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ON FIEDLER'S CHARACTERIZATION OF TRIDIAGONAL MATRICES OVER ARBITRARY FIELDS

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Dedicated with friendship and admiration to Graciano N. de Oliveira.

ABSTRACT: M. Fiedler proved in [1] that the set of *real* n-by-n symmetric matrices A such that $\operatorname{rank}(A+D) \geq n-1$ for any real diagonal matrix D is the set of matrices PTP^T where P is a permutation matrix and T an irreducible tridiagonal matrix. We show that this result remains valid for arbitrary fields with some exceptions for 5-by-5 matrices over $\mathbb{Z}_3$.

1. Introduction

Let A be an n-by-n, irreducible, tridiagonal matrix with elements in a field $\mathbb{K}$. It is well known and easy to prove that $\operatorname{rank} A \geq n-1$: just delete the first row and the last column of A to obtain an upper triangular $(n-1)$-by-$(n-1)$ submatrix of A with nonzero diagonal elements. For every n-by-n diagonal matrix D, with elements in $\mathbb{K}$, $A+D$ is again an irreducible tridiagonal matrix and so A has the following property, which we call *Fiedler's Property*:
For every diagonal matrix D, rank$(A+D) \geq n-1$.

M. Fiedler proved in [1] the interesting fact that, over the reals, the only real symmetric matrices which have that property are, up to permutation of rows and the same permutation of columns, the irreducible tridiagonal matrices:

Theorem 1 (Fiedler's Characterization of Tridiagonal Matrices). *Let A be an n-by-n real symmetric matrix. We have rank$(A+D) \geq n-1$, for every n-by-n real diagonal matrix D, if and only if A is, up to permutation of rows and the same permutation of columns, an irreducible tridiagonal matrix.*

Note that permuting the rows of A is equivalent to pre-multiplication by an appropriate permutation matrix P while doing the same permutation of columns is equivalent to post-multiplication by P^T. Moreover the set of matrices P^TTP, where P is an n-by-n permutation matrix and T an irreducible tridiagonal matrix, is precisely the set of matrices whose graph is

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a path on n vertices. Therefore we can restate Theorem 1 saying that a real symmetric matrix A has the Fiedler's Property if and only if *there exists an n-by-n permutation matrix P such that PAP^T is an irreducible tridiagonal matrix* or equivalently if and only if *the graph of A is a path.*

Fiedler's Proof of Theorem 1 is highly analytical but W. C. Reinbolt and R. A. Shepherd in [4] presented a purely algebraic and combinatorial proof of that theorem (actually they presented two proofs of that sort). Although the authors of [4] state Theorem 1 in terms of real matrices, actually their proof is valid for any infinite field (in fact for fields with a sufficiently large number of elements). Our purpose here is to discuss the case of finite fields: Is Fiedler's Characterization of Tridiagonal Matrices valid for matrices over finite fields?

The answer to the above question is *no*! Over $\mathbb{Z}_3$ each one of the following matrices has Fiedler's Property:

$$F_1 = \begin{bmatrix} a_{11} & 1 & 1 & 1 & 1 \\ 1 & a_{22} & 1 & 2 & 2 \\ 1 & 1 & a_{33} & 1 & 2 \\ 1 & 2 & 1 & a_{44} & 1 \\ 1 & 2 & 2 & 1 & a_{55} \end{bmatrix}, \quad F_2 = \begin{bmatrix} a_{11} & 2 & 2 & 2 & 2 \\ 2 & a_{22} & 2 & 1 & 1 \\ 2 & 2 & a_{33} & 2 & 1 \\ 2 & 1 & 2 & a_{44} & 2 \\ 2 & 1 & 1 & 2 & a_{55} \end{bmatrix}, \quad (1)$$

$$F_3 = \begin{bmatrix} a_{11} & 1 & 1 & 1 & 2 \\ 1 & a_{22} & 2 & 2 & 2 \\ 1 & 2 & a_{33} & 1 & 1 \\ 1 & 2 & 1 & a_{44} & 2 \\ 2 & 2 & 1 & 2 & a_{55} \end{bmatrix}, \quad F_4 = \begin{bmatrix} a_{11} & 1 & 1 & 2 & 2 \\ 1 & a_{22} & 2 & 1 & 2 \\ 1 & 2 & a_{33} & 2 & 1 \\ 2 & 1 & 2 & a_{44} & 1 \\ 2 & 2 & 1 & 1 & a_{55} \end{bmatrix}. \quad (2)$$

for any choice of the diagonal elements a_{11}, a_{22}, a_{33}, a_{44}, $a_{55} \in \mathbb{Z}_3$ (see Sec. 5). (Note that the diagonal elements of each F_i are arbitrary and so we should have written something like $F_i(a_{11}, a_{22}, a_{33}, a_{44}, a_{55})$ instead of just F_i, but for simplicity we usually write F_i.)

But, amazingly, these are essentially the only exceptions; in fact our main result is the following:

Theorem 2. *Let $\mathbb{K}$ be any field and A an n-by-n symmetric matrix with elements in $\mathbb{K}$. We have rank$(A + D) \geq n - 1$ for any n-by-n diagonal matrix D with elements in $\mathbb{K}$ if and only if the graph of A is a path or*

$\mathbb{K} = \mathbb{Z}_3$, $n = 5$ and $A = PF_iP^T$, where P is a 5-by-5 permutation matrix and F_i, $i = 1,\ 2,\ 3,\ 4$, is one of the matrices of (1) and (2).

Like the proofs given in [4], our proof of Theorem 2 will be by induction; we have to treat first the case of some small values of n ($n \leq 4$, for arbitrary $\mathbb{K}$, and $n \leq 6$ for $\mathbb{K} = \mathbb{Z}_3$): this will be done in Proposition 3 and in Sections 4, 5 and 6; in the next section we present some basic properties of matrices with Fiedler's Property and in the Section 3 we consider matrices with zero non-diagonal elements. Finally in section 7 we complete the proof of Theorem 2.

We shall use the following notation:

We denote the set of all n-by-n matrices with elements in a field $\mathbb{K}$ by $\mathcal{M}_n(\mathbb{K})$ and the set of all n-by-n symmetric matrices with elements in $\mathbb{K}$ by $\mathcal{S}_n(\mathbb{K})$. The set of matrices in $\mathcal{S}_n(\mathbb{K})$ verifying Fiedler's Property will be denoted by $\mathcal{F}_n(\mathbb{K})$, that is

$$\mathcal{F}_n(\mathbb{K}) = \{A \in \mathcal{S}_n(\mathbb{K}) : \operatorname{rank}(A+D) \geq n-1 \text{ for any diagonal matrix } D \in \mathcal{M}_n(\mathbb{K})\}.$$

Sometimes we will refer to the elements of $\mathcal{F}_n(\mathbb{K})$ as Fiedler's matrices.

Let $i_1,\ i_2, \ldots,\ i_k$ integers with $1 \leq i_1 < i_2 < \cdots < i_k \leq n$ and $A \in \mathcal{M}_n(\mathbb{K})$. We denote by $A[i_1\, i_2 \ldots i_k]$ (respectively $A(i_1\, i_2 \ldots i_k)$) the submatrix of A contained in rows and columns $i_1,\ i_2, \ldots,\ i_k$ (respectively obtained from A by deleting rows and columns $i_1,\ i_2, \ldots,\ i_k$). Finally, for $A \in \mathcal{S}_n(\mathbb{K})$, we denote by $G(A)$ the (undirected) graph of A.

Theorems 1 and 2 may be seen as results about *Completion Problems* (see *e g* [2], [3]): by a *partial matrix* we mean a matrix in which some of the entries are specified elements of a certain set S, while others are independent indeterminate variables over S (the unspecified elements). A *completion* of a partial matrix is the matrix with elements in S obtained form the partial matrix when we specify for each of these variables a value of S. So if $A = (a_{ij})$ is a partial matrix a completion of A will be any matrix $B = (b_{ij})$ with elements in S, the same dimensions of A, and such that if a_{ij} is specified in A, $b_{ij} = a_{ij}$. A matrix completion problem asks whether any (in some problems, at least one) completion of a partial matrix has a completion with certain properties.

In our case the specified entries of A are the non-diagonal ones, while we may see the diagonal entries as free variables. We want that for any completion B of A, B is a symmetric n-by-n matrix with $\operatorname{rank} B \geq n-1$. What Theorem 2 says is that this is only possible if A is symmetric and its graph is a path or, when $\mathbb{K} = \mathbb{Z}_3$, A is up to a permutation similarity, one of the matrices F_i. In sequel we sometimes use these kind of ideas and think of an $A \in \mathcal{F}_n(\mathbb{K})$ as a partial matrix; for instance we often refer to the a choice of a particular diagonal element of A.

We would like to note that Theorems 1 and 2 are theorems about *symmetric matrices*; they fail for general matrices, namely for Hermitian matrices as the following example shows.

Let A be the following complex Hermitian matrix:

$$\begin{bmatrix} 0 & 1 & i \\ 1 & 0 & 1 \\ -i & 1 & 0 \end{bmatrix},$$

and $D = \operatorname{diag}(d_1,\ d_2,\ d_3)$ any complex 3-by-3 diagonal matrix. We have $\operatorname{rank}(A+D) \geq 2$. In fact if $d_1 = 0$ the minor

$$\begin{vmatrix} d_1 & 1 \\ 1 & d_2 \end{vmatrix},$$

of $A+D$ is nonzero. If $d_1 \neq 0$ at least one of the minors

$$\begin{vmatrix} 1 & i \\ d_2 & 1 \end{vmatrix} \quad \begin{vmatrix} d_2 & 1 \\ -i & 1 \end{vmatrix},$$

of $A+D$ is nonzero.

2. Basic Properties of $\mathcal{F}_n(\mathbb{K})$

We present in this section some basic facts about the set $\mathcal{F}_n(\mathbb{K})$ that we will need later; although these results are given in [1], [4] and the proofs in [4] are valid for arbitrary fields (with the exception of n. 2 of our Proposition 2), for completeness we include here also the proofs.

Lemma 1. *Let $A \in \mathcal{M}_n(\mathbb{K})$. There exists a diagonal matrix $D \in \mathcal{M}_n(\mathbb{K})$ such that rank$(A+D) < n$.*

<u>Proof.</u> Let $A = [a_{ij}]$ and $d_j = -\sum_{i=1}^{n} a_{ij},\ j = 1,\ \ldots,\ n$. Take $D = \operatorname{diag}(d_1,\ \ldots,\ d_n)$ and let $C_1,\ \ldots,\ C_n$ be the columns of $A+D$. We have $C_1 + \cdots + C_n = 0$ and so the columns of $A+D$ are linearly dependent, that is $\operatorname{rank}(A+D) < n$. ∎

Proposition 2. *Let A be a n-by-n symmetric matrix with elements in $\mathbb{K}$.*
1. For any diagonal matrix $D \in \mathcal{M}_n(\mathbb{K})$ and any permutation matrix $P \in \mathcal{M}_n(\mathbb{K})$ the following are equivalent:

i) $A \in \mathcal{F}_n(\mathbb{K})$;
ii) $-A \in \mathcal{F}_n(\mathbb{K})$;
iii) $A + D \in \mathcal{F}_n(\mathbb{K})$;
iv) $P^T AP \in \mathcal{F}_n(\mathbb{K})$.

2. If $A \in \mathcal{F}_n(\mathbb{K})$ then A is irreducible.

Proof. 1 follows immediately from the definition of $\mathcal{F}_n(\mathbb{K})$. To prove 2, suppose that A is reducible, say $A = A_1 \oplus A_2$ where A_1 has order k, $1 < k < n$. By Lemma 1 there exist diagonal matrices $D_1 \in \mathcal{M}_k(\mathbb{K})$ and $D_2 \in \mathcal{M}_{n-k}(\mathbb{K})$ such that rank $(A_1 + D_1) < k$ and rank $(A_2 + D_2) < n - k$. Let $D = D_1 \oplus D_2$; then rank $(A + D) < n - 1$ and so A can not be an element of $\mathcal{F}_n(\mathbb{K})$. ■

Proposition 3. *Let $A \in \mathcal{S}_3(\mathbb{K})$. We have $A \in \mathcal{F}_3(\mathbb{K})$ if and only is the graph of A is a path.*

Proof. If the graph of $A = [a_{ij}]$ is connected (i e A irreducible) and its graph is not a path then it is the complete graph, that is $a_{ij} \neq 0$ for $i \neq j$. Choose $a_{11} = \dfrac{a_{12}\ a_{13}}{a_{23}}$, $a_{22} = \dfrac{a_{12}\ a_{23}}{a_{13}}$, $a_{33} = \dfrac{a_{13}\ a_{23}}{a_{12}}$. A will have rank one. ■

The next Proposition, due to W. C. R Rheinbolt and R. A. Shepherd, [4], is crucial for the induction procedure in the proof of Theorem 2.

Proposition 4. *Let $A \in \mathcal{F}_n(\mathbb{K})$ and i an integer, $1 \leq i \leq n$. Choose $a_{ii} \neq 0$ and apply Gaussian elimination along the i-the row and column of A. Let A' be the resulting matrix. Then $A'(i) \in \mathcal{F}_{n-1}(\mathbb{K})$.*

Proof. By Proposition 1 we may suppose, with out loss of generality, $i = 1$. Suppose A partitioned in the following way:

$$A = \begin{bmatrix} a_{11} & b^T \\ b & A(1) \end{bmatrix}.$$

Now elimination along the first column is just pre-multiplying by the matrix:

$$A = \begin{bmatrix} 1 & 0 \\ -a_{11}^{-1} b & I_{n-1} \end{bmatrix},$$

while elimination along the first row is just post multiplying A by E^T. We have then $A' = EAE^T$. Take an arbitrary diagonal matrix $D_1 \in \mathcal{M}_{n-1}(\mathbb{K})$ and let $D = 0 \oplus D_1$. Now $\operatorname{rank} E(A+D)E^T = \operatorname{rank}(A+D) \geq n-1$ and

$$E(A+D)E^T = \left[\begin{array}{cc} a_{11} & 0 \\ 0 & -a_{11}^{-1}bb^T + A(1) + D_1 \end{array}\right],$$

But $-a_{11}^{-1}bb^T + A(1) = A'(1)$ and so $\operatorname{rank}(A'(1) + D_1) \geq n-2$; this means that $A'(1) \in \mathcal{F}_{n-1}(\mathbb{K})$.

■

Corollary 5. *Let $A = [a_{ij}] \in \mathcal{F}_n(\mathbb{K})$; denote by L_i the i-th row of A and by L'_i the row vector obtained from L_i by deleting the element in the column i, $i = 1, \ldots, n$. Then, for any i, j, $1 \leq i < j \leq n$, L'_i and L'_j are linearly independent.*

Proof. If L'_i and L'_j are linearly dependent then is is possible to choose nonzero diagonal elements a_{ii} and a_{jj} such that L_i and L_j are linearly dependent. Eliminate along the row and column i; let A' be the resulting matrix. All the elements of row and column j of A' are 0 and so $A'(i)$ is reducible. Therefore $A'(i) \notin \mathcal{F}_{n-1}(\mathbb{K})$, which contradicts Proposition 4.

■

The next Proposition is due to Fiedler, [1]; our proof follows that in [4].

Recall that a *cycle* on n vertices is a (undirected) connected graph in which any vertex has degree two; or equivalently for some ordering of the vertices, say $v_1, v_2, \ldots, v_n$, the vertex v_i is connected with v_{i-1} and v_{i+1}, $i = 1, 2, \ldots, n$, with the convention $v_0 = v_n$, $v_{n+1} = v_1$.

Proposition 6. *Let $A \in \mathcal{S}_n(\mathbb{K})$, $n > 2$. If the graph of A is a cycle then $A \notin \mathcal{F}_n(\mathbb{K})$.*

Proof. The proof is by induction on n. For $n = 3$, by Proposition 3, the result is certainly true.

Assume the result is true for $n-1$ and let us prove that it remains true for n ($n > 3$).

Let $A \in \mathcal{S}_n(\mathbb{K})$ be a matrix whose graph is a cycle. By Proposition 2, we may assume, without loss of generality

$$A = \begin{bmatrix} a_{11} & a_{12} & 0 & \dots & 0 & a_{1n} \\ a_{12} & a_{22} & a_{23} & \dots & 0 & 0 \\ 0 & a_{23} & a_{33} & \dots & 0 & 0 \\ \dots & \dots & \dots & \dots & \dots & \dots \\ 0 & 0 & 0 & \dots & a_{n-1\,n-1} & a_{n\,n-1} \\ a_{1n} & 0 & 0 & \dots & a_{n\,n-1} & a_{nn} \end{bmatrix}.$$

Choose $a_{11} \neq 0$ and eliminate along the first row and column of A. Let A' be the resulting matrix. The graph of $A'(1)$ is a cycle on $n-1$ vertices. If $A \in \mathcal{F}_n(\mathbb{K})$ we will have, by Proposition 4, $A'(1) \in \mathcal{F}_{n-1}(\mathbb{K})$ which contradicts the induction hypothesis. Therefore $A \notin \mathcal{F}_n(\mathbb{K})$. ■

3. Fiedler matrices with zero non-diagonal elements

In this section we will prove our main result for matrices that have some matrices that have some zero nondiagonal elements.

Proposition 7. *Let $A \in \mathcal{F}_n(\mathbb{K})$. If A has a zero non-diagonal element then the graph of A is a path.*

Proof. We will use induction over n. For $n < 3$ there is nothing to prove and the case $n = 3$ was proved in Proposition 3.

We will prove the case $n = 4$. Let $A \in \mathcal{F}_4(\mathbb{K})$.

Suppose first that G(A) has a vertex of degree one; without loss of generality we may assume that vertex one has degree one and and moreover that $\{1, 2\}$ is an edge of $G(A)$. We have then

$$A = \begin{bmatrix} a_{11} & a_{12} & 0 & 0 \\ a_{12} & a_{22} & a_{23} & a_{24} \\ 0 & a_{23} & a_{33} & a_{34} \\ 0 & a_{24} & a_{34} & a_{44} \end{bmatrix}$$

with $a_{12} \neq 0$. Choose $a_{11} \neq 0$ and eliminate along the first row and column; as this does not change the graph of $A(1)$, it follows from Propositions 3 and 4 that the graph of $A(1)$ is a path and so either $G(A)$ is a star (a four vertex tree with a vertex of degree three) or each vertex of $G(A)$ has degree less than 3. The first case is clearly impossible: take all diagonal elements of A

equal to 0: A will have rank two. In the second case Proposition 6 implies that $G(A)$ is a path.

Suppose now that all vertices of $G(A)$ have degree at least two and that A has a zero non-diagonal element.

Suppose $\mathbb{K} \neq \mathbb{Z}_2$. Without loss of generality we may suppose A in the following form:

$$A = \begin{bmatrix} a_{11} & a_{12} & a_{13} & 0 \\ a_{12} & a_{22} & a_{23} & a_{24} \\ a_{13} & a_{23} & a_{33} & a_{34} \\ 0 & a_{24} & a_{34} & a_{44} \end{bmatrix}.$$

Choose $a_{11} \neq 0$; the Gaussian elimination along the first row and column of A does not change the elements in the last row and column; moreover we are supposing $a_{24} \neq 0$, $a_{34} \neq 0$. So A must be transformed in the following matrix

$$\begin{bmatrix} a_{11} & 0 & 0 & 0 \\ 0 & a'_{22} & 0 & a_{24} \\ 0 & 0 & a'_{33} & a_{34} \\ 0 & a_{24} & a_{34} & a_{44} \end{bmatrix},$$

that is the element in position (2, 3) must be transformed into zero *for any choice of a non-zero diagonal element* a_{11}; but this is impossible as $a_{13} \neq 0$ and $\mathbb{K}$ has at least two non-zero elements.

Suppose $\mathbb{K} = \mathbb{Z}_2$. If all vertices of $G(A)$ have degree less than three then, by Proposition 6, G(A) must be a path. Suppose that $G(A)$ has a vertex of degree three, say

$$A = \begin{bmatrix} 1 & 1 & 1 & 1 \\ 1 & a_{22} & a_{23} & a_{24} \\ 1 & a_{23} & a_{33} & a_{34} \\ 1 & a_{24} & a_{34} & a_{44} \end{bmatrix}.$$

The elimination along the first row and column of A changes the zero non-diagonal elements of $A(1)$ to 1 and vice-versa; so, by Proposition 3, $G(A(1))$ has only one edge and $G(A)$ has a vertex of degree one, contrary of what we are supposing. So, for any field, $G(A)$ can not have all the vertices with degree at two or more, and we have already proved that, if there is a vertex of degree one, then $G(A)$ is a path.

Assume now that the theorem is true for integers less than n, $n > 4$, and let us prove that it still holds for n.

Let $A = [a_{ij}] \in \mathcal{F}_n(\mathbb{K})$. We will begin by proving that if, for a vertex i of $G(A)$, $\deg i < n-1$ then $\deg i \leq 2$. We may assume, without loss of generality, that $i = n$ and that, for a certain k $(1 < k < n)$, $a_{12} \neq 0, \ldots, a_{1k} \neq 0$ $a_{1\,k+1} = \cdots = a_{1n} = 0$.

Suppose first that $A[2, \ldots, k]$ is diagonal (including the case of $k = 2$). Choose $a_{11} \neq 0$ and eliminate along the first row and column. Let $A' = [a'_{ij}]$ be the resulting matrix. In order to apply the induction hypothesis we must prove first that $A'(1)$ has a zero non-diagonal element; this clearly happens if $k > 3$. For $k \leq 3$ we will show that at least one of the elements $a_{k\,k+1}, \ldots, a_{k\,n}, a_{k+1\,k+2}, \ldots, a_{k+1\,n}$ must be zero. In fact, if all those elements are nonzero, choose $a_{kk} = \dfrac{a_{k\,k+1}\ a_{k\,k+2}}{a_{k+1\,k+2}}$ and eliminate along the k-th row and column of A. Let $A'' = [a''_{ij}]$ be the resulting matrix. Note that, for $i > 1$, we have $a''_{1i} \neq 0$, and so, in $G(A''(k))$ we have $\deg 1 = n-2 \geq 3$. Moreover, as $a''_{k+1\,k+2} = 0$, we may apply the induction hypothesis to $A''(k)$ and therefore the graph of that matrix is a path; but this is impossible because we have just observed that, in $G(A''(k))$, $\deg 1 \geq 3$. Hence some of the elements $a_{k\,k+1}, \ldots, a_{k\,n}, a_{k+1\,k+2}, \ldots, a_{k+1\,n}$ must be zero. Now observe that the elimination along the first row and column of A only changes the elements of $A[2, \ldots, k]$, and so the matrix $A'(1)$ does have a zero non-diagonal element. Therefore we may apply the induction hypothesis to $A'(1)$ to conclude that the graph of that matrix is a path. This shows that, in $G(A)$, $\deg n \leq 2$.

Suppose that $A[2, \ldots, k]$ is not diagonal; then we will have $a_{rs} \neq 0$ for some r, s, $2 \leq r \leq k$, $2 \leq s \leq k$, $r \neq s$. Choose $a_{11} = \dfrac{a_{1s}a_{1r}}{a_{rs}}$ and eliminate along the first row and column. Let $A' = [a'_{ij}]$ be the resulting matrix. Note that, $a'_{rs} = 0$, and so we may apply the induction hypothesis to $A'(1)$; therefore $G(A'(1))$ is a path. As before, the elimination process did not change the last row and column of A and so we have, in $G(A)$, $\deg n \leq 2$. This finish the proof that if, for a vertex i of $G(A)$, $\deg i < n-1$ then $\deg i \leq 2$.

Let us prove that, if $A = [a_{ij}] \in \mathcal{F}_n(\mathbb{K})$ has a zero non-diagonal element, then the graph of A is in fact a path. As above we may assume that $a_{1n} = 0$; then in $G(A)$ we will have $\deg 1 \leq 2$, $\deg n \leq 2$; doing, if necessary, an appropriate permutation of rows and the same permutation of columns we may also assume that $a_{14} = a_{15} = \cdots = a_{1n} = 0$; hence we will also have

$\deg j \leq 2$ for $j \geq 4$. Moreover among the elements $a_{24}, a_{25}, \ldots, a_{2n}$ there are at most two different from zero; to see that this happen choose $a_{11} \neq 0$ and eliminate along the first row and column; this will change only the elements of the submatrix $A[1, 2, 3]$. Let $A' = [a'_{ij}]$ be the resulting matrix. Clearly $A'(1)$ must have a zero non-diagonal element and so, by the induction hypothesis, the graph of $A'(1)$ is a path; as we have $a'_{2j} = a_{2j}$ for $j \geq 4$, this proves our claim. The same argument also shows that among the elements $a_{34}, a_{35}, \ldots, a_{3n}$ there are at most two different from zero. Hence, if $n > 5$, rows (and columns) two and three will also have zero elements and so we will have $\deg j \leq 2$ for $1 \leq j \leq n$. By Proposition 6 the graph of A is a path and we are done.

It remains to examine the case of $n = 5$. Let $A = [a_{ij}] \in \mathcal{F}_5(\mathbb{K})$; we will assume that $a_{14} = a_{15} = 0$. Choose $a_{11} \neq 0$ and eliminate along the first row and column of A; let A' be the resulting matrix; we will have

$$A' = \begin{bmatrix} a_{11} & 0 & 0 & 0 & 0 \\ 0 & a'_{22} & a'_{23} & a_{24} & a_{25} \\ 0 & a'_{23} & {}'a_{33} & a_{34} & a_{35} \\ 0 & a_{24} & a_{34} & a_{44} & a_{45} \\ 0 & a_{25} & a_{35} & a_{44} & a_{45} \end{bmatrix}.$$

By Proposition 4, $A'(1) \in \mathcal{F}_4(\mathbb{K})$; from $a_{15} = 0$ follows that, in $G(A)$, $\deg 5 \leq 2$ and therefore at least one of the non-diagonal elements of the last row and column of $A'(1)$ is zero; by induction hypothesis $G(A'(1))$ is a path and so at least one of the elements a_{24}, a_{25}, a_{34} and a_{35} must be zero; there is no loss of generality in assuming $a_{35} = 0$; then we will have $\deg 3 \leq 2$ and so at least one of the elements a_{13}, a_{23} or a_{34} must be zero. If $a_{13} = 0$ then elimination along the first row and column does not change $G(A(1))$ and so this graph is a path; hence each row and column of A has a zero non-diagonal element; it follows that $\deg i \leq 2$ for every i, $1 \leq i \leq 5$; by Proposition 6, G(A) is a path. If $a_{23} = 0$ we have again $\deg i \leq 2$ for every i, $1 \leq i \leq 5$ and the same conclusion follows. Finally if $a_{34} = 0$ then $G(A'(1))$ can not be a path unless either $a_{24} = 0$ or $a_{25} = 0$; but then again $\deg i \leq 2$ for every i and $G(A)$ is a path. ∎

4. The set $\mathcal{F}_4(\mathbb{K})$

Proposition 8. *For any field $\mathbb{K}$ and any matrix $A \in \mathcal{S}_4(\mathbb{K})$ we have $A \in \mathcal{F}_4(\mathbb{K})$ if and only if the graph of A is a path.*

Proof. The sufficiency part is already known. For necessity we have only to prove that if a matrix $A \in \mathcal{M}_4(\mathbb{K})$ has all its non-diagonal elements different from zero then $A \notin \mathcal{F}_4(\mathbb{K})$; to prove this we show that, for an appropriate choice of the diagonal elements A will have rank less than three. Let

$$A = \begin{bmatrix} a_{11} & a_{12} & a_{13} & a_{14} \\ a_{12} & a_{22} & a_{23} & a_{24} \\ a_{13} & a_{23} & a_{33} & a_{34} \\ a_{14} & a_{24} & a_{34} & a_{44} \end{bmatrix} .$$

Choose

$$a_{11} = 0, \quad a_{22} = \frac{a_{12}a_{23}}{a_{13}} - \frac{a_{12}^2 a_{34}}{a_{13}a_{14}} + \frac{a_{24}a_{12}}{a_{14}},$$

$$a_{33} = \frac{a_{13}a_{34}}{a_{14}} - \frac{a_{13}^2 a_{24}}{a_{12}a_{14}} + \frac{a_{13}a_{23}}{a_{12}}, \quad a_{44} = \frac{a_{14}a_{34}}{a_{13}} - \frac{a_{14}^2 a_{23}}{a_{12}a_{13}} + \frac{a_{14}a_{24}}{a_{12}} .$$

Let C_i be the i-th column of A, $i = 1, 2, 3, 4$; then C_1 and C_2 are linearly independent while

$$C_3 = \left(\frac{a_{34}}{a_{14}} - \frac{a_{13}a_{24}}{a_{12}a_{14}} \right) C_1 + \frac{a_{13}}{a_{12}} C_2 ,$$

$$C_4 = \left(\frac{a_{34}}{a_{13}} - \frac{a_{14}a_{23}}{a_{12}a_{13}} \right) C_1 + \frac{a_{14}}{a_{12}} C_2 .$$

Therefore A has rank two. ∎

5. The set $\mathcal{F}_5(\mathbb{Z}_3)$

Now we are going to see that the set $\mathcal{F}_5(\mathbb{Z}_3)$ does contains matrices with all non-diagonal elements different from zero.

Proposition 9. *Let $A \in \mathcal{S}_5(\mathbb{Z}_3)$. We have $A \in \mathcal{F}_5(\mathbb{Z}_3)$ if and only if the graph $G(A)$ is a path or there exists a permutation matrix $P \in \mathcal{M}_5(\mathbb{Z}_3)$ such that A is one of the matrices PF_iP^T, where F_i, $i = 1, 2, 3, 4$, is one of the matrices of (1) and (2).*

Proof. We need only to focus on matrices with all the non-diagonal elements different from zero. We begin by the "only if" part. So let $A = [a_{ij}] \in \mathcal{F}_5(\mathbb{Z}_3)$; We consider several cases:
CASE 1: A has a row with equal non-diagonal elements. By Proposition 2

we may suppose that this row is the first one. If this elements are equal to 1 we will have:

$$A = \begin{bmatrix} a_{11} & 1 & 1 & 1 & 1 \\ 1 & a_{22} & a_{23} & a_{24} & a_{25} \\ 1 & a_{23} & a_{33} & a_{34} & a_{35} \\ 1 & a_{24} & a_{34} & a_{44} & a_{45} \\ 1 & a_{25} & a_{35} & a_{45} & a_{55} \end{bmatrix}.$$

Take $a_{11} = 2$ and eliminate along the fist row and column; we will obtain the following matrix:

$$A' = \begin{bmatrix} 2 & 0 & 0 & 0 & 0 \\ 0 & a_{22}+1 & a_{23}+1 & a_{24}+1 & a_{25}+1 \\ 0 & a_{23}+1 & a_{33}+1 & a_{34}+1 & a_{35}+1 \\ 0 & a_{24}+1 & a_{34}+1 & a_{44}+1 & a_{45}+1 \\ 0 & a_{25}+1 & a_{35}+1 & a_{45}+1 & a_{55}+1 \end{bmatrix}.$$

According to Proposition 4 and 8 the graph of $A'(1)$ must be a path; there is no loss of generality in assuming that $A'(1)$ is actually an irreducible tridiagonal matrix. This is only possible if

$$A = \begin{bmatrix} a_{11} & 1 & 1 & 1 & 1 \\ 1 & a_{22} & 1 & 2 & 2 \\ 1 & 1 & a_{33} & 1 & 2 \\ 1 & 2 & 1 & a_{44} & 1 \\ 1 & 2 & 2 & 1 & a_{55} \end{bmatrix}.$$

Now if the non-diagonal elements in the first row of A are equal to 2 we just apply the previous reasoning to the the matrix $-A$ (which is also an element $\mathcal{F}_5(\mathbb{Z}_3)$). So in Case 1 A is, up to row permutation and the same permutation of column, either F_1 or F_2.

CASE 2: There is a row of A with three equal non-diagonal elements and no row with four equal non-diagonal elements. By Proposition 2 we may suppose that the row with the three non diagonal elements is the first one and moreover that $a_{12} = a_{13} = a_{14} \neq a_{15}$. Suppose first that these equal elements are 1. So we will have

$$A = \begin{bmatrix} a_{11} & 1 & 1 & 1 & 2 \\ 1 & a_{22} & a_{23} & a_{24} & a_{25} \\ 1 & a_{23} & a_{33} & a_{34} & a_{35} \\ 1 & a_{24} & a_{34} & a_{44} & a_{45} \\ 2 & a_{25} & a_{35} & a_{45} & a_{55} \end{bmatrix}.$$

Choose $a_{11} = 2$ and eliminate along the first row and column. Let A' be the resulting matrix; by Propositions 4 and 8 $G(A'(1))$ is a path. Note that we may permute the rows two to four of A and do the same permutations of columns two to four, without changing the first row and column of A. This means that, up to row and columns permutations that do not change the first row and column of A, $A'(1)$ must be one of the following matrices:

$$\begin{bmatrix} a'_{22} & 0 & a'_{24} & 0 \\ 0 & a'_{33} & a'_{34} & a'_{35} \\ a'_{24} & a'_{34} & a'_{44} & 0 \\ 0 & a'_{35} & 0 & a'_{55} \end{bmatrix}, \quad \begin{bmatrix} a'_{22} & 0 & 0 & a'_{25} \\ 0 & a'_{33} & a'_{34} & 0 \\ 0 & a'_{34} & a'_{44} & a'_{45} \\ a'_{25} & 0 & a'_{45} & a'_{55} \end{bmatrix},$$

$$\begin{bmatrix} a'_{22} & a'_{23} & 0 & 0 \\ a'_{23} & a'_{33} & 0 & a'_{35} \\ 0 & 0 & a'_{44} & a'_{45} \\ 0 & a'_{35} & a'_{45} & a'_{55} \end{bmatrix}, \quad \begin{bmatrix} a'_{22} & a'_{23} & 0 & 0 \\ a'_{23} & a'_{33} & a'_{34} & 0 \\ 0 & a'_{34} & a'_{44} & a'_{45} \\ 0 & 0 & a'_{45} & a'_{55} \end{bmatrix}.$$

If $A'(1)$ is the first of these matrices then the non-diagonal elements of the fourth row of A will be all equal to 1 while if $A'(1)$ is the fourth of these matrices the non-diagonal elements of the third row of A will be all equal to 1; recall that, in CASE 2, we are supposing that in any row of A the non-diagonal elements are not all equal and so $A'(1)$ must be either the second or the third of the above matrices; this gives the following two hypothesis for A:

$$\begin{bmatrix} a_{11} & 1 & 1 & 1 & 2 \\ 1 & a_{22} & 2 & 2 & 2 \\ 1 & 2 & a_{33} & 1 & 1 \\ 1 & 2 & 1 & a_{44} & 2 \\ 2 & 2 & 1 & 2 & a_{55} \end{bmatrix}, \quad \begin{bmatrix} a_{11} & 1 & 1 & 1 & 2 \\ 1 & a_{22} & 1 & 2 & 1 \\ 1 & 1 & a_{33} & 2 & 2 \\ 1 & 2 & 2 & a_{44} & 2 \\ 2 & 1 & 2 & 2 & a_{55} \end{bmatrix}.$$

The first hypothesis is precisely F_3. The second hypothesis is similar to F_3 via a permutation matrix: This can be easily seen if we draw the *graph of ones* of both matrices: the graph on five vertices with an edge between vertices i and j if and only if the matrix considered has a 1 in position $(i,\, j)$. It is easily seen that the two above matrix have the same graph of ones: in fact the second matrix can be obtained from F_3 by just moving the third column of F_3 to the second position, the fourth column to the third position and the second column to the third position and doing the same permutation with the rows of F_3.

Suppose now that A has in the first row three 2's and one 1. The matrix $-A$ is also an element of $\mathcal{F}_5(\mathbb{Z}_3)$ and is in the previous conditions. So $-A$ must be, up to row permutations and the same permutations of columns, the matrix F_3. Therefore we have (again up to row permutations and the same permutations of columns)

$$A = \begin{bmatrix} a_{11} & 2 & 2 & 2 & 1 \\ 2 & a_{22} & 1 & 1 & 1 \\ 2 & 1 & a_{33} & 2 & 2 \\ 2 & 1 & 2 & a_{44} & 1 \\ 1 & 1 & 2 & 1 & a_{55} \end{bmatrix}.$$

But again the above matrix is similar to F_3 via a permutation matrix: to obtain the above matrix from F_3 by just move the second column of F_3 to the first position, the third column of F_3 to the second position, the fifth column to the third position and the first column to the fifth position and doing the same permutation with the rows of F_3.

CASE 3: Each row of A has at most two equal non-diagonal elements. As we are considering just nonzero non-diagonal elements, this means that each row of A has exactly two 1's and two 2's in non-diagonal positions. We may also suppose that

$$A = \begin{bmatrix} a_{11} & 1 & 1 & 2 & 2 \\ 1 & a_{22} & a_{23} & a_{24} & a_{25} \\ 1 & a_{23} & a_{33} & a_{34} & a_{35} \\ 2 & a_{24} & a_{34} & a_{44} & a_{45} \\ 2 & a_{25} & a_{35} & a_{45} & a_{55} \end{bmatrix}.$$

Now by Corollary 5, applied to rows one and two, the second row of A must be $[1\ a_{22}\ 2\ 1\ 2]$ or $[1\ a_{22}\ 2\ 2\ 1]$. The second hypothesis may be reduced to the first just by permuting the fourth column of A with the fifth and doing the same permutation on the rows of A; note that this permutation do not change the first row and column of A. So we have only to consider the first hypothesis. Repeated use of Corollary 5 (together with the fact that there are two 1's and two 2's in non-diagonal positions in each row of A gives us the following matrix:

$$\begin{bmatrix} a_{11} & 1 & 1 & 2 & 2 \\ 1 & a_{22} & 2 & 1 & 2 \\ 1 & 2 & a_{33} & 2 & 1 \\ 2 & 1 & 2 & a_{44} & 1 \\ 2 & 2 & 1 & 1 & a_{55} \end{bmatrix},$$

which is precisely F_4.

To finished the proof we have only to show that $F_i \in \mathcal{F}_5(\mathbb{Z}_3)$, $i = 1, 2, 3, 4$. To prove this, it is sufficient to show first that for any j, $1 \le j \le 5$, and any nonzero choice of the diagonal element a_{jj} of F_i, if we eliminate along the row and column j and then delete the row and column j we will obtain a 4-by-4 matrix whose graph is a path (this means that, for that choice of a_{jj} $\operatorname{rank} F_i \ge 4$). Next we must verify that when all the diagonal elements of F_i are zero the rank of F_i is also greater than three (it is actually four). These are straightforward calculations and we omit them.

Note that, once we establish, for a certain i, say $i = 1$, that $F_1 \in \mathcal{F}_5(\mathbb{Z}_3)$ the fact that, for $j > 1$, $F_j \in \mathcal{F}_5(\mathbb{Z}_3)$, follows immediately from the following relations (note that actually, for any i, j, any matrix of type F_i is similar to some matrix of type F_j):

$$\begin{aligned} F_2(a_{11}, a_{22}, a_{33}, a_{44}, a_{55}) &= -F_1(-a_{11}, -a_{22}, -a_{33}, -a_{44}, -a_{55}) = \\ &= [e_2\, e_3\, e_4\, e_1\, e_5]^T \times \operatorname{diag}(1, 2, 1, 2, 1) \times F_1(a_{22}, a_{33}, a_{44}, a_{11}, a_{55}) \times \\ &\qquad \times \operatorname{diag}(1, 2, 1, 2, 1) \times [e_2\, e_3\, e_4\, e_1\, e_5]\ , \end{aligned}$$

$$\begin{aligned} F_3(a_{11}, a_{22}, a_{33}, a_{44}, a_{55}) &= \\ &= [e_1\, e_2\, e_5\, e_4\, e_3]^T \times \operatorname{diag}(1, 1, 2, 1, 1) \times F_1(a_{11}, a_{22}, a_{55}, a_{44}, a_{33}) \times \\ &\qquad \times \operatorname{diag}(1, 1, 2, 1, 1) \times [e_1\, e_2\, e_5\, e_4\, e_3]\ , \end{aligned}$$

$$\begin{aligned} F_4(a_{11}, a_{22}, a_{33}, a_{44}, a_{55}) &= \\ \operatorname{diag}(1, 1, 1, 2, 1) &\times F_3(a_{11}, a_{22}, a_{33}, a_{44}, a_{55}) \times \operatorname{diag}(1, 1, 1, 2, 1) \times = \\ &= [e_1\, e_2\, e_5\, e_4\, e_3]^T \times \operatorname{diag}(1, 1, 2, 2, 1) \times F_1(a_{11}, a_{22}, a_{55}, a_{44}, a_{33}) \times \\ &\qquad \times \operatorname{diag}(1, 1, 2, 2, 1) \times [e_1\, e_2\, e_5\, e_4\, e_3]\ , \end{aligned}$$

where e_i is the i-th column of the 5-by-5 identity matrix. ■

6. The set $\mathcal{F}_6(\mathbb{Z}_3)$

In order to prove, by induction, that for $n > 5$, if $A \in \mathcal{F}_n(\mathbb{Z}_3)$ then $G(A)$ is a path, we have to prove it first for $n = 6$. This will be done in the next proposition.

Proposition 10. *Let A be a 6-by-6 symmetric matrix over $\mathbb{Z}_3$. We have $A \in \mathcal{F}_6(\mathbb{Z}_3)$ if and only if the graph $G(A)$ is a path.*

Proof. We have only to show that, if the non-diagonal elements of a 6-by-6 symmetric matrix, $A = [a_{ij}]$, over $\mathbb{Z}_3$, are all nonzero, then $A \notin \mathcal{F}_6(\mathbb{Z}_3)$.

assume that $A \in \mathcal{F}_6(\mathbb{Z}_3)$. Let A' (respectively A'') the matrix obtained from A by choosing $a_{11} = 1$ (respectively $a_{11} = 2$) and eliminating along the first row and column of A. By Proposition 4 $A'(1)$, $A''(1) \in \mathcal{F}_5(\mathbb{Z}_3)$. We will show that, in fact, the graphs of these submatrices are the path on 5 vertices. In fact if the graph of $A'(1)$ is not a path then it must be the complete graph on five vertices. The elements of the first row of A' are $a_{2i} + 2a_{12}a_{1i}$, $i = 2, \ldots, 6$; so we have $a_{2i} \neq a_{12}a_{1i}$, $i = 3, \ldots, 6$. As $\mathbb{Z}_3$ has only two nonzero elements we have $a_{2i} = 2a_{12}a_{1i}$, that is $a_{2i} + a_{12}a_{1i} = 0$, $i = 3, \ldots, 6$; but $a_{2i} + a_{12}a_{1i}$, $i = 2, \ldots, 6$ are precisely the elements in the first row of A'' and so this matrix would be reducible, which is impossible. Therefore $A'(1)$ must have a zero in a non-diagonal position and so, by Proposition 7, $G(A'(1))$ is a path. An analogous argument shows that $G(A''(1))$ is also a path.

Now note that, as the non-diagonal elements of the first row and column of A are nonzero, the elimination process along the first row and column will change *every* element of $A(1)$; moreover if, for $a_{11} = 1$, the elimination process transforms an element of $A(1)$ into zero then, for $a_{11} = 2$, that element will be transformed into a nonzero element (and vice-versa); this means that $G(A''(1))$ contains the complement of the graph $G(A'(1))$; but that complement has cycles and so $G(A''(1))$ could not be a path. therefore if $G(A)$ is the complete graph then $A \notin \mathcal{F}_6(\mathbb{Z}_3)$. ■

Remark 11. *We note that the argument used in the above proof does not work for 5-by-5 matrices. This is due to the fact that the complement of the 4-path is again the four path and so it may be possible that, for any of the two nonzero choices of the diagonal elements of A, the graphs of the submatrices corresponding to A' and A'' in the above proof are both path. This is precisely what happens for each of the matrices F_1, F_2, F_3 and F_4 of Proposition 9.*

7. The main Theorem

We are now going to complete the proof of our main result, the Theorem 2.

Theorem 2. *Let $\mathbb{K}$ be any field and A an n-by-n symmetric matrix with elements in $\mathbb{K}$. We have $\operatorname{rank}(A+D) \geq n-1$ for any n-by-n diagonal matrix D with elements in $\mathbb{K}$ if and only if the graph of A is a path or $\mathbb{K} = \mathbb{Z}_3$, $n = 5$ and $A = PF_iP^T$, where P is a 5-by-5 permutation matrix and F_i, $i = 1, 2, 3, 4$, is one of the matrices of (1) and (2).*

Proof. We have only to prove the "only if" part, the "if" part being already established.

We proceed by induction; for $n \leq 4$ (respectively, $n \leq 6$ if $\mathbb{K} = \mathbb{Z}_3$) the result has already been proven. Now let $n > 4$ (respectively, $n > 6$ if $\mathbb{K} = \mathbb{Z}_3$) and suppose that the result is true for matrices of order less than n.

Let $A \in \mathcal{F}_n(\mathbb{K})$. By Proposition 7 we may suppose that all the non-diagonal elements of A are non-zero. Choose $a_{11} \neq 0$ and eliminate along the first row and column. Let A' be the resulting matrix. By Proposition 4 $A'(1) \in \mathcal{F}_{n-1}(\mathbb{K})$ and so, by the induction hypothesis, the graph of $A'(1)$ is a path.

Now note that, as the non-diagonal elements of the first row and column of A are nonzero, the elimination process along the first row and column will change *every* element of $A(1)$. So, if $\mathbb{K} = \mathbb{Z}_2$ all the non-diagonal elements of $A(1)$ will be transformed into zero, that is $A'(1)$ is a diagonal matrix; this contradicts the fact that $G(A'(1))$ is a path.

For $\mathbb{K} \neq \mathbb{Z}_2$, choose another nonzero element in position (1,1) (different from the previous one) and eliminate along the first row and column. Let A'' be the resulting matrix; an element of $A(1)$ that, for the first choice of a_{11}, was transformed into zero will be, for this second choice transformed into a non-zero element; this means that $G(A''(1))$ contains the complement of the graph $G(A'(1))$; but $G(A''(1))$ must also be a path while the complement of a path has cycles if $n > 5$. So for $n > 5$ we got a contradiction! For $n = 5$ we just choose a third nonzero element (note that for n=5 we are supposing that $\mathbb{K} \neq \mathbb{Z}_3$) in position (1,1) and again eliminate along the first row and column of A. Let A''' be the resulting matrix. A non-diagonal element of $A(1)$ that, for one the two previous choices of a_{11} was transformed into zero by the elimination procedure will be now transformed into a nonzero element of $A'''(1)$. Therefore the graph of $A'''(1)$ must be the complete graph on four vertices. But by Propositions 4 and 8 that graph is a path; once again we get a contradiction, finishing the proof. ∎

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