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THE SEMICONTINUOUS QUASI-UNIFORMITY OF A FRAME

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ABSTRACT: The aim of this note is to show how various facts in classical topology connected with semicontinuous functions and the semicontinuous quasi-uniformity have their counterparts in pointfree topology. In particular, we introduce the localic semicontinuous quasi-uniformity, which generalizes the semicontinuous quasi-uniformity of a topological space (known to be one of the most important examples of transitive compatible quasi-uniformities). We show that it can be characterized in terms of the so called spectrum covers, via a construction introduced by the authors in a previous paper. Several consequences are derived.

KEYWORDS: frames, biframes, Weil entourages, quasi-uniform frames, transitive quasi-uniformities, totally bounded quasi-uniformities, functorial quasi-uniformities, Fletcher construction, spectrum covers, biframe of reals, semicontinuous real functions.
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A familiar construction, due to Fletcher [6], of compatible quasi-uniformities of a given topological space X, starts from a suitable collection of interior-preserving open covers of X and constructs a subbase for a compatible quasi-uniformity $\mathcal{E}$ on X, that is, a quasi-uniform space $(X, \mathcal{E})$ inducing as the first topology the topology $\mathcal{O}X$ of the given space X. This procedure determines a well-known class of compatible quasi-uniformities, and all transitive compatible quasi-uniformities of a space are obtained by it [4].

Locales (also frames) have been recognized as an important generalization of (sober) topological spaces, which allows the study of topological questions in contexts where intuitionistic logic rather than Boolean logic prevails and in which spaces without points occur naturally (cf. [16]).

The construction referred to above seems to require a space as its starting point, since it depends seriously on the fact that the subobject lattice of a space is a Boolean algebra (and, on the other hand, the sublocale lattice is not, in general, Boolean). Another obstacle in the pointfree setting, is the

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fact that, contrary to the classical case, infinite intersections of entourages are not necessarily entourages. Recently, however, the authors were able to establish that the Fletcher construction has a natural counterpart in pointfree topology [5], solving a problem posed by G.C.L. Brümmer. The paper [5] ended with the observation that, starting with the collection of all spectrum covers of the frame, we get, via that construction, the so called semicontinuous quasi-uniformity. The present note aims to prove just that. For this, we need to work with the upper and lower frames of reals (more precisely, with the biframe of reals) and to develop the theory of localic semicontinuous functions. It is probably worth mention that the theory of localic semicontinuous functions is more general and interesting than that in the classical case (even when the frame is spatial) [17], but has been rarely used ([17] is the only reference we know about the subject).

The paper is divided into five sections. In the first, we recall the basic facts about frames and quasi-uniform frames that are pertinent for our approach, and in the second we introduce upper and lower semicontinuous real functions and semicontinuous characteristic functions on a frame. Section 3 deals with a natural quasi-uniformity $\mathcal{Q}$ on the frame of reals $\mathcal{L}(\mathbb{R})$, which induces the biframe of reals $(\mathcal{L}(\mathbb{R}), \mathcal{L}_u(\mathbb{R}), \mathcal{L}_l(\mathbb{R}))$ as its biframe structure. Section 4 then presents the semicontinuous quasi-uniformity of a frame, induced by a family of upper semicontinuous real functions. We show that this is the coarsest quasi-uniformity $\mathcal{E}$ on the congruence frame $\mathfrak{C}L$ for which each biframe homomorphism $h : (\mathcal{L}(\mathbb{R}), \mathcal{Q}) \to \mathfrak{C}L$ is uniform, and that it can be characterized in terms of spectrum covers, via our general construction from [5]. This gives us the pointfree version of a theorem of Fletcher and Lindgren [7] on transitive quasi-uniformities. Finally, the last section offers a few consequences on totally boundedness and bounded upper semicontinuous real functions.

1. Preliminaries

A *frame* (or *locale*) is a complete lattice L satisfying the infinite distributive law

$$x \wedge \bigvee S = \bigvee \{x \wedge s \mid s \in S\}$$

for every $x \in L$ and every $S \subseteq L$. The category Frm has as maps the homomorphisms which preserve the respective operations $\wedge$ (including the top element 1) and $\bigvee$ (including the bottom element 0). The lattice $\mathcal{O}X$ of open sets of a topological space X is a frame. For further information concerning frames see Johnstone [15] or Vickers [21].

If L is a frame and $x \in L$ then $x^* := \bigvee\{a \in L \mid a \wedge x = 0\}$ is the *pseudocomplement* of x. Obviously, if $x \vee x^* = 1$, x is complemented and we denote the complement x^* by $\neg x$. Note that, in any frame, the first De Morgan law

$$(\bigvee_{i \in I} x_i)^* = \bigwedge_{i \in I} x_i^* \tag{1.1}$$

holds but for meets we have only the trivial inequality $\bigvee_{i \in I} x_i^* \leq (\bigwedge_{i \in I} x_i)^*$. A *cover* A of L is a subset $A \subseteq L$ such that $\bigvee A = 1$.

Concerning frame homomorphisms, a frame homomorphism $h : L \to M$ is called *dense* if $h(x) = 0$ implies $x = 0$. For any frame homomorphism $h : L \to M$, there is its *right adjoint* $h_* : M \to L$ such that $h(x) \leq y$ iff $x \leq h_*(y)$, explicitly given by $h_*(y) = \bigvee\{a \in L \mid h(a) \leq y\}$.

Recall also that a *biframe* is a triple (L_0, L_1, L_2) where L_1 and L_2 are subframes of the frame L_0, which together generate L_0. A *biframe homomorphism,* $h : (L_0, L_1, L_2) \longrightarrow (M_0, M_1, M_2)$, is a frame homomorphism $h : L_0 \longrightarrow M_0$ which maps L_i into M_i $(i = 1, 2)$ and BiFrm denotes the resulting category. Further, a biframe (L_0, L_1, L_2) is *strictly zero-dimensional* [2] if it satisfies the following condition or its counterpart with L_1 and L_2 reversed: each $x \in L_1$ is complemented in L_0, with complement in L_2, and L_2 is generated by these complements. Along this paper, when referring to a strictly zero-dimensional biframe, we always assume that it satisfies the condition above, not its counterpart with L_1 and L_2 reversed. Additional information concerning biframes may be found in [2] and [3].

For a frame L consider the frame $\mathcal{D}(L \times L)$ of all non-void decreasing subsets of $L \times L$, ordered by inclusion. The coproduct $L \oplus L$ will be represented as usual (cf. [15]), as the subset of $\mathcal{D}(L \times L)$ consisting of all *C-ideals*, that is, of those sets A for which

$$(x, \bigvee S) \in A \text{ whenever } \{x\} \times S \subseteq A$$

and

$$(\bigvee S, y) \in A \text{ whenever } S \times \{y\} \subseteq A.$$

Since the premise is trivially satisfied if $S = \emptyset$, each C-ideal A contains $\mathbf{O} := \{(0,a),(a,0) \mid a \in L\}$, and $\mathbf{O}$ is the zero of $L \oplus L$. Obviously, each

$$x \oplus y = {\downarrow}(x,y) \ \cup \{(0,a),(a,0) \mid a \in L\}$$

is a C-ideal and for each C-ideal A one has $A = \bigvee\{x \oplus y \mid (x,y) \in A\}$. The coproduct injections $u_i^L : L \to L \oplus L$ are defined by $u_1^L(x) = x \oplus 1$ and $u_2^L(x) = 1 \oplus x$ so that $x \oplus y = u_1^L(x) \wedge u_2^L(y)$.

For any frame homomorphism $h : L \longrightarrow M$, the coproduct definition ensures us the existence (and uniqueness) of a frame homomorphism $h \oplus h : L \oplus L \longrightarrow M \oplus M$ such that $(h \oplus h) \cdot u_i^L = u_i^M \cdot h$ $(i = 1, 2)$.

If $A, B \in \mathcal{D}(L \times L)$ then

$$A \circ B := \bigvee\{x \oplus y \mid \exists z \in L \setminus \{0\} : (x,z) \in A,\ (z,y) \in B\}.$$

Note that

$$(h \oplus h)(A) \circ (h \oplus h)(B) \subseteq (h \oplus h)(A \circ B)$$

for every frame homomorphism h.

A *(Weil) entourage* [18] on L is just an element E of $L \oplus L$ for which

$$\bigvee\{x \in L \mid (x,x) \in E\} = 1.$$

For every entourage E, $E \subseteq E \circ E$. A Weil entourage E is called

transitive if $E \circ E = E$,

finite if there exists a finite cover $\{x_1, \ldots, x_n\}$ of L such that $\bigvee_{i=1}^n (x_i \oplus x_i) \subseteq E$.

Given $A \in \mathcal{D}(L \times L)$, we denote by $<A>$ the C-ideal generated by A. The following properties, taken from [18], are decisive in our approach:

Lemma 1.1. *For any $A, B \in \mathcal{D}(L \times L)$, we have:*

(a) $< A > \circ < B >= A \circ B$.

(b) $< A > \cap < B >= A \cap B$.

For a system $\mathcal{E}$ of Weil entourages of a frame L (always understood to be non-void) we write $x \overset{\mathcal{E}}{\lhd}_1 y$ if there exists $E \in \mathcal{E}$ such that

$$st_1(x,E) := \bigvee\{a \in L \mid (a,b) \in E, b \wedge x \neq 0\} \leq y. \tag{1.2}$$

Similarly, we write $x \overset{\mathcal{E}}{\lhd}_2 y$ if there exists $E \in \mathcal{E}$ such that

$$st_2(x,E) := \bigvee\{b \in L \mid (a,b) \in E, a \wedge x \neq 0\} \leq y. \tag{1.3}$$

The elements $st_i(x, E)$ $(i = 1, 2)$ satisfy the following properties, for every $x, y \in L$ and every $A, B \in \mathcal{D}(L \times L)$ [18]:

(S1) $x \le y \Rightarrow st_i(x, A) \le st_i(y, A)$;
(S2) $st_i(x, <\!A\!>) = st_i(x, A)$.
(S3) If E is a Weil entourage then $x \le st_1(x, E) \wedge st_2(x, E)$.
(S4) For each frame map $h : L \to M$ and each $E \in L \oplus L$,

$$st_i(h(x), h \oplus h(E)) \le h(st_i(x, E)).$$

$\mathcal{E}$ is called *admissible* if, for every $x \in L$,

$$x = \bigvee\{y \in L \mid y \overset{\overline{\mathcal{E}}}{\triangleleft_1} x\}, \tag{1.4}$$

where $\overline{\mathcal{E}}$ stands for $\mathcal{E} \cup \{E^{-1} \mid E \in \mathcal{E}\}$.

An admissible filter $\mathcal{E}$ of $WEnt(L)$ is a *(Weil) quasi-uniformity* on L if , for each $E \in \mathcal{E}$ there exists $F \in \mathcal{E}$ such that $F \circ F \subseteq E$. Further, a *(Weil) quasi-uniform frame* is a pair $(L, \mathcal{E})$ where L is a frame and $\mathcal{E}$ is a quasi-uniformity on L.

Concerning special types of maps between quasi-uniform frames, a frame homomorphism $h : (L, \mathcal{E}) \to (M, \mathcal{F})$ between quasi-uniform frames is called

uniform if $(h \oplus h)(E) \in \mathcal{F}$ for each $E \in \mathcal{E}$,
a *surjection* if it is onto and the $(h_* \oplus h_*)(F)$, $F \in \mathcal{F}$, are entourages generating $\mathcal{E}$.

Obviously, any surjection is a uniform homomorphism. Quasi-uniform frames and uniform homomorphisms form the category QUFrm of quasi-uniform frames.

Each quasi-uniformity $\mathcal{E}$ on L defines two subframes of L:

$$\mathcal{L}_1(\mathcal{E}) := \left\{x \in L \mid x = \bigvee\{y \in L \mid y \overset{\mathcal{E}}{\triangleleft_1} x\}\right\},$$

$$\mathcal{L}_2(\mathcal{E}) := \left\{x \in L \mid x = \bigvee\{y \in L \mid y \overset{\mathcal{E}}{\triangleleft_2} x\}\right\}.$$

The admissibility condition (1.4) is equivalent to saying that the triple

$$(L, \mathcal{L}_1(\mathcal{E}), \mathcal{L}_2(\mathcal{E}))$$

is a biframe [19]. This is the pointfree expression of the classical fact that each quasi-uniform space $(X, \mathcal{E})$ induces a bitopological structure $(\mathcal{T}_1(\mathcal{E}), \mathcal{T}_2(\mathcal{E}))$ on X.

Regarding quasi-uniform frames, we shall need the following notions: a quasi-uniform frame $(L, \mathcal{E})$ is called

transitive if $\mathcal{E}$ has a base consisting of transitive entourages,
totally bounded if $\mathcal{E}$ has a base consisting of finite entourages.

For more information on transitive quasi-uniformities and totally bounded quasi-uniformities we refer to [14] and [13], respectively.

The lattice of frame congruences on L under set inclusion is a frame, denoted by $\mathfrak{C}L$. A good presentation of the congruence frame is given by Frith [10]. Here, we shall need the following properties:

(1) For any $x \in L$, $\nabla_x = \{(a,b) \in L \times L \mid a \vee x = b \vee x\}$ is the least congruence containing $(0,x)$; $\Delta_x = \{(a,b) \in L \times L \mid a \wedge x = b \wedge x\}$ is the least congruence containing $(1,x)$. The ∇_x are called *closed* and the Δ_x *open*.
(2) Each ∇_x is complemented in $\mathfrak{C}L$ with complement Δ_x.
(3) $\nabla L = \{\nabla_x \mid x \in L\}$ is a subframe of $\mathfrak{C}L$. Let ΔL denote the subframe of $\mathfrak{C}L$ generated by $\{\Delta_x \mid x \in L\}$. Since $\theta = \bigvee\{\nabla_y \wedge \Delta_x \mid (x,y) \in \theta, x \leq y\}$, for every $\theta \in \mathfrak{C}L$, the triple $(\mathfrak{C}L, \nabla L, \Delta L)$ is a biframe, usually referred to as the *Skula biframe* of L [10]. Clearly, this is a strictly zero-dimensional biframe.
(4) The map $x \mapsto \nabla_x$ is a frame isomorphism $L \to \nabla L$, whereas the map $x \mapsto \Delta_x$ is a dual poset embedding $L \to \Delta L$ taking finitary meets to finitary joins and arbitrary joins to arbitrary meets.

2. Semicontinuous real functions

Throughout the paper we denote by $\mathbb{Q}$ the totally ordered set of rational numbers.

In pointfree topology it is natural and useful to introduce the reals in a pointfree way, independent of any notion of real number. The *frame of reals* [1] is the frame $\mathfrak{L}(\mathbb{R})$ generated by all ordered pairs (p,q) where $p, q \in \mathbb{Q}$, subject to the relations

(R1) $(p,q) \wedge (r,s) = (p \vee r, q \wedge s)$,
(R2) $(p,q) \vee (r,s) = (p,s)$ whenever $p \leq r < q \leq s$,
(R3) $(p,q) = \bigvee\{(r,s) \mid p < s < t < q\}$,
(R4) $1 = \bigvee\{(p,q) \mid p, q \in \mathbb{Q}\}$.

Classically, this is just the interval topology of the real line, but under the point of view of constructiveness, these two notions are not the same (see [1], [9]). The definition of $\mathfrak{L}(\mathbb{R})$ immediately implies that, for any frame L, a map into L from the set of all pairs (p,q) determines a (unique) homomorphism $\varphi : \mathfrak{L}(\mathbb{R}) \to L$ if and only if it transforms the above relations into identities in L.

Let $\mathfrak{L}_u(\mathbb{R})$ be the frame generated, inside $\mathfrak{L}(\mathbb{R})$, by elements

$$(-,q) = \bigvee\{(p,q) \mid p \in \mathbb{Q}\},$$

or equivalently, subject to the relations

(U1) $p \le q \Rightarrow (-,p) \le (-,q)$,
(U2) $\bigvee_{q<p} (-,q) = (-,p)$,
(U3) $\bigvee_{q\in\mathbb{Q}} (-,q) = 1$,

and, dually, let $\mathfrak{L}_l(\mathbb{R})$ be the frame generated by elements

$$(p,-) = \bigvee\{(p,q) \mid q \in \mathbb{Q}\},$$

subject to the relations

(L1) $p \le q \Rightarrow (p,-) \ge (q,-)$,
(L2) $\bigvee_{p>q}(p,-) = (q,-)$,
(L3) $\bigvee_{p\in\mathbb{Q}} (p,-) = 1$.

In addition, note that [1]:

(a) For any p,q, $(p,-) \wedge (-,q) = (p,q)$.
(b) For any $p<q$, $(p,-) \vee (-,q) = 1$.
(c) For any $p<q$, $(p,q)^* = (-,p) \vee (q,-)$.
(d) For any p, $(-,p)^* = (p,-)$ and $(p,-)^* = (-,p)$.

Further, applying De Morgan law (1.1) to (U2) and (L2), respectively, we get

$$\bigwedge_{p<q}(p,-) = (q,-) \text{ and } \bigwedge_{q>p}(-,q) = (-,p).$$

The triple $(\mathfrak{L}(\mathbb{R}), \mathfrak{L}_u(\mathbb{R}), \mathfrak{L}_l(\mathbb{R}))$ is a biframe (the *biframe of reals*) and a map from the set of generators of $\mathfrak{L}(\mathbb{R})$ into a biframe L determines a (unique) biframe homomorphism $\varphi : (\mathfrak{L}(\mathbb{R}), \mathfrak{L}_u(\mathbb{R}), \mathfrak{L}_l(\mathbb{R})) \to (L, L_1, L_2)$ if and only if it transforms the above relations into identities in L and takes the generators of $\mathfrak{L}_u(\mathbb{R})$ to L_1 and the generators of $\mathfrak{L}_l(\mathbb{R})$ to L_2.

Classically, $\mathfrak{L}_u(\mathbb{R})$ and $\mathfrak{L}_l(\mathbb{R})$ are, respectively, the upper and lower topologies of the real line, but not in the sense of the constructive view.

Recall that, for a space X, a map $f : X \to \mathbb{R}$ is *upper semi-continuous* if $f : X \to \mathbb{R}_u$ is continuous, where $\mathbb{R}_u$ denotes the space of reals with the upper topology $\{]-\infty, a[\,:\, a \in \mathbb{R}\}$. It is straightforward to check that, for any space X, there is a bijection

$$\mathsf{Frm}(\mathfrak{L}_u(\mathbb{R}), \mathcal{O}X) \to \mathsf{Top}(X, \mathbb{R}_u)$$

given by the correspondence $\varphi \mapsto \tilde{\varphi}$ such that $p < \tilde{\varphi}(x)$ iff $x \in \varphi(-, p)$ for every $p \in \mathbb{Q}$. This justifies to adopt the following definition:

Definition 2.1. An *upper semicontinuous real function* on a frame L is a frame homomorphism $\mathfrak{L}_u(\mathbb{R}) \to L$. Dually, a *lower semicontinuous real function* is a frame homomorphism $\mathfrak{L}_l(\mathbb{R}) \to L$.

Remark 2.2. In classical topology, a map f from a space X to the space of reals (with euclidean topology) is continuous iff both $f : X \to \mathbb{R}_u$ and $f : X \to \mathbb{R}_l$ are continuous, that is, iff f is both upper and lower semicontinuous. The extension of this to the pointfree setting may fail: there are maps $h : \mathcal{L}(\mathbb{R}) \to L$ whose restrictions $h : \mathcal{L}_u(\mathbb{R}) \to L$ and $h : \mathcal{L}_l(\mathbb{R}) \to L$ are frame homomorphisms but h itself is not (as we shall see below, this is true only if $h(-, q) \vee h(p, -) = 1$ whenever $p < q$ and $h(-, p) \wedge h(q, -) = 0$ whenever $p \leq q$).

In order to have a pointfree counterpart for that result we have to look at it from the bitopological point of view. Classically, $f : X \to \mathbb{R}$ is continuous iff $f : (X, \mathcal{O}X, \mathcal{O}X) \to (\mathbb{R}, \mathbb{R}_u, \mathbb{R}_l)$ is a bicontinuous map; now, in the frame setting, we also have that $h : \mathcal{L}(\mathbb{R}) \to L$ is a frame homomorphism iff $h : (\mathcal{L}(\mathbb{R}), \mathcal{L}_u(\mathbb{R}), \mathcal{L}_l(\mathbb{R})) \to (L, L, L)$ is a biframe homomorphism.

By the isomorphism $\nabla_L : L \to \nabla L$, we may look at upper semicontinuous real functions on L as frame homomorphism $h : \mathcal{L}_u(\mathbb{R}) \to \nabla L$ and then, since each element of ∇L is complemented in $\mathfrak{C}L$ with complement in ΔL, we have a map $\mathcal{L}_l(\mathbb{R}) \to \Delta L$ given by $(p, -) \mapsto \neg h(-, p)$. We need to know the conditions under which this defines a lower semicontinuous real function on $\mathfrak{C}L$ which extends to a biframe map $\mathcal{L}(\mathbb{R}) \to \mathfrak{C}L$.

Recall from [1] that a *trail* in L is a map $t : \mathbb{Q} \to L$ such that $t(p) \prec t(q)$ (that is, $t(p)^* \vee t(q) = 1$) whenever $p < q$, and

$$\bigvee\{t(p) \mid p \in \mathbb{Q}\} = 1 = \bigvee\{t(p)^* \mid p \in \mathbb{Q}\}.$$

Similarly, a *descending trail* in L is a map $t : \mathbb{Q} \to L$ such that $t(q) \prec t(p)$ whenever $p < q$, and the same join condition holds.

By Lemma 2 of Banaschewski [1], for any trail t in L (resp. descending trail t),

$$h(p,q) = \bigvee\{t(p')^* \wedge t(q') \mid p < p' < q' < q\} \tag{2.5}$$

(resp.

$$h(p,q) = \bigvee\{t(p') \wedge t(q')^* \mid p < p' < q' < q\}) \tag{2.6}$$

defines a homomorphism $h : \mathcal{L}(\mathbb{R}) \to L$. However this correspondence from trails to continuous real functions is not a bijection. In order to describe continuous real functions on L in terms of trails we need to introduce trail pairs.

We say that a trail (resp. descending trail) is *continuous* if $\bigvee_{q<p} t(q) = t(p)$ (resp. $\bigvee_{q>p} t(q) = t(p)$) for any $p \in \mathbb{Q}$.

Remarks 2.3. (1) For an example of a trail which is not continuous consider the Boolean algebra $L = \mathcal{P}(\mathbb{Q})$ and $t(p) =]-\infty, p]$.
(2) Any trail $t : \mathbb{Q} \to L$ induces a continuous trail $\overline{t} : \mathbb{Q} \to L$ by $\overline{t}(p) = \bigvee_{q<p} t(q)$.

We say that a pair (t_1, t_2), where t_1 is a continuous trail and t_2 is a continuous descending trail, is a *trail pair* in L if

(T1) $p < q \Rightarrow t_1(q) \vee t_2(p) = 1$,
(T2) $\forall\, p \in \mathbb{Q},\ t_1(p) \wedge t_2(p) = 0$.

Then, a map $h : \mathcal{L}(\mathbb{R}) \to L$ is a frame homomorphism if and only if (t_1, t_2), where $t_1(p) = h(-,p)$ and $t_2(p) = h(p,-)$, is a trail pair in L (the proof is straightforward and we omit it). It is then easy to check that this correspondence $h \mapsto (t_1, t_2)$ defines a bijection between frame homomorphisms $\mathcal{L}(\mathbb{R}) \to L$ and trail pairs in L. The inverse correspondence assigns to each trail pair (t_1, t_2) the homomorphism of Banaschewski given by (2.5) applied to t_1 or, which is the same in the case of a trail pair, by (2.6) applied to t_2.

Now, for any upper semicontinuous real function $h : \mathcal{L}_u(\mathbb{R}) \to L \cong \nabla L$, we have:

Lemma 2.4. *$\overline{h}(-,q) = h(-,q) \in \nabla L$ and $\overline{h}(p,-) = \bigvee_{q>p} \neg h(-,q) \in \Delta L$ define a (bi)frame homomorphism $\overline{h} : \mathcal{L}(\mathbb{R}) \to \mathfrak{C}L$ if and only if*

$$\bigvee_{p \in \mathbb{Q}} \neg h(-,p) = 1.$$

Proof. If $\overline{h}$ is a frame map then

$$1 = \overline{h}(1) = \overline{h}\Big(\bigvee_{p\in\mathbb{Q}}(p,-)\Big) = \bigvee_{p\in\mathbb{Q}}\overline{h}(p,-) = \bigvee_{p\in\mathbb{Q}}\bigvee_{q>p}\neg h(-,q) = \bigvee_{p\in\mathbb{Q}}\neg h(-,p).$$

Conversely, as we observed above, it suffices to check that

$$\Big((h(-,q))_{q\in\mathbb{Q}}, (\bigvee_{q>p}\neg h(-,q))_{p\in Q}\Big)$$

is a trail pair, which is an easy exercise. □

Proposition 2.5. *There is an extension of h to a biframe homomorphism $\overline{h}:\mathcal{L}(\mathbb{R})\to\mathfrak{C}L$ if and only if $\bigvee_{q\in\mathbb{Q}}\neg h(-,q)=1$.*

Proof. Let $\overline{h}:\mathcal{L}(\mathbb{R})\to\mathfrak{C}L$ be a biframe map that extends h. Then, as we observed before the lemma, the restrictions of $\overline{h}$ to, respectively, $\mathcal{L}_u(\mathbb{R})$ and $\mathcal{L}_l(\mathbb{R})$ define a trail pair in $\mathfrak{C}L$. In particular, this means that $\bigvee_{q\in\mathbb{Q}}\neg h(-,q)=1$.

The converse follows from the lemma. □

Remark 2.6. It can be shown that if we assume, besides the condition $\bigvee_{p\in\mathbb{Q}}\neg h(-,p)$, that $\bigvee_{p>q}\neg h(-,p)=\neg h(-,q)$, then the extension of h to a biframe map is unique.

For each $x\in L$, consider the upper semicontinuous real function

$$\begin{array}{rccl} h_x^u: & \mathfrak{L}_u(\mathbb{R}) & \to & L \\ & (-,p) & \mapsto & \begin{cases} 1 \text{ if } 1<p \\ x \text{ if } 0<p\le 1 \\ 0 \text{ if } p\le 0 \end{cases} \end{array}$$

and the lower semicontinuous real function

$$\begin{array}{rccl} h_x^l: & \mathfrak{L}_l(\mathbb{R}) & \to & L \\ & (p,-) & \mapsto & \begin{cases} 1 \text{ if } p<0 \\ x \text{ if } 0\le p< 1 \\ 0 \text{ if } 1\le p \end{cases} \end{array}$$

These are the *upper and lower characteristic functions* on x. In the sequel we shall see that the h_x^u (resp. h_x^l) play the role, in the pointfree setting, of the characteristic functions f_C, on a closed set C, of classical topology (resp. the characteristic functions f_A on a open A).

It is an easy exercise to observe that the pair (h_x^u, h_y^l) is a trail pair if and only if $y=\neg x$. So, when our frame is the congruence frame $\mathfrak{C}L$ of a frame,

$(h^u_{\nabla_x}, h^l_{\Delta_x})$ is a trail pair for every $x \in L$, which thus induces a biframe map $h_x : \mathcal{L}(\mathbb{R}) \to \mathfrak{C}L$.

3. The quasi-metric quasi-uniformity of the reals

The frame $\mathfrak{L}(\mathbb{R})$ carries a natural quasi-uniformity compatible with the upper frame of reals $\mathfrak{L}_u(\mathbb{R})$, its *quasi-metric quasi-uniformity* $\mathcal{Q}$, generated by the entourages

$$Q_n = \bigvee \Big\{ (-,q) \oplus (p,-) \mid p,q \in \mathbb{Q}, 0 < q-p < \frac{1}{n} \Big\} \qquad (n \in \mathbb{N}),$$

as we are going to see.

Remark 3.1. Note that

$$Q_n = <\bigcup \Big\{ (-,q) \oplus (p,-) \mid p,q \in \mathbb{Q}, 0 < q-p < \frac{1}{n} \Big\}>$$

and, as can be easily proved using the fact that $\mathbb{Q}$ is dense in itself,

$$((r,s),(t,u)) \in \bigcup_{0<q-p<\frac{1}{n}} (-,q) \oplus (p,-)$$

if and only if $s - t < \frac{1}{n}$. In the sequel we shall denote this union by Q'_n (so $Q_n = <Q'_n>$).

Lemma 3.2. *Let $p < q$. Then, for each natural $n > \frac{1}{q-p}$, we have:*

(a) $st_1((-,p), Q_n) \le (-,q)$;
(b) $st_2((q,-), Q_n) \le (p,-)$.

Proof. (a) By (S2) and Remark 3.1 we only need to show that

$$\bigvee \{ (r,s) \mid ((r,s),(\alpha,\beta)) \in Q'_n, (\alpha,\beta) \wedge (-,p) \neq 0 \} \le (-,q).$$

So let $((r,s),(\alpha,\beta)) \in Q'_n$ such that $(\alpha,\beta) \wedge (-,p) \neq 0$. This means that $s - \alpha < \frac{1}{n}$ and $\alpha < p$. Therefore, $q - \alpha > q - p > \frac{1}{n} > s - \alpha$, which implies $s < q$.
(b) Similar. □

Lemma 3.3. *Let $p_i < q_i$ $(i \in I)$. Then, for each $n \in \mathbb{N}$, we have:*

(a) $st_1(\bigvee_{i\in I}(p_i,q_i), Q_n) = st_1(\bigvee_{i\in I}(-,q_i), Q_n)$.
(b) $st_2(\bigvee_{i\in I}(p_i,q_i), Q_n) = st_2(\bigvee_{i\in I}(p_i,-), Q_n)$.

Proof. (a) The inequality "$\leq$" is obvious by (S1). The reverse inequality follows from the fact that, for any $((r,s),(\alpha,\beta)) \in Q'_n$ satisfying

$$(\alpha,\beta) \wedge \bigvee_{i\in I}(-,q_i) \neq 0,$$

there exists $j \in I$ such that $\alpha < q_j$, and therefore $((r,s),(\alpha,q_j))$, which belongs to $Q'_n \subseteq Q_n$, is such that

$$(\alpha,q_j) \wedge \bigvee_{i\in I}(p_i,q_i) \geq (\alpha,q_j) \wedge (p_j,q_j) \neq 0.$$

(b) Similar. □

Proposition 3.4. *For each $n \in \mathbb{N}$, we have:*

(a) *Q_n is an entourage of $\mathfrak{L}(\mathbb{R})$.*
(b) *$Q_{n+1} \subseteq Q_n$.*
(c) *$Q_{2n} \circ Q_{2n} \subseteq Q_n$.*

Proof. (a) Since $((p,q),(p,q)) \in Q_n$ whenever $0 < q-p < \frac{1}{n}$, it suffices to check that

$$\bigvee\{(p,q) \mid 0 < q-p < \frac{1}{n}\} = 1.$$

By (R2), any (r,s) is the join of some $(p_1,q_1),\ldots,(p_m,q_m)$ where

$$p_1 = r < p_2 < q_1 < p_3 < \cdots < p_m < q_{m-1} < q_m = s$$

and $0 < q_i - p_i < \frac{1}{n}$. Thus, by (R4),

$$1 = \bigvee_{r,s\in\mathbb{Q}}(r,s) = \bigvee_{0<q-p<\frac{1}{n}}(p,q).$$

(b) Trivial.
(c) By Lemma 1.1 it suffices to check that $Q'_{2n} \circ Q'_{2n} \subseteq Q_n$. Let $((p_1,q_1),(p_2,q_2))$ and $((p_2,q_2),(p_3,q_3))$ belong to Q'_{2n} with $p_2 < q_2$. Then, by Remark 3.1, $q_1 - p_2 < \frac{1}{2n}$ and $q_2 - p_3 < \frac{1}{2n}$. Therefore $q_1 - p_2 + q_2 - p_3 < \frac{1}{n}$ which implies that $((-,q_1-p_2+q_2),(p_3,-)) \in Q'_n$. But

$$((p_1,q_1),(p_3,q_3)) \leq ((-,q_1-p_2+q_2),(p_3,-)),$$

since $q_2 - p_2 > 0$. Hence $((p_1,q_1),(p_3,q_3)) \in Q'_n \subseteq Q_n$. □

By (a) and (b) of the above proposition, the Q_n $(n \in \mathbb{N})$ form a filter base of entourages of $\mathfrak{L}(\mathbb{R})$, which, by (c), satisfies the square refinement property. Let $\mathcal{Q}$ be the corresponding filter.

Corollary 3.5. $(\mathfrak{L}(\mathbb{R}), \mathcal{Q})$ *is a quasi-uniform frame whose underlying biframe is the biframe of reals.*

Proof. It remains to prove the admissibility, or equivalently, that

$$(\mathfrak{L}(\mathbb{R}), \mathfrak{L}_1(\mathbb{R}), \mathfrak{L}_2(\mathbb{R})) \in \mathsf{BiFrm}.$$

We prove this by showing that $\mathfrak{L}_1(\mathbb{R}) = \mathfrak{L}_u(\mathbb{R})$ and $\mathfrak{L}_2(\mathbb{R}) = \mathfrak{L}_l(\mathbb{R})$.

By Lemma 3.2(a), $(-,p) \overset{\mathcal{Q}}{\lhd}_1 (-,q)$ whenever $p < q$, so, for each $(-,q) \in \mathfrak{L}_u(\mathbb{R})$, we have

$$(-,q) = \bigvee_{p<q} (-,p) \leq \bigvee\{x \in \mathfrak{L}(\mathbb{R}) \mid x \overset{\mathcal{Q}}{\lhd}_1 (-,q)\} \leq (-,q).$$

This shows the inclusion $\mathfrak{L}_u(\mathbb{R}) \subseteq \mathfrak{L}_1(\mathbb{R})$.

In order to show the reverse inclusion let $x \in \mathfrak{L}_1(\mathbb{R})$. Then

$$x = \bigvee\{y \in \mathfrak{L}(\mathbb{R}) \mid y \overset{\mathcal{Q}}{\lhd}_1 x\}.$$

But $y = \bigvee_{i\in I}(p_i, q_i)$ for some pairs (p_i, q_i), $p_i < q_i$. Therefore, by Lemma 3.3(a), $y \leq \bigvee_{i\in I}(-,q_i) \overset{\mathcal{Q}}{\lhd}_1 x$, and consequently,

$$x \leq \bigvee\{z \in \mathfrak{L}_u(\mathbb{R}) \mid z \overset{\mathcal{Q}}{\lhd}_1 x\} \leq x,$$

which shows that $x \in \mathfrak{L}_u(\mathbb{R})$.

Finally, the equality $\mathfrak{L}_2(\mathbb{R}) = \mathfrak{L}_l(\mathbb{R})$ may be shown analogously, using assertions (b) of Lemmas 3.2 and 3.3. □

Recall from [11] that a quasi-uniform frame $(L, \mathcal{E})$ is *complete* if every dense surjection $(M, \mathcal{F}) \to (L, \mathcal{E})$ is an an isomorphism. We end this section with the proof that $\mathfrak{L}(\mathbb{R})$ is complete in its quasi-metric quasi-uniformity.

Proposition 3.6. $(\mathfrak{L}(\mathbb{R}), \mathcal{Q})$ *is complete.*

Proof. Let

$$h : (M, \mathcal{E}) \to (\mathfrak{L}(\mathbb{R}), \mathcal{Q})$$

be a dense uniform surjection. Since h is dense, we show h is an isomorphism by simply exhibiting a right inverse g for it. Let $g(p,q) = h_*(p,q)$, h_* the right adjoint of h. By the properties of h_* and of dense surjections, this turns the conditions (R1)-(R4) into identities in M (we omit the details, that are straightforward) and therefore, it defines a frame homomorphism $g : \mathfrak{L}(\mathbb{R}) \to M$. This gives us the right inverse for h, as $hh_* = id$ because h is onto. □

4. The semicontinuous quasi-uniformity of a frame

Let L be a frame and let $\mathcal{S}$ be the collection of all upper semicontinuous functions

$$h : \mathcal{L}_u(\mathbb{R}) \to \nabla L$$

such that $\bigvee_{p\in\mathbb{Q}} \neg h(-,p) = 1$. (Obviously, each upper characteristic function h_x^u belongs to $\mathcal{S}$.) Then, by Proposition 2.5, each h extends to a biframe map $\overline{h} : \mathcal{L}(\mathbb{R}) \to \mathfrak{C}L$ given by

$$\begin{aligned} \overline{h}(-,p) &= h(-,p), \\ \overline{h}(p,-) &= \bigvee_{q>p} \neg h(-,q). \end{aligned}$$

For each $h \in \mathcal{S}$ and each $n \in \mathbb{N}$ let

$$E_{h,n} = \bigvee_{0<q-p<\frac{1}{n}} \overline{h}(-,q) \oplus \overline{h}(p,-).$$

Lemma 4.1. *For any $h_1, \ldots, h_k \in \mathcal{S}$, $n_1, \ldots, n_k \in \mathbb{N}$ and $\theta \in \mathfrak{C}L$,*

$$st_1(\theta, \bigcap_{i=1}^{k} E_{h_i,n_i}) \in \nabla L.$$

Proof. By Lemma 1.1(b), we have

$$\begin{aligned} st_1(\theta, \bigcap_{i=1}^{k} E_{h_i,n_i}) &= st_1(\theta, \bigcap_{i=1}^{k} E'_{h_i,n_i}) \\ &= \bigvee\{\alpha \mid (\alpha,\beta) \in \bigcap_{i=1}^{k} E'_{h_i,n_i}, \beta \wedge \theta \neq 0\}, \end{aligned}$$

where E'_{h_i,n_i} denotes the union $\bigcup_{0<q-p<\frac{1}{n_i}} (\overline{h_i}(-,q) \oplus \overline{h_i}(p,-))$. But for each such pair (α,β) we have:

$$(\alpha,\beta) \leq \overline{h_i}(-,q_i) \oplus \overline{h_i}(p_i,-)$$

for some $0 < q_i - p_i < \frac{1}{n_i}$, thus $\alpha \leq \bigwedge_{i=1}^{k} \overline{h_i}(-,q_i)$; on the other hand,

$$(\bigwedge_{i=1}^{k} \overline{h_i}(-,q_i), \beta) \in \bigcap_{i=1}^{k} E'_{h_i,n_i}.$$

Hence $st_1(\theta, \bigcap_{i=1}^{k} E_{h_i,n_i})$ is equal to

$$\bigvee\{\bigwedge_{i=1}^{k}\overline{h_i}(-,q_i) \mid \exists\, p_i \in \mathbb{Q} : 0 < q_i - p_i < \frac{1}{n_i}, \bigwedge_{i=1}^{k}\overline{h_i}(-,q_i) \wedge \theta \neq 0\},$$

which clearly belongs to ∇L. $\square$

Further, for each upper characteristic function h_x^u ($x \in L$) and each $n \in \mathbb{N}$, we have:

Lemma 4.2.

(a) $st_1(\nabla_x, E_{h_x^u,n}) = \nabla_x$;
(b) $st_2(\Delta_x, E_{h_x^u,n}) = \Delta_x$.

Proof. The proof follows immediately from the definition of h_x^u and properties (S2) and (S3) in Section 1. $\square$

Proposition 4.3. $\{E_{h,n} \mid h \in \mathcal{S}, n \in \mathbb{N}\}$ *is a subbase for a compatible quasi-uniformity* $\mathcal{E}_{\mathcal{S}}$ *on* $\mathfrak{C}L$.

Proof. Since each $E_{h,n}$ coincides with $(\overline{h} \oplus \overline{h})(Q_n)$, it follows immediately from Proposition 3.4 that the $E_{h,n}$ form a filter base of entourages satisfying the square refinement property.

Let $\theta \in \mathfrak{C}L$. Then $\theta = \bigvee_{i\in I}(\nabla_{x_i} \wedge \Delta_{y_i})$ for some $x_i, y_i \in L$. Lemma 4.2 implies that $\nabla_{x_i} \overset{\mathcal{E}_{\mathcal{S}}}{\triangleleft_1} \nabla_{x_i}$ and $\Delta_{y_i} \overset{\mathcal{E}_{\mathcal{S}}}{\triangleleft_2} \Delta_{y_i}$. Consequently, each ∇_{x_i} belongs to $(\mathfrak{C}L)_1$ and each Δ_{y_i} belongs to $(\mathfrak{C}L)_2$ and we may conclude that $\mathfrak{C}L$ is generated by $(\mathfrak{C}L)_1$ and $(\mathfrak{C}L)_2$, that is, $(\mathfrak{C}L, (\mathfrak{C}L)_1, (\mathfrak{C}L)_2) \in \mathsf{BiFrm}$, and we have the admissibility condition.

Finally, the compatibility: $\nabla L \subseteq (\mathfrak{C}L)_1$ again by Lemma 4.2(a). The reverse inclusion $(\mathfrak{C}L)_1 \subseteq \nabla L$ follows immediately from Lemma 4.1, because for each $\theta \in (\mathfrak{C}L)_1$,

$$\theta = \bigvee\{\alpha \in \mathfrak{C}L \mid \alpha \overset{\mathcal{E}_{\mathcal{S}}}{\triangleleft_1} \theta\}$$

and $\alpha \overset{\mathcal{E}_{\mathcal{S}}}{\triangleleft_1} \theta$ means that there exist $h_1, \ldots, h_k \in \mathcal{S}$ and $n_1, \ldots, n_k \in \mathbb{N}$ such that $\alpha \leq st_1(\alpha, \bigcap_{i=1}^{k} E_{h_i,n_i}) \leq \theta$. $\square$

It is obvious that $\mathcal{E}_{\mathcal{S}}$ is the coarsest quasi-uniformity on $\mathfrak{C}L$ for which each biframe map $h : \mathcal{L}(\mathbb{R}) \to \mathfrak{C}L$ is a uniform homomorphism $h : (\mathcal{L}(\mathbb{R}), \mathcal{Q}) \to (\mathfrak{C}L, \mathcal{E})$. We call it the *semicontinuous quasi-uniformity* for L and denote it by $\mathcal{SC}$.

Corollary 4.4. *Let $h : \mathfrak{C}L \to \mathfrak{C}M$ be a biframe homomorphism, let $\mathcal{SC}(L)$ denote the semicontinuous quasi-uniformity on L and let $\mathcal{SC}(M)$ be the corresponding quasi-uniformity on M. Then h is a uniform homomorphism.*

Proof. Let $E_{g,n} \in \mathcal{SC}(L)$, for some upper semicontinuous real function g on L and $n \in \mathbb{N}$. Then, we have

$$\begin{aligned}(h \oplus h)(E_{g,n}) &= (h \oplus h)\Big(\bigvee_{0<q-p<\frac{1}{n}} \overline{g}(-,q) \oplus \overline{g}(p,-)\Big) \\ &= \Big(\bigvee_{0<q-p<\frac{1}{n}} h\overline{g}(-,q) \oplus h\overline{g}(p,-)\Big).\end{aligned}$$

But, evidently, $h\overline{g}$ is the biframe extension $\overline{h_1 g}$ of the upper semicontinuous real function $h_1 g : \mathcal{L}(\mathbb{R}) \to M$ (where h_1 denotes the restriction of h to ∇L, regarded as a frame homomorphism from L to M). Hence

$$(h \oplus h)(E_{g,n}) = E_{h_1 g,n} \in \mathcal{SC}(M). \qquad \square$$

It is also clear from its proof that Proposition 4.3 may be generalized to any collection $\mathcal{S}$ containing all upper characteristic functions.

Corollary 4.5. *Let $\mathcal{S}$ be a collection of upper semicontinuous real functions $h : \mathcal{L}_u(\mathbb{R}) \to \nabla L$ such that $\bigvee_{p\in\mathbb{Q}} \neg h(-,p) = 1$, containing all upper characteristic functions h_x^u ($x \in L$). Then $\{E_{h,n} \mid h \in \mathcal{S}, n \in \mathbb{N}\}$ is a subbase for a compatible quasi-uniformity on $\mathfrak{C}L$.* $\square$

Proposition 4.3 and its Corollary 4.5 are the pointfree version of results from [20].

Example 4.6. Recall from [5] that the *Frith quasi-uniformity* $\mathcal{F}$ of $\mathfrak{C}L$ is the quasi-uniformity with subbase $\{(\nabla_x \oplus 1) \vee (1 \oplus \Delta_x) \mid x \in L\}$. This is the localic analogue of the Pervin quasi-uniformity. For each upper characteristic function h_x^u,

$$E_{h_x^u,n} = \bigvee_{0<q-p<\frac{1}{n}} (h_x^u(-,q) \oplus h_x^l(p,-)) = (\nabla_x \oplus 1) \vee (1 \oplus \Delta_x).$$

Indeed: $h_x^u(-,q) = 1$ implies $h_x^l(p,-) \leq \Delta_x$ and $h_x^l(p,-) = 1$ implies $h_x^u(-,q) \leq \nabla_x$, for every those p, q; on the other hand, there exist p, q for which

$$h_x^u(-,q) \oplus h_x^l(p,-) = \nabla_x \oplus 1$$

and there exist p, q for which

$$h^u_x(-,q) \oplus h^l_x(p,-) = 1 \oplus \Delta_x.$$

Thus, for $\mathcal{S} = \{h^u_x \mid x \in L\}$, $\mathcal{E}_\mathcal{S}$ and the Frith quasi-uniformity $\mathcal{F}$ have a common subbase. Hence $\mathcal{E}_\mathcal{S} = \mathcal{F}$.

We end this section by showing that $\mathcal{SC}$ is transitive and that it can be obtained by our construction of [5]. This is the pointfree version of a theorem of Fletcher and Lindgren [7].

Recall from [5] that a *spectrum cover* of L is a cover $A = \{a_n \mid n \in \mathbb{Z}\}$ of L such that $a_n \leq a_{n+1}$ for each $n \in \mathbb{Z}$, and $\bigvee_{n\in\mathbb{Z}} \Delta_{a_n} = 1$ (which implies, in particular, that $\bigwedge_{n\in\mathbb{Z}} a_n = 0$). As we proved in [5], the collection $\mathcal{A}$ of spectrum covers of L is an example of a family of interior-preserving covers, for which the following general procedure works. For each $A \in \mathcal{A}$, let

$$R_A := \bigcap_{a\in A} (\nabla_a \oplus 1) \vee (1 \oplus \Delta_a)$$

and let $\mathcal{E}_\mathcal{A}$ be the filter of entourages of $\mathfrak{C}L$ generated by $\{R_A \mid A \in \mathcal{A}\}$. Then $\mathcal{E}_\mathcal{A}$ is a compatible quasi-uniformity on $\mathfrak{C}L$, satisfying $\mathcal{L}_1(\mathcal{E}_\mathcal{A}) = \nabla L$ and $\mathcal{L}_2(\mathcal{E}_\mathcal{A}) = \Delta L$.

Here is a proof of the result announced in [5] that this quasi-uniformity is precisely the semicontinuous quasi-uniformity.

Theorem 4.7. *Let $\mathcal{A}$ be the collection of all spectrum covers of L. Then $\mathcal{E}_\mathcal{A} = \mathcal{SC}$.*

Proof. Let $\mathcal{S}$ be the collection of all upper semicontinuous real functions $h : \mathcal{L}_u(\mathbb{R}) \to \nabla L$ such that $\bigvee_{p\in\mathbb{Q}} \neg h(-,p) = 1$. It suffices to show that $\{R_A \mid A \in \mathcal{A}\}$ and $\{E_{h,n} \mid h \in \mathcal{S}, n \in \mathbb{N}\}$ are equivalent subbases.

Let $A = \{a_n \mid n \in \mathbb{Z}\} \in \mathcal{A}$. For each $p \in \mathbb{Q}$ let $n(p)$ be the largest integer contained in q. Then, immediately, $h_A : \mathcal{L}_u(\mathbb{R}) \to \mathfrak{C}L$ given by $h_A(-,p) = \nabla_{a_{n(p)}}$ belongs to $\mathcal{S}$. It is also easy to see that

$$E_{h_A,1} = \bigvee_{n\in\mathbb{Z}} (\nabla_{a_n} \oplus \Delta_{a_{n-1}}) \subseteq \bigcap_{n\in\mathbb{Z}} ((\nabla_{a_n} \oplus 1) \vee (1 \oplus \Delta_{a_{n-1}})) = R_A.$$

Let $E_{h,m} \in \mathcal{SC}$. Then $\bigvee_{n\in\mathbb{Z}} \neg h(-,\frac{n}{2m}) = 1$. Therefore, considering, for each $n \in \mathbb{Z}$, the $a_n \in L$ such that $h(-,\frac{n}{2m}) = \nabla_{a_n}$ we get a spectrum cover

$A = \{a_n \mid n \in \mathbb{Z}\}$ of L. Now it suffices to check that $R_A \subseteq E_{h,m}$. So, let

$$(\alpha, \beta) \in R_A = \bigcap_{n\in\mathbb{Z}} \Big((h(-, \frac{n}{2m}) \oplus 1) \cup (1 \oplus \neg h(-, \frac{n}{2m})) \Big).$$

This means that, for some partition $\mathbb{Z}_1 \cup \mathbb{Z}_2$ of $\mathbb{Z}$, we have

$$\alpha \leq \bigwedge_{n\in\mathbb{Z}_1} h(-, \frac{n}{2m})$$

and

$$\beta \leq \bigwedge_{n\in\mathbb{Z}_2} \neg h(-, \frac{n}{2m}) = \neg(\bigvee_{n\in\mathbb{Z}_2} h(-, \frac{n}{2m})) = \neg h(\bigvee_{n\in\mathbb{Z}_2} (-, \frac{n}{2m})).$$

Then, in order to prove that $(\alpha, \beta) \in E_{h,m}$, it remains to show that

$$\bigwedge_{n\in\mathbb{Z}_1} h(-, \frac{n}{2m}) \leq h(-, q)$$

and

$$\neg h(\bigvee_{n\in\mathbb{Z}_2} (-, \frac{n}{2m})) \leq h(p, -),$$

for some (p, q) such that $0 < q - p < \frac{1}{m}$.

If $\mathbb{Z}_2$ has a greatest element $\overline{n}$, $\overline{n} + 1 \in \mathbb{Z}_1$ and

$$\bigwedge_{n\in\mathbb{Z}_1} h(-, \frac{n}{2m}) \leq h(-, \frac{\overline{n}+1}{2m}).$$

Take $q = \frac{\overline{n}+1}{2m}$ and $p = \frac{\overline{n}}{2m} - \epsilon$ for some rational $\epsilon \in]0, \frac{1}{2m}[$. Clearly, $0 < q - p < \frac{1}{m}$ and

$$\neg h(\bigvee_{n\in\mathbb{Z}_2} (-, \frac{n}{2m})) = \neg h(-, \frac{\overline{n}}{2m}) \leq h(p, -)$$

because

$$h(p, -) \vee h(-, \frac{\overline{n}}{2m}) = h((p, -) \vee (-, \frac{\overline{n}}{2m}) = h(1) = 1.$$

If $\mathbb{Z}_2$ has no greatest element, we have $\neg h(\bigvee_{n\in\mathbb{Z}_2}(-, \frac{n}{2m})) = \neg h(1) = 0$, which implies $\beta = 0$. Then $(\alpha, \beta) \in E_{h,m}$ trivially. □

Corollary 4.8. *$\mathcal{SC}$ is a transitive quasi-uniformity.* □

Remark 4.9. It is clear that we can substitute the Skula biframe $\mathfrak{C}L$, in every result of this section, by a general strictly zero-dimensional biframe $B = (L_0, L_1, L_2)$ (the proofs could be effected in a perfect similar way). For instance, if L_1 is the part of B whose elements are all complemented with complements in L_2, Corollary 4.5 could be formulated in the following way:

Let $\mathcal{S}$ be a collection of upper semicontinuous real functions $h : \mathcal{L}_u(\mathbb{R}) \to L_1$ such that $\bigvee_{p\in\mathbb{Q}} \neg h(-,p) = 1$, containing all upper characteristic functions h^u_x ($x \in L_1$). Then $\{E_{h,n} \mid h \in \mathcal{S}, n \in \mathbb{N}\}$ is a subbase for a quasi-uniformity on L_0, compatible with L_1.

5. Some consequences

We say that an upper semicontinuous real function $h : \mathcal{L}_u(\mathbb{R}) \to L$ is *bounded* if $h(-,p) = 1$ for some $p \in \mathbb{Q}$. Since each upper characteristic function is bounded, the discussion in Example 4.6 immediately leads to the following result:

Proposition 5.1. *Let $\mathcal{S}$ be the collection of all bounded upper semicontinuous real functions on L. Then $\{E_{h,n} \mid h \in \mathcal{S}, n \in \mathbb{N}\}$ is a subbase for $\mathcal{F}$.* □

Remark 5.2. Note that the proof of the corresponding classical result (in [12], Theorem 2, or [8], Proposition 2.10) is not so direct and simple as the proof above is.

Lemma 5.3. *If $(\mathfrak{C}L, \mathcal{E})$ is a totally bounded quasi-uniform frame then every uniform homomorphism $h : (\mathcal{L}(\mathbb{R}), \mathcal{Q}) \to (\mathfrak{C}L, \mathcal{E})$ is bounded.*

Proof. Let $h : (\mathcal{L}(\mathbb{R}), \mathcal{Q}) \to (\mathfrak{C}L, \mathcal{E})$ be a uniform homomorphism. For each $n \in \mathbb{N}$, $(h \oplus h)(Q_n) \in \mathcal{E}$, so there exists a finite cover $\{\alpha_1, \ldots, \alpha_k\}$ of $\mathfrak{C}L$ such that

$$\bigvee_{i=1}^{k} (\alpha_i \oplus \alpha_i) \le (h \oplus h)(Q_n).$$

For each $i \in \{1, \ldots, k\}$, $\bigvee_{p\in\mathbb{Q}} (h(-,p) \wedge \alpha_i) = \alpha_i \neq 0$, thus there exists $p_i \in \mathbb{Q}$ such that $h(-,p_i) \wedge \alpha_i \neq 0$. Consequently,

$$\alpha_i \le st_1(h(-,p_i), (h \oplus h)(Q_n)),$$

which, using property (S4) of Section 1, implies that

$$\alpha_i \le h(st_1((-,p_i), Q_n)) \le h(-,q_i)$$

for every $q_i > p_i$. Hence $1 \leq \bigvee_{i=1}^{k} h(-, q_i)$ for every $q_i > p_i$. Choose $q_i \in \mathbb{Q}$ ($i = 1, \ldots, k$) such that $q_i > p_i$ and let $q \in \mathbb{Q}$ be the largest of these q_i. Immediately, $h(-, q) = 1$. Similarly, we may guarantee the existence of $p \in \mathbb{Q}$ such that $h(p, -) = 1$. Then $h(p, q) = h(p, -) \wedge h(-, q) = 1$ and h is bounded. □

This allows us to get the pointfree counterpart of a theorem of Hunsaker and Lindgren [12].

Theorem 5.4. *Let $(\mathfrak{C}L, \mathcal{E})$ be a totally bounded quasi-uniform frame. Then there exists a collection $\mathcal{S}$ of bounded upper semicontinuous real functions $h : \mathfrak{L}_u(\mathbb{R}) \to L$ such that $\{E_{h,n} \mid h \in \mathcal{S}, n \in \mathbb{N}\}$ is a subbase for $\mathcal{E}$.*

Proof. Let $(\mathfrak{C}L, \mathcal{E})$ be a totally bounded quasi-uniform frame. Every uniform homomorphism

$$g : (\mathcal{L}(\mathbb{R}), \mathcal{Q}) \to (\mathfrak{C}L, \mathcal{E}),$$

which is bounded by Lemma 5.3, restricts to a bounded upper semicontinuous $h : \mathfrak{L}_u(\mathbb{R}) \to L$. Let $\mathcal{S}$ be the collection of all these maps. By Corollary 4.5, $\{E_{h,n} \mid h_1 \in \mathcal{S}, n \in \mathbb{N}\}$ is a subbase for a quasi-uniformity $\mathcal{E}_{\mathcal{S}}$ on $\mathfrak{C}L$. Evidently, $E_{h,n} = (g \oplus g)(Q_n) \in \mathcal{E}$, because g is uniform, thus

$$\{E_{h,n} \mid h \in \mathcal{S}, n \in \mathbb{N}\}$$

is a subbase for $\mathcal{E}$. □

Remark 5.5. Again, by putting any strictly zero-dimensional biframe (L_0, L_1, L_2) in the place of the Skula biframe $\mathfrak{C}L$, we could get, similarly, the following:

If $(L_0, \mathcal{E})$ is a totally bounded quasi-uniform frame, whose induced biframe (L_0, L_1, L_2) is strictly zero-dimensional, then there exists a collection $\mathcal{S}$ of bounded upper semicontinuous real functions $h : \mathfrak{L}_u(\mathbb{R}) \to L_1$ such that $\{E_{h,n} \mid h \in \mathcal{S}, n \in \mathbb{N}\}$ is a subbase for $\mathcal{E}$.

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