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DIRAC-NIJENHUIS STRUCTURES

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ABSTRACT: We introduce the concept of Dirac-Nijenhuis structures as those manifolds carrying a Dirac structure and admitting a deformation by Nijenhuis operators which is compatible with it. This concept generalizes the notion of Poisson-Nijenhuis structure and can be adapted to include the Jacobi-Nijenhuis case.

KEYWORDS: Dirac structure, Nijenhuis operator, Lie algebroid, Lie bialgebroid, Courant algebroid.
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1. Introduction

Dirac structures were introduced by Courant and Weinstein and Dorfman by the end of the eighties [3, 4]. They are a generalization of Poisson structures which, roughly speaking, replace the canonical symplectic foliation of Poisson manifold by a presymplectic one. Roughly speaking still, we can think on Dirac structures as a Poisson manifold endowed with a distinguished distribution which, speaking in mechanical terms, defines a set of implicit constraints. Mechanical systems with constraints, singular Lagrangian systems, and many engineering systems are naturally described by a Dirac manifold. These applications in engineering are defined through the theory of Port Controlled Hamiltonian Systems introduced by van der Schaft, Maschke and coworkers (see for instance [13, 16] and references therein).

¿From a geometrical point of view, Dirac structures are intimately related to Lie algebroids and bialgebroids [2, 9, 8]. A Dirac structure on a manifold M was defined in [3, 2] as a subbundle D of the Whitney sum $TM \oplus T^*M$ satisfying certain properties, which correspond to the definition of a Lie algebroid structure. Later, the concept was generalized to similar subbundles defined on Whitney sums of the form $A \oplus A^*$ where (A, A^*) is a Lie bialgebroid [9].

The deformation of structures by using Nijenhuis operators is a concept often used in the Literature. Originally proposed within the framework of integrable systems (see the introduction and references of [11]), it allows a deformation of Lie algebra structures defined on different types of manifolds.

It has been recently extended to the Lie algebroid case, and therefore a very interesting example seems to be the study of the deformation of Lie algebroid structure which corresponds to a Dirac manifold. In [5], the problem was discussed for the case of Poisson manifolds (corresponding to the case of Poisson-Nijenhuis manifolds [6, 7]). Within the Lie algebroid domain, the Jacobi-Nijenhuis case (i.e. the deformation associated to a Jacobi manifold) was also studied in [12, 14]. We propose a generalization of the concept to general Dirac structures, which includes the Poisson-Nijenhuis case as a particular example, and can be adapted to the Jacobi-Nijenhuis framework (see [1]). Besides the intrinsic interest of the concept, we also have in mind eventual applications to Engineering within the framework of Port Controlled Hamiltonian systems.

The structure of the paper is as follows. In Section 2 we introduce the main properties of Lie algebroids which we are going to use in the paper, from the definition itself to the exterior algebra and the Schouten bracket. We also present the definition of Dirac structures in the simplest situations, the constant case and the definition as a subbundle of $TM \oplus T^*M$. In Section 3 the notion of Dirac structure is generalized to general Lie bialgebroids: we discuss the definition of a Lie bialgebroid, and some properties and examples, as well as the notion of Courant algebroids. Then, Dirac structures are defined as suitable subbundles within that framework. The notion of the characteristic pair of a Lie algebroid, which will be very important for us later, is also discussed. The last sections are devoted to the deformation of the structures presented so far: Section 4 studies the deformation of Lie algebroids and Lie bialgebroids, since they are necessary to define the deformation of Dirac structures themselves. This is done in two steps: Section 5 studies the deformation of the Dirac bundle alone, via transformations that do not affect the underlying Lie bialgebroid. Finally, Section 6 presents the conditions for a deformation of a Lie bialgebroid to define also a deformation of a Dirac structure defined on it.

2. Dirac structures and Lie algebroids

2.1. Lie algebroids. The idea behind Lie algebroids has been used in the last fifty years in the algebraic geometric framework, under different names but the first proper definition, from the point of view of Differential Geometry, is due to Pradines [15].

Definition 2.1. *A* **Lie algebroid** *on a manifold M is a vector bundle $A \to M$, in whose space of sections we define a Lie algebra structure $(\Gamma A, [\cdot,\cdot]_A)$, and a mapping $\rho : A \to TM$ which is a homomorphism for this structure in relation with the natural Lie algebra structure of the set of vector fields $(\mathfrak{X}(M), [\cdot,\cdot]_{TM})$. We have therefore:*

$$\rho([X,Y]_A) = [\rho(X), \rho(Y)]_{TM} \quad \forall\, X, Y \in \Gamma A,$$

Besides, the structure allows a derivation-type property for the module of sections:

$$[X, fY]_A = [X,Y]_A + (\rho(X)f)\,Y \quad \forall\, X, Y \in \Gamma A, \quad \forall\, f \in C^\infty(M)\,. \tag{1}$$

The main idea we have to keep in mind is that a Lie algebroid is a geometrical object very similar to a tangent bundle. The sections of the bundle A play the role of vector fields. The other basic objects of differential calculus on TM can be defined for A as well. The sections of the dual bundle A^* play the role of one-forms, and the sections of its skew-symmetrized tensor product that of the p-forms. There is a cohomology defined by an exterior differential as follows:

- For functions $d : C^\infty(M) \to \Gamma A^*$ such that:

$$\langle df, X\rangle = \rho(X)f \quad \forall f \in C^\infty(M), X \in \Gamma A.$$

- For higher orders forms we take the direct analogue of the usual definition $d : \Gamma\bigwedge^p(A) \to \Gamma\bigwedge^{p+1}(A)$:

$$d\theta(\sigma_1, \ldots, \sigma_{p+1}) = \sum_{i=1}^{p+1} (-1)^{i+1} \rho(\sigma_i)\theta(\sigma_1, \ldots, \hat{\sigma}_i, \ldots \sigma_{p+1})$$
$$+\sum_{i<j} (-1)^{i+j} \theta([\sigma_i, \sigma_j], \sigma_1, \ldots, \hat{\sigma}_i, \ldots, \hat{\sigma}_j, \ldots \sigma_{p+1}) \quad \forall \sigma_i \in \Gamma A \quad \theta \in \Gamma\bigwedge^p A^*, \tag{2}$$

where by the symbol $\hat{\sigma}_i$ we mean that the corresponding section is omitted.

Lemma 2.1. *The differential for A can be related to the de Rham differential of the base manifold M as:*

$$d_A = \rho^* \circ d. \tag{3}$$

Proof: It follows directly from the definition of d_A and the homomorphism property of the anchor mapping. ∎

Note: In the following, we will denote the differential of A as d unless some confusion with the usual de Rham differential arises.

The action of an A-section X on a p-form ω is defined in the natural way, as the $p-1$ form given as:

$$i_X\omega(X_1, \cdots, X_{p-1}) = \omega(X, X_1, \cdots, X_{p-1}) \qquad \forall X_1, \cdots, X_{p-1} \in \Gamma A.$$

And finally, the Lie derivative for sections of the dual bundle can be defined by generalizing Cartan's formula:

$$\mathcal{L}^A_\sigma = i_\sigma \circ d + d \circ i_\sigma \qquad \sigma \in \Gamma A.$$

This definition leads to the following result, that some authors use as the definition of the Lie derivative:

Lemma 2.2. *Let A be a Lie algebroid, with anchor ρ and differential d. Then, the following property holds:*

$$\langle Y, \mathcal{L}_X\alpha\rangle = \rho(X)\langle Y, \alpha\rangle - \alpha([X, Y]). \tag{4}$$

Proof: ¿From the definition of the Lie derivative:

$$\mathcal{L}_X\alpha = i_X \circ d\alpha + di_X\alpha.$$

Then,

$$\langle Y, \mathcal{L}_X\alpha\rangle = i_Y\mathcal{L}_X\alpha = i_Y(i_x \circ d\alpha + di_X\alpha) = \rho(Y)\langle X, \alpha\rangle + (d\alpha)(X, Y) =$$
$$\rho(Y)\langle X, \alpha\rangle + \rho(X)\langle Y, \alpha\rangle - \rho(Y)\langle X, \alpha\rangle - \alpha([X, Y]) = \rho(X)\langle Y, \alpha\rangle - \alpha([X, Y]).$$

■

The concept of Lie derivative applied to sections of $\Gamma\bigwedge^p A$ is a bit more involved. It requires the definition of the Schouten bracket as a extension of the Lie bracket $[\cdot, \cdot]_A$ to A–multivectors. The procedure is completely analogous to the usual case, though (see [10]).

2.2. Simple Dirac structures. The simplest example of Dirac structure is defined on vector spaces. Let V be a vector space and consider also its dual space V^* with respect to the dual inner product $\langle \cdot, \cdot\rangle$.

Consider a bilinear operation $(\cdot, \cdot)_+$ defined on $V \times V^*$ as:

$$((v_1, w_1), (v_2, w_2))_+ = \langle v_1, w_2\rangle + \langle v_2, w_1\rangle \quad \forall (v_1, w_1), (v_2, w_2) \in V \times V^*. \tag{5}$$

We can consider a subset D of the space $V \times V^*$ which is maximally isotropic with respect to $(\cdot,\cdot)_+$, i.e. such that $D^\perp = D$ where,

$$D^\perp = \{(w_1, w_2) \in V \times V^* | \langle v_1, w_2 \rangle + \langle w_1, v_2 \rangle = 0 \quad \forall (v_1, v_2) \in D \subset V \times V^*\}.$$

We will say that this subspace D is a **constant Dirac structure**. Any subspace $D \subset V \times V^*$ of dimension $n = \mathrm{dim} V$ such that:

$$\langle v, v^* \rangle = 0 \quad \forall (v, v^*) \in D \tag{6}$$

will define a constant Dirac structure.

The definition of a constant Dirac structure can also be generalized to the non constant case, when we consider that this vector space V is the tangent space to a manifold M in one point. A constant Dirac structure on the point $p \in M$ is defined on T_pM as a subspace $D \subset T_pM \times T_p^*M$ verifying the condition above ($D^\perp = D$). We define a global structure making this D depend on the point $p \in M$, where now the vectors become vector fields as well as the co-vectors become one forms. The inner product is the natural pairing $\langle \cdot,\cdot \rangle : \mathfrak{X}(M) \times \bigwedge^1(M) \to C^\infty(M)$ (we denote by $\mathfrak{X}(M)$ the set of vector fields on M and by $\bigwedge^1(M)$ the set of one forms) and the definition of $D^\perp$ becomes now:

$$\begin{aligned} D^\perp = \{(Y, \beta) \in TM \oplus T^*M | \langle X, \beta \rangle + \langle Y, \alpha \rangle = 0 \\ \forall (X, \alpha) \in D \subset TM \oplus T^*M\}. \end{aligned}$$

And the definition of $(\cdot,\cdot)_+$ is now:

$$((X_1, \alpha_1), (X_2, \alpha_2))_+ = i_{X_1}\alpha_2 + i_{X_2}\alpha_1. \tag{7}$$

The condition to be satisfied by the set D (actually it must be a subbundle of $TM \oplus T^*M$) is still the same:

Definition 2.2. *A subbundle $D \subset TM \oplus T^*M$ is said to be a* **generalized Dirac structure** *defined on a manifold M if and only if it is maximally isotropic with respect to (7), i.e.*

$$D^\perp = D.$$

Equivalently, D can be defined as the subbundle where the operation $(\cdot,\cdot)_+$ is identically zero.

To define a closed Dirac structure (or simply a Dirac structure) it is necessary to ensure that the skew-symmetric operation (11) is inner in the space of sections of the subbundle D, i.e. that given any two sections $\sigma_1, \sigma_2 \in \Gamma D$,

$$[\sigma_1, \sigma_2] \in \Gamma D \ .$$

This can be achieved by imposing two different conditions: the plain condition above or the following one (see [2, 16]):

Definition 2.3. *A generalized Dirac structure is said to be* **closed** *if for any three sections $\sigma_1 \equiv (X_1, \alpha_1), \sigma_2 \equiv (X_2, \alpha_2), \sigma_3 \equiv (X_3, \alpha_3)$ the following property holds:*

$$\langle X_1, \mathcal{L}_{X_2}\alpha_3\rangle + \langle X_3, \mathcal{L}_{X_1}\alpha_2\rangle + \langle X_2, \mathcal{L}_{X_3}\alpha_1\rangle = 0. \tag{8}$$

2.3. Dirac structures as Lie algebroids. Consider now again the manifold M and the Whitney sum $TM \oplus T^*M$. A closed Dirac structure defines an integrable subbundle on it:

$$D \subset TM \oplus T^*M, \tag{9}$$

which yields a Lie algebroid structure on it. The original definition is due to Courant in [2] and is as follows:

- The vector bundle is, of course, the bundle D and the base the manifold M.
- The anchor mapping is the natural projection:

$$\rho : D \to TM. \tag{10}$$

- The Lie algebra structure is defined as:

$$[(X_1, \alpha_1), (X_2, \alpha_2)] = \left([X_1, X_2], \mathcal{L}_{X_1}\alpha_2 - \mathcal{L}_{X_2}\alpha_1 - \frac{1}{2} d(i_{X_1}\alpha_2 - i_{X_2}\alpha_1) \right). \tag{11}$$

Being a Lie algebroid, the cohomology complex of D can be considered, and its coboundary operator d_D.

Equivalently, we can formulate the integrability condition in the following way:

Theorem 2.1 (Courant)**.** *A Dirac structure is closed if and only if it is a Lie algebroid, with the natural projection $\rho : D \to TM$ as anchor mapping and (11) as the Lie algebra structure on the space of sections.*

2.4. Examples.

Example 2.1. *Given a symplectic manifold (M, ω), the symplectic form defines a mapping:*

$$\begin{aligned} \hat{\omega} &: TM \to T^*M \\ \hat{\omega}(X)(Y) &= \omega(X, Y) \quad \forall Y \in \mathfrak{X}(M). \end{aligned} \tag{12}$$

It is straightforward to check that the subbundle defined in $TM \oplus T^*M$ by the graph of $\hat{\omega}$ defines a generalized Dirac structure:

$$D = \{(X, \hat{\omega}(X)) \, |X \in \mathfrak{X}(M)\}.$$

The proof is trivial and based in the antisymmetry of the symplectic form. It is immediate that this D has the right dimension (one half of the dimension of $TM \oplus T^*M$) and that it satisfies (6):

$$\langle X, \hat{\omega}(X) \rangle = \omega(X, X) = 0.$$

Example 2.2. *Another very interesting example is the case of a Poisson manifold (M, J) (we denote by J the Poisson tensor). In this case, it is well known that there is a mapping equivalent to (12), but in the other direction, and particularly useful when applied to exact one forms:*

$$\begin{aligned} \hat{J} &: T^*M \to TM \\ \hat{J}(df)(dg) &= J(df, dg) = \{f, g\}. \end{aligned}$$

It is again straightforward to check that the subbundle defined by the graph of the mapping $\hat{J}$ defines on M a generalized Dirac structure:

$$D = \{(\hat{J}(\alpha), \alpha) | \, \alpha \in T^*M\}. \tag{13}$$

The proof is again based in the antisymmetry of the tensor J, and it follows step by step the previous one.

3. More general definitions: Lie bialgebroids and Courant algebroids

The definitions above represent the simplest characterization of Dirac structures from a geometrical point of view. There exist generalizations of the concept of Dirac structures and generalized Dirac structures, which represent the usual framework for the research carried out in the differential geometric side of Dirac structures nowadays.

3.1. Lie bialgebroids. Let us consider first the integrable case. Let us assume that $(A, [\cdot,\cdot], \rho)$ is a Lie algebroid over a base manifold M. Consider also the corresponding dual bundle $A^* \to M$. Assume that it is also possible to define a Lie algebroid structure for this bundle, i.e. there exist a Lie algebra structure $[\cdot,\cdot]_*$ and an anchor mapping $\rho_* : A^* \to M$ which satisfies the conditions specified in Definition 2.1. In both cases, there exist cohomology operators, which we denote by d and d_*. As the bundles are dual, each operator acts on the set of sections of the other bundle. If these actions are derivations with respect to the Lie algebra structure of the sections, the pair (A, A^*) is called a Lie bialgebroid:

Definition 3.1. *A* **Lie bialgebroid** *is a pair of dual Lie algebroids (A, A^*) such that the differential d is a derivation of the Schouten-bracket of A^*, i.e.*

$$d[\alpha_1, \alpha_2] = [d\alpha_1, \alpha_2] + [\alpha_1, d\alpha_2] \quad \forall \alpha_1, \alpha_2 \in \Gamma A^*. \tag{14}$$

Analogously, d_* is also a derivation for the commutator of the sections of A:

$$d_*[X_1, X_2] = [d_* X_1, X_2] + [X_1, d_* X_2]. \tag{15}$$

There are trivial examples of Lie bialgebroids provided by Lie algebroids. Take a Lie algebroid A and consider its dual bundle A^*. We can endow A^* with a Lie algebroid structure by choosing an abelian algebra structure and a null anchor mapping (i.e. all the sections of $A^* \to M$ go to the zero section of TM). It is trivial to see that such a choice fulfills the conditions above and then (A, A^*) becomes a Lie bialgebroid. This is the case, for instance, of the trivial Lie algebroid structure of TM, which hence allows $TM \oplus T^*M$ to be considered as a Lie bialgebroid.

3.2. Dirac structures on Lie bialgebroids. Given a Lie bialgebroid (A, A^*), we can consider the Whitney sum $B \equiv A \oplus A^*$, the duality between the two bundles can be used to define two canonical forms, one symmetric $\langle\langle \cdot,\cdot \rangle\rangle_+ : B \times B \to B$ and one skew-symmetric $\langle\langle \cdot,\cdot \rangle\rangle_- : B \times B \to B$:

$$((X_1, \alpha_1), (X_2, \alpha_2))_\pm = \langle X_1, \alpha_2 \rangle \pm \langle X_2, \alpha_2 \rangle \quad \forall (X_1, \alpha_1), (X_2, \alpha_2) \in B. \tag{16}$$

The symmetric one is precisely the product (7) for the case of $TM \oplus T^*M$. Hence, it is trivial to see how the concept of Dirac structure is trivially extended to the case of more general Lie bialgebroids:

Definition 3.2. *Consider a Lie bialgebroid* (A, A^*). *A subbundle of* $D \subset A \oplus A^* \to M$ *is called a generalized Dirac structure on* M *if it is maximally isotropic with respect to the symmetric operation (16).*

For the definition of the Lie algebra structure on the space of sections of D, the structures of both Lie algebroids are used. The operation is defined on the sum $A \oplus A^*$, but defines a Lie algebra operation only on the sections of the subbundle D:

Definition 3.3. *Consider the Whitney sum bundle* $B = A \oplus A^*$. *We can endow the set of sections of* B *with a bilinear, skew-symmetric operation, in the form:*

$$[(X_1, \alpha_1), (X_2, \alpha_2)]_\diamond = ([X_1, X_2]_A + [X, Y]_{\mathcal{L}^{A^*}}, [\alpha, \beta]_{\mathcal{L}^A} + [\alpha_1, \alpha_2]_{A^*}), \quad (17)$$

with

$$[X, Y]_{\mathcal{L}^{A^*}} = \mathcal{L}^{A^*}_{\alpha_1} X_2 - \mathcal{L}^{A^*}_{\alpha_2} X_1 - \frac{1}{2} d_*(i_{X_1}\alpha_2 - i_{X_2}\alpha_1)$$

and

$$[\alpha, \beta]_{\mathcal{L}^A_N} = \mathcal{L}^A_{X_1}\alpha_2 - \mathcal{L}^A_{X_2}\alpha_1 + \frac{1}{2} d(i_{X_1}\alpha_2 - i_{X_2}\alpha_1),$$

where $\mathcal{L}^A_X, \mathcal{L}^{A^*}_\alpha$ *correspond to the Lie derivatives on* A *and* A^* *(acting on* A*-forms or* A^**-forms, i.e., sections of* A^* *or* A*), i.e.:*

$$\mathcal{L}^A_X = i_X \circ d + d \circ i_X \qquad \mathcal{L}^{A^*}_\alpha = i_\alpha \circ d_* + d_* \circ i_\alpha.$$

For this operation, bi-linearity and skew-symmetry are trivial to prove. On the other hand, Jacobi identity is not satisfied on the whole space of sections of B. It is possible, though, that the property holds for the space of sections of a generalized Dirac structure $D \subset B$.

Definition 3.4. *A generalized Dirac structure* $D \subset B$ *is a Dirac structure if the operation above defines a Lie algebra structure on the space of its sections.*

Analogously to the $TM \oplus T^*M$ case, Dirac structures and Lie algebroids are deeply related:

Proposition 3.1. *Consider the Whitney sum bundle* $B = A \oplus A^*$ *for the Lie bialgebroid* (A, A^*). *Consider also the operation* $[\cdot, \cdot]_\diamond$ *and the mapping* $\rho_B = \rho \oplus \rho_* : B \to TM$. *A subbundle* $D \subset B$ *is a Dirac structure if and only if* $(D, [\cdot, \cdot]_\diamond, \rho|_D)$ *is a Lie algebroid.*

Proof: The operation $[\cdot,\cdot]_\diamond$ is trivially skew-symmetric. Bilinearity follows from the bilinearity of the commutators $[\cdot,\cdot]_A$ and $[\cdot,\cdot]_{A^*}$ and the properties of the Lie derivative (as it happens in the $TM\oplus T^*M$ case). Jacobi identity is assumed by the statement for the sections of D, thus providing a Lie algebra structure for ΓD. The only points to be proved are:

- Homomorphism condition for $\rho|_D$. We must prove that:
$$\rho[(X,\alpha),(Y,\beta)]_\diamond = [\rho(X,\alpha),\rho(Y,\beta)] \qquad \forall (X,\alpha),(Y,\beta)\in\Gamma D.$$
As $[\cdot,\cdot]$ and $[\cdot,\cdot]_*$ trivially satisfy the condition, the only thing to prove is that:
$$\rho[X,Y]^{\mathcal{L}_{A^*}} + \rho_*[\alpha,\beta]^{\mathcal{L}^A} = [\rho X,\rho_*\beta] + [\rho Y,\rho_*\alpha].$$
The proof can be found in Proposition 4.2 of [9].
- Derivation property for the module:
$$[(X,\alpha), f(Y,\beta)]_\diamond = f[(X,\alpha),(Y,\beta)]_\diamond + (\rho(X,\alpha)f)(Y,\beta).$$
This point is simple, because $[\cdot,\cdot]$ and $[\cdot,\cdot]_*$ satisfy this property, and the brackets $[\cdot,\cdot]_{\mathcal{L}^{A^*}}$ and $[\cdot,\cdot]_{\mathcal{L}^A}$ also satisfy it because of the properties of the Lie derivatives (the proof is completely analogous to the proof of the cotangent bundle of a Poisson manifold carrying a canonical Lie algebroid structure).

■

3.3. Characteristic pairs. The characterization of Dirac structures can be done in terms of subbundles of A and suitable A-tensors. This generalizes the description proposed in [16] for the case of Dirac structures defined on $TM\oplus T^*M$. We will follow Liu's construction, described in [8].

Definition 3.5. *Consider a Lie bialgebroid (A,A^*) and a maximally isotropic subbundle of its Whitney sum $D\subset A\oplus A^*$. Any pair of a smooth subbundle $I\subset A$ and a bivector $\Omega\in\Gamma(\bigwedge^2 A)$ corresponds to a maximally isotropic subbundle of $A\oplus A^*$ with respect to the symmetric product in (16). The* **characteristic pair** *of the Dirac structure D is a pair (I,Ω) which corresponds to it. The subbundle $I\subset A$ is called the* **characteristic bundle***. The expression of the Dirac structure is as follows:*
$$D=\left\{(X+\Omega^\#\alpha,\alpha)\,|\,\forall X\in I,\,\forall\alpha\in I^\perp\right\} = I\oplus \mathrm{graph}\,(\Omega^\#|I^\perp), \tag{18}$$
where $I^\perp\subset A^*$ stands for the co-normal bundle of I.

For the sake of simplicity, we will assume hereafter that the intersection $D \cap A$ is of constant rank.

Lemma 3.1. *Given a Dirac structure $D \subset A \oplus A^*$ and a subbundle $I \subset A$ which belongs to D, the existence of the bundle map $\Omega^\#$ restricted to $I^\perp$ is equivalent to a bivector field on the quotient bundle A/I.*

Proof: Consider an A–tensor Ω on M defined by the Dirac structure, i.e.

$$(\Omega^\#(\alpha), \alpha) \in D \quad \alpha \in A^*.$$

Skewsymmetry of Ω follows from the definition of the Dirac structure. Obviously the mapping $\Omega^\#$ is well defined only on $I^\perp$, since the image of the null section of A^* is I. This leads to a natural restriction:

$$\Omega^\# : I^\perp \to A/I.$$

∎

Therefore, we can define an equivalence relation on the space of characteristic pairs, by claiming that two characteristic pairs $(I_1, \Omega_1), (I_2, \Omega_2)$ are equivalent if and only if :

$$\begin{cases} I_1 = I_2 \equiv I \\ \Omega_1^\#(\alpha) - \Omega_2^\#(\alpha) \in I \quad \forall \alpha \in I^\perp. \end{cases} \tag{19}$$

This definition leads then to the equivalence of the equivalence classes with the set of Dirac structures of a given Lie bialgebroid.

Note 3.1. *The equivalence class of characteristic pairs thus defines the concept of a generalized Dirac structure for the general Lie bialgebroid case. The main point here is that the concept associated to the characteristic pair is merely the existence of a maximally isotropic subbundle of the Lie bialgebroid with respect to the product (16). The closeness of the Lie algebra structure (17) restricted to the subbundle is not required yet.*

But these characteristic pairs are subject to some conditions in order to represent a Dirac structure. We saw above how, for the $TM \oplus T^*M$ case, closeness of the Dirac structure was equivalent to the Schouten bracket condition for the Poisson tensor (in Poisson manifolds) or to the closeness of the symplectic form (for symplectic ones). For the general case, a similar condition arises:

Theorem 3.1 (Liu). *Let (A, A^*) be a Lie bialgebroid and D a subbundle of $A \oplus A^*$ maximally isotropic with respect to the symmetric product (16), and corresponding to the characteristic pair class (I, Ω). Then, D is a Dirac structure if and only if:*

- *I is a Lie subalgebroid.*
- *Ω satisfies the Maurer-Cartan type of equation:*

$$d_*\Omega + \frac{1}{2}[\Omega, \Omega] = 0 \ mod\ I \ . \tag{20}$$

- *The following bracket is closed on $\Gamma I^\perp$:*

$$[\alpha, \beta] = [\alpha, \beta]_{A^*} + \mathcal{L}^A_{\Omega^\# \alpha}\beta - \mathcal{L}^A_{\Omega^\# \beta}\alpha - d(\Omega(\alpha, \beta)) \qquad \forall \alpha, \beta \in \Gamma I^\perp. \tag{21}$$

We also present some technical lemmas which will be useful in the next sections:

Lemma 3.2. *Let D be a generalized Dirac structure with characteristic pair (I, Ω). Then,*

$$\mathcal{L}_X\alpha \in I^\perp \qquad \forall X \in I \, \forall \alpha \in I^\perp.$$

Proof: We have to check that, for any $Y \in I$:

$$\langle Y, \mathcal{L}_X\alpha \rangle = 0.$$

By using Cartan identity $\mathcal{L}_X = i_x \circ d + d \circ i_X$ and the fact that $\langle X, \alpha \rangle = 0$ we obtain:

$$\langle Y, \mathcal{L}_X\alpha \rangle = \langle Y, i_X d\alpha \rangle = i_Y i_X d\alpha = d\alpha(X, Y) = \\ X(\alpha(Y)) - Y(\alpha(X)) - \alpha([X, Y]).$$

As $X, Y \in I$, the first two terms vanish, and hence:

$$\langle Y, \mathcal{L}_X\alpha \rangle = -\alpha([X, Y]).$$

Hence, if $(I, [\cdot, \cdot])$ is integrable, the conclusion follows. ∎

Corollary 3.1. *For the $TM \oplus T^*M$ case, the theorem above reads as follows: a generalized Dirac structure defined by the characteristic pair (I, Ω) is a Dirac structure iff I is a Lie subalgebroid and Ω defines a Poisson structure on the quotient space $\Omega^\#(\Gamma I^\perp)/I$.*

Lemma 3.3. *Consider a Lie bialgebroid (A, A^*) and a generalized Dirac structure described by the equivalence class of characteristic pairs $[(I, \Omega)]$. Then, if one representant satisfies the conditions of Theorem 3.1, so do all others.*

Proof: As we saw above, two representants (I_1, Ω_1), (I_2, Ω_2) of the same generalized Dirac structure are related as:

$$\begin{cases} I_1 = I_2 \equiv I \\ \Omega_1^{\#}(\alpha) - \Omega_2^{\#}(\alpha) \in I \quad \forall \alpha \in I^{\perp}. \end{cases}$$

This implies that given a one form $\alpha \in I^{\perp}$, we can write

$$\Omega_2^{\#}(\alpha) = \Omega_1^{\#}(\alpha) + X^{\alpha} \qquad X^{\alpha} \in I. \tag{22}$$

Consider now the three conditions of Theorem 3.1. Assume that the pair (I, Ω_1) does define a Dirac structure, i.e. it satisfies Maurer-Cartan-type equation, and defines a closed Lie algebra structure on $I^{\perp}$. Let us verify that the pair (I, Ω_2) also does.

- The subbundle I is also closed.
- For the second point, we follow Liu's proof in [8]. Given two elements $(\Omega_1^{\#}(\alpha), \alpha), (\Omega_1^{\#}(\beta), \beta) \in D \subset A \oplus A^*$ the commutator

$$\begin{aligned} [(\Omega_1^{\#}(\alpha), \alpha), (\Omega_1^{\#}(\beta), \beta)] = (d_*\Omega + \frac{1}{2}[\Omega, \Omega])(\alpha, \beta) + \\ \Omega_1^{\#}\left([\alpha, \beta]_* + \mathcal{L}^A_{\Omega_1^{\#}\alpha}\beta - \mathcal{L}^A_{\Omega_1^{\#}\beta}\alpha - d(\Omega_1(\alpha, \beta))\right) + \\ \left([\alpha, \beta]_* + \mathcal{L}^A_{\Omega_1^{\#}\alpha}\beta - \mathcal{L}^A_{\Omega_1^{\#}\beta}\alpha - d(\Omega_1(\alpha, \beta))\right) \end{aligned}$$

is supposed to belong to D. This implies Maurer-Cartan equation. Now, assuming that this relation holds, we have to verify that replacing Ω_1 by Ω_2 the relation is also satisfied.

If we use relation (22), we obtain:

$$\begin{aligned} [(\Omega_2^{\#}(\alpha), \alpha), (\Omega_2^{\#}(\beta), \beta)] = [(\Omega_1^{\#}(\alpha), \alpha), (\Omega_1^{\#}(\beta), \beta)] + [X^{\alpha}, (\Omega_1^{\#}(\beta), \beta)] + \\ [(\Omega_1^{\#}(\alpha), \alpha), X^{\beta}] + [(X^{\alpha}, 0), (X^{\beta}, 0)]. \end{aligned}$$

Hence, as Ω_1 is assumed to satisfy Maurer-Cartan equation, the first term does belong to D. As I is an integrable subbundle, the last term also belongs to D. And the two middle terms, correspond to a commutator of an element of I and an element of D defined by the graph of $\Omega_1^{\#}$, which belongs to D as it is proved in Theorem 3.1.

- For the last point, we must proof that the bracket (21) defined by Ω_2 is closed on $\Gamma I^\perp$ if the one defined by Ω_1 is. We can write then:

$$[\alpha,\beta]=[\alpha,\beta]_*+\mathcal{L}^A_{\Omega_2^\#\alpha}\beta-\mathcal{L}^A_{\Omega_2^\#\beta}\alpha-d(\Omega_2(\alpha,\beta))\qquad\forall\alpha,\beta\in\Gamma I^\perp.$$

  But rewriting $\Omega_2^\#$ as above we get:

$$[\alpha,\beta]=[\alpha,\beta]_*+\mathcal{L}^A_{\Omega_1^\#\alpha}\beta-\mathcal{L}^A_{\Omega_1^\#\beta}\alpha-d(\Omega_1(\alpha,\beta))+\mathcal{L}^A_{X^\alpha}\beta-\mathcal{L}^A_{X^\beta}\alpha,$$

  where we have used that $\Omega_2(\alpha,\beta)=\langle\Omega_2^\#(\alpha),\beta\rangle=\langle\Omega_1^\#(\alpha),\beta\rangle=\Omega_1(\alpha,\beta)$ because $\alpha,\beta\in\Gamma I^\perp$.

  As we are assuming that the bracket with Ω_1 is closed, we have to prove just that:

$$\mathcal{L}^A_{X^\alpha}\beta-\mathcal{L}^A_{X^\beta}\alpha\in\Gamma I^\perp.$$

  But this has been proved in Lemma 3.2. This completes the proof.

■

3.4. The dual Dirac bundle. Consider a Dirac structure $D\subset A\oplus A^*$ defined from a Lie bialgebroid (A,A^*). We are interested now in its dual bundle D^*.

It is simple to see that D^* is a subbundle of the Whitney sum $A^*\oplus A$. But we have the following result:

Theorem 3.2 (Kosmann-Schwarzbach)**.** *If (A,A^*) is a Lie bialgebroid, so is (A^*,A).*

Taking this into account, it will be helpful to define a bundle isomorphism between the two Whitney sums:

Definition 3.6. *Consider a Lie bialgebroid (A,A^*) and the corresponding Lie bialgebroid (A^*,A). Consider also the two Whitney sums*

$$B=A\oplus A^*\qquad B^*=A^*\oplus A.$$

Then, we can define a bundle isomorphism $\theta:B^*\to B$ as

$$\theta(\alpha,X)=(X,\alpha),\qquad\forall X\in A\,\forall\alpha\in A^*.\tag{23}$$

It is immediate, that the transformation is nilpotent, i.e.

$$\theta^2(X,\alpha)=(X,\alpha)\qquad\forall(X,\alpha)\in B^*.$$

With this result, we can try to define a Dirac structure to represent the bundle D^*. To do that, we need some previous results:

Lemma 3.4. *Consider a Dirac structure $D \subset A \oplus A^*$ as above. Then,*

$$A \oplus A^* = D \oplus \theta(D^*). \tag{24}$$

Proof: It is clear that both bundles D and $\theta(D^*)$ have the same dimension, equal to the dimension of the bundles A or A^*. The only thing to prove is that

$$D \cap \theta(D^*) = \emptyset.$$

Consider a point $(Y, \beta) \in D$. If $Y = 0$ or $\beta = 0$ the dual element in D^* has the other element in the pair null. If both elements in the pair are different from zero, there must be some element $(\alpha, X) \in D^*$, such that

$$\langle (Y, \beta), (\alpha, X) \rangle = \langle Y, \alpha \rangle \langle X, \beta \rangle = 1.$$

If we consider now $\theta(\alpha, X) \in D$, we must have:

$$((Y, \beta), (X, \alpha))_+ = \langle Y, \alpha \rangle + \langle X, \beta \rangle = 0.$$

But then,

$$\langle Y, \alpha \rangle^2 = \langle X, \beta \rangle^2 = -1$$

which is absurd. Hence, we conclude that $(X, \alpha) \notin D$. But obviously $(X, \alpha) \in \theta(D^*)$. Therefore,

$$D \cap \theta(D^*) = \emptyset.$$

■

Corollary 3.2. *Consider a Lie bialgebroid (A, A^*) and a generalized Dirac structure $D \subset A \oplus A^*$. Obviously, the bundle $\theta(D) \subset A^* \oplus A$ is a generalized Dirac structure. But from the lemma above we conclude also that the bundle D^* is a generalized Dirac structure too, and $\theta(D^*)$ as well.*

3.5. Courant algebroids.

Definition 3.7. *A* **Courant algebroid** *is a vector bundle $E \to M$, endowed with a non-degenerate symmetric bilinear form $(\cdot, \cdot)$, a skew-symmetric bracket $[\cdot, \cdot]$ and a mapping $\rho : E \to TM$ with the following properties:*

- *For any three sections $e_1, e_2, e_3 \in \Gamma E$, the obstruction to the Jacobi identity to hold can be computed as:*

$$[e_1, [e_2, e_3]] + [e_3, [e_1, e_2]] + [e_2, [e_3, e_1]] = \mathfrak{d} T(e_1, e_2, e_3), \tag{25}$$

where

$$T(e_1, e_2, e_3) = \frac{1}{3}\left((e_1, [e_2, e_3]) + (e_3, [e_1, e_2]) + (e_2, [e_3, e_1])\right) \tag{26}$$

and $\mathfrak{d} : C^\infty(M) \to \Gamma E$ *is the map defined by*

$$\mathfrak{d} = \frac{1}{2}\beta^{-1}\rho^* d \tag{27}$$

with d *the usual exterior differential of the base manifold* M *and* β *is the isomorphism* $\beta : E \to E^*$ *given by the bilinear form* $(\cdot, \cdot)$*.*

- *For any two sections* $e_1, e_2 \in \Gamma E$*,*

$$\rho([e_1, e_2]) = [\rho(e_1), \rho(e_2)], \tag{28}$$

where the last bracket is the commutator of vector fields.

- *For any to sections* $e_1, e_2 \in \Gamma E$ *and any function* $f \in C^\infty(M)$

$$[e_1, fe_2] = f[e_1, e_2] + (\rho(e_1)f)e_2 - (e_1, e_2)\mathfrak{d}f. \tag{29}$$

- *The operator* $\mathfrak{d}$ *is in the kernel of* ρ*, i.e.* $\rho \circ \mathfrak{d} = 0$*. This implies that, given any two functions* $f, g \in C^\infty(M)$*,* $(\mathfrak{d}f, \mathfrak{d}g) = 0$*.*
- *Given any three sections,* $e_, h_1, h_2 \in \Gamma E$*,*

$$\rho(e)(h_1, h_2) = ([e, h_1] + \mathfrak{d}(e, h_1), h_2) + (h_1, [e, h_2] + \mathfrak{d}(e, h_2)). \tag{30}$$

It is possible to see that any Lie bialgebroid (A, A^*) yields a Courant algebroid structure on the Whitney sum $A \oplus A^*$ with the choices $\rho = \rho + \rho_*$, the natural symmetric bilinear form and the bracket (17).

It is easy to identify in the definitions above many similarities with the definition of a Lie algebroid, and interesting relations with the definition of closed Dirac structures. Given then the canonical Lie bialgebroid (TM, T^*M), we endow the sum $TM \oplus T^*M$ with a Courant algebroid structure. We can consider a maximally isotropic subbundle of $TM \oplus T^*M$ with respect to the bilinear form $(\cdot, \cdot)$. Such a subbundle is a generalized Dirac structure, which, can be formulated in the more general context of Courant algebroids:

Definition 3.8. *Let* $(E, \rho, (\cdot, \cdot), [\cdot, \cdot])$ *be a Courant algebroid. A* **generalized Dirac structure** D *is a subbundle of* E*, which is maximally isotropic with respect to the bilinear form* $(\cdot, \cdot)$*.*

It is clear from the first axiom above, that on the module of sections of a Courant algebroid, the bracket $[\cdot, \cdot]$ does not define a Lie algebraic structure, since it does not satisfy Jacobi identity. The obstruction for this to happen is equal to the action of the differential $\mathfrak{d}$ on T. If we look more carefully to the definition of T, we observe that, for the case of the canonical structure on $TM \oplus T^*M$:

$$3T(e_1, e_2, e_3) = \langle X_1, \mathcal{L}_{X_2}\alpha_3\rangle + \langle X_3, \mathcal{L}_{X_1}\alpha_2\rangle + \langle X_2, \mathcal{L}_{X_3}\alpha_1\rangle$$
$$\forall e_1 \equiv (X_1, \alpha_1), e_2 \equiv (X_2, \alpha_2), e_3 \equiv (X_3, \alpha_3) \in \Gamma(TM \oplus T^*M). \quad (31)$$

Hence, the condition for the sections of the bundle D to become a Lie algebra is just

Definition 3.9. *Let the Courant algebroid E be defined as above. Let $D \subset E$ be a generalized Dirac structure in E. We say that D is a* **(closed) Dirac structure** *if it is closed for the product (17).*

Lemma 3.5. *Let the Courant algebroid E be defined as above. Let $D \subset E$ be a generalized Dirac structure in E. Then D is a closed Dirac structure if*

$$T(e_1, e_2, e_3) = 0 \quad \forall e_1, e_2, e_3 \in \Gamma D. \quad (32)$$

The subbundle whose sections satisfy the condition above can be endowed with a canonical Lie algebroid structure as $(D, \rho, [\cdot, \cdot])$. The condition holds trivially for the Whitney sum of any Lie bialgebroid.

4. Deformations of Lie algebroids and Lie bialgebroids.

4.1. Nijenhuis operators on Lie algebroids.

4.1.1. *Definition.* The first concept to consider is a Nijenhuis transformation on a Lie algebroid. The concept is well known ([14, 12, 7]) as a simple generalization of the usual concept for vector fields.

Definition 4.1. *Let $(A, [\cdot, \cdot], \rho)$ be a Lie algebroid. A linear transformation*

$$N : A \to A$$

is said to be a Nijenhuis transformation of A if and only if the torsion tensor $\mathcal{T}_N$ defined as:

$$\mathcal{T}_N(X, Y) = [N(X), N(Y)] - N([X, N(Y)]) - N([N(X), Y]) + N^2([X, Y])$$
$$\forall X, Y \in \Gamma A \quad (33)$$

vanishes.

A Nijenhuis transformation yields a deformation of the Lie algebra structure at the module of sections of A:

Lemma 4.1. *Consider the vector bundle A and a Nijenhuis operator $N : A \to A$. Define the following bracket on the sections of A:*

$$[X,Y]_N = -N([X,Y]) + [X,N(Y)] + [N(X),Y]. \tag{34}$$

Consider the mapping:

$$\hat{N} = \rho \circ N : A \to TM. \tag{35}$$

Then, $(A, [\cdot,\cdot], \hat{N})$ is a Lie algebroid.

Proof: The bracket (34) defines a Lie structure on ΓA which is a cohomologically trivial deformation of the original one (see [5], Theorem 3.1). The mapping $\hat{N}$ defines a homomorphism of Lie algebras, since:

$$\hat{N}[X,Y]_N = \rho(N[X,Y]_N) = \rho([NX,NY]) = [\rho(N(X)),\rho(N(Y))] = \\ [\hat{N}(X),\hat{N}(Y)] \qquad X,Y \in \Gamma A,$$

where we use the fact that ρ is a homomorphism for the original Lie algebra structure $[\cdot,\cdot]$ and the fact that $\mathcal{T}_N(X,Y)$ vanishes.

Finally, the definition of a derivation on the module of sections of A follows also from the properties of the original algebroid structure:

$$\begin{aligned}[X,fY]_N = -N([X,fY]) + [N(X),fY] + [X,N(fY)] = \\ f(-N([X,Y]) + [N(X),Y] + [X,N(Y)]) \\ - N(\rho(X)fY) + \rho(NX)fY + \rho(X)fN(Y) = \\ f[X,Y]_N + (\hat{N}(X)f)Y,\end{aligned}$$

where we used the linearity of all the mappings and the fact that the original structure is a Lie algebroid. ■

4.1.2. *Deformation of the exterior derivative and the Schouten bracket.* The deformation of the Lie structure of the algebroid yields a transformation for both the exterior derivative and the Schouten bracket of A-multivectors. From relation 2 it is clear that a change in the Lie structure must affect the exterior derivative. The effect can be summarized in an operator where i_N is the superderivation of degree zero on the forms $\Gamma\bigwedge^\bullet A^*$ defined as:

$$i_N\theta(X_1,\cdots,X_p) = \sum_{i=1}^{p} \theta(X_1,\cdots,NX_i,X_{i+1},\cdots,X_p). \tag{36}$$

This operator allows us to summarize the effect of the deformation on the exterior derivative in the following way:

Lemma 4.2. *The exterior differentials of the algebroids* $(A, [\cdot,\cdot], \rho)$ *and* $(A, [\cdot,\cdot]_N, \hat{N})$ *(and the same for the dual) are related by:*

$$d^N = [i_N, d] = i_N \circ d - d \circ i_N. \tag{37}$$

Proof: The exterior differential of a Lie algebroid is defined by its action on the ring of functions of the base manifold, and the usual extension to higher order forms. The action of the exterior differential on functions for a Lie algebroid $(A, [\cdot,\cdot], \rho)$ is defined as:

$$i_X df = \rho(X).f \qquad \forall X \in \Gamma A, \forall f \in C^\infty(M). \tag{38}$$

It is simple to see thus that for the Lie algebroid structure $(A, [\cdot,\cdot]_N, \hat{N})$ we have:

$$i_X d^N f = \hat{N}(X)(f) = \langle df, NX\rangle = \langle N^* df, X\rangle = i_X(N^* df) = \\ i_X i_N df \quad \forall f \in C^\infty(M), \forall X \in \Gamma A.$$

The extension to higher order cases is immediate by taking into account the relation between $\hat{N}$ and ρ and the definition of d^N. ■

It is worth noting that this definition does not hold for a deformation $N = \mathrm{Id}$, for which it would give a vanishing exterior derivative (if $N = \mathrm{Id}$, $i_N = \mathrm{Id}$ and $d^N = 0$) instead of the (correct) undeformed one d.

Being defined as an extension of the Lie algebra structure of the sections of the Lie algebroid, the Schouten bracket of A-multivectors heritages the deformation defined by the Nijenhuis operator. In the following, we need the explicit form of this deformation for the computations, hence we present it here. The result can be found in Section 6.4 of [7], where it is presented in the context of differential Lie algebras (the structure defined on the set of sections of the Lie algebroid by the algebroid Lie bracket):

Proposition 4.1. *Let* N *be a Nijenhuis operator on a Lie algebroid* A*. The Schouten bracket defined on* $\Gamma\bigwedge^\bullet A$ *by extension of the deformed bracket* $[\cdot,\cdot]_N$ *satisfies:*

$$[Q, Q']_N = [i_{N^*}Q, Q'] + [Q, i_{N^*}Q'] - i_{N^*}[Q, Q'], \tag{39}$$

where i_{N^*} is the superderivation of degree zero on the sections of $\Gamma \bigwedge^\bullet A$ defined as:

$$i_{N^*}\Psi(\theta_1, \cdots, \theta_p) = \sum_{i=1}^{p} \Psi(\theta_1, \cdots, N^*\theta_i, \theta_{i+1}, \cdots, \theta_p). \tag{40}$$

Proof: If we consider the biduality property $(N^*)^* = N$, it is clear that the expression above is valid for 0-vectors (functions on the base of the algebroid) and sections of A. As the Schouten bracket is defined as a graded derivation of multisections and it is bilinear, the relation is extended to the general case.

■

4.2. Nijenhuis operators on Lie bialgebroids. The next step is to consider the action of a Nijenhuis transformation on a Lie bialgebroid. In principle, we can consider two different transformations applied on each factor, i.e.:

$$\begin{cases} N : A \to A \\ N_* : A^* \to A^*, \end{cases} \tag{41}$$

such that they define two new Lie algebroid structures on the factors, i.e. $(A, [\cdot, \cdot]_N, \hat{N})$ and $(A^*, [\cdot, \cdot]_{N_*}, \hat{N}_*)$ are Lie algebroids. We must verify now whether they define a new Lie bialgebroid, i.e. if the conditions (15) hold. First of all, we must realize that the transformation on the Lie algebroid structures implies a transformation of the Lie algebroid cohomologies, i.e. there are two new exterior differentials d^N and d_*^N which are the ones to consider.

It is important to distinguish two different operators acting on each bundle: the Nijenhuis-like operators N, N_*; and their duals:

$$(N)^* : A^* \to A^* \qquad (N_*)^* : A \to A,$$

which are defined as:

$$\langle X, (N)^*\alpha \rangle = \langle N(X), \alpha \rangle \quad \forall X \in \Gamma A, \ \forall \alpha \in \Gamma A^* \tag{42}$$

and

$$\langle (N_*)^* X, \alpha \rangle = \langle X, N_*\alpha \rangle \quad \forall X \in \Gamma A, \ \forall \alpha \in \Gamma A^*. \tag{43}$$

Definition 4.2. *Consider a Lie bialgebroid (A, A^*) and a transformation of the type (41). We say that the pair (N, N_*) defines deformation of the Lie* *bialgebroid structure if and only if the pair* $\Big((A,[\cdot,\cdot]_N,\hat{N}),(A^*,[\cdot,\cdot]_{N_*},\hat{N}_*)\Big)$ *defines a new Lie bialgebroid structure. The equations to fulfill are:*

$$\begin{cases} d_*^N[X,Y]_N = [d_*^N X,Y]_N + [X,d_*^N Y]_N \\ d^N[\alpha,\beta]_{N_*} = [d^N\alpha,\beta]_{N_*} + [\alpha,d^N\beta]_{N_*}. \end{cases} \tag{44}$$

It is important to realize that in the right hand side of (44), the bracket involved is the deformed Schouten bracket of multisections of A, as defined in (39).

Lemma 4.3. *Given a Lie bialgebroid and a deformation as above:*

$$i_{N_*}\circ NX = N\circ i_{N_*}X \quad \forall X\in\Gamma A \Leftrightarrow (N_*)^*NX = N(N_*)^* \quad \forall X\in\Gamma A.$$

Proof: The definition of $i_{N_*}X$ can be written as:

$$\langle i_{N_*}X,\alpha\rangle = \langle X,N_*\alpha\rangle = \langle (N_*)^*X,\alpha\rangle \quad \forall\alpha\in\Gamma A^*.$$

Then, the expression above turns out to be a section of A whose action on the dual reads:

$$\langle i_{N_*}\circ NX,\alpha\rangle = \langle NX,N_*\alpha\rangle = \langle (N_*)^*NX,\alpha\rangle \quad \forall\alpha\in\Gamma A^*.$$

In the same way:

$$\langle N\circ i_{N_*}X,\alpha\rangle = \langle N(N_*)^*X,\alpha\rangle \quad \forall\alpha\in\Gamma A^*.$$

■

Consider the simplest example of deformation of a Lie bialgebroid: a deformation of the algebroid A which does not involve a deformation of the dual A^*. In such a case, the transformation is as follows:

$$\begin{cases} N\equiv N : A\to A \\ N_*\equiv \mathrm{Id} : A^*\to A^*. \end{cases} \tag{45}$$

In these circumstances, the two new Lie structures are as follows:

$$[X,Y]_N = -N[X,Y]+[NX,Y]+[X,NY] \qquad \forall X,Y\in\Gamma A$$

$$[\alpha,\beta]_{N_*} = [\alpha,\beta] \qquad \forall\alpha,\beta\in\Gamma A^*.$$

And the corresponding exterior derivatives:

$$\begin{cases} d^N = i_N\circ d - d\circ i_N \\ d_*^N = d_*. \end{cases} \tag{46}$$

The condition for (45) to define a deformation of the Lie bialgebroid (A, A^*) is then written as:

$$\begin{cases} d_*[X,Y]_N = [d_*X,Y]_N + [X,d_*Y]_N & \forall X,Y \in \Gamma A \\ d^N[\alpha,\beta] = [d^N\alpha,\beta] + [\alpha, d^N\beta] & \forall \alpha,\beta \in \Gamma A^*. \end{cases} \tag{47}$$

- By using the expression for the deformed bracket we can write:

$$d_*[X,Y]_N = -d_*N[X,Y] + d_*[NX,Y] + d_*[X,NY].$$

For the last two terms, we can use the fact that d_* is a derivation for the original Lie bracket and write:

$$d_*[NX,Y] = [d_*NX,Y] + [NX,d_*Y]$$
$$d_*[X,NY] = [d_*X,NY] + [X,d_*NY].$$

We can write the action of N on the sections of A as:

$$N(X) = i_{N^*}X \qquad \forall X \in \Gamma A.$$

In that case we obtain:

$$d_*[X,Y]_N = -d_*i_{N^*}[X,Y] + [d_*i_{N^*}X,Y] + [i_{N^*}X,d_*Y] + [d_*X,i_{N^*}Y] + [X,d_*i_{N^*}Y].$$

The development of the right hand side of Condition (47) by using the expression of the deformed Schouten bracket leads to:

$$[d_*X,Y]_N + [X,d_*Y]_N = -i_{N^*}[d_*X,Y] - i_{N^*}[X,d_*Y] + [i_{N^*}d_*X,Y] + [d_*X,i_{N^*}Y] + [i_{N^*}X,d_*Y] + [X,i_{N^*}d_*Y].$$

Comparing both sides, many terms cancel and we obtain:

$$-d_*i_{N^*}[X,Y] + [X,d_*i_{N^*}Y] + [d_*i_{N^*}X,Y] = -i_{N^*}d_*[X,Y] + [X,i_{N^*}d_*Y] + [i_{N^*}d_*X,Y].$$

As the Schouten bracket is linear, we conclude that the first relation in (47) holds if and only if $[i_{N^*}, d_*]$ is a derivation for the original Lie bracket of A.

- For the second relation in (47), we have:

$$d_N[\alpha,\beta] = i_N \circ d[\alpha,\beta] - d \circ i_N[\alpha,\beta] = i_N[d\alpha,\beta] + i_N[\alpha,d\beta] - di_N[\alpha,\beta].$$

This relation must hold for any $\alpha, \beta \in \Gamma A^*$. Assume now that we consider closed sections of A^*, i.e. $d\alpha = 0 = d\beta$. Then, the expression above reduces to:

$$d_N[\alpha, \beta] = -di_N[\alpha, \beta].$$

On the other hand, the corresponding right hand side of (47) reads:

$$[d_N\alpha, \beta] + [\alpha, d_N\beta] = [i_N d\alpha, \beta] - [di_N\alpha, \beta] + [\alpha, i_N d\beta] - [\alpha, di_N\beta].$$

Adding and subtracting $[i_N\alpha, d\beta] + [d\alpha, i_N\beta]$, we obtain:

$$[d_N\alpha, \beta] + [\alpha, d_N\beta] =$$
$$[i_N d\alpha, \beta] + [d\alpha, i_N\beta] + [i_N\alpha, d\beta] + [\alpha, i_N d\beta] - d[i_N\alpha, \beta] - d[\alpha, i_N\beta].$$

For closed sections, we obtain:

$$[d_N\alpha, \beta] + [\alpha, d_N\beta] = -d([i_N\alpha, \beta] + [\alpha, i_N\beta]).$$

Hence, we obtain that the second condition in (47) holds if and only if

$$i_N[\alpha, \beta] = [i_N\alpha, \beta] + [\alpha, i_N\beta] + \theta(\alpha, \beta) \qquad \theta(\alpha, \beta) \in Z^1(A).$$

If we assume now the case $d\alpha = 0$ we obtain:

$$i_N[\alpha, d\beta] = [i_N\alpha, d\beta] + [\alpha, i_N d\beta].$$

Hence, we conclude that the relation above must hold for also for the Schouten bracket, and that, therefore, $\theta(\alpha, \beta) = 0$.

By comparing both expressions, and remembering that they must hold for any two sections of A, we conclude:

Theorem 4.1. *An operator* (N, Id) *defines a deformation of the Lie bialgebroid* (A, A^*) *if and only if*

$$\delta_{NA} = i_{N^*} \circ d_* - d_* \circ i_{N^*} \tag{48}$$

is a derivation of the original Schouten bracket on A *and* i_N *is a derivation of the* A^**-Schouten bracket. As* d *is assumed to be also a derivation for it, we conclude that the operator:*

$$\delta_{NA^*} = i_N \circ d - d \circ i_N \tag{49}$$

is also a derivation of the A^**-Schouten bracket.*

The next step is to consider a deformation of the Lie bialgebroid where both factors change. Consider then a deformation of the algebroid A defined by a Nijenhuis operator

$$N : A \to A$$

and the corresponding transformation defined on the dual bundle via the dual mapping, i.e.:

$$N_* = N^* : A^* \to A^*.$$

Proposition 4.2. *Let (A, A^*) be a Lie bialgebroid. Consider the deformation*

$$\begin{cases} N : A \to A \\ N^* : A^* \to A^*. \end{cases} \tag{50}$$

Then, assume that:

- $d^N = i_N \circ d - d \circ i_N$ *is a derivation of the Lie bracket* $[\cdot,\cdot]_*$,
- $d_*^N = i_{N^*} \circ d_* - d_* \circ i_{N^*}$ *is a derivation of* $[\cdot,\cdot]$,
- $[i_{N^*}, d_*^N] = 0 = [i_N, d^N]$.

Then, the deformed structure $((A, N \circ \rho, [\cdot,\cdot]_N), (A^*, N^* \circ \rho_*, [\cdot,\cdot]_{N^*})$ *is a Lie bialgebroid.*

Proof: The conditions to fulfill are:

$$\begin{cases} d_*^N[X,Y]_N = [d_*^N X, Y]_N + [X, d_*^N Y]_N & \forall X, Y \in \Gamma A \\ d^N[\alpha,\beta]_{N^*} = [d^N\alpha, \beta]_{N^*} + [\alpha, d^N\beta]_{N^*} & \forall \alpha, \beta \in \Gamma A^*. \end{cases}$$

Rewriting the expressions above in terms of the undeformed objects and using the conditions above, the result follows. ■

Finally, let us consider again a general deformation of type (41). From what we learned above, we can claim:

Proposition 4.3. *A sufficient condition for (N, N_*) to be a deformation of the Lie bialgebroid (A, A^*) is that i_{N_*} is a derivation for $[,]$ and that $[d_*^N, N] = 0$. Equivalently for the dual objects.*

Proof: Consider the definition of the differential $d_*^N = [i_{N_*}, d_*]$ and the Lie bracket $[\cdot,\cdot]_N$. Writing everything in terms of the undeformed objects, we obtain:

$$d_*^N[X,Y]_N = [i_{N_*}, d_*]\left(-N([X,Y]) + [N(X), Y] + [X, N(Y)]\right).$$

For each factor, we have:

$$\begin{aligned}
&[i_{N_*}, d_*]([N(X), Y]) = \\
&\quad [i_{N_*} d_* N(X), Y] + [d_* N(X), i_{N_*} Y] + [i_{N_*} N(X), d_* Y] + [N(X), i_{N_*} d_* Y] \\
&- [d_* i_{N_*} N(X), Y] - [i_{N_*} N(X), d_* Y] - [d_* N(X), i_{N_*} Y] - [N(X), d_* i_{N_*} Y] = \\
&\qquad\qquad [[i_{N_*}, d_*] N(X), Y] + [N(X), [i_{N_*}, d_*] Y].
\end{aligned}$$

The expression for $[X, N(Y)]$ is analogous. The conditions chosen imply that in the first term, with a similar computation we obtain:

$$[i_{N_*}, d_*] N[X, Y] = N[[i_{N_*}, d_*] X, Y] + N[X, [i_{N_*}, d_*] Y].$$

This concludes the proof. ■

4.3. Application: Poisson-Nijenhuis structures. As an example of deformation of a Lie bialgebroid, let us review Kosmann-Schwartzbach's construction in [6]. The goal is to test our results on a well known situation.

Consider a Poisson manifold (M, Λ). It is well known that the cotangent bundle T^*M can be endowed with a Lie algebroid structure, by using $\Lambda^\# : T^*M \to TM$ as anchor mapping and the Lie product:

$$[\alpha, \beta]_\Lambda = L_{\Lambda^\#(\alpha)} \beta - L_{\Lambda^\#(\beta)} \alpha - d\Lambda(\alpha, \beta) \qquad \forall \alpha, \beta \in \Gamma T^*M. \tag{51}$$

The tangent bundle TM can be endowed trivially with a structure of Lie algebroid, and hence the pair (TM, T^*M) is a pair of dual Lie algebroids. It is also possible to see that the corresponding exterior derivatives are compatible, in such a way that the pair (TM, T^*M) endowed with the two structures above, is a Lie bialgebroid.

We can now consider a deformation of the canonical structure of Lie algebroid on TM by means of a Nijenhuis operator. The effect on the Poisson bivector Λ suggests the introduction of a special type of deformations, the so-called compatible deformations, which yield the concept of Poisson-Nijenhuis manifolds (see [7, 11]):

Definition 4.3. *A* **Poisson-Nijenhuis** *manifold (or PN manifold) is a Poisson manifold (M, Λ) and a Nijenhuis operator $N : TM \to TM$ which is* **compatible** *with the Poisson structure, compatible meaning:*

- $N\Lambda^\# = \Lambda^\# \circ N^*$*, what implies that the tensor $N\Lambda$ is skew-symmetric.*
- *Magri's concomitant (see* [7, 11]*) vanishes, i.e.* $C(\Lambda, N)(\alpha, \beta) \equiv [\alpha, \beta]_{N\Lambda} - ([N^*\alpha, \beta]_\Lambda + [\alpha, N^*\beta]_\Lambda - N^*[\alpha, \beta]_\Lambda) = 0.$

The advantage of PN manifolds is the behavior of the Lie bialgebroid structures on them:

Theorem 4.2 (Kosmann-Schwarzbach). *Consider a Poisson manifold (M, Λ) and a Nijenhuis operator N. The pair (TM, T^*M) endowed with the structures:*

$$TM, \rho = N, [X,Y]_N = [NX, Y] + [X, NY] - N[X,Y] \qquad \forall X, Y \in \Gamma TM$$

$$T^*M, \rho = \Lambda^\#, [\alpha, \beta]_\Lambda = L_{\Lambda^\#(\alpha)}\beta - L_{\Lambda^\#(\beta)}\alpha - d\Lambda(\alpha, \beta) \qquad \forall \alpha, \beta \in \Gamma T^*M,$$

is a Lie bialgebroid if and only if M is a PN manifold.

It is simple to verify that the conditions described above are satisfied in this deformation of Lie bialgebroids. The conditions are the same contained in Proposition 3.2 of [6], but presented in a slightly different way. Roughly speaking, the conditions presented there correspond to the definition of a deformation of a Lie bialgebroid (i.e. the transformed exterior differentials are derivations of the transformed Schouten brackets). We can always test the first condition: is the operator $\delta_\Lambda = i_{N^*} \circ d_\Lambda - d_\Lambda \circ i_{N^*}$ a derivation of the original commutator of vector fields for M?

This condition implies that for any two vector fields $X, Y \in \Gamma TM$,

$$\delta_\Lambda[X,Y] = [\delta_\Lambda X, Y] + [X, \delta_\Lambda Y].$$

But this means:

$$i_{N^*} \circ d_\Lambda[X,Y] - d_\Lambda \circ i_{N^*}[X,Y] = i_{N^*}[\Lambda, [X,Y]] - [\Lambda, N[X,Y]],$$

$$[\delta_\Lambda X, Y] = [(i_{N^*} \circ d_\Lambda - d_\Lambda \circ i_{N^*})X, Y] = [i_{N^*}[\Lambda, X], Y] - [[\Lambda, NX], Y],$$

$$[X, \delta_\Lambda Y] = [X, (i_{N^*} \circ d_\Lambda - d_\Lambda \circ i_{N^*})Y] = [X, i_{N^*}[\Lambda, Y]] - [X, [\Lambda, NY]].$$

But the operator d_Λ is a derivation of the original Schouten bracket. Hence, adapting the proof of Theorem 4.1, we obtain the desired result.

5. Deformation of Dirac structures I: the isolated case.

Let us recall that the characterization of Dirac structures on Lie bialgebroids was done in terms of the characteristic pairs (I, Ω). Our intention now is to establish the conditions on Nijenhuis operators of Dirac structures in terms of these elements. This process directly generalizes the well known constructions for symplectic and Poisson manifolds.

5.1. The transformation. We can consider two different types of transformation yielding a deformation of the Lie structure of the Dirac structure:

- A deformation of the Lie bialgebroid (A, A^*) where the Dirac structure D is defined, i.e.:

$$\mathcal{N} : A \oplus A^* \to A \oplus A^*$$
$$\mathcal{N}(X, \alpha) = (N(X, \alpha), N_*(X, \alpha)),$$

  where the components N and N_* must be bilinear and must define new Lie structures on each sub-factor, i.e. they must be Nijenhuis operators for A and A^* respectively. The Dirac structure heritages that deformation and produce a new Lie structure on the space of its sections. We analyze this case in the next section.
- Directly at the level of the Dirac structure keeping the same structure on the Lie bialgebroid where D is defined. We are going to see how in this case, it is very useful the description of Dirac structures in terms of the characteristic pairs, since it allows us to write the conditions on the transformation without writing its explicit form. In this context we are going to find a situation which has no counterpart in the Poisson case: for Poisson-Nijenhuis manifolds, there is a transformation of the bundle $D \subset A \oplus A^*$ into a different vector bundle, since it is defined through the Poisson tensor, and this is changed in the transformation. In the general Dirac case, though, it makes sense to consider a transformation of the Lie structure which defines the Lie algebroid structure of D, without changing the bundle itself. This case will be interesting in applications, since the bundle structure will be defined by the physical structure of the problem, but there is freedom in the choice of the Lie structure.

For both situations above, we will define the concept of Dirac-Nijenhuis structures:

Definition 5.1. *Let (A, A^*) be a Lie bialgebroid defined on a differentiable manifold M. Consider a Dirac structure D represented by a characteristic pair (I, Ω). Then, M shall be said to be endowed with a* **Dirac-Nijenhuis structure** *with respect to the Lie bialgebroid (A, A^*) if there exists a Nijenhuis operator $\mathcal{N} : A \oplus A^* \to A \oplus A^*$ which is compatible with D in the sense that the transformed characteristic pair (I^N, Ω^N) represents a new Dirac structure on M for the Lie bialgebroid.*

With respect to this definition, the two types of deformations discussed above correspond to situations where the Lie bialgebroid structure is preserved or is transformed by $\mathcal{N}$.

5.2. Transformation at the level of the Dirac structure. Let us consider now a Lie bialgebroid (A, A^*) and a Dirac structure defined on it $D \subset A \oplus A^*$. Consider an operator $\mathcal{N} : D \to D$ which transforms the Lie algebraic structure on the module of sections of D, in such a way that the corresponding Nijenhuis torsion vanishes, i.e.:

$$\mathcal{N}^2[e_1, e_2] - \mathcal{N}[\mathcal{N}e_1, e_2] - \mathcal{N}[e_1, \mathcal{N}e_2] + [\mathcal{N}e_1, \mathcal{N}e_2] = 0 \qquad \forall e_1, e_2 \in \Gamma D. \tag{52}$$

Our purpose now is to obtain the conditions to be satisfied by $\mathcal{N}$, in terms of the characteristic pair which defines the Dirac structure.

In such a framework, the expression of a Nijenhuis operator reads:

$$\mathcal{N}([(I, \Omega)]) = [(I', \Omega')]. \tag{53}$$

Hence, a linear transformation, in general, transforms a characteristic pair in another characteristic pair (since they are geometrical objects). Now, we must impose conditions for it to be a Nijenhuis transformation:

5.2.1. *The transformation of the bundle.* In a Nijenhuis transformation, the bundle involved is preserved, only the Lie algebraic structure is changed. From this point of view, the generalized Dirac structure underlying any Dirac structure must be preserved by the Nijenhuis transformation. This implies that the Nijenhuis transformation maps the original class of characteristic pairs on itself, what implies:

- The subbundle I is preserved, i.e.

$$N(I) = I, \tag{54}$$

  where $N = \pi_1 \circ \mathcal{N}$ and $\pi_1 : D \to A$ is the natural projection.
- The 2-section Ω transforms in a new Ω^N. The new object can be written in terms of the old 2-section by asking (see [7]):

$$(\Omega^N)^\# = N\Omega^\#. \tag{55}$$

Lemma 5.1. *The tensor $(\Omega^N)^\#$ is skew-symmetric if and only if*

$$N\Omega^\# = \Omega^\# N^*. \tag{56}$$

Proof: The skew symmetry condition is written as follows:

$$\Omega^N(\alpha,\beta) = -\Omega^N(\beta,\alpha) \qquad \forall \alpha,\beta \in \Gamma A^*.$$

If we write it with the bundle isomorphism $\Omega^\#$:

$$\Omega^N(\alpha,\beta) = \langle(\Omega^N)^\#(\alpha),\beta\rangle = \langle N\Omega^\#(\alpha),\beta\rangle = \langle \Omega^\#(\alpha), N^*\beta\rangle = \Omega(\alpha, N^*\beta)\,,$$

while

$$\Omega^N(\beta,\alpha) = \langle(\Omega^N)^\#(\beta),\alpha\rangle = \langle N\Omega^\#(\beta),\alpha\rangle = \langle \Omega^\#(\beta), N^*\alpha\rangle = \Omega(\beta, N^*\alpha)\,,$$

As Ω is skew-symmetric we obtain that the skew symmetry of Ω^N is equivalent to:

$$\Omega(N^*\alpha,\beta) = \Omega(\alpha, N^*\beta) \qquad \forall \alpha,\beta \in \Gamma A^*.$$

The condition above can be rewritten as:

$$\langle \Omega^\# N^*(\alpha),\beta\rangle = \langle N\Omega^\#(\alpha),\beta\rangle \qquad \forall \alpha,\beta \in \Gamma A^*.$$

This concludes the proof. ∎

Hence, the transformation $\mathcal{N} : D \to D$ can be rewritten in terms of the characteristic pairs as:

$$\mathcal{N}(I,\Omega) = (N(I),\Omega^N) = (I,\Omega^N), \tag{57}$$

where $(\Omega^N)^\# = N\Omega^\# = \Omega^\# N^*$.

As we want the transformed Dirac bundle to be the same as the original one, we must impose that the equivalence class is preserved, i.e.

$$[(I,\Omega^N)] = [(I,\Omega)]. \tag{58}$$

This implies that:

$$(\Omega^N)^\#(\alpha) - \Omega^\#(\alpha) = N\Omega^\#(\alpha) - \Omega^\#(\alpha) = \Omega^\#(N^*(\alpha)) - \Omega^\#(\alpha) \in I \qquad \forall \alpha \in \Gamma A^*.$$

These condition can be rewritten on the bundle A/I asking the tensor Ω^N to coincide with Ω.

5.2.2. *The transformation of the Lie structure.* ¿From the result above, we conclude that a Nijenhuis transformation maps the subbundle I on itself, and the transforms the tensor Ω in new tensor which coincides with the original one when projected on the quotient bundle A/I.

By using now Liu's theorem, we obtain some extra properties of the transformation $\mathcal{N}$:

- As the transformed subbundle $\mathcal{N}(I) = I$ must be a Lie subalgebroid of A (whose algebroid structure is assumed to not be modified by the transformation), we conclude that the restriction of $\mathcal{N}$ to the subbundle I must be a (normal) Nijenhuis operator, with vanishing Nijenhuis torsion:

$$[N(X), N(Y)] - N[X, N(Y)] - N[N(X), Y] + N^2([X, Y]) = 0 \qquad \forall X, Y \in \Gamma I, \tag{59}$$

where $N = \pi_1 \circ \mathcal{N}$. Then, a new bracket can be defined on I:

$$[X, Y]_N = -N[X, Y] + [X, N(Y)] + [N(X), Y] \qquad \forall X, Y \in \Gamma I. \tag{60}$$

In these circumstances, $(I, [\cdot, \cdot]_N, N \circ \rho)$ defines a Lie subalgebroid of $(A, [\cdot, \cdot], \rho)$.

As a corollary, we also obtain a condition on the action of N^* on the dual bundle:

Lemma 5.2. *$I^\perp$ is stable under the action of N^*, i.e.*

$$N^*(I^\perp) \subset I^\perp. \tag{61}$$

Proof: As I is stable, we have:

$$\langle N(X), \alpha \rangle = 0 \qquad \forall X \in I, \alpha \in I^\perp.$$

But then, by dualizing the action, we obtain:

$$\langle N(X), \alpha \rangle = 0 \Leftrightarrow \langle X, N^*(\alpha) \rangle = 0 \qquad \forall X \in I, \alpha \in I^\perp.$$

Hence, if $\alpha \in I^\perp$, $N^*(\alpha) \in I^\perp$. ■

- In what regards the tensor Ω, we can use the previous results to study the suitable transformation. The two conditions to be fulfilled now are:

$$(d_*\Omega^N - \frac{1}{2}[\Omega^N, \Omega^N])^\#(\alpha) \in I \qquad \forall \alpha \in \Gamma I^\perp$$

$$[\alpha, \beta]_* + [\alpha, \beta]_{\Omega^N} \text{ is closed on } I^\perp.$$

But both conditions are granted by lemma 3.3 if the transformed characteristic pair belongs to the same equivalence class.

6. Deformations of Dirac structures II

There is of course another possible framework to study the deformation of a Dirac structure: consider a deformation of the Lie bialgebroid structure and make the Dirac bundle to heritage the deformation.

Then, consider a Lie bialgebroid (A, A^*), and a Dirac bundle $D \subset A \oplus A^*$. We consider a general transformation of the Lie bialgebroid:

$$\begin{cases} \mathcal{N} : A \oplus A^* \to A \oplus A^* \\ \mathcal{N}(X, \alpha) = (N(X, \alpha), N_*(X, \alpha)), \end{cases} \tag{62}$$

where the components N and N_* must be bilinear and must define new Lie structures on each sub-factor, i.e. they must be Nijenhuis operators for A and A^* respectively. The Dirac structure heritages that deformation and defines a new bundle, with a new Lie algebraic structure on the space of its sections.

6.1. The simplest case: only one of the Lie algebroids is transformed. Consider the simplest case, as we studied it for the Lie bialgebroid case. Let us assume that the transformation in (62) is such that

$$N_* = \mathrm{Id}. \tag{63}$$

Hence we are considering the transformation driven by a Nijenhuis operator for the Lie algebroid A only, the Lie algebroid A^* not being transformed. We know from Theorem 4.1 that such a transformation defines a deformation of the Lie bialgebroid if and only if

$$\delta_{NA} = i_{N^*} \circ d_* - d_* \circ i_{N^*} \tag{64}$$

is a derivation of the original Lie bracket on A and i_N is a derivation of the A^*-Schouten bracket.

We have to study now whether there are further conditions to fulfill in order to define also a deformation of the Dirac structure. As we have Proposition 3.1, the first thing to prove is that the Lie product (17), transformed by the deformation, defines a closed structure on the transformed vector bundle. Then, we have to consider the Lie bracket:

$$[(X_1, \alpha_1), (X_2, \alpha_2)]_N = \left([X_1, X_2]_N + [X, Y]_{\mathcal{L}^{A^*}}, [\alpha, \beta]_{\mathcal{L}^A_N} + [\alpha_1, \alpha_2]_*\right), \tag{65}$$

with

$$[X,Y]_{\mathcal{L}^{A^*}} = \mathcal{L}^{A^*}_{\alpha_1} X_2 - \mathcal{L}^{A^*}_{\alpha_2} X_1 - \frac{1}{2} d_*(i_{X_1}\alpha_2 - i_{X_2}\alpha_1)$$

and

$$[\alpha,\beta]_{\mathcal{L}^A_N} = (\mathcal{L}_N)_{X_1}\alpha_2 - (\mathcal{L}_N)_{X_2}\alpha_1 + \frac{1}{2} d^N(i_{X_1}\alpha_2 - i_{X_2}\alpha_1),$$

where $(\mathcal{L}_N)_X$ and d^N correspond to the deformed Lie derivative and the deformed exterior differential on A respectively.

We have to ask the transformation to be such that the transformed bundle is again a Lie algebroid. This implies that the image must be such that the bracket above defines a Lie structure.

Theorem 6.1. *Consider a Lie bialgebroid (A, A^*), a Dirac structure $D \subset A \oplus A^*$ and a deformation as above. Then, the deformed bundle is a Dirac structure if and only if, for any three elements $e_1 \equiv (X_1, \alpha_1)$,$e_2 \equiv (X_2, \alpha_2)$, $e_3 \equiv (X_3, \alpha_3)$:*

$$\langle [X_1, X_2]_N, \alpha_3\rangle + N \circ \rho(X_3)(e_1, e_2)_- + c.p = \langle [X_1, X_2], \alpha_3\rangle + \rho(X_3)(e_1, e_2)_- + c.p, \tag{66}$$

for $((X_1, \alpha_1), (X_2, \alpha_2))_- = i_{X_1}\alpha_2 - i_{X_2}\alpha_1$.

Proof: As the deformed structure still defines a Lie bialgebroid, we can consider on $A \oplus A^*$ a deformed Courant algebroid structure. Hence, consider the tensor:

$$T^N(e_1, e_2, e_3) = \frac{1}{3}((e_1, [e_2, e_3]_N) + (e_3, [e_1, e_2]_N) + (e_2, [e_3, e_1]_N)) \tag{67}$$

for the deformed case. The result is (see Lemma 3.2 of [9]):

$$\begin{aligned} 2T^N = {} & \langle [X_1, X_2]_N, \alpha_3\rangle + \langle X_3, [\alpha_1, \alpha_2]_*\rangle \\ & + N\rho(X_3)(e_1, e_2)_- - \rho_*(\alpha_3)(e_1, e_2)_- + \ c.p. \end{aligned} \tag{68}$$

¿From Definition 3.9, we know that a Dirac structure on a Lie bialgebroid is equivalent to a vanishing tensor T. The original Dirac structure being closed, we know that:

$$\begin{aligned} 2T = {} & \langle [X_1, X_2], \alpha_3\rangle + \langle X_3, [\alpha_1, \alpha_2]_{A^*}\rangle \\ & + \rho(X_3)(e_1, e_2)_- - \rho_*(\alpha_3)(e_1, e_2)_- + \ c.p = 0. \end{aligned}$$

Hence, we can use this result to replace the second and fourth term in (68) by the undeformed terms. This proves the result. ■

An equivalent formulation can be done in terms of characteristic pairs. Following Liu's construction, and assuming the same scheme above, let us consider the effect that a deformation of the type (63) has on the characteristic pair (I, Ω) of an original Dirac structure D:

- The new subbundle is easily obtained:

$$N : I \to I^N \equiv N(I). \tag{69}$$

- In what regards the bisection, the transformation is clearly given as:

$$N : \Omega^\# \to (\Omega^N)^\# \equiv N\Omega^\#. \tag{70}$$

  In order to ensure the skewsymmetry, we know from Lemma 5.1 that the Nijenhuis operator must be compatible with the bisection as:

$$N\Omega^\# = \Omega^\#(N^*).$$

Hence, we obtain:

Lemma 6.1. *The pair $(N(I), N\Omega^\#)$ defines a generalized Dirac structure on the deformed Lie bialgebroid.*

The conditions for the new characteristic pair to define a closed Dirac structure can be read from Liu's theorem 3.1.

- The bundle $I^N \equiv N(I)$ must be a subalgebra of $(\Gamma TM, [\cdot,\cdot]_N)$. As we know that the Nijenhuis torsion of N vanishes, the unique requirement is that the operation is closed, i.e.:

$$[N(I), N(I)]_N \subset N(I). \tag{71}$$

- Maurer-Cartan equation takes now the form:

$$d_*\Omega^N + [\Omega^N, \Omega^N]_N \in N(I).$$

- The transformed bivector must be such that

$$[\alpha, \beta] + [\alpha, \beta]_{\Omega^N} = [\alpha, \beta] + L_{\Omega^N\#(\alpha)}\beta - L_{\Omega^N\#(\beta)}\alpha - d_A^N\Omega^N(\alpha, \beta),$$

  is closed for any $\alpha, \beta \in (I^N)^\perp$.

6.2. More general deformations. Consider now a general transformation

$$\begin{cases} \mathcal{N} : A \oplus A^* \to A \oplus A^* \\ \mathcal{N}(X, \alpha) = (N(X, \alpha), N_*(X, \alpha)), \end{cases} \tag{72}$$

which still defines a deformation of the Lie bialgebroid (A, A^*).

Again, the condition for the transformation to define a deformation of the Dirac structure is written in terms of the deformation of the tensor T, since the transformed structure is defined on a Courant algebroid.

Theorem 6.2. *Consider a Lie bialgebroid (A, A^*), a Dirac structure $D \subset A \oplus A^*$ and a deformation as (72). Then, the deformed bundle is a Dirac structure if and only if, for any three elements $e_1 \equiv (X_1, \alpha_1)$,$e_2 \equiv (X_2, \alpha_2)$, $e_3 \equiv (X_3, \alpha_3)$, $T_N(e_1, e_2, e_3) = 0$, i.e.*

$$\begin{aligned} &\langle [X_1, X_2]_N, \alpha_3\rangle + \langle X_3, [\alpha_1, \alpha_2]_{N_*}\rangle + \\ &\qquad N \circ \rho(X_3)(e_1, e_2)_- - N_* \circ \rho_*(\alpha_3)(e_1, e_2)_- \ c.p = 0, \end{aligned} \tag{73}$$

for $((X_1, \alpha_1), (X_2, \alpha_2))_- = i_{X_1}\alpha_2 - i_{X_2}\alpha_1$.

Proof: The result follows directly from the properties of Courant algebroids. ∎

Again, another characterization is possible, in terms of characteristic pairs. Denoting as (I, Ω) a representant in the class of characteristic pairs of the original Dirac structure D, and as (I^N, Ω^N) the transformed pair we have:

- The annihilator of the new subbundle, $(I^N)^\perp$ may be different from the image of the original by the operator N_*. Hence, a compatibility condition which arises is:
$$N_*(I^\perp) = (I^N)^\perp = (N(I))^\perp. \tag{74}$$
We can also rewrite this condition as:
$$i_{N_*} N(I) = I, \tag{75}$$
or equivalently
$$i_N N_*(I^\perp) = I^\perp. \tag{76}$$
- In what regards the tensor, condition (56) is still necessary in order to ensure the skew-symmetry of the new tensor. It is important to remark that the condition affects only the operator N.

If the bi-section satisfies condition (56) we have, as in the previous case,

Lemma 6.2. *The pair $(N(I), N\Omega^{\#})$, satisfying the conditions above, defines a generalized Dirac structure on the deformed Lie bialgebroid.*

The conditions for the new pair to define a closed Dirac structure, once again, are read from Theorem 3.1.

- The bundle $I^N \equiv N(I)$ must be a subalgebra of $(\Gamma A, [\cdot,\cdot]_N)$. As we know that the Nijenhuis torsion of N vanishes on A, the unique requirement is that the operation is closed, i.e.:
$$[N(I), N(I)]_N \subset N(I). \tag{77}$$
This condition is again trivially fulfilled because of the properties arising from the deformation of the Lie bialgebroid structure. The proof is completely analogous to the one in the previous case.
- Maurer-Cartan equation takes now the form:
$$d_*^N \Omega^N + [\Omega^N, \Omega^N]_N \in N(I).$$
- The transformed bivector must be such that
$$[\alpha,\beta] + [\alpha,\beta]_{\Omega^N} = [\alpha,\beta] + L_{\Omega^{N\#}(\alpha)}\beta - L_{\Omega^{N\#}(\beta)}\alpha - d^N\Omega^N(\alpha,\beta)$$
is closed for any $\alpha,\beta \in (I^N)^{\perp}$. The fulfillment of condition (74) is necessary for this expression to make sense.

6.3. Examples.

6.3.1. *The simplest case.* As we did above, let us consider first the simplest possible deformation on a Lie bialgebroid. Consider an algebroid $A \to M$ and the trivial algebroid structure on its dual $A^* \to M$ defined by an null anchor mapping and vanishing structure constants, i.e.:
$$\begin{cases} A, \rho, [\cdot,\cdot] \\ A^*, 0, [\alpha,\beta] = 0 \quad \forall \alpha,\beta \in \Gamma A^*. \end{cases} \tag{78}$$

Consider then a Dirac structure defined on $A \oplus A^*$ and a general deformation as (72). As the Lie algebroid structure on A^* is not affected, and the deformed pair:
$$\begin{cases} A, N\circ\rho, [\cdot,\cdot]_N \\ A^*, 0, [\alpha,\beta] = 0 \quad \forall \alpha,\beta \in \Gamma A^* \end{cases}, \tag{79}$$
is still a Lie bialgebroid, we can consider the effect on the Dirac structure D.

As discussed above, D is represented by an equivalence class of characteristic pairs. Consider a representant (I, Ω). The transformation $\mathcal{N}$ transforms the pair as we saw above:

$$\mathcal{N}(I, \Omega) = (I^N, \Omega^N) = \begin{cases} I^N = N(I) \\ \Omega^N = N\Omega. \end{cases} \tag{80}$$

If $N\Omega^\# = \Omega^\# \circ N^*$ the new tensor is skew-symmetric and the pair (I^N, Ω^N) is the characteristic pair of a new generalized Dirac structure. The conditions for it to be closed are:

- Maurer-Cartan type equation: ${[\Omega^N, \Omega^N]_N}^\# \subset I^N$.
- The condition for the algebra defined on $(I^N)^\perp$ is now:

$$[\alpha, \beta]_{\Omega^N} \subset (I^N)^\perp \qquad \forall \alpha, \beta \in (I^N)^\perp.$$

There is also an extra requirement concerning the compatibility of both algebroid deformations. As in principle, A and A^* are deformed independently, we have to ask the transformations to be such that:

$$(N(I))^\perp \subset \text{Im } N_*. \tag{81}$$

If this condition is fulfilled, the two conditions above can be taken directly. If it is not, the second condition must be restricted to the subbundle $(N(I))^\perp \cap \text{Im } N_*$.

6.3.2. *Poisson-Nijenhuis structures.* Consider a simple example of Dirac structure: a Poisson structure Λ defined on a manifold M. We know that in these circumstances we can define a Lie bialgebroid structure on the pair (TM, T^*M) as:

$$\begin{cases} (TM, \rho = \text{Id}, [\cdot, \cdot]) \\ (T^*M, \rho = \Lambda^\#, [\alpha, \beta] = L_{\Lambda^\#(\alpha)}\beta - L_{\Lambda^\#(\beta)}\alpha - \Lambda(\alpha, \beta)), \end{cases} \tag{82}$$

where we denote by $[\cdot, \cdot]$ the usual commutator of vector fields.

We also know [6], that under a deformation, the resulting structure is also a Lie bialgebroid if and only if the manifold is Poisson-Nijenhuis:

$$\begin{cases} (TM, \rho = N, [\cdot, \cdot]_N) \\ (T^*M, \rho = \Lambda^\#, [\alpha, \beta] = L_{\Lambda^\#(\alpha)}\beta - L_{\Lambda^\#(\beta)}\alpha - \Lambda(\alpha, \beta)). \end{cases} \tag{83}$$

We want to study now the new conditions to be imposed to the transformation in order to deform a Dirac structure defined on the first Lie bialgebroid, into another Dirac structure defined on the second.

Consider the Dirac structure defined by the graph of the Poisson tensor on the first bialgebroid, i.e.

$$D = \{(\Lambda^{\#}(\alpha), \alpha) | \alpha \in T^*M\}. \tag{84}$$

Thinking in terms of characteristic pairs, this Dirac structure is represented by $(0, \Lambda)$, and the closeness condition corresponds to the well known condition:

$$d_*\Lambda + [\Lambda, \Lambda] = 2[\Lambda, \Lambda] = 0,$$

where $[\cdot, \cdot]$ is the Schouten bracket on M, and we used that the exterior derivative for the Lie algebroid $(T^*M, \Lambda^{\#}, [\cdot, \cdot]_\Lambda)$ corresponds to $d_\Lambda = [\Lambda, \cdot]$.

Finally, the condition

$$[\alpha, \beta] + [\alpha, \beta]_\Omega = [\alpha, \beta]_\Lambda + [\alpha, \beta]_\Lambda$$

is immediate in this case, since both objects define Lie algebra structures, being $I = \emptyset$.

Now, consider a deformation of the form (63). For the transformed Lie bialgebroid, we know that the image of the points of the graph of the tensor Λ are of the form

$$\mathcal{N}(\Lambda^{\#}(\alpha), \alpha) = (N\Lambda^{\#}(\alpha), \alpha).$$

As we know that the manifold M is Poisson-Nijenhuis, the tensor $N\Lambda$ is also a Poisson tensor, and hence we can consider its graph on the Lie bialgebroid as a candidate to Dirac structure. The situation is slightly different, though, because the Lie algebroid structure on T^*M is defined by using the original Poisson tensor Λ and we assume that this structure is not modified by the deformation. The conditions are as follows:

- Maurer-Cartan equation reads now:

$$d_*(N\Lambda) + \frac{1}{2}[N\Lambda, N\Lambda]_N = [\Lambda, N\Lambda] + \frac{1}{2}[N\Lambda, N\Lambda]_N = 0.$$

- While the second condition corresponds to:

$$[\alpha, \beta] + [\alpha, \beta]_{\Omega^N} = [\alpha, \beta]_\Lambda + [\alpha, \beta]_{N\Lambda}$$

  being closed. This condition is obviously satisfied.

Therefore, we conclude that:

Theorem 6.3. *Let M be a Poisson-Nijenhuis structure. Consider the deformation of the Lie bialgebroid structure (82) into (83) . Then, the condition for a Dirac structure D defined as the graph of the Poisson tensor Λ, to be deformed into a new Dirac structure, corresponding to the graph of the deformed Poisson tensor $N\Lambda$ is:*

$$[\Lambda, N\Lambda] + \frac{1}{2}[N\Lambda, N\Lambda]_N = 0. \tag{85}$$

It is important to realize that the condition of $N\Lambda$ being a Poisson tensor is not enough to ensure that the graph defines a Dirac structure, since this fact depends on the Lie bialgebroid structure encoded in the pair (A, A^*). Only if we consider a trivial Lie algebroid structure on T^*M, with the anchor mapping and the Lie structure vanishing, the condition of $N\Lambda$ being a Poisson tensor ensures the definition of a Dirac structure on its graph.

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