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ON SURFACES OF GENERAL TYPE WITH $p_g = 6$ AND $K^2 = 13$

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ABSTRACT: We prove that the canonical model of a nonsingular surface of general type with $p_g = 6$ and $K^2 = 13$ whose canonical image is not contained in a pencil of quadrics is a complete intersection of four quasihomogeneous forms of degree 2 and two quasihomogeneous forms of degree 1 in the cone over a weighted Grassmannian $\mathbb{P}(1^2)\ltimes\mathbb{G}(\frac{1}{2}^4, \frac{3}{2})$.

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1. Introduction

This work is dedicated to the proof of the following theorem.

Theorem 1.1. *Let S be a nonsingular surface of general type with $p_g = 6$ and $K^2 = 13$ whose image under the canonical map is not contained in a pencil of quadrics. Then,*

(i) *$|K_S|$ is base point free and S is regular;*
(ii) *there exists a map $\rho\colon S \to \mathbb{P}(1^2)\ltimes\mathbb{G}(\frac{1}{2}^4, \frac{3}{2})$ that factors through the pluricanonical morphism $S \to \operatorname{Proj} R(S, K_S)$;*
(iii) *the image of S under ρ is a complete intersection of four quasihomogeneous forms of degree 2 and two quasihomogeneous forms of degree 1 in $\mathbb{P}(1^2)\ltimes\mathbb{G}(\frac{1}{2}^4, \frac{3}{2})$.*

The starting point was the question of Miles Reid whether the work of Mukai on the classification of Gorenstein Fano 3-folds extends to a wider class of Fano 3-folds. In such wider class one would find as first cases Fano 3-folds not necessarily Gorenstein (allowing cyclic quotient singularities) whose anticanonical ring has codimension 3.

Mukai's linear section theorem asserts that an indecomposable Gorenstein Fano 3-fold V with at most Gorenstein canonical singularities of genus 7, 8,

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9 or 10 is isomorphic to a linear section of an homogeneous space X. Fano 3-folds of the type described above have genus $\leqslant 10$ or equal to 12. The linear section theorem tidies the upper end of this range. While for genus $\leqslant 5$ the anticanonical model, $\operatorname{Proj} R(V, -K_V)$, is a complete intersection in (weighted) projective space and for genus 6 it can be related to Buchsbaum–Eisenbud's theorem; for genus $\geqslant 7$, the anticanonical ring is of codimension $\geqslant 4$. I.e, that we know of, the format of the equations of the anticanonical model is not prescribed in any manner by any structure theorem. The main ingredient in Mukai's proof is called the vector bundle method. It consists in constructing on V (or on a linear section $T \in |-K_V|$) the restriction of the tautological vector bundle of X. In a series of articles [6, 7, 9, 10, 11] dedicated to this result, Mukai uses ladders $C \subset T \subset V$ of linear sections of V to set up the embedding $V \hookrightarrow X$; whether using T to construct a bundle on V or using C to construct a bundle on T, embedding T in X and then extending the embedding to V. Though, as Mukai explains in [6], the genus 12 is not a section of an homogeneous space, the vector bundle method still yields a satisfactory description of the anticanonical model.

Reid's question is motivated by the following theorem.

Theorem (Altınok [1])**.** *There are precisely* 69 *families of K3 surfaces with cyclic quotient singularities* $\frac{1}{r}(a, -a)$ *whose general element is a codimension 3 subvariety in weighted projective space given by the* 4×4 *Pfaffians of a skew* 5×5 *matrix.*

Suppose that (T, H) is a polarised K3 surface with a cyclic quotient singularity of type $\frac{1}{2}(1, 1)$ and such that $h^0(T) = 5$. This surface belongs to the first family of Altınok's list. Her result states that for a general K3 with these invariants the ring $R(T, H) = \bigoplus_{n \geqslant 0} H^0(T, nH)$ is isomorphic to a quotient

$$\frac{\mathbb{C}[x_1, \dots, x_5, y]}{\text{Pfaff}}$$

where Pfaff is the ideal generated by the submaximal Pfaffians of a skew matrix

$$\begin{pmatrix} 0 & m_{12} & m_{13} & m_{14} & q_1 \\ & 0 & m_{23} & m_{24} & q_2 \\ & & 0 & m_{34} & q_3 \\ & -\text{sym} & & 0 & q_4 \\ & & & & 0 \end{pmatrix}$$

with $m_{ij} \in \langle x_1, \ldots, x_5 \rangle$ and $q_i \in \mathrm{S}^2 \langle x_1, \ldots, x_5 \rangle \oplus \langle y \rangle$. This suggests that T should be recovered from an embedding $T \hookrightarrow \mathbb{G}(\frac{1}{2}^4, \frac{3}{2})$ as a complete intersection. In fact, such is the assertion of Proposition 2.8 in Corti and Reid's paper [5] for all general K3 surfaces of each family of Altınok's list. Their proof relies on Buchsbaum–Eisenbud's theorem [3] for Gorenstein ideals of codimension 3. In particular they do not give an explicit construction of a bundle (or of an orbi-bundle) on T.

Notice that if the image of $T \hookrightarrow \mathbb{G}(\frac{1}{2}^4, \frac{3}{2})$ is a linear section of $\mathbb{G}(\frac{1}{2}^4, \frac{3}{2})$, in particular, there exists a Fano 3-fold V such that $T \in |-K_V|$. This 3-fold V has a cyclic quotient singularity of type $\frac{1}{2}(1, 1, 1)$ and therefore is not Gorenstein. However a general member of $|-2K_V|$ is a nonsingular surface of general type with $p_g = 6$ and $K^2 = 13$.

Let us explain the nature of our assumptions. It is traditional to start an analysis of a particular birational class of surfaces of general type with $p_g \geqslant 4$ by considering the case with birational canonical morphism. Here we consider surfaces S with the aforementioned invariants whose canonical image is not contained in a pencil of quadrics. However, the canonical image of S must be contained in at least one quadric (see the proof of Proposition 3.1); so that what we require is that S be general in this sense. Additionally, we can also show from our assumptions that the canonical map is a birational morphism. We refer the reader to the work of Ciliberto [4] on regular surfaces with $p_g = 5$ and $K^2 = 10$, where it is shown that there exist special loci in the component of the coarse moduli space of these surfaces corresponding to the birational canonical morphism. In some of these special loci the canonical image of the corresponding surface is contained in one more quadric than expected.

The proof of Theorem 1.1 is half-split in the next two sections. In section 3, Proposition 3.1 deals with the proof of regularity. We establish ρ in section 4 using the Mukai's vector bundle technique. This consists in finding a suitable vector bundle $\mathscr{E}$ on S and using its sections to write ρ. In doing so, we rely on a good knowledge of $R(S, K_S)$ which, in the end, turns out to be the main computation. Indeed there is a nontrivial step that involves showing that a set of 5 relations of the ring $R(S, K_S)$ are all the generators of the canonical ideal I_{K_S}.

2. Preliminaries

An algebraic surface S is a surface of the general type if K_S is a nef divisor and $K^2 > 0$. In particular, by this convention, surfaces of general type are always minimal. As to displaying skew symmetric matrices, we follow the convention of only writing their upper triangle. If M is skew 5×5 matrix, $\mathrm{Pf}_1, \mathrm{Pf}_2, \mathrm{Pf}_3, \mathrm{Pf}_4, \mathrm{Pf}_5$ denote the five submaximal Pfaffians of M, i.e. Pf_i is the Pfaffian of the matrix obtained by removing the line and column i of the matrix M. $\mathrm{Pf}\, M$ denotes the ideal generated by the submaximal Pfaffians of M.

2.1. Cones over a weighted Grassmannian. Weighted Grassmannians were defined by Corti and Reid in [5]. Consider a polynomial ring $\mathbb{C}[m_{ij}]$ where $1 \leqslant i < j \leqslant 5$ and suppose that there exist $c_i \in \frac{1}{2}\mathbb{Z}$ such that $\mathrm{wt}(m_{ij}) = c_i + c_j$. The weighted Grassmannian of weights $c_1, \dots, c_5$ (denoted by $\mathbb{G}(c_1, \dots, c_5)$) is the subscheme of $\mathbb{P}[m_{ij}]$ defined by the ideal generated by the submaximal Pfaffians of the skew matrix:

$$\begin{pmatrix} m_{12} & m_{13} & m_{14} & m_{15} \\ & m_{23} & m_{24} & m_{25} \\ & & m_{34} & m_{35} \\ & & & m_{45} \end{pmatrix}. \tag{1}$$

(Notice that despite the terminology, this definition only generalises the notion of Grassmannian of subspaces of dimension 2 of a fixed 5-dimensional vector space.) Let $\mathrm{aG}(2,5)$ denote the affine cone over the Grassmannian $\mathrm{G}(2,5) \subset \mathbb{P}^9$. In the same way as weighted projective space (w.p.s.) is the quotient of the punctured affine cone over ordinary projective space by a weighted $\mathbb{C}^*$-action, so is the weighted Grassmannian (w.G.) a quotient of punctured $\mathrm{aG}(2,5)$ by a weighted $\mathbb{C}^*$-action. We draw yet another comparison between w.p.s. and w.G., this time within the context of graded rings. Suppose that (X, D) is a pair consisting of an algebraic variety and an ample divisor. Let $x_1, \dots, x_n$ be a choice of generators of the graded ring $R(X, D)$ and consider the epimorphism $\mathrm{ev}\colon \mathbb{C}[x_1, \dots, x_m] \to R(X, D)$. Suppose, additionally, that there exist m quasihomogeneous forms $f_1, \dots, f_m$ in the kernel of the map ev such that $m + \dim X = n$ and such that the image of X under the map $\psi\colon X \longrightarrow \mathbb{P}[X_1, \dots, X_m]$ defined by $p \mapsto (x_1(p), \dots, x_n(p))$ is cut out by $f_1, \dots, f_m$. Then if

$$D^{\dim X} = \deg(\psi(X)) = \frac{\deg(f_1) \cdots \deg(f_m)}{\mathrm{wt}(x_1) \cdots \mathrm{wt}(x_n)}$$

we can conclude that $\psi(X)$ is a complete intersection. Moreover, we deduce that

$$R(X, D) = \frac{\mathbb{C}[x_1, \dots, x_n]}{(f_1, \dots, f_m)}.$$

This argument is tacitly employed when one calculates divisorial rings when they happen to be complete intersections. One important fact to be borne in mind when applying the previous argument is that w.p.s. are arithmetically Cohen-Macaulay schemes. In particular this ensures that a sequence like $f_1, \dots, f_m$ cutting out a "correct-dimensional" subscheme cannot give rise to embedded components. Weighted Grassmannians are suitable varieties in which to draw arguments like the previous, when the ring $R(X, D)$ is no longer a complete intersection, but a quotient by a 5×5 Pfaffian ideal, which is to say, the first nontrivial structure in codimension 3. To make this idea more precise we need introduce the notion of cone over a weighted Grassmannian.

Definition 2.1. Let $\mathbb{C}[m_{ij}]$ with $1 \leqslant i < j \leqslant 5$ be a weighted polynomial ring for which there exist $c_1, \dots, c_5 \in \frac{1}{2}\mathbb{Z}$ such that $\mathrm{wt}(m_{ij}) = c_i + c_j$. Consider an injective graded homomorphism of degree 0 of $\mathbb{C}[m_{ij}]$ into a graded ring $A = \mathbb{C}[\underline{x}]$ whose remaining generators* $z_1, \dots, z_n$ have weights $b_1, \dots, b_n$. We define the cone over a weighted Grassmannian of weights $(b_1, \dots, b_n)$ and $(c_1, \dots, c_5)$ to be the subscheme of $\mathbb{P}[\underline{x}] = \operatorname{Proj} A$ given by the ideal generated by the 5 submaximal Pfaffians of (1). We denote this subscheme by $\mathbb{P}(b_1, \dots, b_n) \ltimes \mathbb{G}(c_1, \dots, c_5)$.

Remark 2.2. Consider $\mathbb{P}[z_1, \dots, z_n]$ and $\mathbb{P}[m_{ij}]$ as linear subspaces of $\mathbb{P}[\underline{x}]$. Then the cone over a weighted Grassmannian $\mathbb{P}(b_1, \dots, b_n) \ltimes \mathbb{G}(c_1, \dots, c_5)$ is the quotient the punctured join in affine space $\mathbb{A}[\underline{x}]$ of $\mathbb{A}[z_1, \dots, z_n]$ and of the affine cone $\mathrm{aG}(2, 5) \subset \mathbb{A}[m_{ij}]$ by the following weighted $\mathbb{C}^*$-action:

$$z_l \mapsto \lambda^{b_l} z_l \text{ and } m_{ij} \mapsto \lambda^{c_i + c_j} m_{ij} \text{ for } 1 \leqslant l \leqslant n \text{ and } 1 \leqslant i < j \leqslant 5.$$

Proposition 2.3. *Let $\mathbb{G}$ be the the cone over a weighted Grassmannian $\mathbb{P}(b_1, \dots, b_n) \ltimes \mathbb{G}(c_1, \dots, c_5)$. Denote $\sum_{1=1}^{5} c_i$ by k. Then*

*we use $\underline{x}$ to denote the collection of all m_{ij} and all z_l.

(1) $\mathbb{G}$ *is a variety of dimension* $(n+6)$*;*

(2) $\omega_{\mathbb{G}} = \mathcal{O}_{\mathbb{G}}(-2k - \sum_i b_i)$*;*

(3) $\deg \mathbb{G} = \frac{\sum \binom{k-c_i}{3} - \sum \binom{k+c_i}{3} + \binom{2k}{3}}{\prod_l b_l \cdot \prod_{i<j}(c_i+c_j)}$*;*

(4) $H^i(\mathcal{O}_{\mathbb{G}}(j)) = 0$ *for all* $0 < i < n+6$*.*

Proof: Item (i) is clear from either the definition or the geometric interpretation of cone over a weighted Grassmannians. Item (ii) is deduced from the projective resolution of the ideal of Pfaffians:

$$A \xleftarrow{(\mathrm{Pf}_i)} \bigoplus_{i=1}^{5} A(c_i - k) \xleftarrow{M} \bigoplus_{i=1}^{5} A(-c_i - k) \xleftarrow{(\mathrm{Pf}_i)^t} A(-2k) \leftarrow 0 \tag{2}$$

where $A = \mathbb{C}[z_1, \dots z_n, m_{ij}]$ with $\mathrm{wt}(z_i) = b_i$ and $\mathrm{wt}(m_{ij}) = c_i + c_j$. The dualising module of $\mathbb{P}$ is given by $\mathcal{O}_{\mathbb{P}}(-4k - \sum_i b_i)$ and $\omega_{\mathbb{G}} = \mathcal{O}_{\mathbb{G}}(-2k - \sum_i b_i)$ follows from Gorenstein adjunction.

Given that R is a quotient of A by a Gorenstein ideal, the computation of the degree can be carried out from (2) by a standard formula in the theory of Hilbert series. Namely we have:

$$\deg \mathbb{G} = \frac{\binom{2k}{3} - \sum_{i=1}^{5} \binom{2k+c_i-k}{3} + \sum_{i=1}^{5} \binom{2k-c_i-k}{3} - \binom{2k-2k}{3}}{\prod_l b_l \cdot \prod_{i<j}(c_i + c_j)}$$

where the numerator is a sum of binomial coefficients of the form "$\{2k +$ twist$\}$ choose $\{\operatorname{codim} \mathbb{G}\}$" whose sign changes as we move a step in (2). Likewise item (iv) follows from a standard computation in the theory of arithmetically Cohen–Macaulay subschemes. We split sequence (2) into short exact sequences and use $H^i(\mathcal{O}_{\mathbb{P}}(j)) = 0$ for $0 < i < 9 + n$. ■

Proposition 2.4. *A general complete intersection of four quasihomogeneous forms of degree* 2 *in* $\mathbb{G}(\frac{1}{2}^4, \frac{3}{2})$ *is a nonsingular regular surface of general type with* $p_g = 6$*,* $K^2 = 13$ *and whose canonical image is not contained in a pencil of quadrics.*

Proof: The weighted Grassmannian $X = \mathbb{G}(\frac{1}{2}^4, \frac{3}{2})$ is a subscheme of weighted projective space $\mathbb{P}(1^6, 2^4) = \mathbb{P}[x_{ij}, y_i]$ where $1 \leqslant i < j \leqslant 5$. We claim that X is nonsingular away from $\mathbb{P}[y_1, \dots, y_4]$. Indeed, since X and $\mathrm{G}(2,5)$ have the same affine cone and $\mathrm{G}(2,5)$ is nonsingular, X is nonsingular away from the locus of points of the affine cone with nontrivial $\mathbb{C}^*$ stabiliser. It is clear that if four quasihomogeneous forms are chosen general enough (i.e. so that it is possible to write them as $y_k = q_2(x_{ij})$ for $k = 1..4$) the subscheme they cut out in X does not meet $\mathbb{P}[y_1, \ldots, y_4]$ and therefore, by an argument of the type of Bertini's theorem this subscheme is nonsingular. In this situation, the intersection of these forms with $\mathbb{G}(\frac{1}{2}^4, \frac{3}{2})$ is a surface, S, with $K_S = \mathcal{O}_X(1)_{|S}$. We calculate its invariants: $p_g = 6$, $q = 0$, $K_S^2 = 2^4 \cdot \frac{13}{4} = 13$. Since the map $\mathrm{sym}^2\colon \mathrm{S}^2 H^0(K_S) \to H^0(2K_S)$ is surjective, we deduce that the canonical image of S is not contained in a pencil of quadrics. ■

3. Generators of the canonical ring of S

Proposition 3.1. *Let S be a nonsingular surface of general type with $p_g = 6$ and $K^2 = 13$ whose canonical image is not contained in a pencil of quadrics. Then the canonical map is a birational morphism. Moreover, S is regular.*

Proof: The canonical image of S is not contained in a pencil of quadrics if and only of the map

$$\mathrm{sym}^2\colon \mathrm{S}^2 H^0(K_S) \to H^0(2K_S)$$

has a kernel of dimension $\leqslant 1$. Now, on one hand, by Riemann–Roch

$$\chi(\mathcal{O}_S(K_S)) = \chi(\mathcal{O}_S) + K_S^2 \iff h^0(2K_S) = 20 - q(S).$$

On the other hand $\dim \mathrm{S}^2 H^0(K_S) = 21$ and accordingly,

$$q(S) \geqslant 1 \iff \dim \mathrm{Ker}(\mathrm{sym}^2) \geqslant 2.$$

Therefore if the canonical image of S is not contained in a pencil of quadrics then $q(S) = 0$; among other things. A classical result of Bombieri [2] states that for a nonsingular surface of general type with $K^2 \geqslant 5$ the linear system $|2K_S|$ is base point free. Hence, for a surface of general type with $K^2 \geqslant 5$ such that sym^2 is surjective, the canonical linear system is base point free. If the morphism φ_{K_S} maps S onto a curve then there exists a positive integer r such that $K_S = E_1 + \cdots + E_r$, where E_i are nonsingular curves, fibres of a factor map of φ_{K_S}. Since E_i are numerically equivalent, denoting by E a particular fibre, we deduce that $13 = K_S^2 = rK_SE$, so that $K_SE = 1$. However by adjunction, $K_SE + E^2 = 2g - 2$ which is a contradiction, since $E^2 = 0$. We deduce that φ_{K_S} is a birational morphism onto a surface of degree 13. ■

As far as the canonical ring is concerned, if sym^2 is surjective, $R(S, K_S)$ needs no new generators in degree 2. It is a theorem of Ciliberto [4] that the degree of the generators of $R(S, K_S)$ is $\leqslant 3$. Hence we should determine whether the map

$$H^0(K_S) \otimes H^0(2K_S) \to H^0(3K_S)$$

is or is not surjective. We will show that $R(S, K_S)$ is generated by $H^0(K_S)$ so the answer is affirmative. Our proof relies on the hyperplane principle and on the geometry of a general canonical curve $\mathcal{C} \in |K_S|$. We recall the statement of the hyperplane principle and leave the proof to the reader.

Proposition 3.2 (Hyperplane principle)**.** *Let R be a graded ring and $x_0 \in R_d$ a nonzero-divisor. Denote by $\widehat{R}$ the quotient $R/(x_0)$. Then, there exists an exact sequence:*

$$0 \to R(-d) \xrightarrow{x_0} R \xrightarrow{\pi} \widehat{R} \to 0$$

of graded homomorphisms of degree 0 such that the following hold:

(1) *If $\widehat{x_1}, \dots, \widehat{x_n} \in \widehat{R}$ generate the ring $\widehat{R}$ then choosing pre-images $x_1, \dots, x_n$ under π of $\widehat{x_1}, \dots, \widehat{x_n}$ the elements $x_0, \dots, x_n \in R$ generate R.*
(2) *If $\widehat{R} = \mathbb{C}[\widehat{x_1}, \dots, \widehat{x_n}]/(f_1, \dots, f_m)$ then there exist $F_1, \dots, F_m$ in $\mathbb{C}[x_0 \dots, x_n]$ such that $R = \mathbb{C}[x_0 \dots, x_m]/(F_1, \dots, F_m)$.*

Suppose that $S \subset \mathbb{P}^n$ is a canonical surface. Then a general hyperplane section is a curve with a halfcanonical divisor given, by adjunction, as the restriction to the curve of the canonical divisor of S. Let $f \in H^0(K_S)$ be an equation of the hyperplane, then under some assumptions, the ring $R(S, K_S)/(f)$ is the halfcanonical ring $R(\mathcal{C}, A)$.

Proposition 3.3. *Let S be a nonsingular regular surface of general type. If $\mathcal{C} \in |K_S|$ is a nonsingular curve cut out by $f \in H^0(K_S)$, then*

$$R(\mathcal{C}, A) = R(S, K_S)/(f).$$

Proof: By regularity and Kodaira vanishing the cohomology space $H^1(nK_S)$ is null for any integer n. Hence from the restriction exact sequence, we deduce that

$$0 \to H^0((n-1)K_S) \xrightarrow{f} H^0(nK_S) \to H^0(\mathcal{C}, nA) \to 0.$$

The direct sum of these sequences implies that $R(\mathcal{C}, A) \simeq R(S, K_S)/(f)$. ■

3.1. The halfcanonical ring of $\mathcal{C} \in |K_S|$. We return to the case of a nonsingular surface of general type with $p_g = 6$ and $K_S^2 = 13$ whose canonical image is not contained in a pencil of quadrics. Recall from Proposition 3.1, that such surface is necessarily regular and its canonical map is a birational morphism. By Bertini's theorem, a general member $\mathcal{C} \in |K_S|$ is a nonsingular reduced curve of genus 14 with a halfcanonical divisor $A = K_{S|\mathcal{C}}$ whose space of global sections $H^0(A)$, by regularity of S, is 5-dimensional. Moreover A is free and the map

$$\mathrm{sym}^2 \colon H^0(A) \to H^0(2A), \tag{3}$$

(which we still denote by sym^2) is surjective. The next proposition in conjunction with the hyperplane principle shows that $R(S, K_S)$ is generated in degree 1.

Proposition 3.4. *Let $\mathcal{C}$ be nonsingular curve of genus 14 with a halfcanonical divisor A such that $h^0(A) = 5$. Assume that sym^2 is surjective. Then $R(\mathcal{C}, A)$ is a codimension 3 ring generated in degree 1.*

Proof: We start by proving the following lemma.

Lemma 3.5. *Let A and B be two divisors on an algebraic curve $\mathcal{C}$. The extension bundles of $\mathcal{O}_{\mathcal{C}}(B)$ by $\mathcal{O}_{\mathcal{C}}(A)$ with maximum number of global sections are parametrised by the cokernel of the multiplication map*

$$H^0(K_{\mathcal{C}} - A) \otimes H^0(B) \to H^0(K_{\mathcal{C}} + B - A).$$

Proof: The group classifying extensions of $\mathcal{O}_{\mathcal{C}}(B)$ by $\mathcal{O}_{\mathcal{C}}(A)$ is $\mathrm{Ext}^1(B, A)$. By Serre duality we have:

$$\mathrm{Ext}^1(B, A) = \mathrm{Ext}^1(K_{\mathcal{C}} + B - A, K_{\mathcal{C}}) \simeq H^0(K_{\mathcal{C}} + B - A)^\vee.$$

On the other hand, extensions of $\mathcal{O}_{\mathcal{C}}(B)$ by $\mathcal{O}_{\mathcal{C}}(A)$ with maximum number of global sections,

$$0 \to \mathcal{O}_{\mathcal{C}}(A) \to \mathscr{F} \to \mathcal{O}_{\mathcal{C}}(B) \to 0$$

have zero connecting homomorphism $H^0(B) \to H^1(A)$, i.e. $\mathscr{F}$ has maximum number of global sections if and only if its class $[\mathscr{F}] \in \mathrm{Ext}^1(B, A)$, under the canonical morphism $\mathrm{Ext}^1(B, A) \to \mathrm{Hom}(H^0(B), H^1(A))$ maps to zero. Which is to say, $\mathscr{F}$ has maximum number of global sections if and only if

$$[\mathscr{F}] \in \mathrm{Ker}\left\{\mathrm{Ext}^1(B, A) \to H^0(B)^\vee \otimes H^1(A)\right\}. \tag{4}$$

Again by Serre duality we have

$$H^1(A) \simeq \mathrm{Ext}^0(A, K_{\mathcal{C}})^\vee = \mathrm{Ext}^0(0, K_{\mathcal{C}} - A)^\vee = H^0(K_{\mathcal{C}} - A)^\vee.$$

Finally, dualising statement (4) we conclude that extension classes corresponding to bundles with maximum number of global sections are in bijection with the cokernel of $H^0(K_{\mathcal{C}} - A) \otimes H^0(B) \to H^0(K_{\mathcal{C}} + B - A)$. ■

The lemma is used in the remainder of this proof to equate the cokernel of the maps

$$H^0(A) \otimes H^0(nA) \to H^0((n+1)A) \tag{5}$$

with certain extension bundles on $\mathcal{C}$. First, let us treat the case $n \geqslant 3$. From Lemma 3.5, showing that (5) is surjective is equivalent to showing that all extension bundles of $\mathcal{O}_{\mathcal{C}}(A)$ by $\mathcal{O}_{\mathcal{C}}((2-n)A)$ with 5 global sections are split extensions. Let $\mathscr{F}$ be such an extension bundle

$$0 \to \mathcal{O}_{\mathcal{C}}((2-n)A) \to \mathscr{F} \to \mathcal{O}_{\mathcal{C}}(A) \to 0. \tag{6}$$

Since $h^0(\mathscr{F}) = 5$, there exists a section of $\mathscr{F}$ with nontrivial divisor of zeros δ. This section yields an embedding $\mathcal{O}_{\mathcal{C}}(\delta) \hookrightarrow \mathscr{F}$, which upon saturation gives:

$$0 \to \mathcal{O}_{\mathcal{C}}(\xi) \to \mathscr{F} \to \mathcal{O}_{\mathcal{C}}((3-n)A - \xi) \to 0.$$

Since ξ is effective and $n \geqslant 3$ we deduce that $h^0(\xi) = h^0(A) = 5$. Besides, as $\mathcal{O}_{\mathcal{C}}(\xi)$ does not embed into $\mathcal{O}_{\mathcal{C}}((2-n)A)$, the composition of $\mathcal{O}_{\mathcal{C}}(\xi) \to \mathscr{F}$ with the map $\mathscr{F} \to \mathcal{O}_{\mathcal{C}}(A)$ of (6) is injective. Since A is free, we conclude that $\xi \simeq A$ and therefore that $\mathscr{F}$ is isomorphic to the split extension.

Let us now show that the map

$$H^0(A) \otimes H^0(2A) \to H^0(3A) \tag{7}$$

is surjective. Let $\mathscr{F}$ be an extension of $\mathcal{O}_{\mathcal{C}}(A)$ by $\mathcal{O}_{\mathcal{C}}$ with 6 global sections,

$$0 \to \mathcal{O}_{\mathcal{C}} \to \mathscr{F} \to \mathcal{O}_{\mathcal{C}}(A) \to 0. \tag{8}$$

Our aim is to show that $\mathscr{F}$ is split. From the dimension of the space of global sections of $\mathscr{F}$ we deduce that for any two $p, q \in \mathcal{C}$ there exists a section of that bundle vanishing on $p + q$. Denote the divisor of zeros of such a section by δ. Saturating the embedding $\mathcal{O}_{\mathcal{C}}(\delta) \hookrightarrow \mathscr{F}$ we obtain

$$0 \to \mathcal{O}_{\mathcal{C}}(\xi) \to \mathscr{F} \to \mathcal{O}_{\mathcal{C}}(A - \xi) \to 0$$

where $\xi \supset \delta$ is an effective divisor. Since p, q can be chosen general enough we have $h^0(A - \xi) \leqslant h^0(A) - 2$ and accordingly $h^0(\xi) \geqslant 3$. Assume for the time being that $\mathcal{C}$ has no g^2_9. Then $\deg(\xi) \geqslant 10$. But then $h^0(A - \xi) \leqslant 1$

since $\nexists g^2_9 \implies \mathrm{gon}(\mathcal{C}) \geqslant 4$. Therefore $h^0(\xi) \geqslant h^0(A)$ and we must have $\xi \subset \mathcal{O}_{\mathcal{C}}(A)$. From the fact that A is free we conclude that $\xi = A$. In other words, an element of the cokernel of (6) corresponds to the split extension, i.e. the cokernel is null. Hence, provided that we can show that $\mathcal{C}$ has no g^2_9 we can show that $R(\mathcal{C}, A)$ is generated in degree 1. We need an auxiliary result.

Lemma 3.6. *Let $\mathcal{C}$ be a nonsingular curve and A a halfcanonical divisor for which* $\mathrm{sym}^2\colon H^0(A) \to H^0(K_{\mathcal{C}})$ *is surjective. Let D be a divisor on $\mathcal{C}$. Denote by d the dimension of $H^0(A)/H^0(A-D)$. Then:*

$$\deg(D) - h^0(D) \leqslant \frac{d(d+1)}{2} - 1.$$

Proof: Since sym^2 is surjective, the induced map

$$\mathrm{S}^2\left(\frac{H^0(A)}{H^0(A-D)}\right) \to \frac{H^0(K_{\mathcal{C}})}{H^0(K_{\mathcal{C}}-D)}$$

is surjective. Hence $h^0(K_{\mathcal{C}}) - h^0(K_{\mathcal{C}} - D) \leqslant \frac{d(d+1)}{2}$. On the other hand, by Riemann-Roch,

$$h^0(K_{\mathcal{C}}) - h^0(K_{\mathcal{C}} - D) = \deg(D) - h^0(D) + 1.$$

■

To show that the surjectivity of sym^2 implies the nonexistence of a g^2_9 we begin by showing that $\mathrm{gon}(\mathcal{C}) \geqslant 6$. Suppose there exists a divisor on $\mathcal{C}$ with $h^0(D) = 2$ and $\deg(D) \leqslant 5$. Since $h^0(A) - h^0(A - (A-D)) = 3$, applying Lemma 3.6, we deduce that $h^0(A-D) \geqslant 8 - \deg(D)$. Hence $h^0(A) - h^0(A-D) \leqslant \deg(D) - 3$. In particular, $\deg(D) = 4$ or 5. By the same lemma, we deduce that if $\deg(D) = 5$, then $\deg(D) - 2 \leqslant 2$ and if $\deg(D) = 4$, that $\deg(D) - 2 \leqslant 0$. A contradiction in both cases. We have shown that $\mathrm{gon}(\mathcal{C}) \geqslant 6$. Finally, assume that for some $d \leqslant 9$, there exists a free g^2_d on $\mathcal{C}$ and let us denote the divisor of the associated complete linear system by D. Since $h^0(A) - h^0(A-(A-D)) \leqslant 2$, from Lemma 3.6 we deduce that $\deg(A-D) - h^0(A-D) \leqslant 2$, i.e. $h^0(A-D) \geqslant 2$, which is a contradiction, since $\mathrm{gon}(\mathcal{C}) \geqslant 6$. We have finished the proof of Proposition 3.4. ■

Corollary 3.7. *Let S be a nonsingular surface of general type with $p_g = 6$ and $K^2 = 13$ whose canonical image is not contained in a pencil of quadrics. Then the canonical ring $R(S, K_S)$ is generated in degree* 1.

Corollary 3.8. *Let S be a nonsingular surface of general type with $p_g = 6$ and $K_S^2 = 13$ whose canonical image is not contained in a pencil of quadrics. Then the canonical morphism factors through $S \to \operatorname{Proj} R(S, K_S)$, the pluricanonical morphism, and therefore the canonical image of S is a surface with at most Du Val singularities.*

4. Generators of the canonical ideal S

From Corollary 3.7 we know that there exists a surjective homomorphism of graded rings $\mathrm{ev}\colon \mathbb{C}[x_1, \dots, x_6] \to R(S, K_S)$. To describe the canonical model of S we need to describe the canonical ideal $\mathcal{I}_{K_S}$, which is the kernel of the map ev. By analogy with the case of complete intersections in projective space, we hope to obtain this information from a "key variety." This variety is the cone over a weighted Grassmannian $X = \mathbb{P}(1^2) \ltimes \mathrm{G}(\frac{1}{2}^4, \frac{3}{2})$.

4.1. A vector bundle on S. As we show in the proof of Proposition 3.4 a general member of $|K_S|$ is a nonsingular curve $\mathcal{C}$ with $\mathrm{gon}(\mathcal{C}) \geqslant 6$. Suppose that there exists a nonsingular member of $|K_S|$ with a (free and complete) g^1_6 that we denote by ξ. Then the evaluation morphism

$$2\mathcal{O}_S \to 2\mathcal{O}_{\mathcal{C}} \to \mathcal{O}_{\mathcal{C}}(\xi)$$

has a locally free kernel of rank 2. Its dual $\mathscr{E}$ is a locally free sheaf of rank 2 (or equivalently a vector bundle of rank 2) fitting in the exact sequence:

$$0 \to 2\mathcal{O}_S \to \mathscr{E} \to \mathscr{E}xt^1(\mathcal{O}_{\mathcal{C}}(\xi), \mathcal{O}_{\mathcal{C}}) \to 0. \tag{9}$$

This is the bundle we use to write the embedding of S into $\mathbb{P}(1^2)\ltimes\mathbb{G}(\frac{1}{2}^4, \frac{3}{2})$. Before we enumerate its properties we show that there exists a nonsingular curve $\mathcal{C} \in |K_S|$ with a g^1_6.

Proposition 4.1. *Let S be a nonsingular surface of general type with $p_g = 6$ and $K_S^2 = 13$ whose canonical image is not contained in a pencil of quadrics. Then there exists a nonsingular curve in $|K_S|$ with a g^1_6.*

Proof: From Corollary 3.8, $\Sigma \subset \mathbb{P}^5$, the canonical image of S, has at most Du Val singularities. By a numerical argument Σ is contained in a quadric Q of rank $\geqslant 3$. To show that there exists a nonsingular curve $\mathcal{C} \in |K_S|$ having a pencil of degree 6, it is enough to show that the hyperplane section of Σ determining $\mathcal{C}$ is contained in a quadric of rank $\leqslant 4$, since the ruling of a quadric of rank 3 induces on the curve a pencil of degree 6 and the rulings of a quadric of rank 4, pencils of degree 6 and 7. If $\operatorname{rank} Q \leqslant 4$ then this is obvious: all hyperplane sections satisfy the requirement. In the remaining cases we must show that there exists a hyperplane H such that $Q \cap H$ is a quadric of rank $\leqslant 4$ and $\Sigma \cap H$ is a nonsingular curve, which is to say, H is not tangent to Σ at any point of $\Sigma \cap H$. Suppose that $\operatorname{rank} Q \geqslant 5$. The variety parametrising tangent hyperplanes to Q (the dual of Q) is a nonsingular quadric in dual projective space of dimension $\operatorname{rank} Q - 2 \geqslant 3$. Its linear span has dimension $\operatorname{rank} Q - 1$ and coincides with the locus of hyperplanes H such that $\operatorname{rank} Q \cap H \leqslant \operatorname{rank} Q - 1$. The dual variety of Σ (containing an open subset parametrising hyperplanes containing tangent planes at nonsingular points of Σ) has dimension $\leqslant 2 + 2 = 4$. Suppose that $\operatorname{rank} Q = 5$, then, since the dual of Σ is not a hyperplane in dual projective space, there exists a hyperplane H not tangent to Σ such that $\operatorname{rank} Q \cap H \leqslant 5 - 1 = 4$. If $\operatorname{rank} Q = 6$ then the dimension of the dual of Q is 4 and since the dual of Σ does not coincide with the dual of Q, there exists a tangent hyperplane to Q not tangent to Σ. For this hyperplane $\operatorname{rank} Q \cap H \leqslant 6 - 2 = 4$. ■

Remark 4.2. As we mentioned before, $\mathcal{C} \in |K_S|$ has a halfcanonical divisor A with $h^0(A) = 5$, given by the restriction of the canonical divisor of S. Notice that by Proposition 3.3 we deduce that the map $\operatorname{sym}^2\colon \operatorname{S}^2 H^0(A) \to H^0(K_{\mathcal{C}})$ is surjective. Indeed this fact holds for any nonsingular curve $\mathcal{C}$ in $|K_S|$. In particular we can use the arguments of the proof of Proposition 3.4 to deduce that $\operatorname{gon}(\mathcal{C}) = 6$ and that $\mathcal{C}$ has no g^2_9.

Proposition 4.3. *Let S be a nonsingular surface of general type with $p_g = 6$ and $K^2 = 13$ whose canonical image not contained in a pencil of quadrics. Then, there exists a bundle $\mathscr{E}$ of rank 2 and determinant K_S such that:*

(1) $\dim H^0(\mathscr{E}) = 4$;
(2) $\bigwedge^2 H^0(\mathscr{E}) \to H^0(K_S)$ *has a kernel of dimension* $\leqslant 2$;
(3) $H^0(K_S) \otimes H^0(\mathscr{E}) \to H^0(\mathscr{E}(K_S))$ *has a 1-dimensional cokernel.*

Proof: Let $\mathcal{C} \in |K_S|$ be a nonsingular curve with a g^1_6. Since the gonality of $\mathcal{C}$ is 6 such linear system is necessarily free and complete. In what follows we make no distinction between ξ and its associated divisor. Use Castelnuovo's free-pencil trick to obtain

$$0 \to \mathcal{O}_{\mathcal{C}}(A - \xi) \to 2\mathcal{O}_{\mathcal{C}}(A) \to \mathcal{O}_{\mathcal{C}}(A + \xi) \to 0.$$

From this we deduce that $h^0(A - \xi) \geqslant h^0(A) - \deg(\xi) = 2$. And since $\mathcal{C}$ has no g^2_9, $h^0(A - \xi) \leqslant 2$. We deduce that $|A - \xi|$ is a g^1_7. (Not necessarily a free one.) Let us denote it by η. Recall that

$$0 \to \mathscr{E}^\vee \to 2\mathcal{O}_S \to \mathcal{O}_\mathcal{C}(\xi) \to 0. \tag{10}$$

Since $\mathscr{E}xt^1_{\mathcal{O}_S}(\mathcal{O}_\mathcal{C}(\xi), \mathcal{O}_S) = \mathcal{O}_\mathcal{C}(A - \xi)$, the dual of the above sequence is

$$0 \to 2\mathcal{O}_S \to \mathscr{E} \to \mathcal{O}_\mathcal{C}(\eta) \to 0. \tag{11}$$

From this and the fact that $\mathcal{C} \in |K_S|$ we deduce that $\det(\mathscr{E}) = K_S$.

Proof of (i). By regularity of S, $\dim H^1(2\mathcal{O}_S) = 0$. Taking global sections of the exact sequence (11) we get $h^0(\mathscr{E}) = h^0(2\mathcal{O}_S) + h^0(\mathcal{O}_\mathcal{C}(\eta)) = 4$.

Proof of (ii). Here, we use an argument involving the restriction of $\mathscr{E}$ to the curve $\mathcal{C} \in |K_S|$ used to construct $\mathscr{E}$. Consider the following diagram:

$$\begin{tikzcd}
0 \arrow[r] & \mathscr{E}^\vee \arrow[r] \arrow[d] & 2\mathcal{O}_S \arrow[r] \arrow[d] & \mathcal{O}_\mathcal{C}(\xi) \arrow[r] \arrow[d] & 0 \\
0 \arrow[r] & \mathscr{E} \arrow[r, "id"] \arrow[d] & \mathscr{E} \arrow[r] \arrow[d] & 0 & \\
& \mathscr{E}_C & \mathcal{O}_\mathcal{C}(\eta) & &
\end{tikzcd}$$

Applying the snake lemma we deduce that

$$0 \to \mathcal{O}_\mathcal{C}(\xi) \to \mathscr{E}_\mathcal{C} \to \mathcal{O}_\mathcal{C}(\eta) \to 0. \tag{12}$$

Since $\mathscr{E}(-K_S) \simeq \mathscr{E}^\vee$ and from (10), $H^0(\mathscr{E}^\vee) = 0$, we deduce that the restriction morphism yields an isomorphism $H^0(\mathscr{E}_\mathcal{C}) \simeq H^0(\mathscr{E})$ on the global sections of $\mathscr{E}_\mathcal{C}$ and $\mathscr{E}$. From

$$0 \to 2\mathcal{O}_S \to \mathscr{E} \to \mathcal{O}_\mathcal{C}(\eta) \to 0$$

we see that there exists a pair of sections $s_1, s_2 \in H^0(\mathscr{E})$ such that $s_1 \wedge s_2 \neq 0$ on S but $s_1 \wedge s_2 = 0$ on $\mathcal{C}$. Therefore, item (ii) of this proposition, follows from the next lemma.

Lemma 4.4. $\bigwedge^2 H^0(\mathscr{E}_\mathcal{C}) \to H^0(A)$ *has a kernel of dimension* $\leqslant 3$.

Proof: Let us denote by W the kernel of the map

$$\bigwedge^2 H^0(\mathscr{E}_\mathcal{C}) \to H^0(A).$$

The projective space $\mathbb{P}[W]$ is a linear subspace of $\mathbb{P}(\bigwedge^2 H^0(\mathscr{E}_\mathcal{C}))$. The variety of skew tensors of rank 2, which we denote by $\mathrm{G}(2, H^0(\mathscr{E}_\mathcal{C}))$, is also contained in $\mathbb{P}(\bigwedge^2 H^0(\mathscr{E}_{\mathcal{C}}))$. Let $a \wedge b$ be an element of $\mathbb{P}[W] \cap \mathrm{G}(2, H^0(\mathscr{E}_{\mathcal{C}}))$. This means that $a, b \in H^0(\mathscr{E}_{\mathcal{C}})$ span a (torsion free) subsheaf of $\mathscr{E}$ of rank 1, that is given by $\mathcal{O}_{\mathcal{C}} \cdot a + \mathcal{O}_{\mathcal{C}} \cdot b$. Since $\mathcal{C}$ is nonsingular this sheaf is invertible. The saturation of $\mathcal{O}_{\mathcal{C}} \cdot a + \mathcal{O}_{\mathcal{C}} \cdot b \subset \mathscr{E}_{\mathcal{C}}$ yields

$$0 \to \mathcal{O}_{\mathcal{C}}(\delta) \to \mathscr{E}_{\mathcal{C}} \to \mathcal{O}_{\mathcal{C}}(A - \delta) \to 0,$$

where $\mathcal{O}_{\mathcal{C}} \cdot a + \mathcal{O}_{\mathcal{C}} \cdot b \subset \mathcal{O}_{\mathcal{C}}(\delta)$. Since $\mathrm{gon}(\mathcal{C}) = 6$ and $h^0(\delta) \geqslant 2$ we deduce that $\deg(\delta) \geqslant 6$. From (12) we deduce that $\mathcal{O}_{\mathcal{C}}(\delta) \hookrightarrow \xi$ or $\mathcal{O}_{\mathcal{C}}(\delta) \hookrightarrow \eta$. The same sequence yields that $h^0(\mathscr{E}_{\mathcal{C}}(-\delta)) \leqslant 2$. Since $h^0(\xi - \delta) \leqslant 1$ and $h^0(\eta - \delta) \leqslant 1$ we conclude that

$$\dim \mathbb{P}[W] \cap \mathrm{G}(2, H^0(\mathscr{E}_{\mathcal{C}})) \leqslant 1.$$

Therefore we must have $\dim W \leqslant 3$. ■

Remark 4.5. If the canonical image of S is contained in a quadric of rank 3, then the image of $\mathcal{C}$ under $\varphi_{|A|}$ is also contained in a quadric of rank 3. Therefore η has a base point. And $\mathscr{E}_{\mathcal{C}}$ can be the split extension, i.e. $\mathscr{E}_{\mathcal{C}} \simeq \mathcal{O}_{\mathcal{C}}(\xi) \oplus \mathcal{O}_{\mathcal{C}}(\eta)$. In this situation, if $\delta \simeq \xi$, $h^0(\mathscr{E}_{\mathcal{C}}(-\delta)) = 2$.

Proof of (iii). Tensoring (11) with $\mathcal{O}_S(K_S)$ we have

$$0 \to 2\mathcal{O}_S(K_S) \to \mathscr{E}(K_S) \to \mathcal{O}_{\mathcal{C}}(A + \eta) \to 0.$$

So that, by RR and Serre duality on $\mathcal{C}$, and using the regularity of S,

$$h^0(\mathscr{E}(K_S)) = 2h^0(K_S) + h^0(\mathcal{O}_{\mathcal{C}}(A + \eta)) = h^0(K_S) + h^1(\mathcal{O}_{\mathcal{C}}(\xi)) = 21.$$

Since $\dim H^0(K_S) \otimes H^0(\mathscr{E}) = 24$ all we have to show is that the kernel of the map:

$$H^0(K_S) \otimes H^0(\mathscr{E}) \to H^0(\mathscr{E}(K_S)) \tag{13}$$

is 4-dimensional. To see this, we identify this map with the map on global sections of a map of sheaves. We choose a 2-dimensional subspace of $H^0(\mathscr{E})$ projecting down to $H^0(\eta)$ and write evaluation maps in the following diagram:

$$\begin{tikzcd}
 & 0 \arrow[r] \arrow[d] & 2\mathcal{O}_S \arrow[r, "\sim"] \arrow[d] & 2\mathcal{O}_S \arrow[d] \\
0 \arrow[r] & \mathscr{N} \arrow[r] & 2\mathcal{O}_S \bigoplus 2\mathcal{O}_S \arrow[r] \arrow[d] & \mathscr{E} \arrow[d] \\
 & & 2\mathcal{O}_S \arrow[r] & \eta
\end{tikzcd}$$

Tensoring the middle sequence with $\mathcal{O}_S(K_S)$, we deduce that the map on global sections of the morphism

$$2\mathcal{O}_S(K_S) \oplus 2\mathcal{O}_S(K_S) \to \mathscr{E}(K_S)$$

is exactly that of (13). Thus its kernel is isomorphic to $H^0(\mathscr{N}(K_S))$. Notice that none of the evaluation maps needs to be surjective, as η (and consequently $\mathscr{E}$) might have base points. The snake lemma applies to the first two rows, giving

$$0 \to \mathscr{N} \to 2\mathcal{O}_S \to \eta. \tag{14}$$

Lemma 4.6. $\dim H^0(\mathscr{N}(K_S)) = 4.$

Proof: By Castelnuovo's free-pencil trick, the map

$$H^0(\mathcal{C}, \mathcal{O}_{\mathcal{C}}(\eta)) \otimes H^0(\mathcal{C}, A) \to H^0(\mathcal{C}, \mathcal{O}_{\mathcal{C}}(A + \eta)) \tag{15}$$

has a kernel isomorphic to $H^0(\mathcal{C}, \mathcal{O}_{\mathcal{C}}(\xi + B))$, where B denotes the base locus of η. Since $\deg B \leqslant 1$ and $\mathcal{C}$ has no g^2_9, $h^0(\xi + B) = 2$. By (14) the dimension of $\mathscr{N}(K_S)$ equals the dimension of the kernel of the map $2H^0(K_S) \to H^0(\mathcal{C}, A + \eta)$ given by multiplication of the two linearly independent sections of $H^0(\eta)$. This map factors through $2H^0(K_S) \to 2H^0(\mathcal{C}, A)$ (whose kernel is 2-dimensional) and the map of (15). We conclude that $\dim H^0(\mathscr{N}(K_S)) = 4$. ■

4.2. The map $\rho\colon S \to X = \mathbb{P}(1^2)\ltimes\mathbb{G}(\frac{1}{2}^4, \frac{3}{2})$. Let s_1, s_2, s_3, s_4 denote a choice of basis for $H^0(\mathscr{E})$. We denote by t a choice of a generator of the cokernel of $H^0(K_S) \otimes H^0(\mathscr{E}) \to H^0(\mathscr{E}(K_S))$. Additionally let $\{u_1, u_2\} \subset H^0(K_S)$ be a choice of 2 (not necessarily linearly independent) generators of the cokernel of $\bigwedge^2(\mathscr{E}) \to H^0(K_S)$. Let X denote the cone over a weighted Grassmannian $\mathbb{P}(1^2)\ltimes\mathbb{G}(\frac{1}{2}^4, \frac{3}{2})$ that (by definition) is a projectively Gorenstein subscheme of $\mathbb{P}(1^2, 1^6, 2^4)$. We fix notation for the variables of this w.p.s. as in $\mathbb{P}[z_1, z_2, m_{ij}, n_i]$. Define a map $\rho\colon S \to X$ in the following way:

$$p \mapsto (u_1(p), u_2(p), s_i \wedge s_j(p), s_i \wedge t(p)) \quad \text{where } 1 \leqslant i < j \leqslant 4.$$

The image of S under ρ is denoted by Γ.

Proposition 4.7. *Let S be a nonsingular surface of general type with $p_g = 6$ and $K_S^2 = 13$ whose canonical image is not contained in a pencil of quadrics. Then there exists a map ρ of S into the cone over a weighted Grassmannian X which is an embedding away from -2-cycles. The image of S under this* *map is contained in the intersection of two quasihomogeneous forms of degree* 1 *and four quasihomogeneous forms of degree* 2*.*

Proof: Let $\mathbb{P}$ denote the w.p.s. $\mathbb{P}(1^2, 1^6, 2^4)$. Since the space $H^0(K_S)$ is generated by $\{u_1, u_2, s_i \wedge s_j \mid 1 \leqslant i < j \leqslant 4\}$, there exists a 6-dimension subspace V of $\langle z_1, z_2, m_{ij}, 1 \leqslant i < j \leqslant 4\rangle \simeq H^0(\mathbb{P}, \mathcal{O}(1))$ such that $V \simeq H^0(K_S)$. Therefore ρ composed with a projection onto the linear subspace $\mathbb{P}[V] \subset \mathbb{P}$ yields the canonical map. By Corollary 3.8 the canonical map is an embedding away from -2-cycles. Likewise, ρ is an embedding away from -2-cycles.

Since the restriction of $H^0(\mathcal{O}_{\mathbb{P}}(1))$ is only 6-dimensional, there are two linear forms of $\mathbb{P}[z_1, z_2, m_{ij}, n_i]$ vanishing on Γ. We denote a choice of two linearly independent quasihomogeneous forms of degree 1 vanishing on Γ by $L_1, L_2 \in \langle z_1, z_2, m_{ij}\rangle$. Since $s_i \wedge t$ is an element of $H^0(2K_S)$ and the map $\mathrm{sym}^2\colon \mathrm{S}^2 H^0(K_S) \to H^0(2K_S)$ is surjective we deduce that there exist $q_1, q_2, q_3, q_4 \in \mathrm{S}^2 \langle z_1, z_2, m_{ij}\rangle$ such that the four quasihomogeneous forms of degree 2, $n_i - q_i$, vanish on Γ. ∎

Since the dimension of $X = \mathbb{P}(1^2) \ltimes \mathrm{G}(\frac{1}{2}^4, \frac{3}{2})$ is 8 we aim to show that Γ is the complete intersection of $L_1, L_2, y_1 - q_1, \ldots, y_4 - q_4$. At the same time this enables the understanding of the structure of the canonical ideal. From now on, we fix a choice of the linearly independent forms vanishing on Γ as given in the previous proposition. Notice that since the map sym^2 has a 1-dimensional kernel, the quadratic forms q_i are only well defined modulo this kernel. Let us also fix a choice of q_i.

Proposition 4.8. *Consider the four quasihomogeneous forms of degree two,* $y_1 - q_1$*,* $y_2 - q_2$*,* $y_3 - q_3$*,* $y_4 - q_4$ *vanishing on* Γ*. Let* $\Upsilon \subset \mathbb{P}[z_1, z_2, m_{ij}]$ *be the variety cut out on* X *by* $y_1 - q_1, \ldots, y_4 - q_4$*. In other words, consider* Υ *in* $\mathbb{P}[z_1, z_2, m_{ij}]$ *cut out by the* 5 *submaximal Pfaffians of*

$$\begin{pmatrix} m_{12} & m_{13} & m_{14} & q_1 \\ & m_{23} & m_{24} & q_2 \\ & & m_{34} & q_3 \\ & & & q_4 \end{pmatrix}.$$

Then, $\mathrm{Pf}_1, \mathrm{Pf}_2, \mathrm{Pf}_3, \mathrm{Pf}_4$ *are linearly independent modulo* $\langle z_1, z_2, m_{ij}\rangle \cdot \mathrm{Pf}_5$*.*

Proof: Given any set of homogeneous elements s, t, u, v of the Serre module of $\mathscr{E}$, we denote by $\mathrm{Pf}(s, t, u, v)$ the Pfaffian:

$$\mathrm{Pf}\begin{pmatrix} s\wedge t & s\wedge u & s\wedge v \\ & t\wedge u & t\wedge v \\ & & u\wedge v \end{pmatrix}.$$

We argue by contradiction. Suppose that there exist $\alpha_i \in \mathbb{C}$ and $L \in \langle z_1, z_2, m_{ij}\rangle$ such that

$$\sum_{i=1}^{4} \alpha_i \mathrm{Pf}_i + L\,\mathrm{Pf}_5 = 0 \tag{16}$$

is a nontrivial linear dependence relation. Let us show that then, there exists a section $s \in H^0(\mathscr{E})$ and a section $u \in H^0(\mathscr{E}(K_S))$ spanning the cokernel of the map $H^0(K_S) \otimes H^0(\mathscr{E}) \to H^0(\mathscr{E}(K_S))$ such that $s \wedge u = 0$. Let i_0 be an integer in $\{1, 2, 3, 4\}$ such that $\alpha_{i_0} \neq 0$. In view of (16) we deduce that

$$\sum_{i=1}^{4} \alpha_i \mathrm{Pf}((s_j)_{j\neq i_0}, t) + L\,\mathrm{Pf}(s_1, s_2, s_3, s_4) = 0.$$

Let $u = t + \frac{L}{\alpha_{i_0}} s_{i_0}$. It is easy to see that $\mathrm{Pf}(s, t, u, v)$ is skew-multilinear in s, t, u, v. Bearing this in mind we deduce that

$$\sum_{i=1}^{4} \alpha_i \mathrm{Pf}((s_j)_{j\neq i_0}, u) = \sum_{i=1}^{4} \alpha_i \mathrm{Pf}((s_j)_{j\neq i_0}, t) + L\,\mathrm{Pf}(s_1, s_2, s_3, s_4) = 0.$$

Consider a new basis of $H^0(\mathscr{E})$ given by

$$a_{i_0} = s_{i_0} \quad \text{and} \quad a_j = \alpha_{i_0} s_j \pm \alpha_j s_{i_0} \quad \text{for} \quad j \neq i_0.$$

The $\pm$ signs can be determined as a function of i_0. Then

$$\mathrm{Pf}((a_j)_{j\neq i_0}, u) = \sum_{i=1}^{4} \alpha_i \mathrm{Pf}((s_i)_{i\neq i_0}, u) = 0.$$

Therefore we can reduce to the case when $\mathrm{Pf}(s_1, s_2, s_3, t)$ is zero. This is

$$\mathrm{Pf}\begin{pmatrix} s_1\wedge s_2 & s_1\wedge s_3 & s_1\wedge t \\ & s_2\wedge s_3 & s_2\wedge t \\ & & s_3\wedge t \end{pmatrix} = 0. \tag{17}$$

Notice that $\mathrm{G}(2, 3) \simeq \mathbb{P}^2$ and therefore all tensors in $\bigwedge^2 \langle s_1, s_2, s_3\rangle$ are decomposable. Accordingly, recalling that the kernel of $\bigwedge^2 H^0(\mathscr{E}) \to H^0(K_S)$ is at most 2-dimensional (see Proposition 4.3), there are three possibilities, enumerated by the dimension of $\mathrm{Ker}\{\bigwedge^2 \langle s_1, s_2, s_3\rangle \to H^0(K_S)\}$. If this kernel is 2 dimensional then, possibly after changing the choice of basis of $\langle s_1, s_2, s_3\rangle$ we can assume that $s_2 \wedge s_3 \neq 0$ and so from (17) we get $(s_2 \wedge s_3)(s_1 \wedge t) = 0$, i.e. $s_1 \wedge t = 0$. If the kernel is 1-dimensional then we can assume that $s_1 \wedge s_2 = 0$ and that $s_1 \wedge s_3$, $s_2 \wedge s_3$ form a regular sequence in $R(S, K_S)$. Then from (17) we deduce that there exists $k \in H^0(K_S)$ such that $s_1 \wedge t = k(s_1 \wedge s_3)$ and $s_2 \wedge t = k(s_2 \wedge s_3)$. Then $s_1 \wedge (t - ks_3) = 0$ and $s_2 \wedge (t - ks_3) = 0$. Finally if that kernel is 0-dimensional and hence $s_1 \wedge s_2$, $s_1 \wedge s_3$, $s_2 \wedge s_3$ form a regular sequence in $R(S, K_S)$, (17) implies that there exist $k_1, k_2, k_3 \in H^0(K_S)$ such that

$$\begin{cases} s_3 \wedge t = k_1(s_1 \wedge s_3) - k_2(s_2 \wedge s_3) \\ s_2 \wedge t = k_1(s_1 \wedge s_2) + k_3(s_2 \wedge s_3) \\ s_3 \wedge t = k_2(s_1 \wedge s_2) + k_3(s_1 \wedge s_3). \end{cases}$$

And then, for example, $s_3 \wedge (t + k_1 s_1 - k_2 s_2) = 0$.

We have shown that a nontrivial linear dependence relation as (16) implies the existence of s in $H^0(\mathscr{E})$ and u in $H^0(\mathscr{E}(K_S))$ spanning the cokernel of the map $H^0(K_S) \otimes H^0(\mathscr{E}) \to H^0(\mathscr{E}(K_S))$ such that $s \wedge u = 0$. The next lemma draws the contradiction.

Lemma 4.9. *Let $u \in H^0(\mathscr{E}(K_S))$ be a section spanning the cokernel of the map $H^0(K_S) \otimes H^0(\mathscr{E}) \to H^0(\mathscr{E}(K_S))$. Then the map $H^0(\mathscr{E}) \xrightarrow{\wedge t} H^0(2K_S)$ is injective.*

Proof: The map $\mathscr{E} \to \mathcal{O}_S(2K_S)$ given by wedging sections of $\mathscr{E}$ with u may not be surjective. In any case its image is a torsion free sheaf that we write as $\mathcal{O}_S(2K_S - D) \otimes \mathcal{I}_\delta$ for some divisor D and for some 0-dimensional subscheme δ of S. Since $\mathscr{E}$ is globally spanned except a possibly at a 0-dimension locus of S, D is the divisor of zeros u. We have

$$0 \to \mathcal{O}_{\mathcal{C}}(D - K_S) \otimes \mathcal{I}_{\delta'} \to \mathscr{E} \to \mathcal{O}_S(2K_S - D) \otimes \mathcal{I}_\delta \to 0$$

for some 0-dimensional subscheme δ' of S. We need to show that the dimension of $H^0(\mathcal{O}_S(D-K_S)) \otimes \mathcal{I}_{\delta'})$ is zero.This must hold, otherwise, $h^0(D-K_S) \geqslant 0$ implies that $u \in H^0(\mathscr{E}(K_S) \otimes \mathcal{O}_S(-K_S))$ and hence $u \in H^0(K_S) \otimes H^0(\mathscr{E})$, which not true. ■

Corollary 4.10. *Let $\Upsilon \subset \mathbb{P}^7 = \mathbb{P}[z_1, z_2, m_{ij}]$ be the variety defined in the previous proposition as the zero locus of the* 5 *submaximal Pfaffians of*

$$\begin{pmatrix} m_{12} & m_{13} & m_{14} & q_1 \\ & m_{23} & m_{24} & q_2 \\ & & m_{34} & q_3 \\ & & & q_4 \end{pmatrix}$$

Then, Υ is of pure dimension 4.

Proof: To show that the components of Υ have dimension $\leqslant 4$ we argue by contradiction. Suppose that Υ has a component Z of dimension $\geqslant 5$. Then a general $\mathbb{P}^5$ section is an (irreducible) variety Z' of dimension $\geqslant 3$ defined by the 5 submaximal Pfaffians of

$$\begin{pmatrix} m_{12} & m_{13} & m_{14} & q'_1 \\ & m_{23} & m_{24} & q'_2 \\ & & m_{34} & q'_3 \\ & & & q'_4 \end{pmatrix}$$

where $q'_i \in \mathrm{S}^2 \langle m_{ij} \rangle$ and such that Z' is contained in a single quadric hypersurface (given by Pf_5 of the above display) and contained in 4 cubic hypersurfaces whose equations— $\mathrm{Pf}_1, \ldots, \mathrm{Pf}_4$ —are linearly independent modulo Pf_5. Since

$$Z' \subset \mathrm{G}(2,4) = (\mathrm{Pf}_5 = 0) \subset \mathbb{P}^5[m_{ij}]$$

there are only two possibilities. Either Z' is 4-dimensional or it is 3-dimensional. In the first instance $Z' = \mathrm{G}(2,4)$ and then there would not be any cubic hypersurfaces through Z', which are not multiples of Pf_5. We deduce that we must have $\dim Z' = 5$. Since $\mathrm{G}(2,4)$ is a nonsingular 4-dimensional hypersurface of $\mathbb{P}^5$ its Picard group is free of rank 1. This implies that there exists $d \geqslant 1$ such that Z' is the complete intersection of $\mathrm{G}(2,4)$ and an hypersurface of degree d. Since Z' is already contained in a cubic hypersurface which does not vanish on $\mathrm{G}(2,4)$, d has to equal 3. But then Z' is a complete intersection of type $(2,3)$ in $\mathbb{P}^5$ and as such is not contained in four cubic hypersurfaces whose equations are linearly independent modulo the quadric equation. This, again, is a contradiction. We deduce that Υ has components of dimension $\leqslant 4$. However, notice that Υ is also the intersection of $X = \mathbb{P}(1^2) \ltimes \mathrm{G}(\frac{1}{2}^4, \frac{3}{2})$ with four quasihomogeneous forms $y_1 - q_1, \ldots, y_4 - q_4$ and since X is 8-dimensional this implies that the components of Υ have dimension $\geqslant 4$. ∎

Proposition 4.11. *Let S be a nonsingular surface of general type with $p_g = 6$ and $K^2 = 13$ whose canonical image is not contained in a pencil of quadrics. Let $\rho\colon S \to X$ be the map of S into a cone over a weighted Grassmannian of Proposition 4.7. Then, Γ is a complete intersection in X of 6 quasihomogeneous forms; two of degree 1 and 4 of degree 2.*

Proof: We know from Proposition 4.7 that $\Gamma \subset X$ is contained in the intersection of two quasihomogeneous forms of degree 1 (L_1 and L_2) and of four quasihomogeneous forms of degree 2 ($y_1 - q_1$, $y_2 - q_2$, $y_3 - q_3$, $y_4 - q_4$, with $q_i \in \mathrm{S}^2 \langle z_1, z_2, m_{ij}\rangle$). Consider $\mathbb{C}[X]$ the homogeneous ring of $X = \mathbb{P}(1^2) \ltimes \mathrm{G}(\frac{1}{2}^4, \frac{3}{2})$ in $\mathbb{P}(1^2, 2^6, 2^4)$ and contained in it, the homogeneous ideal of Γ, which we denote by I. Saying that Γ is the complete intersection of 6 quasihomogeneous forms in X is saying that

$$I = (L_1, L_2, y_1 - q_1, y_2 - q_2, y_3 - q_3, y_4 - q_4).$$

Consider the ideal $J \subset \mathbb{C}[X]$ given by

$$J = (y_1 - q_1, y_2 - q_2, y_3 - q_3, y_4 - q_4).$$

Lemma 4.12. *The ideal J is the homogeneous ideal of the variety Υ viewed as a subvariety of X, in other words, Υ is a complete intersection in X.*

Proof: By Corollary 4.10 Υ has pure dimension 4, which is to say that $J \subset \mathbb{C}[X]$ has codimension 4. Since X is arithmetically Cohen-Macaulay, by the theorem of Unmixedness the radical ideal of each primary ideal in a primary decomposition of J:

$$J = P_1 \cap \cdots \cap P_n$$

is minimal over J. In particular this means that for some integer i, $\operatorname{Rad} P_i$ is the homogeneous ideal of the component of Υ containing Γ. Since the canonical image of S is isomorphic to the intersection of Υ with two hyperplanes, the degree of that component of Υ, is 13. In other words, the degree of $\operatorname{Rad} P_i$ is 13. Since J is generated by 4 quasihomogeneous forms of degree 2 we deduce that $\deg J = 2^4 \deg(X)$. By Proposition 2.3, $\deg(X) = \frac{13}{2^4}$ and accordingly $\deg J = 13$. We have:

$$\deg J = 13 = \sum \deg P_i \geqslant \sum \operatorname{Rad} P_i \geqslant 13.$$

Hence $\deg P_i = \deg \operatorname{Rad} P_i$ (therefore P_i equals $\operatorname{Rad} P_i$) and $n = 1$. In other words, J is prime. ∎

From this lemma, we deduce that in order to show that I is the homogeneous ideal of Γ, it is enough to show that Υ is not contained in any 5-dimensional linear space. This is easily verified since Υ is in complete intersection in X and we know from Proposition 2.3 that the cohomology groups $H^i(\mathcal{O}_X(j))$ on X vanish for all $0 < j < 8$. ■

With the last proposition we have finished the proof of Theorem 1.1. We can easily derive the following corollary which is a finner description of $R(S, K_S)$ than that given by Buchsbaum–Eisenbud's theorem.

Corollary 4.13. *Let S be a nonsingular (regular) surface of general type with $p_g = 6$ and $K^2 = 13$ whose canonical image is not contained in a pencil of quadrics. Then, the canonical ring $R(S, K_S)$ is a codimension 3 ring, generated in degree 1 and the canonical ideal I_{K_S} is generated by the 5 submaximal Pfaffians of a skew matrix*

$$\begin{pmatrix} m_{12} & m_{13} & m_{14} & q_1 \\ & m_{23} & m_{24} & q_2 \\ & & m_{34} & q_3 \\ & & & q_4 \end{pmatrix}$$

where $m_{ij} \in H^0(K_S)$, span a subspace of $H^0(K_S)$ of dimension $\geqslant 4$ and $q_i \in \mathrm{S}^2\, H^0(K_S)$ are general quadric forms.

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