Pré-Publicações do Departamento de Matemática
Universidade de Coimbra
Preprint Number 03–28

A $q-$SAMPLING THEOREM RELATED TO THE $q-$HANKEL TRANSFORM

L. D. ABREU

ABSTRACT: A $q-$version of the sampling theorem is derived using the $q-$Hankel transform introduced by Koornwinder and Swarttouw. The sampling points are the zeros of the third Jackson $q-$Bessel function.

KEYWORDS: Sampling theorem, reproducing kernel, $q-$Bessel functions, $q-$Hankel transform.
AMS SUBJECT CLASSIFICATION (2000): Primary 33D15, 33D05; Secondary 94A20.

1. Introduction

The classical sampling theorem asserts that every function f in the Paley-Wiener space defined by

$$PW = \left\{ f \in L^2(\mathbf{R}) : f(x) = \frac{1}{\sqrt{2\pi}} \int_{-\pi}^{\pi} e^{ixt} u(t)\, dt, u \in L^2(0,1) \right\}$$

can be represented by the interpolation series

$$f(x) = \sum_{n=-\infty}^{\infty} f(n) \frac{\sin \pi (x-n)}{\pi (x-n)}$$

Hardy´s proof of this fact [4] used properties from the kernel of the Fourier transform. Relying on properties of the Hankel´s transform kernel, Higgins [5] used the theory of reproducing kernels to obtain a sampling theorem where the sampling points are the zeros of the Bessel function. In this note, a $q-$Bessel analogue of the sampling theorem is derived by considering the kernel of the $q-$Hankel transform, H_q^ν, introduced by Koornwinder and Swarttouw [8]

$$\left(H_q^\nu f\right)(x) = \int_0^\infty (xt)^{\frac{1}{2}} J_\nu^{(3)}\left(xt; q^2\right) f(t)\, d_q t$$

Partial financial assistance by Fundação para a Ciência e Tecnologia and Centro de Matemática da Universidade de Coimbra.

where $J_\nu^{(3)}$ denotes the third Jackson $q-$Bessel function defined by the power series

$$J_\nu^{(3)}(x;q)=\frac{(q^{\nu+1};q)_\infty}{(q;q)_\infty}x^\nu\sum_{n=0}^{\infty}(-1)^n\frac{q^{\frac{n(n+1)}{2}}}{(q^{\nu+1};q)_n(q;q)_n}x^{2n} \tag{1.1}$$

with $0<q<1$, $(a;q)_n=(1-a)(1-aq)\ldots\left(1-aq^{n-1}\right)$ and $(a;q)_\infty=\lim\limits_{n\to\infty}(a;q)_n$. We are using the definition of the $q-$integral. The $q-$integral in the interval $(0,1)$ is defined as

$$\int_0^1 f(t)\,d_qt=(1-q)\sum_{n=0}^{\infty}f(q^n)\,q^n \tag{1.2}$$

and in the interval $(0,\infty)$ as

$$\int_0^\infty f(t)\,d_qt=(1-q)\sum_{n=-\infty}^{\infty}f(q^n)\,q^n \tag{1.3}$$

The sampling points will turn out to be $qj_{n\nu}\left(q^2\right)$, where $j_{n\nu}\left(q^2\right)$ is the n^{th} zero of $J_\nu^{(3)}\left(x;q^2\right)$. In [2] it was proved that $j_{n\nu}\left(q^2\right)=q^{-n+\epsilon_n},0<\epsilon_n<1$. This shows how big is the spacing between the sampling points.

2. Preliminaries on reproducing kernels

Let H be a class of complex valued functions, defined in a set $X\subset \mathbf{C}$, such that X is a Hilbert space with the norm of $L^2(X,\mu)$. $g(s,x)$ is a *reproducing kernel* to H if

i) $g(t,x)\in H$ for every $x\in X$;

ii) $f(x)=\langle f(t),g(t,x)\rangle$ for every $f\in H$, $x\in X$.

The next result lists the properties of Hilbert spaces with reproducing kernel that will be used in the remainder. Properties (a), (c) and (d) are proved in [5]. Property (b) is a well known property of the reproducing kernels, of primary importance, because it relates two different kinds of convergence. A proof of (b) can be found in [10], together with an introduction to the general theory.

Proposition 1. *In the Hilbert space $L^2[(a,b),\mu]$, an operator is defined by*

$$Ku=\langle K(x,t),u(t)\rangle_{L^2[(a,b),\mu]}$$

The following properties hold:

(a) If K^{-1} is bounded, the range of K, denoted by N, is a Hilbert space with reproducing kernel.

(b) If the sequence $\{f_n\}$ converges strongly to f in the norm of H, with reproducing kernel g, then $\{f_n\}$ converges pointwise in X to f. The convergence is uniform in every set of X where $g(x,x)$ is bounded.

(c) If K is an isometry, then $g(s,x) = \langle K(s,t), K(x,t)\rangle_{L^2[(a,b),\mu]}$.

(d) Let $\{f_n\}$ be a complete ortogonal sequence in H and (x_n) such that $f_n(x_m) = \delta_{nm}$. Then

$$f_n(t) = \frac{g(t,x_n)}{g(x_n,x_n)}$$

We will suppose $N \subset L^2(X,\mu)$, and this implies K bounded. K^{-1} is a transformation of N over $L^2[(a,b),\mu]$, also bounded.

3. A q-sampling theorem

We introduce a $q-$Bessel version of the Paley-Wiener space, and call it PW_q^ν:

$$PW_q^\nu = \left\{ f \in L_q^2(0,\infty) : f(x) = \int_0^1 (tx)^{\frac{1}{2}} J_\nu^{(3)}\left(xt;q^2\right) u(t)\, d_qt, u \in L_q^2(0,1) \right\} \tag{3.1}$$

The notation $L_q^2(0,1)$ stands for the Hilbert space associated to the measure of the $q-$integral in $(0,1)$. In [8] it was proved the inversion formula

$$f(t) = \int_0^\infty (xt)^{\frac{1}{2}} \left(H_q^\nu f\right)(x)\, J_\nu^{(3)}\left(xt;q^2\right) d_qx = \left(H_q^\nu\left(H_q^\nu f\right)\right)(t) \tag{3.2}$$

Let $f \in L_q^2(0,\infty)$ such that $\left(H_q^\nu f\right)(q^{-n}) = 0$, $n = 1,2,....$ Then $f \in PW_q^\nu$. To see this use the formula (3.2) and compare (1.2) and (1.3) to write f as an element of PW_q^ν.

Now, in the language of the preceding section, consider $X = (0,\infty)$, $(a,b) = (0,1)$ and the kernel $K(x,t) = (xt)^{\frac{1}{2}} J_\nu^{(3)}\left(xt;q^2\right)$. The corresponding operator K is

$$(Ku)(x) = \langle K(x,t), u(t)\rangle_{L_q^2(0,1)} = \int_0^1 (xt)^{\frac{1}{2}} J_\nu^{(3)}\left(xt;q^2\right) u(t)\, d_qt$$

By (3.2), H_q^ν is a self-inverse operator and consequently, an isometry. Thus, K is also an isometry. The range of K, N, is the set of functions $f \in L_q^2(0,\infty)$

such that $f = Ku$ for some $u \in L_q^2(0,1)$. By (3.1), $N = PW_q^\nu$. In the next Lemma, the reproducing kernel of the space PW_q^ν is evaluated.

Lemma 1. *The set PW_q^ν is a Hilbert space with reproducing kernel given by*

$$g(s,x) = (1-q)\,q^{v}\frac{(xs)^{\frac{1}{2}}\left[xJ_{\nu+1}^{(3)}\left(x;q^2\right)J_\nu^{(3)}\left(sq^{-1};q^2\right) - sJ_{\nu+1}^{(3)}\left(s;q^2\right)J_\nu^{(3)}\left(xq^{-1};q^2\right)\right]}{x^2-s^2} \tag{3.3}$$

Proof: By Proposition 1 (a), PW_q^ν is a space with reproducing kernel $g(s,x)$. From Proposition 1 (c), since K is an isometry,

$$g(s,x) = \langle K(s,t), K(x,t)\rangle_{L_q^2(0,1)} = \int_0^1 t\,(xs)^{\frac{1}{2}} J_\nu^{(3)}\left(xt;q^2\right)J_\nu^{(3)}\left(st;q^2\right)d_qt$$

In [7], the following formula was proved

$$\left(a^2-b^2\right)\int_0^z tJ_\nu^{(3)}\left(aqt;q^2\right)J_\nu^{(3)}\left(bqt;q^2\right)d_qt \tag{3.4}$$

$$= (1-q)\,q^{\nu-1}z\left[aJ_{\nu+1}^{(3)}\left(aqz;q^2\right)J_\nu^{(3)}\left(bz;q^2\right) - bJ_{\nu+1}^{(3)}\left(bqz;q^2\right)J_\nu^{(3)}\left(az;q^2\right)\right]$$

Setting $z=1$, $a = xq^{-1}$ and $b = sq^{-1}$ in (3.4), (3.3) follows. ■

The $q-$sampling theorem can now be stated and proved.

Theorem 1. *If $f \in PW_q^\nu$ then f has the unique representation*

$$f(x) = \sum_{n=1}^{\infty} f\left(qj_{n\nu}\left(q^2\right)\right)\frac{2\left(xqj_{n\nu}\left(q^2\right)\right)^{\frac{1}{2}}J_\nu^{(3)}\left(x;q^2\right)}{\frac{d}{dx}\left[J_\nu^{(3)}\left(x;q^2\right)\right]_{x=qj_{n\nu}(q^2)}\left(x^2-q^2j_{n\nu}^2\left(q^2\right)\right)} \tag{3.5}$$

where $\left(j_{n\nu}\left(q^2\right)\right)$ denotes the sequence of positive zeros of $J_\nu^{(3)}\left(x;q^2\right)$. The series converges uniformly in compact subsets of $(0,\infty)$.

Proof: Consider the sequence $\{f_n(x)\}$ defined by

$$f_n(x) = \left(xqj_{n\nu}\left(q^2\right)\right)^{\frac{1}{2}}J_\nu^{(3)}\left(qxj_{n\nu}\left(q^2\right);q^2\right)$$

It was proved in [1] that $\{f_n(x)\}$ is a complete orthogonal sequence in $L_q^2(0,1)$. Taking into account that K is an isometry, the sequence $(Kf_n)(x)$ is also orthogonal and complete in PW_q^ν. Now set

$$F_n(x) = \frac{(Kf_n)(x)}{(Kf_n)\left(qj_{n\nu}\left(q^2\right)\right)}$$

The orthogonality of $\{f_n(x)\}$ implies

$$F_n\left(qj_{m\nu}\left(q^2\right)\right)=\delta_{nm} \tag{3.6}$$

Proposition 1 (d) allows to write

$$F_n(x)=\frac{g\left(x,qj_{n\nu}\left(q^2\right)\right)}{g\left(qj_{n\nu}\left(q^2\right),qj_{n\nu}\left(q^2\right)\right)}$$

Substituting in (3.3) yields

$$F_n(x)=\frac{2\left(xqj_{n\nu}(q^2)\right)^{\frac{1}{2}}J_\nu^{(3)}\left(x;q^2\right)}{\frac{d}{dx}\left[J_\nu^{(3)}\left(x;q^2\right)\right]_{x=qj_{n\nu}(q^2)}\left(x^2-q^2j_{n\nu}^2(q^2)\right)}$$

$F_n(x)$ is an orthonormal complete sequence in N. Thus, every $f\in PW_q^\nu$ has a unique series expansion in the form

$$f(x)=\sum_{n=1}^{\infty}a_nF_n(x) \tag{3.7}$$

where a_n are the Fourier coefficients of f in $\{F_n(x)\}$. The series in (3.7) is convergent in the norm of $L_q^2(0,1)$ and also in the norm of PW_q^ν. The real-valued function $g(x,x)$ is continuous, thus bounded in every compact subset of $(0,\infty)$. It follows from Proposition 1 (b) that (3.7) converges uniformly in compact subsets of $(0,\infty)$. Finally, setting $x=qj_{n\nu}\left(q^2\right)$ in (3.7), (3.6) implies $f\left(qj_{m\nu}\left(q^2\right)\right)=a_m$ and thus, (3.7) can be written in the form (3.5). ∎

4. Application

The following formula is a consequence of the product representation for the classical Bessel function

$$\frac{\frac{d}{dx}J_\nu(x)}{J_\nu(x)}=2x\sum_{n=1}^{\infty}\frac{1}{j_{n\nu}^2-x^2}+\frac{\nu}{x} \tag{4.1}$$

Using the recurrence $x\frac{d}{dx}J_\nu(x)-\nu J_\nu(x)=-xJ_{\nu+1}(x)$, (4.1) becomes

$$\frac{J_{\nu+1}(x)}{J_\nu(x)}=-2x\sum_{n=1}^{\infty}\frac{1}{j_{n\nu}^2-x^2} \tag{4.2}$$

where $j_{n\nu}$ stands for the zeros of $J_\nu(x)$. In the case of the q-analogues of the Bessel function, this analysis cannot be done, for there are no formulas to establish a simple relation between a q-Bessel function and its derivative. While the $q-$analogue of (4.1) is very simple to derive from the Hadamard factorization theorem or using residues, the q-analogue of (4.2) is harder to obtain. In [6], Ismail studied the second Jackson q-Bessel function, $J_{\nu}^{(2)}(x;q)$, and found such $q-$analogue using the orthogonality measure of the modified q-Lommel polynomials associated to $J_{\nu}^{(2)}(x;q)$. Kvitsinsky [9] found a recurrence relation for the coefficients h_n in the identity

$$\frac{J_{\nu+1}^{(3)}(x;q)}{J_{\nu}^{(3)}(x;q)} = \sum_{n=1}^{\infty} h_n x^{2n-1} \tag{4.3}$$

In this section an explicit formula for the coefficients h_n will be obtained as a special case of the expansion of a particular function as a sampling series. Preliminary to this expansion, a $q-$integral formula connecting two $q-$Bessel functions of different order is established .

Lemma 2. *For $y>0$, $\nu>-\frac{1}{2}$ and $x\in\mathbf{R}$, the following relation holds*

$$\frac{(q;q)_\infty}{(q^y;q)_\infty} x^{-y} J_{\nu+y}^{(3)}(x;q) = \int_0^1 t^{\frac{\nu}{2}} \frac{(tq;q)_\infty}{(tq^y;q)_\infty} J_{\nu}^{(3)}(xt^{\frac{1}{2}};q)d_qt \tag{4.4}$$

Proof: The $q-$analogues of the gamma and the beta function will be critical in the proof. According to [3, 1.10], the $q-gama$ function, $\Gamma_q(x)$, is defined by

$$\Gamma_q(x) = \frac{(q;q)_\infty}{(q^x;q)_\infty}(1-q)^{1-x} \tag{4.5}$$

and the $q-$beta function, $\beta_q(x,y)$ by

$$\beta_q(x,y) = \frac{\Gamma_q(x)\Gamma_q(y)}{\Gamma_q(x+y)} \tag{4.6}$$

The $q-$beta function has the q-integral representation

$$\beta_q(x,y) = \int_0^1 t^{x-1}\frac{(tq;q)_\infty}{(tq^y;q)_\infty} d_qt,\ Re(x)>0,\ y\neq 0,-1,-2,... \tag{4.7}$$

Using the series representation (1.1) and the $q-$integral representation (4.7) it is easy to see that, if $\nu>-\frac{1}{2}$ and $y>0$,

$$\int_0^1 t^{\frac{\nu}{2}} \frac{(tq;q)_\infty}{(tq^y;q)_\infty} J_{\nu}^{(3)}(xt^{\frac{1}{2}};q)d_qt = x^{\nu}\sum_{k=0}^{\infty}(-1)^k \frac{q^{\frac{k(k+1)}{2}}}{(q;q)_k(q^{\nu+1};q)_k} x^{2k}\beta_q(k+\nu+1,y) \tag{4.8}$$

Now use (4.5) and (4.6) to express $\beta_q(k+\nu+1, y)$ as a quotient of infinite products. Then, some algebraic manipulations using the formula $(a;q)_\infty = (a;q)_n\,(aq^n;q)_\infty$ allow us to see that the right hand member of (4.8) is equal to the left hand member of identity (4.4). ■

Before moving to the next Theorem, it is convenient to point out that, from the definition (1.2), one can verify the relation:

$$\int_0^1 f(t^{\frac{1}{2}})d_{q^2}t = (1+q)\int_0^1 tf\,(t)\,d_qt \tag{4.9}$$

Theorem 2. *If $u > \nu > -\frac{1}{2}$, the following identity holds*

$$x^{\nu-u}\frac{J_u^{(3)}\left(x;q^2\right)}{J_\nu^{(3)}\left(x;q^2\right)} = -2\sum_{n=1}^{\infty}\frac{\left(qj_{n\nu}\left(q^2\right)\right)^{\nu-u+1}J_u^{(3)}\left(qj_{n\nu}\left(q^2\right);q^2\right)}{\frac{d}{dx}\left[J_\nu^{(3)}\left(x;q^2\right)\right]_{x=qj_{n\nu}(q^2)}\left(q^2j_{n\nu}^2\left(q^2\right)-x^2\right)} \tag{4.10}$$

Proof: Setting $y = u-\nu$ in (4.4) and replacing q by q^2, the result is, if $u > \nu$,

$$\frac{(q^2;q^2)_\infty}{(q^{2u-2\nu};q^2)_\infty}x^{\nu-u}J_u^{(3)}(x;q^2) = \int_0^1 t^{\frac{\nu}{2}}\frac{(tq^2;q^2)_\infty}{(tq^{2u-2\nu};q^2)_\infty}J_\nu^{(3)}(xt^{\frac{1}{2}};q^2)d_{q^2}t$$

Taking (4.9) into account, this can be rewritten as

$$\frac{(q^2;q^2)_\infty}{(q^{2u-2\nu};q^2)_\infty}x^{\nu-u}J_u^{(3)}(x;q^2) = (1+q)\int_0^1 t^{\nu+1}\frac{(t^2q^2;q^2)_\infty}{(t^2q^{2u-2\nu};q^2)_\infty}J_\nu^{(3)}(xt;q^2)d_qt \tag{4.11}$$

Considering

$$u\,(t) = t^{\nu+\frac{1}{2}}\frac{(1+q)\left(q^{2u-2\nu};q^2\right)_\infty\left(t^2q^2;q^2\right)_\infty}{\left(q^2;q^2\right)_\infty\left(t^2q^{2u-2\nu};q^2\right)_\infty}$$

relation (4.11) yields

$$x^{\nu-u+\frac{1}{2}}J_u^{(3)}\left(x;q^2\right) = \int_0^1 (tx)^{\frac{1}{2}}\,J_\nu^{(3)}(xt;q^2)u\,(t)\,d_{q^2}t$$

Thus,

$$f\,(x) = x^{\nu-u+\frac{1}{2}}J_u^{(3)}\left(x;q^2\right) \in PW_q^\nu$$

And it is possible to apply Theorem 1 to f. The result of this application is (4.10). ■

Taking $u = \nu + 1$ in (4.10) and replacing q^2 by q, the result is the analogue of (4.2) previously mentioned:

$$\frac{J_{\nu+1}^{(3)}(x;q)}{J_{\nu}^{(3)}(x;q)} = -2x\sum_{n=1}^{\infty}\frac{J_{\nu+1}^{(3)}\left(q^{\frac{1}{2}}j_{n\nu}(q);q\right)}{\frac{d}{dx}\left[J_{\nu}^{(3)}(x;q)\right]_{x=q^{\frac{1}{2}}j_{n\nu}(q)}}\frac{1}{qj_{n\nu}^{2}(q)-x^{2}} \tag{4.12}$$

Expanding $1/\left(j_{n\nu}^2(q) - x^2\right)$ in power series of x and substituting in (4.12), the coefficients h_n in (4.3) can be seen to be

$$h_n = \sum_{k=1}^{\infty}\frac{J_{\nu+1}^{(3)}\left(q^{\frac{1}{2}}j_{k\nu}(q);q\right)}{\frac{d}{dx}\left[J_{\nu}^{(3)}(x;q)\right]_{x=q^{\frac{1}{2}}j_{n\nu}(q)}}\left(\frac{1}{qj_{k\nu}^{2}(q)}\right)^{2n}$$

References

[1] L. D. Abreu, J. Bustoz *Complete sets of q-Bessel functions*, to appear in "Theory and Applications of Special Functions. A volume dedicated to Mizan Rahman", (eds. M. E. H. Ismail and E. Koelink), Developments in Mathematics, Kluwer Acad. Publ.

[2] L. D. Abreu, J. Bustoz, J. L. Cardoso, *The roots of the third Jackson q-Bessel function*, to appear in Internat. J. Math. Math. Sci.

[3] G. Gasper and M. Rahman, "*Basic Hypergeometric Series*," Cambridge University Press, Cambridge, UK, 1990.

[4] G. H. Hardy, *Notes on special sistems of orthogonal functions IV*, Proc. Cambridge Phil. Soc., 37 (1941), 331-348.

[5] J. R. Higgins, *An interpolation series associated with the Bessel-Hankel transform* , J. Lond. Math. Soc. **5**, (1972) 707-714

[6] M. E. H. Ismail, *The Zeros of Basic Bessel functions, the functions $J_{\nu+ax}(x)$, and associated orthogonal polynomials*, J. Math. Anal. Appl. **86** (1982), 1-19.

[7] H. T. Koelink, R. F. Swartouw , *On the zeros of the Hahn-Exton q-Bessel Function and associated q-Lommel polynomials,* J. Math. Anal. Appl. **186**, (1994), 690-710.

[8] T. H. Koornwinder, R. F. Swarttouw, *On q-analogues of the Fourier and Hankel transforms.* Trans. Amer. Math. Soc. 333 (1992), no. 1, 445–461.

[9] A. A. Kvitsinsky, *Spectral zeta functions for q-Bessel equations*, J. Phys. A: Math. Gen. **28**, 1753-1764 (1995), no. 6

[10] L. Máté, "*Hilbert space methods in science and engineering*". Adam Hilger, Ltd., Bristol, 1989.

L. D. Abreu

Departamento de Matemática, Universidade de Coimbra, 3001-454 Coimbra, PORTUGAL

E-mail address: daniel@mat.uc.pt