,y) &= (1+\alpha_5)\,[[[x,y],z],y] \\ &\quad + (-1-\alpha_5+\alpha_6+\alpha_7-\alpha_{10}-\alpha_{11})\,[[[x,y],y],z] \\ &\quad + (-\alpha_6-\alpha_7+\alpha_{10}+\alpha_{11})\,[[[x,z],y],y] \\ &\quad + (\alpha_6-\alpha_{10})\,[[[y,z],x],y] \\ &\quad + (\alpha_7-\alpha_{11})\,[[[y,z],y],x] + (\alpha_{12}-\alpha_{14})\,[[x,y],[z,y]]; \end{aligned} \tag{3.22}$$

$$\begin{aligned} f(x,y,z,z) &= (\alpha_6+\alpha_7+\alpha_8+\alpha_9)\,[[[x,y],z],z] \\ &\quad + (-\alpha_3-\alpha_5-\alpha_6-\alpha_7-\alpha_8-\alpha_9)\,[[[x,z],y],z] \\ &\quad + (\alpha_3+\alpha_5)\,[[[x,z],z],y] + (\alpha_6+\alpha_8)\,[[[y,z],x],z] \\ &\quad + (\alpha_7+\alpha_9)\,[[[y,z],z],x] + (\alpha_{13}+\alpha_{14})\,[[x,z],[y,z]]. \end{aligned} \tag{3.23}$$

Taking $x = e_1 \wedge e_2$, $z = e_1 \wedge e_3$ and $w = e_2 \wedge e_5$ in (3.18), the identity $f(x,x,z,w) = 0$ implies

$$\left\{ \begin{array}{c} -2\alpha_3 + 2\alpha_5 - 2\alpha_7 + 2\alpha_9 + \alpha_{10} + \alpha_{11} - \alpha_{13} + \alpha_{14} = 0 \\ -\alpha_3 + \alpha_5 - \alpha_7 + \alpha_9 + 2\alpha_{10} + 2\alpha_{11} + \alpha_{13} - \alpha_{14} = 0 \end{array} \right. . \tag{3.24}$$

Further, taking the same elements $e_1 \wedge e_2$, $e_1 \wedge e_3$ and $e_2 \wedge e_5$, *mutatis mutandis*, in (3.19)-(3.23), we obtain

$$\left\{ \begin{array}{rcr} -2\alpha_5 - 2\alpha_6 - 3\alpha_8 - 3\alpha_9 - 2\alpha_{10} - \alpha_{12} - \alpha_{14} & = & -2 \\ -\alpha_5 - \alpha_6 - \alpha_{10} + \alpha_{12} + \alpha_{14} & = & -1 \\ -2\alpha_3 - \alpha_6 - \alpha_7 + 2\alpha_9 + 2\alpha_{10} + \alpha_{12} - \alpha_{13} & = & 2 \\ -\alpha_3 + \alpha_6 + \alpha_7 + \alpha_9 + \alpha_{10} - \alpha_{12} + \alpha_{13} & = & 1 \\ 2\alpha_3 + 2\alpha_6 + 2\alpha_7 + 3\alpha_8 + \alpha_9 + \alpha_{10} - \alpha_{11} + \alpha_{12} + \alpha_{13} & = & 2 \\ \alpha_3 + \alpha_6 + \alpha_7 - \alpha_9 - \alpha_{10} - 2\alpha_{11} - \alpha_{12} - \alpha_{13} & = & 1 \\ 2\alpha_5 + \alpha_6 - \alpha_7 - \alpha_{10} + \alpha_{11} - \alpha_{12} + \alpha_{14} & = & -2 \\ \alpha_5 - \alpha_6 - 2\alpha_7 + \alpha_{10} + 2\alpha_{11} + \alpha_{12} - \alpha_{14} & = & -1 \\ -2\alpha_3 - 2\alpha_5 - 3\alpha_6 - \alpha_7 - 3\alpha_8 - \alpha_9 - \alpha_{13} - \alpha_{14} & = & 0 \\ -\alpha_3 - \alpha_5 + \alpha_7 + \alpha_9 + \alpha_{13} + \alpha_{14} & = & 0 \end{array} \right. . \tag{3.25}$$

The linear system of equations (3.24) and (3.25) is undetermined and equivalent to:

$$\left\{ \begin{array}{lcl} \alpha_3 & = & \alpha_9 + \alpha_{14} \\ \alpha_5 & = & \alpha_7 + \alpha_{13} \\ \alpha_6 & = & -\alpha_7 + \alpha_{12} - \alpha_{13} \\ \alpha_8 & = & -\alpha_9 - \alpha_{12} - \alpha_{14} \\ \alpha_{10} & = & 1 + \alpha_{14} \\ \alpha_{11} & = & -1 - \alpha_{13} \end{array} \right.$$

with $\alpha_7, \alpha_9, \alpha_{12}, \alpha_{13}, \alpha_{14} \in \Phi$. After replacing these coefficients in (3.17) we obtain:

$$\begin{aligned} f(x,y,z,w) = {} & [[[x,y],z],w] + (-1-\alpha_{13}-\alpha_{14})[[[x,y],w],z] \\ & + (-\alpha_9-\alpha_{12})[[[x,z],y],w] + (\alpha_9+\alpha_{14})[[[x,z],w],y] \\ & + (-\alpha_7+\alpha_{12})[[[x,w],y],z] + (\alpha_7+\alpha_{13})[[[x,w],z],y] \\ & + (-\alpha_7+\alpha_{12}-\alpha_{13})[[[y,z],x],w] + \alpha_7[[[y,z],w],x] \\ & + (-\alpha_9-\alpha_{12}-\alpha_{14})[[[y,w],x],z] + \alpha_9[[[y,w],z],x] \\ & + (1+\alpha_{14})[[[z,w],x],y] + (-1-\alpha_{13})[[[z,w],y],x] \\ & + \alpha_{12}[[x,y],[z,w]] + \alpha_{13}[[x,z],[y,w]] + \alpha_{14}[[x,w],[y,z]] . \end{aligned} \tag{3.26}$$

Putting $z = w$ in (3.18), from (3.26) we have:

$$f(x,x,z,z) = (2\alpha_7 + 2\alpha_9 + \alpha_{13} + \alpha_{14})([[[x,z],z],x] - [[[x,z],x],z]) .$$

Considering $x = e_1 \wedge e_2$, $z = e_1 \wedge e_3 + e_4 \wedge e_6$, from $f(x, x, z, z) = 0$ we obtain:

$$2\alpha_7 + 2\alpha_9 + \alpha_{13} + \alpha_{14} = 0. \tag{3.27}$$

Taking $y = w$ in (3.19), after (3.26) we obtain:

$$f(x, y, x, y) = (2 + 2\alpha_7 - \alpha_{12} + 2\alpha_{13} + \alpha_{14}) \left([[[x, y], x], y] - [[[x, y], y], x] \right).$$

Considering $x = e_1 \wedge e_2$, $y = e_1 \wedge e_3 + e_4 \wedge e_6$, the identity $f(x, y, x, y) = 0$ implies:

$$2 + 2\alpha_7 - \alpha_{12} + 2\alpha_{13} + \alpha_{14} = 0. \tag{3.28}$$

Considering $y = z$ in (3.20), by (3.26) we get:

$$f(x, y, y, x) = (-2 + 2\alpha_9 + \alpha_{12} - \alpha_{13}) \left([[[x, y], x], y] - [[[x, y], y], x] \right).$$

Considering $x = e_1 \wedge e_2$, $y = e_1 \wedge e_3 + e_4 \wedge e_6$, the identity $f(x, y, y, x) = 0$ implies:

$$-2 + 2\alpha_9 + \alpha_{12} - \alpha_{13} = 0. \tag{3.29}$$

¿From (3.27)-(3.29) we have

$$\begin{cases} \alpha_{12} = 2 - 2\alpha_9 + \alpha_{13}, \\ \alpha_{14} = -2\alpha_7 - 2\alpha_9 - \alpha_{13}. \end{cases}$$

Thus:

$$\begin{aligned} f(x, y, z, w) = {} & [[[x, y], z], w] + (-1 + 2\alpha_7 + 2\alpha_9) [[[x, y], w], z] \\ & + (-2 + \alpha_9 - \alpha_{13}) [[[x, z], y], w] \\ & + (-2\alpha_7 - \alpha_9 - \alpha_{13}) [[[x, z], w], y] \\ & + (2 - \alpha_7 - 2\alpha_9 + \alpha_{13}) [[[x, w], y], z] \\ & + (\alpha_7 + \alpha_{13}) [[[x, w], z], y] \\ & + (2 - \alpha_7 - 2\alpha_9) [[[y, z], x], w] + \alpha_7 [[[y, z], w], x] \\ & + (-2 + 2\alpha_7 + 3\alpha_9) [[[y, w], x], z] + \alpha_9 [[[y, w], z], x] \\ & + (1 - 2\alpha_7 - 2\alpha_9 - \alpha_{13}) [[[z, w], x], y] \\ & + (-1 - \alpha_{13}) [[[z, w], y], x] \\ & + (2 - 2\alpha_9 + \alpha_{13}) [[x, y], [z, w]] + \alpha_{13} [[x, z], [y, w]] \\ & + (-2\alpha_7 - 2\alpha_9 - \alpha_{13}) [[x, w], [y, z]] \end{aligned} \tag{3.30}$$

with $\alpha_7, \alpha_9, \alpha_{13} \in \Phi$.

Making $x = y$ in (3.30), we have:

$$f(x,x,z,w) = (\alpha_7 + \alpha_9 + \alpha_{13})\,(-[[[x,z],x],w] - [[[x,z],w],x] + [[[x,w],x],z] + [[[x,w],z],x] + 2\,[[x,z],[x,w]] - 2\,[[[z,w],x],x]).$$

¿From this, if $f(x,x,z,w) = 0$ then

$$\alpha_7 + \alpha_9 + \alpha_{13} = 0, \tag{3.31}$$

by considering $x = e_1 \wedge e_2$, $y = e_1 \wedge e_3 + e_4 \wedge e_6$, and $w = e_2 \wedge e_5$.

If we put $x = z$ in (3.30), we have:

$$f(x,y,x,w) = (-1 + \alpha_7 + 2\alpha_9)\,([[[x,y],x],w] + [[[x,y],w],x] - [[[x,w],x],y] - [[[x,w],y],x] + 2\,[[[y,w],x],x] - 2\,[[x,y],[x,w]]).$$

After this, the identity $f(x,y,x,w) = 0$ implies

$$-1 + \alpha_7 + 2\alpha_9 = 0, \tag{3.32}$$

by taking $x = e_1 \wedge e_2$, $y = e_1 \wedge e_3 + e_4 \wedge e_6$, and $w = e_2 \wedge e_5$.

Analogously, if we have $x = w$ in (3.30), we obtain:

$$f(x,y,z,x) = (1 - \alpha_9)\,([[[x,y],x],z] + [[[x,y],z],x] - [[[x,z],x],y] - [[[x,z],y],x] + 2\,[[[y,z],x],x] - 2\,[[x,y],[x,z]]).$$

Thus, if we take $x = e_1 \wedge e_2$, $y = e_1 \wedge e_3 + e_4 \wedge e_6$, and $z = e_2 \wedge e_5$, the identity $f(x,y,x,w) = 0$ implies

$$1 - \alpha_9 = 0. \tag{3.33}$$

¿From (3.31)-(3.33), we get $\alpha_9 = 1$, $\alpha_7 = -1$ and $\alpha_{13} = 0$. Whence, the only possible identity of fourth degree in $A^{(-)}$ is $f(x,y,z,w) = 0$, where

$$\begin{aligned} f(x,y,z,w) \;=\; & [[[x,y],z],w] - [[[x,y],w],z] - [[[x,z],y],w] \\ & + [[[x,z],w],y] + [[[x,w],y],z] - [[[x,w],z],y] \\ & + [[[y,z],x],w] - [[[y,z],w],x] - [[[y,w],x],z] \\ & + [[[y,w],z],x] + [[[z,w],x],y] - [[[z,w],y],x]. \end{aligned}$$

However, considering $x = e_1 \wedge e_2$, $y = e_1 \wedge e_3$, $z = e_2 \wedge e_5 + e_3 \wedge e_5$ and $w = e_2 \wedge e_6 + e_3 \wedge e_7$ in this development, we have

$$\begin{aligned} f(x, y, z, w) &= 18e_1 \wedge e_2 - 12e_1 \wedge e_3 + 12e_2 \wedge e_4 + 18e_3 \wedge e_4 \\ &\quad +6e_5 \wedge e_6 - 12e_5 \wedge e_7 - 12e_6 \wedge e_8 - 6e_7 \wedge e_8, \end{aligned}$$

which is nonzero. Therefore, there are no identities of fourth degree in $A^{(-)}$. ∎

Definition 3.6. If

$$f = \sum_{\sigma \in S_5} \alpha_\sigma \left[\left[x_{\sigma(1)}, x_{\sigma(2)}, x_{\sigma(3)} \right], x_{\sigma(4)}, x_{\sigma(5)} \right]$$

is an identity for a ternary algebra A, then we call it *identity of degree* 2.

Lemma 3.7. *There are no identities of degree* 2 *for* M_8.

Proof: Let f be such an identity. Then, for some $\alpha_i \in \Phi$ we have:

$$\begin{aligned} f(x_1, x_2, x_3, x_4, x_5) = \alpha_1 [[x_1, x_2, x_3], x_4, x_5] + \alpha_2 [[x_1, x_2, x_4], x_3, x_5] \\ +\alpha_3 [[x_1, x_3, x_4], x_2, x_5] + \alpha_4 [[x_2, x_3, x_4], x_1, x_5] \\ +\alpha_5 [[x_1, x_2, x_5], x_3, x_4] + \alpha_6 [[x_1, x_3, x_5], x_2, x_4] \\ +\alpha_7 [[x_2, x_3, x_5], x_1, x_4] + \alpha_8 [[x_1, x_4, x_5], x_2, x_3] \\ +\alpha_9 [[x_2, x_4, x_5], x_1, x_3] + \alpha_{10} [[x_3, x_4, x_5], x_1, x_2]. \end{aligned}$$

It is easy to notice the following implications:

$$\begin{aligned} f(e_1, e_2, e_3, e_4, e_5) = 0 &\Rightarrow -\alpha_5 + \alpha_6 - \alpha_7 - \alpha_8 + \alpha_9 - \alpha_{10} = 0; \\ f(e_1, e_2, e_3, e_5, e_6) = 0 &\Rightarrow -\alpha_1 - \alpha_3 + \alpha_4 + \alpha_6 - \alpha_7 - \alpha_{10} = 0; \\ f(e_1, e_2, e_3, e_6, e_7) = 0 &\Rightarrow \alpha_1 - \alpha_2 + \alpha_3 + \alpha_5 - \alpha_6 + \alpha_8 = 0; \\ f(e_1, e_2, e_4, e_6, e_7) = 0 &\Rightarrow -\alpha_1 + \alpha_2 + \alpha_4 - \alpha_5 - \alpha_7 + \alpha_9 = 0; \\ f(e_1, e_3, e_5, e_6, e_7) = 0 &\Rightarrow -\alpha_2 + \alpha_3 - \alpha_4 + \alpha_8 - \alpha_9 + \alpha_{10} = 0. \end{aligned}$$

Therefore, for $\alpha_6, \alpha_7, \alpha_8, \alpha_9, \alpha_{10} \in \Phi$, we obtain:

$$\begin{aligned} \alpha_1 &= \alpha_8 - \alpha_9 + \alpha_{10}, \quad \alpha_2 = \alpha_6 - \alpha_7 - \alpha_{10}, \quad \alpha_3 = \alpha_6 - \alpha_8 - \alpha_{10}, \\ \alpha_4 &= \alpha_7 - \alpha_9 + \alpha_{10}, \quad \alpha_5 = \alpha_6 - \alpha_7 - \alpha_8 + \alpha_9 - \alpha_{10}, \end{aligned}$$

and thus

$$\begin{aligned} f(x_1,x_2,x_3,x_4,x_5) &= (\alpha_8-\alpha_9+\alpha_{10})\left[\left[x_1,x_2,x_3\right],x_4,x_5\right] \\ &+(\alpha_6-\alpha_7-\alpha_{10})\left[\left[x_1,x_2,x_4\right],x_3,x_5\right] \\ &+(\alpha_6-\alpha_8-\alpha_{10})\left[\left[x_1,x_3,x_4\right],x_2,x_5\right] \\ &+(\alpha_7-\alpha_9+\alpha_{10})\left[\left[x_2,x_3,x_4\right],x_1,x_5\right] \\ &+(\alpha_6-\alpha_7-\alpha_8+\alpha_9-\alpha_{10})\left[\left[x_1,x_2,x_5\right],x_3,x_4\right] \\ &+\alpha_6\left[\left[x_1,x_3,x_5\right],x_2,x_4\right] \\ &+\alpha_7\left[\left[x_2,x_3,x_5\right],x_1,x_4\right]+\alpha_8\left[\left[x_1,x_4,x_5\right],x_2,x_3\right] \\ &+\alpha_9\left[\left[x_2,x_4,x_5\right],x_1,x_3\right]+\alpha_{10}\left[\left[x_3,x_4,x_5\right],x_1,x_2\right]. \end{aligned}$$

¿From this development, it is easy to see that:

$$f(x_1,x_1,x_3,x_3,x_5) = (\alpha_6+\alpha_7+\alpha_8+\alpha_9)\left[\left[x_1,x_3,x_5\right],x_1,x_3\right].$$

Considering $x_1,x_3,x_5\in\varepsilon$, we get $\left[\left[x_1,x_3,x_5\right],x_1,x_3\right]=-x_5$ and thus

$$f(x_1,x_1,x_3,x_3,x_5) = -\left(\alpha_6+\alpha_7+\alpha_8+\alpha_9\right)x_5.$$

This way, *e.g.*, $f(e_1,e_1,e_2,e_2,e_3)=0$ implies

$$\alpha_6+\alpha_7+\alpha_8+\alpha_9=0.$$

Computing $f(x_1,x_1,x_3,x_4,x_3)$, $f(x_1,x_2,x_1,x_2,x_5)$, $f(x_1,x_2,x_1,x_4,x_2)$, $f(x_1,x_2,x_3,x_1,x_2)$, and proceeding (*mutatis mutandis*) as above, we obtain, respectively, the equalities:

$$\begin{aligned} \alpha_6+\alpha_7-2\alpha_8-2\alpha_9 &= 0; \\ \alpha_6-2\alpha_7-2\alpha_8+\alpha_9 &= 0; \\ \alpha_6-2\alpha_7+\alpha_8+\alpha_9-3\alpha_{10} &= 0; \\ \alpha_6+\alpha_7+\alpha_8-2\alpha_9+3\alpha_{10} &= 0. \end{aligned}$$

The linear system consisting on these five equalities has one only trivial solution. Thus, $\alpha_6=\alpha_7=\alpha_8=\alpha_9=\alpha_{10}=0$ and, consequently, $\alpha_i=0$ for all $i=1,...,10$. ■

4. On weight spaces of a ternary Malcev algebra

Let L be a ternary Malcev algebra. We call a right multiplication $R_{x,y}$ *regular*, if in the Fitting decomposition of $L=L_0\oplus L_1$ relative to

$R_{x,y}$ the dimension of the zero component, L_0, is minimal. In this case, we call L_0 a *Cartan subspace* of L.

The field Φ is called *quadratically closed* if, for any $\alpha \in \Phi$, there exists $\sqrt{\alpha} \in \Phi$ such that $(\sqrt{\alpha})^2 = \alpha$. In what follows, we assume that the ground field satisfies this property.

Lemma 4.1. *Let $x, y \in M$. Then $R_{x,y}$ is regular if and only if $n_x n_y \neq (x,y)^2$.*

Proof: Suppose that $R_{x,y}$ is non-regular. Then there exists $z \in M$ such that $\dim \langle x,y,z\rangle_\Phi = 3$ and $zR_{x,y} = 0$. Observing that

$$zR_{x,y} = xR_{y,z} = x\bar{y}z - (y,z)x + (x,z)y - (x,y)z,$$

that identity implies $(x\bar{y} - (x,y))z = (y,z)x - (x,z)y$. If we have $n_{x\bar{y}-(x,y)} \neq 0$ then $z \in \langle x,y\rangle_\Phi$, which gives a contradiction. Thus, $n_{x\bar{y}-(x,y)} = n_x n_y - (x,y)^2 = 0$.

Conversely, suppose that $n_x n_y = (x,y)^2$ (note that in this case $n_{x'} n_y = (x',y)^2$ for any $x' = \alpha x + \beta y$).

Consider the case $1 \notin \langle x,y\rangle_\Phi$.

Let $n_x = n_y = 0$ and $(x,y) = 0$. If $xy = 0$ then $1R_{x,y} = -xR_{1,y} \in \langle x,y\rangle_\Phi$ and $R_{x,y}$ is nilpotent on $\langle 1,x,y\rangle_\Phi$. If $xy \neq 0$ and $xy = \alpha x + \beta y$ then $\alpha x\bar{y} = 0$. If $\alpha \neq 0$ then

$$y = \gamma + y',\ \gamma \neq 0,\ \bar{y} = \gamma - y',\ x\gamma = xy' \text{ and } xy = 2\gamma x.$$

Therefore, $1R_{x,y} \in \langle x,y\rangle_\Phi$, and $R_{x,y}$ is nilpotent on $\langle 1,x,y\rangle_\Phi$. If $xy \neq 0$ and $xy \notin \langle x,y\rangle_\Phi$ then $xyR_{x,y} = 0$.

Let $n_x = 0$, $n_y \neq 0$ and $(x,y) = 0$. Note that $xy \neq 0$. Indeed, otherwise $xR_{1,y} \in \langle x,y\rangle_\Phi$, and $R_{x,y}$ is nilpotent on $\langle 1,x,y\rangle_\Phi$. If $xy = \alpha x + \beta y$ then again $1R_{x,y} \in \langle x,y\rangle_\Phi$. Thus, $xy \notin \langle x,y\rangle_\Phi$, $xyR_{x,y} \in \langle x,y\rangle_\Phi$, and $R_{x,y}$ is nilpotent on $\langle x,y,xy\rangle_\Phi$.

If $n_x \neq 0$ and $n_y \neq 0$ then, taking $x + \alpha y$ instead of x, for some α, we come to the case considered above.

Let us consider the case $1 \in \langle x,y\rangle_\Phi$.

In this case we may suppose that $R = R_{1,x}$, where $n_x = 0$ and $(1,x) = 0$. Then there exists z such that $zx \in \langle x\rangle_\Phi$ and $y \notin \langle 1,x\rangle_\Phi$. Therefore, $zR_{1,x} \in \langle 1,x\rangle_\Phi$, and $R_{1,x}$ is nilpotent on $\langle 1,x,z\rangle_\Phi$. ∎

Lemma 4.2. *Let M be the simple 8-dimensional ternary Malcev algebra over a quadratically closed field Φ. Let $R_{x,y}$ be a regular element and*

$M = M_0 \oplus M_1$ the Fitting decomposition of M relative to $R_{x,y}$. Then M_0 is a two-dimensional abelian subalgebra of M, and we have the following Cartan decomposition of M

$$M = M_0 \oplus M_\alpha \oplus M_{-\alpha},$$

where $\alpha \in \Phi$ such that $vR_{x,y} = \pm\alpha v$ for any $v \in M_{\pm\alpha}$.

Proof: Let $R_{x,y}$ be a regular element, where $1 \notin \langle x, y\rangle_\Phi$. We may assume that $n_x = n_y = 1$, and $(x, y) = 0$. Let $z \in \langle 1, x, y, xy\rangle_\Phi^\perp$ and $n_z \neq 0$. Then

$$M = M_0 \oplus M_i \oplus M_{-i},$$

where

$$M_0 = \langle x, y\rangle_\Phi, \quad M_{\pm i} = \langle 1 \pm ixy, z \pm ixyz, yz \pm ixz\rangle_\Phi .$$

Let $R = R_{x,y}$ be again a regular element but now $1 \in \langle x, y\rangle_\Phi$. We may assume that $R = R_{1,x}$, where $n_x = 1$ and $(1, x) = 0$. Let $y \in \langle 1, x\rangle_\Phi^\perp$, $n_y \neq 0$, $z \in \langle 1, x, y, xy\rangle_\Phi^\perp$ and $n_z \neq 0$. Then

$$M = M_0 \oplus M_i \oplus M_{-i},$$

where

$$M_0 = \langle 1, x\rangle_\Phi, \quad M_{\pm i} = \langle y \pm ixy, z \pm ixz, xyz \pm iyz\rangle_\Phi .$$

Let $R = R_{1+x,y}$ be a regular element, $(1, x) = (1, y) = 0$ and $n_y = 0$. We may assume that $n_x = 0$ and $(x, y) = 1$. Then

$$(1 - y)R = -(1 + xy) \text{ and } (1 + xy)R = -(1 - y).$$

Let $V = \langle 1, x, y, xy\rangle_\Phi$. It is easy to notice that $\dim V = 4$. Let $Z = V^\perp$ and $z_1 \in Z$. If $z_1 y = 0$ then $z_1 \in M_1$ (if $z_1 y \neq 0$ then $z_1 yy = 0$), $z_1 xy = -2z_1$ and $z_1 x \neq 0$. It is easy to see that z_1 and $z_1(1 + x)$ are linearly independent modulo V, and $z_1(1 + x) \in M_{-1}$. Choose $z_2 \in Z$ such that $z_2 \notin \langle z_1, z_1(1 + x)\rangle_\Phi$. If $z_2 y = 0$ then $z_2 \in M_1$ and $z_2 x \neq 0$. Suppose that $z_2(1 + x) \in \langle z_1, z_1(1 + x), z_2\rangle_\Phi \oplus V$. Acting scalarly with $(\cdot, x), (\cdot, y), (\cdot, xy), (\cdot, 1)$ and multiplying on x, we come to the conclusion that $(z_2 - \alpha z_1)x = 0$, contradicting the fact that $(z_2 - \alpha z_1)y = 0$. If $z_2 y \neq 0$ then we may consider $z_2 y$ instead of z_2. If $z_2 y \in \langle z_1, z_1(1 + x)\rangle_\Phi \oplus V$ then, as above, we obtain $z_2 y = \alpha_1 z_1$. Take

z_3 instead of z_2 (with the same properties). Then $z_3y = \alpha_2 z_1$, and the element $\alpha_2 z_2 - \alpha_1 z_3$ is required. Thus,

$$M = M_0 \oplus M_1 \oplus M_{-1},$$

where

$$M_0 = \langle 1+x, y\rangle_\Phi \,, \; M_1 = \langle y+xy, z_1, z_2\rangle_\Phi$$

and

$$M_{-1} = \langle 2-y+xy, z_1(1+x), z_2(1+x)\rangle_\Phi \,.$$

Let $R = R_{1+x,y}$ be a regular element, $(1,x) = (1,y) = 0$ and $n_y \neq 0$. We may assume that $n_y = 1$, $n_{1+x} = \alpha \neq 0$ and $(x,y) = 0$. Let $a = 1 - \alpha^{-1}(1+x)$ and $b = xy$. Then

$$aR = -b, \; bR = \alpha a \quad \text{and} \quad b \pm i\sqrt{\alpha}a \in M_{\mp i\sqrt{\alpha}}.$$

Let $U = \langle 1, x, y, xy\rangle_\Phi^\perp$ and $z \in U$ be an eigenvector for R. Then

$$zR_{1+x,y} = z(1-x)y = zy - zxy = \beta z$$

for some $\beta \in \Phi^*$. Since $zxy = -z(xy)$, this is equivalent to $z(-\beta + y + xy) = 0$. From the last equality we obtain

$$n_{-\beta+y+xy} = \beta^2 + 1 + n_x = 0 \text{ and } \beta = \pm i\sqrt{\alpha}.$$

Conversely, if $z \in U$ satisfies $z(-\beta+y+xy) = 0$ for some $\beta \in \Phi^*$, then $z \in M_\beta$. We next show that it is possible to choose linearly independent $u_1, u_2, u_3, u_4 \in U$ such that $u_1, u_2 \in M_{i\sqrt{\alpha}}$ and $u_3, u_4 \in M_{-i\sqrt{\alpha}}$. Let $v_\alpha = -i\sqrt{\alpha} + y + xy$. Take $u_1 \in U$ such that $u_1 v_\alpha = 0$. Choose $u_2 \in U$ such that $u_2 \notin \langle u_1\rangle_\Phi$. If $u_2 v_\alpha = 0$ then we found the required elements u_1 and u_2. If $u_2 v_\alpha \neq 0$ and $u_2 v_\alpha \notin \langle u_1\rangle_\Phi \oplus V$, then we can take $u_2 v_\alpha$ instead of u_2. If $u_2 v_\alpha \neq 0$ and $u_2 v_\alpha \in \langle u_1\rangle_\Phi \oplus V$, then it is easy to see that $u_2 v_\alpha = \gamma_1 u_1$. Consider an element $u_3 \in U$ such that $u_3 \notin \langle u_1, u_2\rangle_\Phi$. If $u_3 v_\alpha = 0$ then we can take u_3 instead of u_2. If $u_3 v_\alpha \neq 0$ and $u_3 v_\alpha \in \langle u_1\rangle_\Phi \oplus V$, then it is easy to see that $u_3 v_\alpha = \gamma_2 u_1$. Thus, we can consider $\gamma_2 u_2 - \gamma_1 u_3$ instead of u_2. We may apply an analogous procedure to find the elements u_3 and u_4. Suppose that $\sum_{i=1}^4 \gamma_i u_i = 0$. Since

$$u_k(y+xy) = i\sqrt{\alpha}u_k, \quad k = 1,2, \quad \text{and} \quad u_k(y+xy) = -i\sqrt{\alpha}u_k, k = 3,4,$$

we conclude that either u_1 and u_2 are linearly dependent, or u_3 and u_4 are linearly dependent, which is impossible. Thus, we have

$$M = M_0 \oplus M_{i\sqrt{\alpha}} \oplus M_{-i\sqrt{\alpha}},$$

where

$$M_0 = \langle 1 + x, y\rangle_\Phi\,, \quad M_{i\sqrt{\alpha}} = \left\langle xy - i\sqrt{\alpha} + i\sqrt{\alpha}\alpha^{-1}(1 + x), u_1, u_2\right\rangle_\Phi$$

and

$$M_{-i\sqrt{\alpha}} = \left\langle xy + i\sqrt{\alpha} - i\sqrt{\alpha}\alpha^{-1}(1 + x), u_3, u_4\right\rangle_\Phi\,.$$

■

Remark 4.3. According to the proof of the lemma, for any regular element R, we can construct explicitly the Cartan decomposition of M relative to R.

Lemma 4.4. *Let* $V = \langle 1, x, y, xy\rangle_\Phi$, $U = V^\perp$ *and* $M_\pm = M_{\pm\alpha} \cap U$. *Then*

1. $(M_\pm M_\pm, M_\mp) = 0$;
2. $M_\pm M_\pm \subseteq \langle \alpha x \pm n_x y \mp xy\rangle_\Phi\,$;
3. $M_+M_- \subseteq \langle -\alpha + y + xy\rangle_\Phi\,$;
4. $M_+M_+ + M_+M_- + M_-M_- \subseteq V$.

Proof: 1. Let $v = (u_1u_2, u_3)$, where $u_1, u_2 \in M_\pm$, $u_3 \in M_\mp$. Since $\pm\alpha u_2 = u_2(y + xy)$, we have

$$\begin{aligned} \pm\alpha v &= (u_1(u_2(y + xy)), u_3) = (u_1((\overline{y + xy})u_2), u_3) \\ &= -((y + xy)(\overline{u}_1u_2), u_3) = -(u_1u_2, (y + xy)u_3) \\ &= (u_1u_2, u_3(y + xy)) = \mp\alpha(u_1u_2, u_3). \end{aligned}$$

2. By 1., $u_1u_2 \in V$. Therefore, $u_1u_2 = \alpha_1 + \alpha_2x + \alpha_3y + \alpha_4xy$. Applying $(\cdot, 1), (\cdot, x), (\cdot, y), (\cdot, xy)$ and using $u_i(\mp\alpha + y + xy) = 0$, we obtain the required inclusion.

3. The proof is analogous to the proof of the item 2.

4. It easily follows from 1., 2. and 3. ■

Lemma 4.5. *Denote* $M_{\pm\alpha}$ *by* $M_{\pm 1}$. *Then*

$$[M_i, M_j, M_k] \subseteq M_{i+j+k},$$

where the sum $i + j + k$ *is considered modulo* 3.

Proof: Consider the decomposition with respect to a regular element $R = R_{1+x,y}$, where

$$(1, x) = (1, y) = 0,\ n_y = 1,\ n_{1+x} = \kappa \neq 0 \text{ and } (x, y) = 0.$$

Put $\alpha = i\sqrt{\kappa}$. First, we prove the inclusion $[M_0, M_{\pm1}, M_{\pm1}] \subseteq M_{\mp1}$. In order to do it, we begin to show that

$$W = [M_0, \pm n_x \mp x + \alpha xy, M_{\pm1}] \subseteq M_{\mp1}.$$

Let $w \in W$. Then

$$w = [\pm n_x \mp x + \alpha xy, \beta + \beta x + \gamma y, u],$$

where $u(\mp\alpha + y + xy) = 0$, $u \in M_{\pm1}$. Thus the inclusion $w \in V^{\perp}$ follows from

$$\begin{aligned} w &= [(\pm n_x \mp x + \alpha xy)(\beta - \beta x - \gamma y)]u \\ &= (\pm\beta + \gamma/\delta)(\alpha x \mp n_x y \pm xy)u. \end{aligned} \tag{4.1}$$

We have the following criterium

$$v \in M_{\mp\alpha} \iff v(\pm\alpha + y + xy) = 0. \tag{4.2}$$

It is easy to see that $w(\pm\alpha + y + xy) = 0$, *i.e.*, $w \in M_{\mp\alpha}$. The inclusion

$$[M_0, M_{\pm1}, M_{\pm1}] \subseteq M_{\mp1}$$

easily follows from the previous conclusion and from Lemma 4.4.

Using 2. of Lemma 4.4, we obtain $[M_{\pm1}, M_{\pm1}, M_{\pm1}] \subseteq M_0$.

Using 3. of Lemma 4.4., (4.1) and the standard computations, we obtain the inclusion $[M_0, M_1, M_{-1}] \subseteq M_0$.

We now show that $[M_1, M_1, M_{-1}] \subseteq M_1$.

1. Let $u_1, u_2 \in M_+,\ u \in M_{-1}$. We have $[u_1, u_2, u] \subseteq \langle (u_1u_2)u\rangle_\Phi + M_1$. By 2 of Lemma 4.4., $\langle (u_1u_2)u\rangle_\Phi \subseteq \langle (\alpha x + n_x y - xy)u\rangle_\Phi$. Using direct computations (and (4.2) if $u \in M_-$), we obtain the required inclusion.

2. Let $u_\pm \in M_\pm$. Then

$$[n_x - x + \alpha xy, u_+, u_-] \subseteq \langle (u_+u_-)(n_x - x + \alpha xy)\rangle_\Phi + M_1$$

and it is enough to apply 3. of Lemma 4.4.

3. $[n_x - x + \alpha xy, u_+, n_x - x - \alpha xy] \subseteq \langle (-\alpha + y + xy)u_+\rangle_\Phi$, and it is enough to apply (4.2).

The case $[M_1, M_{-1}, M_{-1}] \subseteq M_1$ can be analyzed analogously.

Further, concerning the remaining regular elements, we can adopt analogous procedures. ■

Corollary 4.6. *The ternary Malcev algebra M_8 can be equipped with a non-trivial Z_3-grading.*

References

[1] R.E.Block, (1969), *Determination of the differentiably simple rings with a minimal ideal*, Ann. Math., 90, N3, 433-459.
[2] R.B.Brown and A.Gray, (1967), *Vector cross products*, Comment. Math. Helv., 42, 222–236.
[3] S. Eilenberg, (1948), *Extensions of general algebras*, Ann. Soc. Polon. Math. N21, 125-134.
[4] V.T.Filippov, (1985), *n-Lie algebras*, Sib. Math. J., 26, N6, 879-891 (translation from Sib. Mat. Zh., 26, N6 (1985), 126-140 (Russian)).
[5] V.T.Filippov, (2000), *On δ-derivations of prime alternative and Mal'tsev algebras*, Algebra and Logic 39, N5, 354–358.
[6] Sh. M. Kasymov and E. N. Kuz'min, (1987), *Classification of representations of a splitting, simple n-Lie algebras of type A_1*, preprint N23, Institute of Mathematics, Novosibirsk, 1-21.
[7] A.P.Pozhidaev, (2001), *n-Ary Mal'tsev algebras*, Algebra and Logic, 40, N3, 309-329.
[8] P.Saraiva, (2003), *On some generalizations of Malcev algebras*, Int. J. of Math., Game Th. and Algebra, 13, N2, 89-108.
[9] I.P.Shestakov, A.I.Shirshov, A.M.Slin'ko and K.A.Zhevlakov, (1978), *Rings that are Nearly Associative*, M.: Nauka (English translation: Acad. Press N. Y. 1982).

Alexandre P. Pojidaev
Sobolev Institute of Mathematics, Novosibirsk, pr. Koptyuga 4, 630090, Russia
E-mail address: app@math.nsc.ru

Paulo Saraiva
Faculdade de Economia, Universidade de Coimbra, 3004-512 Coimbra, Portugal
E-mail address: psaraiva@sonata.fe.uc.pt