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# COMPUTING TIME DEPENDENT WAITING TIME PROBABILITIES IN NONSTATIONARY MARKOVIAN QUEUEING SYSTEMS

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Abstract: In this paper we present algorithms that compute, exactly or approximately, time dependent waiting time tail probabilities and the time dependent expected waiting time in $M(t)/M/s(t)$ queueing systems.

## 1. Introduction

In many service systems, the performance measure of interest is a function of the tail probability of the waiting time. For example, in many telephone call centers, the service target is a maximum fraction of customers delayed for more than a given number of seconds, e.g. the probability that a customer waits more than twenty seconds is less than fifteen percent. In many of these systems, like call centers, the customer arrival rate varies over the day, and managers vary the staffing over the day in order to meet the desired performance standard.

In this paper we consider an $M(t)/M/s(t)$ queueing system with periodic arrival rate $\{\lambda(t), t > 0\}$, service rate $\mu$, and the number of servers at time $t$, $s(t)$. Let $W_q(t)$ denote the waiting time in queue of a customer that arrives to the system at time $t$. We are interested in computing the tail probability $P(W_q(t) > x)$. When $x = 0$, this reduces to the probability of delay, which is dependent only on the number of servers at time $t$. But when $x > 0$, the derivation is complicated by the fact that the event '$W_q(t) > x$' depends not only on $s(t)$, the number of servers available at time $t$, but also on the number of servers available after $t$, i.e., $s(u), u \in (t, t+x]$. Similarly, the derivation of the expected waiting time in queue, denoted $E(W_q(t))$, is problematic since it depends upon the tail probabilities.

In our derivations we assume that the infinite dimensional vector $p(t) = [p_n(t)]$, where $p_n(t)$ denotes the probability of $n$ customers in the system at

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time $t$, is known. For example, this vector $p(t)$ may have been obtained numerically as the solution of the Chapman-Kolmogorov differential equations that describe the queueing system at hand (see e.g. [2]). Let $W_q^n(t)$ denote the waiting time before service of a customer that arrives at time $t$ and sees $n$ people in the system. Then

$$P(W_q(t) > x) = \sum_{n=s(t)}^{+\infty} P(W_q^n(t) > x) p_n(t), \tag{1}$$

and

$$E(W_q(t)) = \sum_{n=s(t)}^{+\infty} E(W_q^n(t)) p_n(t). \tag{2}$$

In this paper, we present exact expressions for $P(W_q^n(t) > x)$ in the important special case where the number of servers changes at most once in the interval $(t, t+x)$, and we present an algorithm for the general case. "Easy-to-compute" lower and upper bounds are also derived for the general case. We do a similar analysis with $E(W_q^n(t))$, for any $n$, so that the desired quantities follow from (1) and (2).

Since the departure process behaves as an non-homogeneous Poisson process with rate $\mu s(u)$, for $u \geq t$, (assuming an infinite queue at time $t$) the number of departures over the time period $[t, t+x]$ is Poisson distributed with mean

$$a = \mu \int_t^{t+x} s(u)\, du. \tag{3}$$

Thus, when $n \geq s(t)$, we may be tempted to say that the event '$W_q^n(t) > x$' is equivalent to the event '$n - s(t)$ or fewer departures over $[t, t+x]$' so that $P(W_q^n(t) > x)$ would be given by

$$P(\text{'}n - s(t) \text{ or fewer departures over } [t, t+x]\text{'}) = \sum_{j=0}^{n-s(t)} \frac{a^j e^{-a}}{j!}. \tag{4}$$

This is not true in general. For example, suppose the number of servers changes exactly once over the time period $[t, t+x]$ at the epoch $t + \Delta t$, and the resulting number of servers is reduced to a level that is less than the number of customers in service. Then the "excess" customers being served at the epoch $t + \Delta t$ will have to join the queue so that the $(n+1)$st customer at time $t$ may have to see more than $n - s(t)$ departures over

$[t, t+x]$ before starting service. Thus, $P(W_q^n(t) > x)$ is not always given by (4), contrary to the exposition in the appendix of [1]. Similarly, if the number of servers increases during $[t, t+x]$, fewer than $n - s(t)$ departures may result in $W_q^n(t) < x$.

In Section 2, we derive precise and simple formulae for $P(W_q^n(t) > x)$ when the number of servers changes at most once in the interval $[t, t+x]$. In many actual settings this is a valid assumption. For example, in a call center, $x$ is likely to be a number of seconds, while staffing levels are typically changed in intervals ranging from 15 minutes to 2 hours. In Section 3, we study the general case, i.e., when the number of servers changes finitely many times. Then, in Sections 4 and 5, we do a similar analysis for $E(W_q^n(t))$.

## 2. The simplest cases for $P(W_q^n(t) > x)$

Assume that the number of servers does not change during the time period $[t, t+x]$, i.e., $s(u) = s_0, u \in [t, t+x]$. Then, $a = \mu s_0 x$ and $W_q^n(t)$ is either zero, when $n < s_0$, or the sum of $n - s_0 + 1$ independent and identically distributed (i.i.d.) exponential random variables with mean $1/(\mu s_0)$, when $n \geq s_0$. Mathematically,

$$P(W_q^n(t) > x) = \begin{cases} \displaystyle\sum_{i=0}^{n-s_0} \frac{a^i e^{-a}}{i!} & \text{if } n \geq s_0, \\ 0 & \text{if } n < s_0. \end{cases} \tag{5}$$

Now, assume that the number of servers changes exactly once in $[t, t+x]$, i.e., there exists some $\Delta t \leq x$ such that

$$s(u) = \begin{cases} s_0 & \text{if } u \in [t, t+\Delta t), \\ s_1 & \text{if } u \in [t+\Delta t, t+x]. \end{cases} \tag{6}$$

In this case, $a = y_0 + y_1$ where $y_0 = \mu s_0 \Delta t$ and $y_1 = \mu s_1 (x - \Delta t)$. Notice that $P(W_q^n(t) > x) = 0$ when $n < \max(s_0, s_1)$ because the $(n+1)$st customer will begin service no later than time $t + \Delta t$. When $n \geq \max(s_0, s_1)$ then $P(W_q^n(t) > x)$ will have a positive value computed in the following way, distinctly for when the number of servers increases or decreases at time $t+\Delta t$.

Assume $s_0 < s_1$, i.e., the number of servers increases at time $t + \Delta t$. Then, for $n \geq s_1$, the events '$W_q^n(t) > x$' and '$n - s_1$ or fewer departures over

$[t, t+x]$' are equivalent. Thus,

$$P(W_q^n(t) > x) = \begin{cases} \displaystyle\sum_{j=0}^{n-s_1} \frac{a^j e^{-a}}{j!} & \text{if } n \geq s_1, \\ 0 & \text{if } n < s_1. \end{cases} \tag{7}$$

Now, assume that $s_0 > s_1$, i.e., the number of servers decreases at time $t + \Delta t$. For any $n \geq s_0$, let $K_{n,t}(u)$ denote the number of departures over $[t, t+u)$. The event '$W_q^n(t) > x$' can be expressed as the union of two disjoint events: '$K_{n,t}(x) \leq n - s_0$' and '$K_{n,t}(\Delta t) \leq n - s_0, n - s_0 < K_{n,t}(x) \leq n - s_1$'. Putting together the probabilities of these two events we get

$$P(W_q^n(t) > x) = \tag{8}$$
$$= \begin{cases} \displaystyle\sum_{j=0}^{n-s_0} \frac{a^j e^{-a}}{j!} + \sum_{j=0}^{n-s_0} \left( \left( \frac{y_0^j e^{-y_0}}{j!} \right) \left( \sum_{i=n-j-s_0+1}^{n-j-s_1} \frac{y_1^i e^{-y_1}}{i!} \right) \right) & \text{if } n \geq s_0, \\ 0 & \text{if } n < s_0. \end{cases}$$

Or, by using the combinatorial identity

$$\sum_{j=0}^{n} \frac{a^j}{j!} = \sum_{j=0}^{n} \left( \frac{y_0^j}{j!} \sum_{i=0}^{n-j} \frac{y_1^i}{i!} \right), \tag{9}$$

which is generally valid for any pair of real numbers $y_0, y_1$ such that $a = y_0 + y_1$ (see Lemma 1 in the appendix), we get the following equivalent expression for (8)

$$P(W_q^n(t) > x) = \tag{10}$$
$$= \begin{cases} \displaystyle\sum_{j=0}^{n-s_1} \frac{a^j e^{-a}}{j!} - \sum_{j=n-s_0+1}^{n-s_1} \left( \left( \frac{y_0^j e^{-y_0}}{j!} \right) \left( \sum_{i=0}^{n-j-s_1} \frac{y_1^i e^{-y_1}}{i!} \right) \right) & \text{if } n \geq s_0, \\ 0 & \text{if } n < s_0, \end{cases}$$

that allows for the computation of $P(W_q^{n+1}(t) > x)$ from $P(W_q^n(t) > x)$ in $\mathcal{O}(s_0 - s_1)$ operations.

## 3. The general case for $P(W_q^n(t) > x)$

In general, $s(u), u \in [t, t+x]$ is piecewise constant, i.e., for some finite $K$, there are $K+1$ positive integers $s_0, s_1, \ldots, s_K$ and $K+1$ real numbers satisfying $0 = \Delta t_0 < \Delta t_1 < \Delta t_2 < \ldots < \Delta t_K \leq \Delta t_{K+1} = x$ such that, for every $u \in [t, t+x]$,

$$s(u) = \begin{cases} s_0 & \text{if } u \in [t + \Delta t_0, t + \Delta t_1), \\ s_1 & \text{if } u \in [t + \Delta t_1, t + \Delta t_2), \\ \ldots & \\ s_K & \text{if } u \in [t + \Delta t_K, t + \Delta t_{K+1}]. \end{cases}$$

Define the following quantities that will be used in the results that follow. For each $i = 1, 2, \ldots, K$, let

$$\begin{aligned} S_i &= \max\{s_i, s_{i+1}, \ldots, s_K\}, \\ y_i &= \mu s_i \left(\Delta t_{i+1} - \Delta t_i\right) \\ a_i &= \mu \int_{t+\Delta t_i}^{t+x} s(u)\, du = \sum_{j=i}^{K} \mu s_j \left(\Delta t_{j+1} - \Delta t_j\right) = \sum_{j=i}^{K} y_j, \end{aligned}$$

so that $a_0$ equals the value of $a$ defined in (3). Theorem 1 below provides a characterization of $P(W_q^n(t) > x)$ from which an "easy-to-compute" lower bound can be derived. Theorem 2 provides a similar characterization for an "easy-to-compute" upper bound.

**Theorem 1.** *For every $i = 0, 1, \ldots, K$,*

$$P(W_q^n(t + \Delta t_i) > x - \Delta t_i) = \begin{cases} e^{-a_i} \left( \displaystyle\sum_{j=0}^{n-S_i} \frac{a_i^j}{j!} + \sigma_i(n) \right) & \text{if } n \geq S_i, \\ 0 & \text{if } n < S_i, \end{cases} \tag{11}$$

*where, $\sigma_K(n) = 0$, for every $n$, and, for every $i = 0, 1, \ldots, K-1$, and $n \geq S_i$,*

$$\begin{aligned} \sigma_i(n) &= \rho_i(n) + \sum_{j=0}^{n-S_i} \frac{y_i^j}{j!} \sigma_{i+1}(n-j) \\ \rho_i(n) &= \sum_{j=0}^{n-S_i} \frac{y_i^j}{j!} \left( \sum_{k=n-j-S_i+1}^{n-j-S_{i+1}} \frac{a_{i+1}^k}{k!} \right). \end{aligned}$$

*Proof*: We prove this statement by mathematical induction. When $i = K$, (5) applies, so that

$$P(W_q^n(t+\Delta t_K) > x - \Delta t_K) = \begin{cases} \displaystyle\sum_{i=0}^{n-s_K} \frac{y_K^i e^{-y_K}}{i!} & \text{if } n \geq s_K, \\ 0 & \text{if } n < s_K. \end{cases}$$

Since $a_K = y_K$ and $S_K = s_K$, we conclude that the statement is true when $i = K$. Now, suppose that (11) holds for some $i+1$. We will prove that it also holds for $i$. Conditioning on the system state at time $t + \Delta t_{i+1}$,

$$P(W_q^n(t+\Delta t_i) > x - \Delta t_i) =$$
$$= \begin{cases} \displaystyle\sum_{j=0}^{n-s_i} \frac{y_i^j e^{-y_i}}{j!} P(W_q^{n-j}(t+\Delta t_{i+1}) > x - \Delta t_{i+1}) & \text{if } n \geq s_i, \\ 0 & \text{if } n < s_i, \end{cases} \tag{12}$$

because the number of service completions over $[t + \Delta t_i, t + \Delta t_{i+1})$ is Poisson distributed with parameter $y_i$.

First, assume $s_i > S_{i+1}$ which, in particular, implies $s_i = S_i > S_{i+1}$. For every $n < S_i = s_i$, $P(W_q^n(t+\Delta t_i) > x - \Delta t_i) = 0$ follows from (12). For every $n \geq S_i$, we have that $n - j \geq S_i > S_{i+1}$, for every $j = 0, 1, \ldots, n - S_i$, so that, from (12), from the induction hypothesis and from the combinatorial identity (9),

$$P(W_q^n(t+\Delta t_i) > x - \Delta t_i) =$$
$$= \sum_{j=0}^{n-S_i} \left( \frac{y_i^j e^{-y_i}}{j!} \left( e^{-a_{i+1}} \left( \sum_{k=0}^{n-j-S_{i+1}} \frac{a_{i+1}^k}{k!} + \sigma_{i+1}(n-j) \right)\right)\right) \tag{13}$$
$$= e^{-a_i} \left( \sum_{j=0}^{n-S_i} \frac{y_i^j}{j!} \left( \sum_{k=0}^{n-j-S_{i+1}} \frac{a_{i+1}^k}{k!} \right) + \sum_{j=0}^{n-S_i} \frac{y_i^j}{j!} \sigma_{i+1}(n-j) \right) \tag{14}$$
$$= e^{-a_i} \left( \sum_{j=0}^{n-S_i} \frac{a_i^j}{j!} + \sum_{j=0}^{n-S_i} \frac{y_i^j}{j!} \left( \sum_{k=n-j-S_i+1}^{n-j-S_{i+1}} \frac{a_{i+1}^k}{k!} \right) + \sum_{j=0}^{n-S_i} \frac{y_i^j}{j!} \sigma_{i+1}(n-j) \right) \tag{15}$$
$$= e^{-a_i} \left( \sum_{j=0}^{n-S_i} \frac{a_i^j}{j!} + \sigma_i(n) \right). \tag{16}$$

Thus, (11) follows when $s_i > S_{i+1}$. Finally, assume $s_i \le S_{i+1}$, which implies $s_i \le S_i = S_{i+1}$. Then, from the induction hypothesis,

$$\begin{aligned} n < S_i &\Rightarrow n - j \le n < S_{i+1} && (j = 0, 1, \ldots, n - s_i) \\ &\Rightarrow P(W_q^{n-j}(t + \Delta t_{i+1} > x - \Delta t_{i+1})) = 0 && (j = 0, 1, \ldots, n - s_i), \end{aligned}$$

so that $P(W_q^n(t + \Delta t_i) > x - \Delta t_i) = 0$, follows from (12). Moreover, $n \ge S_i$ implies

$$n - j \le S_i - 1 < S_{i+1} \quad (j = n - S_i + 1, n - S_i + 2, \ldots, n - s_i)$$

which, in turn, also implies

$$P(W_q^{n-j}(t + \Delta t_{i+1} > x - \Delta t_{i+1})) = 0 \quad (j = n - S_i + 1, n - S_i + 2, \ldots, n - s_i).$$

Furthermore,

$$n \ge S_i \;\Rightarrow\; n - j \ge S_i = S_{i+1} \;\; (j = 0, 1, \ldots, n - S_i).$$

Thus, (13) and the chain of equalities (14)-(16) is again valid so that the desired result follows also when when $s_i \le S_{i+1}$. ■

Theorem 1 implies that $P(W_q^n(t + \Delta t_i) > x - \Delta t_i) \ge l_i(n)$ where, for any $i = 0, 1, \ldots, K$ and $n \ge S_i$,

$$\begin{aligned} l_i(n) &\equiv e^{-a_i} \left( \sum_{j=0}^{n-S_i} \frac{a_i^j}{j!} + \rho_i(n) \right) \\ &= e^{-a_i} \left( \sum_{j=0}^{n-S_{i+1}} \frac{a_i^j}{j!} - \sum_{j=n-S_i+1}^{n-S_{i+1}} \frac{y_i^j}{j!} \left( \sum_{k=0}^{n-j-S_{i+1}} \frac{a_{i+1}^k}{k!} \right) \right), \end{aligned}$$

which is an interesting expression because it allows for the computation of $l_i(n+1)$ from $l_i(n)$ in $\mathcal{O}(\max\{S_i - S_{i+1}, 1\})$ operations. A necessary and sufficient condition for this lower bound to be tight for any $n$ is that $S_{i+1} = S_K$.

**Theorem 2.** *For every $i = 0, 1, \ldots, K$,*

$$P(W_q^n(t + \Delta t_i) > x - \Delta t_i) = \begin{cases} e^{-a_i} \left( \displaystyle\sum_{j=0}^{n-S_K} \frac{a_i^j}{j!} - \epsilon_i(n) \right) & \text{if } n \ge S_i, \\ 0 & \text{if } n < S_i, \end{cases} \tag{17}$$

*where,* $\epsilon_K(n) = 0$, *for every* $n$, *and, for every* $i = 0, 1, \ldots, K-1$, *and* $n \geq S_i$,

$$\begin{aligned}\epsilon_i(n) &= \delta_i(n) + \sum_{j=0}^{n-S_i} \frac{y_i^j}{j!} \epsilon_{i+1}(n-j) \\ \delta_i(n) &= \sum_{j=n-S_i+1}^{n-S_K} \frac{y_i^j}{j!} \left( \sum_{k=0}^{n-j-S_K} \frac{a_{i+1}^k}{k!} \right).\end{aligned}$$

*Proof*: As before, we prove this statement through mathematical induction. The statement is true when $i = K$, as shown in the proof of Theorem 1. Now, suppose that the statement is true for some $i+1$ and we prove that it is also true for $i$. As in (12), the exact value of $P(W_q^n(t+\Delta t_i) > x - \Delta t_i)$ can be derived through conditioning on the system state at time $t + \Delta t_{i+1}$.

First, assume $s_i > S_{i+1}$ which, in particular, implies $s_i = S_i > S_{i+1} \geq S_K$. For every $n < S_i = s_i$, $P(W_q^n(t+\Delta t_i) > x - \Delta t_i) = 0$ follows from (12). For every $n \geq S_i$, we have that $n - j \geq S_i > S_{i+1}$, for every $j = 0, 1, \ldots, n - S_i$ so that, from (12), from the induction hypothesis and from (9),

$$\begin{aligned}
&P(W_q^n(t+\Delta t_i) > x - \Delta t_i) = \\
&= e^{-a_i} \left( \sum_{j=0}^{n-S_i} \frac{y_i^j}{j!} \left( \sum_{k=0}^{n-j-S_K} \frac{a_{i+1}^k}{k!} - \epsilon_{i+1}(n-j) \right) \right) &(18)\\
&= e^{-a_i} \left( \sum_{j=0}^{n-S_i} \frac{y_i^j}{j!} \left( \sum_{k=0}^{n-j-S_K} \frac{a_{i+1}^k}{k!} \right) - \sum_{j=0}^{n-S_i} \frac{y_i^j}{j!} \epsilon_{i+1}(n-j) \right) &(19)\\
&= e^{-a_i} \left( \sum_{j=0}^{n-S_K} \frac{a_i^j}{j!} - \sum_{j=n-S_i+1}^{n-S_K} \frac{y_i^j}{j!} \left( \sum_{k=0}^{n-j-S_K} \frac{a_{i+1}^k}{k!} \right) - \sum_{j=0}^{n-S_i} \frac{y_i^j}{j!} \epsilon_{i+1}(n-j) \right) &(20)\\
&= e^{-a_i} \left( \sum_{j=0}^{n-S_K} \frac{a_i^j}{j!} - \epsilon_i(n) \right). &(21)
\end{aligned}$$

Thus, (17) follows when $s_i > S_{i+1}$. Finally, assume $s_i \leq S_{i+1}$, which implies $s_i \leq S_i = S_{i+1} \geq S_K$. Then, from the induction hypothesis,

$$\begin{aligned}
n < S_i &\Rightarrow n - j \leq n < S_{i+1} && (j = 0, 1, \ldots, n - s_i) \\
&\Rightarrow P(W_q^{n-j}(t + \Delta t_{i+1} > x - \Delta t_{i+1})) = 0 && (j = 0, 1, \ldots, n - s_i),
\end{aligned}$$

so that $P(W_q^n(t+\Delta t_i) > x - \Delta t_i) = 0$, follows from (12). Moreover, $n \geq S_i$ implies

$$n - j \leq S_i - 1 < S_{i+1} \qquad (j = n - S_i + 1, n - S_i + 2, \ldots, n - s_i)$$

which, in turn, implies

$$P(W_q^{n-j}(t+\Delta t_{i+1} > x - \Delta t_{i+1})) = 0 \qquad (j = n - S_i + 1, n - S_i + 2, \ldots, n - s_i).$$

Thus, the chain of equalities (18)-(20) is again valid so that the desired result follows also when $s_i \leq S_{i+1}$. ■

Theorem 2 implies that $P(W_q^n(t+\Delta t_i) > x - \Delta t_i) \leq u_i(n)$ where, for any $i = 0, 1, \ldots, K$ and $n \geq S_i$,

$$\begin{aligned} u_i(n) &\equiv e^{-a_i}\left(\sum_{j=0}^{n-S_K} \frac{a_i^j}{j!} - \delta_i(n)\right) \\ &= e^{-a_i}\left(\sum_{j=0}^{n-S_K} \frac{a_i^j}{j!} - \sum_{j=n-S_i+1}^{n-S_K} \frac{y_i^j}{j!}\left(\sum_{k=0}^{n-j-S_K} \frac{a_{i+1}^k}{k!}\right)\right). \end{aligned}$$

As before, this expression is interesting because it allows for the computation of $u_i(n+1)$ from $u_i(n)$ in $\mathcal{O}(\max\{S_i - S_K, 1\})$ operations. A necessary and sufficient condition for this upper bound to be tight for any $n$ is that $S_{i+1} = S_K$.

## 4. The simplest cases for $E(W_q^n(t))$

If $s(u) = s_0$, for every $u \in [t, \infty)$ then $W_q^n(t)$ is either zero, when $n < s_0$, or the sum of $n - s_0 + 1$ independent and identically distributed exponential random variables with mean $1/(\mu s_0)$, when $n \geq s_0$. Thus,

$$E(W_q^n(t)) = \begin{cases} \dfrac{n - s_0 + 1}{\mu s_0} & \text{if } n \geq s_0, \\ 0 & \text{if } n < s_0. \end{cases} \tag{22}$$

Now, assume that the number of servers changes exactly once in $[t, \infty)$. I.e., there exists some $\Delta t$ such that

$$s(u) = \begin{cases} s_0 & \text{if } u \in [t, t+\Delta t), \\ s_1 & \text{if } u \in [t+\Delta t, \infty). \end{cases}$$

Our general expressions use the expressions of $P(W_q^n(t) > x)$ derived in Section 2, distinctly for the cases of $s_0 < s_1$ and $s_0 > s_1$, and the fact that

$$E(W_q^n(t)) = \int_0^{\Delta t} P(W_q^n(t) > x)\, dx + \int_{\Delta t}^{+\infty} P(W_q^n(t) > x)\, dx.$$

If $s_0 < s_1$ then $P(W_q^n(t) > x)$ is given by expression (5) when $x \leq \Delta t$, and by expression (7) when $x > \Delta t$. Thus, for each $n < s_0$, $E(W_q^n(t)) = 0$. For each $n \in \{s_0, s_0+1, \ldots, s_1-1\}$, $E(W_q^n(t))$ equals

$$\begin{aligned}\int_0^{\Delta t} P(W_q^n(t) > x)\, dx &= \int_0^{\Delta t} \left( \sum_{i=0}^{n-s_0} \frac{(\mu s_0 x)^i e^{-(\mu s_0 x)}}{i!} \right) dx \\ &= \frac{1}{\mu s_0} \int_0^{y_0} \left( \sum_{i=0}^{n-s_0} \frac{y^i e^{-y}}{i!} \right) dy,\end{aligned}$$

where $y_0 = \mu s_0 \Delta t$, which, from Lemma 2, equals

$$\frac{n-s_0+1}{\mu s_0} \left( 1 - \sum_{j=0}^{n-s_0} \frac{y_0^j e^{-y_0}}{j!} \right) + \Delta t \left( \sum_{j=0}^{n-s_0-1} \frac{y_0^j e^{-y_0}}{j!} \right). \tag{23}$$

For any $n \geq s_1$, $E(W_q^n(t))$ equals (23) plus

$$\begin{aligned}&\int_{\Delta t}^{+\infty} P(W_q^n(t) > x)\, dx = \\ &= \int_{\Delta t}^{+\infty} \left( \sum_{i=0}^{n-s_1} \frac{(\mu s_0 \Delta t + \mu s_1 (x - \Delta t))^i\, e^{-(\mu s_0 \Delta t + \mu s_1 (x-\Delta t))}}{i!} \right) dx \\ &= \frac{1}{\mu s_1} \int_{y_0}^{\infty} \left( \sum_{i=0}^{n-s_1} \frac{y^i e^{-y}}{i!} \right) dy\end{aligned}$$

which, from Lemma 2 of the appendix, equals

$$\frac{n-s_1+1}{\mu s_1} \left( \sum_{j=0}^{n-s_1} \frac{y_0^j e^{-y_0}}{j!} \right) - \frac{s_0 \Delta t}{s_1} \left( \sum_{j=0}^{n-s_1-1} \frac{y_0^j e^{-y_0}}{j!} \right). \tag{24}$$

If $s_0 > s_1$ then $P(W_q^n(t) > x)$ is given by expression (5) when $x \leq \Delta t$, and by expression (8) when $x > \Delta t$. Thus, for any $n < s_0$, $E(W_q^n(t)) = 0$. For any $n \geq s_0$, $E(W_q^n(t))$ equals (23) plus

$$\frac{n-s_0+1}{\mu s_1}\left(\sum_{j=0}^{n-s_0}\frac{y_0^j e^{-y_0}}{j!}\right)-\frac{s_0\Delta t}{s_1}\left(\sum_{j=0}^{n-s_0-1}\frac{y_0^j e^{-y_0}}{j!}\right). \tag{25}$$

plus

$$\begin{aligned}
&\sum_{j=0}^{n-s_0}\left(\frac{y_0^j e^{-y_0}}{j!}\right)\int_{\Delta t}^{+\infty}\left(\sum_{i=n-j-s_0+1}^{n-j-s_1}\left(\frac{(\mu s_1(x-\Delta t))^i e^{-(\mu s_1(x-\Delta t))}}{i!}\right)\right)dx = \\
&= \frac{1}{\mu s_1}\sum_{j=0}^{n-s_0}\left(\frac{y_0^j e^{-y_0}}{j!}\right)\int_0^{+\infty}\left(\sum_{i=n-j-s_0+1}^{n-j-s_1}\left(\frac{y^i e^{-y}}{i!}\right)\right)dy \\
&= \frac{s_0-s_1}{\mu s_1}\left(\sum_{j=0}^{n-s_0}\frac{y_0^j e^{-y_0}}{j!}\right)
\end{aligned}$$

through Lemma 2 of the appendix. We note that, in any case, $E(W_q^{n+1}(t))$ can be computed from $E(W_q^n(t))$ in $\mathcal{O}(1)$ operations.

## 5. The general case for $E(W_q^n(t))$

Now, we assume a more general case. Suppose that, for some $K$ finite, there are $K+1$ positive integers $s_0, s_1, \ldots, s_K$, and $K+1$ real numbers satisfying $0 = \Delta t_0 < \Delta t_1 < \Delta t_2 < \ldots < \Delta t_K$ such that, for every $u \in [t, \infty)$,

$$s(u) = \begin{cases} s_0 & \text{if } u \in [t+\Delta t_0, t+\Delta t_1), \\ s_1 & \text{if } u \in [t+\Delta t_1, t+\Delta t_2), \\ \ldots & \\ s_K & \text{if } u \in [t+\Delta t_K, \infty). \end{cases}$$

The quantities $E(W_q^n(t))$, for every $n$, can be computed recursively. We use conditioning on the number of service completions to derive the recursion formula. The formula is initiated, for $i = K$, with

$$E(W_q^n(t+\Delta t_K)) = \begin{cases} \dfrac{n-s_K+1}{\mu s_K} & \text{if } n \geq s_K, \\ 0 & \text{if } n < s_K. \end{cases} \tag{26}$$

For a generic $i \in \{0, 1, \ldots, K-1\}$ we have the following derivation. For any $n < s_i$, $E(W_q^n(t+\Delta t_i)) = 0$, while, for any $n \geq s_i$,

$E(W_q^n(t+\Delta t_i)) =$

$$= A_{in}(t)\left(1-\sum_{j=0}^{n-s_i}\frac{y_i^j e^{-y_i}}{j!}\right)+\sum_{j=0}^{n-s_i}\frac{y_i^j e^{-y_i}}{j!}\left(\Delta t_{i+1}-\Delta t_i+E(W_q^{n-j}(t+\Delta t_{i+1}))\right),$$

where $y_i=\mu s_i(\Delta t_{i+1}-\Delta t_i)$ and $A_{in}(t)$ denotes the expected value of $W_q^n(t+\Delta t_i)$ given that there are more than $n-s_i$ service completions in the interval $[t+\Delta t_i, t+\Delta t_{i+1})$. Thus,

$$A_{in}(t)=E(X|X\leq \Delta t_{i+1}-\Delta t_i),$$

where $X$ is the sum of $n-s_i+1$ i.i.d. exponential random variables of parameter $\mu s_i$. From Lemma 3 in the appendix,

$$A_{in}(t)=\frac{n-s_1+1}{\mu s_i}\left(1-\left(\frac{\frac{y_i^{n-s_i+1}e^{-y_i}}{(n-s_i+1)!}}{1-\sum_{j=0}^{n-s_i}\frac{y_i^j e^{-y_i}}{j!}}\right)\right) \tag{27}$$

and so, for any $n\geq s_i$,

$$\begin{aligned}&E(W_q^n(t+\Delta t_i))=\\ &=\frac{n-s_i+1}{\mu s_i}\left(1-\sum_{j=0}^{n-s_i+1}\frac{y_i^j e^{-y_i}}{j!}\right)+(\Delta t_{i+1}-\Delta t_i)\left(\sum_{j=0}^{n-s_i}\frac{y_i^j e^{-y_i}}{j!}\right)+\\ &\qquad+\sum_{j=0}^{n-s_i}\frac{y_i^j e^{-y_i}}{j!}E(W_q^{n-j}(t+\Delta t_{i+1})),\end{aligned}$$

In the end of a backward application of this recursion formula we obtain $E(W_q^n(t))\equiv E(W_q^n(t+\Delta t_0))$, for any $n$.

## References

[1] A. Ingolfsson, M. Haque and A. Umnikov. Accounting for time-varying queueing effects in workforce scheduling. *European Journal of Operational Research*, 139:585–597, 2002.

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# 6. Appendix

**Lemma 1.** *For any pair of reals $y_0, y_1$, and integer $n \geq 0$,*

$$\sum_{j=0}^{n} \frac{(y_0+y_1)^j}{j!} = \sum_{j=0}^{n} \frac{y_0^j}{j!} \left( \sum_{i=0}^{n-j} \frac{y_1^i}{i!} \right) \tag{28}$$

*Proof*: From the binomial expansion, for any integer $n \geq 0$,

$$(y_0+y_1)^n = \sum_{i=0}^{n} \frac{n!}{i!(n-i)!} y_0^i y_1^{n-i}.$$

Thus, for any integer $n \geq 0$,

$$\sum_{j=0}^{n} \frac{(y_0+y_1)^j}{j!} = \sum_{j=0}^{n} \sum_{i=0}^{j} \frac{y_0^i y_1^{j-i}}{i!(j-i)!} = \sum_{i=0}^{n} \sum_{j=i}^{n} \frac{y_0^i y_1^{j-i}}{i!(j-i)!} = \sum_{i=0}^{n} \frac{y_0^i}{i!} \left( \sum_{j=0}^{n-i} \frac{y_1^j}{j!} \right),$$

from where (28) follows. ■

**Lemma 2.** *For any real $y_0 \geq 0$ and integer $n \geq 0$,*

$$\int_0^{y_0} \left( \sum_{i=0}^{n} \frac{y^i e^{-y}}{i!} \right) dy = (n+1) \left( 1 - \sum_{j=0}^{n} \frac{y_0^j e^{-y_0}}{j!} \right) + y_0 \left( \sum_{j=0}^{n-1} \frac{y_0^j e^{-y_0}}{j!} \right) \tag{29}$$

$$\int_{y_0}^{\infty} \left( \sum_{i=0}^{n} \frac{y^i e^{-y}}{i!} \right) dy = (n+1) \left( \sum_{j=0}^{n} \frac{y_0^j e^{-y_0}}{j!} \right) - y_0 \left( \sum_{j=0}^{n-1} \frac{y_0^j e^{-y_0}}{j!} \right) \tag{30}$$

*and, for any pair of integers $m, n \geq 0$ such that $m \geq n$,*

$$\int_0^{\infty} \left( \sum_{i=n}^{m} \frac{y^i e^{-y}}{i!} \right) dy = m - n + 1 \tag{31}$$

*Proof*: As it may be checked

$$\frac{d}{dy} \left( -(n+1) \left( \sum_{j=0}^{n} \frac{y^j e^{-y}}{j!} \right) + y \left( \sum_{j=0}^{n-1} \frac{y^j e^{-y}}{j!} \right) \right) = \sum_{i=0}^{n} \frac{y^i e^{-y}}{i!}.$$

Thus, (29) and (30) follow from the Fundamental Theorem of Calculus. Since the sum of (29) and (30) is $n+1$, (31) follows trivially. ■

**Lemma 3.** *If $\{E_i, i = 1, 2, \ldots, n\}$ are $n$ i.i.d. exponential random variables with mean (or parameter) $b$ and $X \equiv \sum_{i=1}^{n} E_i$ then, for every $t > 0$,*

$$E(X \mid X \leq t) = nb \left( 1 - \left( \frac{\dfrac{(t/b)^n e^{-t/b}}{n!}}{1 - \displaystyle\sum_{i=0}^{n-1} \dfrac{(t/b)^i e^{-t/b}}{i!}} \right) \right). \tag{32}$$

*Proof*:

$$\begin{aligned}
E(X \mid X \leq t) &= \frac{\displaystyle\int_0^t F_X(t) - F_X(x)\, dx}{F_X(t)} \\
&= t - \frac{\displaystyle\int_0^t \left( 1 - \exp\left(-\frac{x}{b}\right) \left( \sum_{i=0}^{n-1} \frac{x^i}{b^i i!} \right) \right) dx}{F_X(t)} \\
&= t - \frac{t - \displaystyle\sum_{i=0}^{n-1} \int_0^t \exp\left(-\frac{x}{b}\right) \frac{x^i}{b^i i!}\, dx}{F_X(t)} \\
&= t - \frac{t - b \displaystyle\sum_{i=0}^{n-1} \left( \int_0^{t/b} \frac{y^i e^{-y}}{i!}\, dy \right)}{F_X(t)} \\
&= t - \frac{t - b \displaystyle\sum_{i=0}^{n-1} \left( 1 - \exp\left(-\frac{t}{b}\right) \sum_{j=0}^{i} \frac{t^j}{b^j j!} \right)}{F_X(t)} \\
&= t - \frac{t - bn + b \exp\left(-\dfrac{t}{b}\right) \displaystyle\sum_{i=0}^{n-1} \sum_{j=0}^{i} \frac{t^j}{b^j j!}}{F_X(t)} \\
&= t - \frac{t - bn + b \exp\left(-\dfrac{t}{b}\right) \displaystyle\sum_{i=0}^{n-1} \frac{t^i}{b^i i!} (n - i)}{F_X(t)}
\end{aligned}$$

$$
\begin{aligned}
&= t - \frac{t - bn + b\exp\left(-\frac{t}{b}\right)\left(n\sum_{i=0}^{n-1}\frac{t^i}{b^i i!} - \frac{t}{b}\sum_{i=0}^{n-2}\frac{t^i}{b^i i!}\right)}{1 - \exp\left(-\frac{t}{b}\right)\sum_{i=0}^{n-1}\frac{t^i}{b^i i!}} \\
&= t - \frac{t\left(1 - \exp\left(-\frac{t}{b}\right)\sum_{i=0}^{n-2}\frac{t^i}{b^i i!}\right) - bn\left(1 - \exp\left(-\frac{t}{b}\right)\sum_{i=0}^{n-1}\frac{t^i}{b^i i!}\right)}{1 - \exp\left(-\frac{t}{b}\right)\sum_{i=0}^{n-1}\frac{t^i}{b^i i!}} \\
&= bn - t\left(\frac{\exp\left(-\frac{t}{b}\right)\frac{t^{n-1}}{b^{n-1}(n-1)!}}{1 - \exp\left(-\frac{t}{b}\right)\sum_{i=0}^{n-1}\frac{t^i}{b^i i!}}\right) \\
&= bn\left(1 - \left(\frac{\exp\left(-\frac{t}{b}\right)\frac{t^n}{b^n n!}}{1 - \exp\left(-\frac{t}{b}\right)\sum_{i=0}^{n-1}\frac{t^i}{b^i i!}}\right)\right).
\end{aligned}
$$

■

When $n = 1$, (32) becomes

$$E(X \mid X \leq t) = b - t\left(\frac{e^{-t/b}}{1 - e^{-t/b}}\right).$$

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