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A PIEZOELECTRIC ANISOTROPIC PLATE MODEL

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Abstract: We mathematically justify a reduced piezoelectric plate model. This is achieved considering the three-dimensional static equations of piezoelectricity, for a nonhomogeneous anisotropic thin plate, and using the asymptotic analysis to compute the limit of the displacement vector and electric potential, as the thickness of the plate approaches zero. We prove that the three-dimensional displacement vector converges to a Kirchhoff-Love displacement, that solves a two-dimensional piezoelectric plate model, defined on the middle surface of the plate. Moreover, the three-dimensional electric potential converges to a scalar function that is a second order polynomial with respect to the thickness variable, with coefficients that depend on the transverse component of the Kirchhoff-Love displacement. We remark that the results of this paper generalize a previous work of A. Sene (2001) for homogeneous and isotropic materials.

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1. Introduction

Piezoelectric plates and more generally structures involving piezoelectric materials are widely used in real-life applications, namely as sensors or actuators. The justification is that piezoelectric materials are characterized by the fact that a mechanical deformation generates an electric field on the material and, vice-versa, the application to the material of an electric field produces a deformation (cf. Ikeda [9]).

In the literature there are several recent papers concerning the modelling of piezoelectric structures. We refer in particular the papers by Bernadou and Haenel [1, 2, 3], for the modelization and numerical approximation of piezoelectric thin shells, Collard and Miara [6] for the justification of geometrically nonlinear thin piezoelectric shells models, Raoult and Sene [11], for the modelling of piezoelectric plates including magnetic effects, Sene [13], for the modelling of piezoelectric static thin plates, Sabu [12], for the modelling of eigenvalues problems for thin piezoelectric shells. The technique used in Collard and Miara [6], Raoult and Sene [11], Sene [13] and Sabu [12] is the asymptotic analysis (cf. Ciarlet [4, 5] for a description of the asymptotic

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analysis procedure applied to elastic plates and shells) and the materials are homogeneous and isotropic.

The present paper is inspired in Sene [13], and we use again the asymptotic analysis to derive a reduced piezoelectric plate model from the static three-dimensional piezoelectric system, but for a nonhomogeneous and anisotropic material, with nonhomogeneous piezoelectric and dielectric coefficients. In particular, we remark that the results of this paper generalize those of Sene [13], obtained for homogeneous isotropic piezoelectric plates, with constant piezoelectric and dielectric coefficients.

We briefly summarize now the main results of the paper. Let $h > 0$ be a small parameter, and for each h we consider the static three-dimensional piezoelectric problem, for a nonhomogeneous and anisotropic (more specifically monoclinic) plate $\bar{\Omega}^h = \bar{\omega} \times [-h, h]$, with middle surface $\bar{\omega}$ and thickness $2h$. Applying the asymptotic analysis technique, we redefine this three-dimensional piezoelectric problem into an equivalent problem posed over the set $\bar{\Omega} = \bar{\omega} \times [-1, 1]$, that is independent of h. Denoting by $(u(h), \varphi(h))$ the solution of this latter problem, where $u(h)$ is the displacement vector and $\varphi(h)$ is the electric potential of the plate Ω, we prove that the pair $(u(h), \varphi(h))$ converges weakly and also strongly, when $h \to 0^+$, to the pair (u, φ), in appropriate functional spaces independent of h. The limit displacement vector u is a Kirchhoff-Love displacement and the unique solution of a two-dimensional piezoelectric plate model, that is totally independent of φ, but depends on the mechanical and electric data of the three-dimensional piezoelectric problem. The limit electric potential is a second order polynomial, with respect to the thickness variable, whose coefficients depend on the transverse component of the Kirchhoff-Love displacement u, the elastic, piezoelectric and dielectric coefficients, and the electric potential data applied on the upper and lower faces of the plate (cf. (58).

The theorem 3.4 describes the variational formulation of the two-dimensional piezoelectric plate model and the expression of φ. The theorem 4.1 presents the boundary value problem formulation of the two-dimensional piezoelectric plate model. In particular this model is a generalization of the usual elasticity plate model for anisotropic plates (cf. Green and Zerna [8] and Destuynder [7]): the stress resultants and the stress couples differ from the pure elasticity stress resultants and the stress couples because they include terms that depend on the piezoelectric and dielectric coefficients (cf. $\hat{N}_{\alpha\beta}$ and $\hat{M}_{\alpha\beta}$ defined by (63) or by (67) in theorem 4.1). In fact, comparing the equations

(64)-(65) that define the two-dimensional piezoelectric plate model, with the usual elasticity plate model for anisotropic plates, it appears on the left-hand sides of (64)-(65) an additional fourth order term and an additional second order term, respectively (cf. expressions of $M_{\alpha\beta}$ and $N_{\alpha\beta}$ defined by (62)), and on the corresponding right-hand sides there are terms that depend on the electric potential data applied on the upper and lower faces of the plate, and that act like a force in the two-dimensional piezoelectric plate model.

For a better physical understanding of the limit problem, the theorem 4.2 presents the formulation of the two-dimensional piezoelectric plate model and the expression of φ, with respect to the original plate $\bar{\Omega}^h$.

Finally let us shortly describe the contents of this paper. After this introduction we recall in section 2 the three-dimensional piezoelectric plate model. The asymptotic analysis is done in section 3 and it involves the definitions of the scalings for the unknowns, the assumptions on the data, the calculus of the limit of $(u(h), \varphi(h))$, when h approaches zero, and the variational formulation of the limit problem. The section 4 concerns the formulation of the two-dimensional piezoelectric plate model as a boundary value problem. In section 5 we make some observations.

2. The three-dimensional piezoelectric plate model

In this section we first introduce some notations. Then, we recall the static three-dimensional piezoelectric model, for a nonhomogeneous anisotropic thin plate, and we describe its formulation as a boundary value problem and the variational formulation.

2.1. Notations. Let $\omega \subset I\!R^2$ be a bounded domain with a Lipschitz continuous boundary $\partial\omega$ and γ_0, γ_e subsets of $\partial\omega$, such that, $meas(\gamma_0) > 0$ and $meas(\gamma_e) > 0$. We also define $\gamma_1 = \partial\omega \setminus \gamma_0$, $\gamma_s = \partial\omega \setminus \gamma_e$.

For each $0 < h \leq 1$ we consider the sets

$$\begin{aligned}
&\Omega^h = \omega \times (-h, h), \quad \Gamma_\pm^h = \omega \times \{\pm h\}, \quad \Gamma_D^h = \gamma_0 \times (-h, h),\\
&\Gamma_1^h = \gamma_1 \times (-h, h), \quad \Gamma_N^h = \Gamma_1^h \bigcup \Gamma_\pm^h, \\
&\Gamma_{eN}^h = \gamma_s \times (-h, h), \quad \Gamma_{eD}^h = \Gamma_\pm^h \bigcup \big(\gamma_e \times (-h, h)\big),
\end{aligned} \tag{1}$$

where Ω^h is a plate with middle surface ω and thickness $2h$, Γ_+^h and Γ_-^h are, respectively, the upper and lower faces of the plate Ω^h, the sets Γ_D^h, Γ_1^h and Γ_{eN}^h are portions of the lateral surface $\partial\omega \times (-h, h)$ of Ω^h, and finally Γ_N^h and Γ_{eD}^h are portions of the boundary $\partial\Omega^h$ of Ω^h. An arbitrary point of Ω^h

is denoted by $x^h = (x_1^h, x_2^h, x_3^h)$, where the first two components $(x_1^h, x_2^h) = (x_1, x_2) \in \omega$ are independent of h and $x_3^h \in (-h, h)$. We also denote by $\nu^h = (\nu_1^h, \nu_2^h, \nu_3^h)$ the outward unit normal vector to $\partial\Omega^h$.

Throughout the paper, the latin indices i, j, k, l... belong to the set $\{1, 2, 3\}$, the greek indices α, β, μ... vary in the set $\{1, 2\}$ and the summation convention with respect to repeated indices is employed, that is, $a_i b_i = \sum_{i=1}^{3} a_i b_i$. Moreover we denote by $a \cdot b = a_i b_i$ the inner product of the vectors $a = (a_i)$ and $b = (b_i)$, by $Ce = (C_{ijkl} e_{kl})$ the contraction of a fourth order tensor $C = (C_{ijkl})$ with a second order tensor $e = (e_{kl})$ and by $Ce : d = C_{ijkl} e_{kl} d_{ij}$ the inner product of the tensors Ce and $d = (d_{ij})$. Given a function $\theta(x^h)$ defined in Ω^h we denote by $\partial_i^h \theta$ its partial derivative with respect to x_i^h, that is, $\partial_i^h \theta = \frac{\partial \theta}{\partial x_i^h}$.

2.2. Boundary value problem. In the sequel we consider a static three-dimensional piezoelectric plate model, disregarding the thermal and magnetic effects, and for the case of small deformations and linear piezoelectricity. This is a model with three groups of equations and boundary conditions, that is described below.

Mechanical equilibrium equations

$$\left[\begin{array}{l} -div\sigma^h(u^h, \varphi^h) = f^h \iff -\partial_j^h \sigma_{ij}^h(u^h, \varphi^h) = f_i^h, \quad \text{in} \quad \Omega^h, \\ \sigma^h(u^h, \varphi^h)\, \nu^h = g^h \quad \iff \sigma_{ij}^h(u^h, \varphi^h)\, \nu_j^h = g_i^h, \quad \text{on} \quad \Gamma_N^h, \\ u^h = 0, \quad \text{on} \quad \Gamma_D^h, \end{array}\right. \tag{2}$$

Constitutive equations

$$\left[\begin{array}{l} \sigma^h(u^h, \varphi^h) = C^h e^h(u^h) - P^h E^h(\varphi^h), \quad \text{in} \quad \Omega^h, \\ D^h(u^h, \varphi^h) = P^h e^h(u^h) + \varepsilon^h E^h(\varphi^h), \quad \text{in} \quad \Omega^h, \end{array}\right. \tag{3}$$

Maxwell-Gauss equations

$$\left[\begin{array}{l} divD^h(u^h, \varphi^h) = 0 \iff \partial_i^h D_i^h(u^h, \varphi^h) = 0, \quad \text{in} \quad \Omega^h, \\ D^h(u^h, \varphi^h)\nu^h = 0 \iff D_i^h(u^h, \varphi^h)\nu_i^h = 0, \quad \text{on} \quad \Gamma_{eN}^h, \\ \varphi^h = \varphi_0^h, \quad \text{on} \quad \Gamma_{eD}^h. \end{array}\right. \tag{4}$$

The unknown of the piezoelectric plate model (2)-(4) is the pair (u^h, φ^h), where $u^h : \Omega^h \to I\!R^3$ denotes the displacement vector field and $\varphi^h : \Omega^h \to I\!R$

is the electric potential, that is a scalar field. For each point $x^h \in \bar{\Omega}^h$, the vector $u^h(x^h)$ represents the displacement that the point x^h undergoes and $\varphi^h(x^h)$ represents the electric potential at x^h.

The mechanical equilibrium equations express the balance of mechanical loads and internal stresses. We suppose that f^h is the density of the applied body forces acting on the plate Ω^h, g^h is the density of applied surface forces on Γ_N^h and we also assume that the plate is clamped along Γ_D^h.

The constitutive equations represent the electromechanical interaction that characterizes piezoelectricity. It is a relation between the stress tensor $\sigma^h : \Omega^h \to I\!R^9$, the electric displacement vector $D^h : \Omega^h \to I\!R^3$, the linear strain tensor $e^h(u^h)$ and the electric field vector $E^h(\varphi^h)$, where

$$e^h(u^h) = \frac{1}{2}\big(\nabla^h u^h + (\nabla^h u^h)^T\big) \quad \text{and} \quad E^h(\varphi^h) = -\nabla^h \varphi^h, \tag{5}$$

or, equivalently, componentwise

$$e^h_{ij}(u^h) = \frac{1}{2}\big(\partial^h_i u^h_j + \partial^h_j u^h_i\big) \quad \text{and} \quad E^h_i(\varphi^h) = -\partial^h_i \varphi^h. \tag{6}$$

Moreover $C^h = (C^h_{ijkl})$ is the elastic fourth order tensor field, $P^h = (P^h_{ijk})$ is the piezoelectric third order tensor field, and $\varepsilon^h = (\varepsilon^h_{ij})$ is the dielectric second order tensor field.

Finally the Maxwell-Gauss equations are the equations that govern the electric displacement vector field D^h. We assume that φ^h_0 is the electric potential applied on Γ^h_{eD} and there is no electric loading in Ω^h nor on Γ^h_{eN}.

In addition we suppose the following hypotheses on the data: the applied forces and applied electric potential have the regularity

$$f^h \in \big[L^2(\Omega^h)\big]^3, \qquad g^h \in \big[L^2(\Gamma^h_N)\big]^3, \qquad \varphi^h_0 \in H^{\frac{1}{2}}(\Gamma^h_{eD}), \tag{7}$$

and the three tensor fields $C^h = (C^h_{ijkl})$, $P^h = (P^h_{ijk})$ and $\varepsilon^h = (\varepsilon^h_{ij})$ are independent of the thickness h but depend on $x = (x_1, x_2, x_3) \in \bar{\omega} \times [-1, 1]$, that is, there exist tensor fields $C = (C_{ijkl})$, $P = (P_{ijk})$ and $\varepsilon = (\varepsilon_{ij})$, such that, for any $x^h = (x_1, x_2, hx_3) \in \bar{\Omega}^h$ with $x_3 \in [-1, 1]$, then

$$\left[\begin{array}{l} C^h_{ijkl}(x^h) = C_{ijkl}(x) \\ P^h_{ijk}(x^h) = P_{ijk}(x) \\ \varepsilon^h_{ij}(x^h) = \varepsilon_{ij}(x) \end{array}\right. , \quad \text{and} \quad x = (x_1, x_2, x_3) \in \bar{\omega} \times [-1, 1], \tag{8}$$

where C_{ijkl}, P_{ijk} and ε_{ij} are independent of h. We also assume that C_{ijkl}, P_{ijk}, ε_{ij} are smooth enough functions defined in $\bar{\omega} \times [-1, 1]$, that verify the following symmetric and coercive properties:

- in $\bar{\omega} \times [-1, 1]$

$$\begin{aligned} & C_{ijkl} = C_{jikl} = C_{klij}, \qquad C_{\alpha\beta\gamma 3} = 0 = C_{\alpha 333}, \\ & P_{ijk} = P_{ikj}, \qquad \varepsilon_{ij} = \varepsilon_{ji}, \end{aligned} \tag{9}$$

- there exists constants $c_1 > 0$ and $c_2 > 0$, such that

$$C_{ijkl}(x) M_{kl} M_{ij} \geq c_1 \sum_{i,j=1}^{3} (M_{ij})^2 \quad \text{and} \quad \varepsilon_{ij}(x)\theta_i\theta_j \geq c_2 \sum_{i=1}^{3} \theta_i^2 \tag{10}$$

  for every symmetric 3×3 real matrix M and any vector $\theta \in I\!R^3$ and for every $x \in \bar{\omega} \times [-1, 1]$.

The hypotheses $C_{\alpha\beta\gamma 3} = 0 = C_{\alpha 333}$ are usually assumed for plates and physically they mean that the plate has elastic symmetry with respect to the plane $x_3 = 0$ (cf. Green and Zerna [8]). Consequently the material is monoclinic and the number of independent elastic coefficients C_{ijkl} is equal to 13.

In particular, we also remark that for a homogeneous and isotropic material the elasticity coefficients C_{ijkl} are constants defined by

$$C_{ijkl} = \lambda \delta_{ij} \delta_{kl} + \mu(\delta_{ik}\delta_{jl} + \delta_{il}\delta_{jk}) \tag{11}$$

where λ and μ are the Lamé constants, and $\delta_{ij} = 0$ if $i \neq j$, and $\delta_{ij} = 1$ if $i = j$.

2.3. Variational formulation. We define the space of admissible displacements

$$V^h = \left\{ v^h \in [H^1(\Omega^h)]^3 : \quad v^h_{|\Gamma^h_D} = 0 \right\} \tag{12}$$

with the norm $\|v^h\|_{V^h} = \|\nabla v^h\|_{[L^2(\Omega)]^9}$, and the space of admissible electric potentials

$$\Psi^h = \left\{ \psi^h \in H^1(\Omega^h) : \quad \psi^h_{|\Gamma^h_{eD}} = 0 \right\} \tag{13}$$

with the norm $\|\psi^h\|_{\Psi^h} = \|\nabla \psi^h\|_{[L^2(\Omega)]^3}$. Let $\bar{\varphi}^h = \varphi^h - \varphi^h_0$, where φ^h_0 is a trace lifting in $H^1(\Omega^h)$ of the boundary potential acting on Γ^h_{eD} (cf. (4)). Then, the variational formulation of the system (2)-(4) is defined by

$$\left\{\begin{array}{l} \text{Find} \quad (u^h, \bar{\varphi}^h) \in V^h \times \Psi^h \quad \text{such that:} \\ a^h\big((u^h, \bar{\varphi}^h), (v^h, \psi^h)\big) = l^h(v^h, \psi^h), \quad \forall (v^h, \psi^h) \in V^h \times \Psi^h, \end{array}\right. \tag{14}$$

where

$$\left[\begin{array}{l} a^h\big((u^h, \bar{\varphi}^h), (v^h, \psi^h)\big) = \\ \int_{\Omega^h} C e^h(u^h) : e^h(v^h)\, dx^h + \int_{\Omega^h} \varepsilon_{ij}\, \partial_i^h \bar{\varphi}^h\, \partial_j^h \psi^h\, dx^h \\ + \int_{\Omega^h} P_{ijk} \left(\partial_i^h \bar{\varphi}^h e_{jk}^h(v^h) - \partial_i^h \psi^h e_{jk}^h(u^h)\right) dx^h \\ \end{array}\right. \tag{15}$$

and

$$\left[\begin{array}{l} l^h(v^h, \psi^h) = \int_{\Omega^h} f^h \cdot v^h\, dx^h + \int_{\Gamma_N^h} g^h \cdot v^h\, d\Gamma_N^h \\ - \int_{\Omega^h} \varepsilon_{ij}\, \partial_i^h \varphi_0^h\, \partial_j^h \psi^h\, dx^h - \int_{\Omega^h} P_{ijk}\, \partial_i^h \varphi_0^h\, e_{jk}^h(v^h)\, dx^h. \end{array}\right. \tag{16}$$

To obtain (14) we do the inner product of the first equation of (2) by $v^h \in V^h$ and we multiply the first equation of (4) by $\psi^h \in \Psi^h$. Afterwards we add the two resulting equations, we integrate in Ω, we use the Green's formula, all the boundary conditions and the constitutive equations defined in (2)-(4).

With the definition of $a^h(.,.)$ it is clear that

$$\left[\begin{array}{l} a^h\big((v^h, \psi^h), (v^h, \psi^h)\big) = \\ \int_{\Omega^h} C e^h(v^h) : e^h(v^h)\, dx^h + \int_{\Omega^h} \varepsilon_{ij}\, \partial_i^h \psi^h\, \partial_j^h \psi^h\, dx^h, \end{array}\right. \tag{17}$$

and therefore, by the coercive properties (10) and the Lax-Milgram lemma, the variational problem (14) has a unique solution.

3. Asymptotic analysis

In this section we apply the asymptotic analysis procedure (as developed by Ciarlet [4]) to the variational problem (14). We first transform the three-dimensional piezoelectric plate problem (14), into an equivalent problem depending on h, but posed over a set $\Omega = \omega \times (-1, 1)$ independent of h, using appropriate scalings of the unknowns u^h, φ^h and convenient assumptions on the data. Then we study the behavior of the scaled displacements, electric potentials, stresses and electric vectors as the thickness $h \to 0^+$. The theorem 3.4 gives a characterization of the limit displacement vector and the limit electric potential.

3.1. The scaled three-dimensional problem. We redefine here the three-dimensional variational problem (14) in the domain $\Omega = \omega \times (-1,1)$ independent of h. To each $x = (x_1, x_2, x_3) \in \Omega$ we associate the element $x^h = (x_1, x_2, hx_3) \in \Omega^h$. We also consider the subsets defined in (1) for the choice $h = 1$, that is

$$\begin{aligned} &\Gamma_\pm = \omega \times \{\pm 1\}, \quad \Gamma_D = \gamma_0 \times (-1,1), \\ &\Gamma_1 = \gamma_1 \times (-1,1), \quad \Gamma_N = \Gamma_1 \bigcup \Gamma_\pm, \\ &\Gamma_{eN} = \gamma_s \times (-1,1), \quad \Gamma_{eD} = \Gamma_\pm \bigcup \big(\gamma_e \times (-1,1)\big). \end{aligned} \tag{18}$$

We denote by $\nu = (\nu_1, \nu_2) = (\nu_\alpha)$ the unit outer normal vector along $\partial\omega$, by $\tau = (\tau_1, \tau_2) = (\tau_\alpha)$, with $\tau_1 = -\nu_2$ and $\tau_2 = \nu_1$, the unit tangent vector along $\partial\omega$, by $\frac{\partial\theta}{\partial\nu} = \nu_\alpha \partial_\alpha \theta$ the outer normal derivative of the scalar function θ along $\partial\omega$, and by $\partial_\tau\theta = \tau_\alpha\partial_\alpha\theta$ the tangential derivative of the scalar function θ along $\partial\omega$. We also denote by $\partial_i = \frac{\partial}{\partial x_i}$ and $\partial_{ij} = \frac{\partial^2}{\partial x_i \partial x_j}$, the first and second partial derivatives with respect to x_i and x_j.

We follow exactly the same choices of Sene [13] for the assumptions on the data and the scalings of the unknowns and the constitutive equations. We suppose that the data verify the hypotheses

$$\begin{aligned} &f_\alpha^h(x^h) = h^2 f_\alpha(x), \quad f_3^h(x^h) = h^3 f_3(x), \quad x \in \Omega, \\ &g_\alpha^h(x^h) = h^2 g_\alpha(x), \quad g_3^h(x^h) = h^3 g_3(x), \quad x \in \Gamma_1, \\ &g_\alpha^h(x^h) = h^3 g_\alpha(x), \quad g_3^h(x^h) = h^4 g_3(x), \quad x \in \Gamma_\pm, \\ &\varphi_0^h(x^h) = h^3 \varphi_0(x), \quad x \in \bar{\Omega}, \end{aligned} \tag{19}$$

where $f_\alpha \in H^1(\Omega)$, $f_3 \in L^2(\Omega)$, $g_\alpha \in H^1(\Gamma_N)$, $g_3 \in L^2(\Gamma_N)$, $\varphi_0 \in H^1(\Omega)$. In addition we denote by $g_3^+ = g_{3|\Gamma_+}$, $g_3^- = g_{3|\Gamma_-}$, $\varphi_0^+ = \varphi_{0|\Gamma_+}$, $\varphi_0^- = \varphi_{0|\Gamma_-}$ and we assume that $\varphi_0^+ - \varphi_0^- \in H^1(\omega)$. For the unknowns we define the scalings

$$\begin{aligned} &u_\alpha^h(x^h) = h^2 u_\alpha(h)(x), \quad u_\alpha^3(x^h) = h u_3(h)(x), \quad x \in \Omega, \\ &\varphi^h(x^h) = h^3 \varphi(h)(x), \quad x \in \Omega. \end{aligned} \tag{20}$$

The scalings of the stress tensor and the electric displacement vector are induced by (20) and are defined by

$$\sigma_{ij}(h)(u(h), \varphi(h)) = h^{-2}\sigma_{ij}^h(u^h, \varphi^h), \quad D_i(u(h), \varphi(h)) = h^{-2} D_i^h(u^h, \varphi^h), \tag{21}$$

where

$$\begin{aligned}\sigma_{ij}(h)(u(h),\varphi(h)) &= C_{ijlm}\,\kappa_{lm}(h) + h\,P_{\alpha ij}\,\partial_\alpha\varphi(h) + P_{3ij}\,\partial_3\varphi(h),\\ D_i(h)(u(h),\varphi(h)) &= P_{ilm}\,\kappa_{lm}(h) - h\,\varepsilon_{i\alpha}\,\partial_\alpha\varphi(h) - \varepsilon_{i3}\,\partial_3\varphi(h).\end{aligned} \tag{22}$$

We also introduce the scaled admissible displacement space V and the scaled admissible electric potential space Ψ

$$\begin{aligned} V &= \Big\{v \in [H^1(\Omega)]^3 : \quad v_{|\Gamma_D} = 0\Big\}\\ \Psi &= \Big\{\psi \in H^1(\Omega) : \quad \psi_{|\Gamma_{eD}} = 0\Big\},\end{aligned} \tag{23}$$

equipped with the norms $\|v\|_V = \|\nabla v\|_{(L^2(\Omega))^9}$ and $\|\psi\|_\Psi = \|\nabla\psi\|_{(L^2(\Omega))^3}$, respectively.

For any $v \in V$ we define the second order symmetric tensor field $\kappa(h)(v) = (\kappa_{ij}(h)(v))$ by

$$\begin{aligned}\kappa_{\alpha\beta}(h)(v) &= e_{\alpha\beta}(v) = \tfrac{1}{2}(\partial_\beta v_\alpha + \partial_\alpha v_\beta),\\ \kappa_{\alpha 3}(h)(v) &= \tfrac{1}{h}e_{\alpha 3}(v) = \tfrac{1}{2h}(\partial_3 v_\alpha + \partial_\alpha v_3),\\ \kappa_{33}(h)(v) &= \tfrac{1}{h^2}e_{33}(v) = \tfrac{1}{h^2}\partial_3 v_3.\end{aligned} \tag{24}$$

In particular, when $v = u(h)$ we set $\kappa(h) = \kappa(h)(u(h))$. As a consequence of the scalings (20) we have

$$e^h(u^h) = h^2\kappa(h)(u(h)) = h^2\kappa(h) \quad \text{and} \quad e^h(v^h) = h^2\kappa(h)(v). \tag{25}$$

Using all the scalings and assumptions on the data, we conclude that (14) is equivalent to the following scaled three-dimensional variational problem

$$\begin{cases}\text{Find} \quad (u(h),\bar\varphi(h)) \in V\times\Psi \quad \text{such that:}\\ a(h)\big((u(h),\bar\varphi(h)),(v,\psi)\big) = l(h)(v,\psi), \quad \forall (v,\psi)\in V\times\Psi,\end{cases} \tag{26}$$

where

$$\left[\begin{aligned} & a(h)\big((u(h),\bar\varphi(h)),(v,\psi)\big) = \textstyle\int_\Omega C\kappa(h) : \kappa(h)(v)\,dx + \int_\Omega \varepsilon_{33}\,\partial_3\bar\varphi(h)\,\partial_3\psi\,dx\\ & + \textstyle\int_\Omega P_{3jk}\left[\partial_3\bar\varphi(h)\,\kappa_{jk}(h)(v) - \partial_3\psi\,\kappa_{jk}(h)\right]dx\\ & + h\textstyle\int_\Omega \varepsilon_{3\alpha}\left[\partial_\alpha\bar\varphi(h)\,\partial_3\psi + \partial_3\bar\varphi(h)\,\partial_\alpha\psi\right]dx\\ & + h\textstyle\int_\Omega P_{\alpha jk}\left[\partial_\alpha\bar\varphi(h)\,\kappa_{jk}(h)(v) - \partial_\alpha\psi\,\kappa_{jk}(h)\right]dx + h^2\int_\Omega \varepsilon_{\alpha\beta}\,\partial_\alpha\bar\varphi(h)\,\partial_\beta\psi\,dx,\end{aligned}\right. \tag{27}$$

and

$$\left[\begin{array}{l} l(h)(v,\psi) = \int_\Omega f\cdot v\,dx + \int_{\Gamma_N} g\cdot v\,d\Gamma_N - \int_\Omega \varepsilon_{33}\,\partial_3\varphi_0\,\partial_3\psi\,dx \\ -h\int_\Omega \varepsilon_{\alpha 3}\left[\partial_\alpha\varphi_0\,\partial_3\psi + \partial_3\varphi_0\,\partial_\alpha\psi\right]dx - h^2\int_\Omega \varepsilon_{\alpha\beta}\,\partial_\alpha\varphi_0\,\partial_\beta\psi\,dx \\ -\int_\Omega P_{3ij}\,\partial_3\varphi_0\,\kappa_{ij}(h)(v)\,dx - h\int_\Omega P_{\alpha ij}\,\partial_\alpha\varphi_0\,\kappa_{ij}(h)(v)\,dx. \end{array}\right. \tag{28}$$

3.2. The limit problem. We essentially compute here the limit of the scaled displacements and scaled electric potentials $(u(h),\varphi(h))$, when $h\to 0^+$. We identify also the limit problem, that we call reduced piezoelectric plate model.

Let

$$\begin{aligned} &V_{KL} = \{v\in[H^1(\Omega)]^3: \quad v_{|\Gamma_D}=0, \quad e_{i3}(v)=0\},\\ &\Psi_l = \{\psi\in L^2(\Omega): \quad \partial_3\psi\in L^2(\Omega)\},\\ &\Psi_{l0} = \{\psi\in L^2(\Omega): \quad \partial_3\psi\in L^2(\Omega), \quad \psi_{|\Gamma_\pm}=0\}. \end{aligned} \tag{29}$$

The space V_{KL}, which is called the Kirchhoff-Love displacement space, is also defined by

$$\begin{aligned} V_{KL} = \{v\in[H^1(\Omega)]^3: \quad &\exists(\eta_1,\eta_2)\in V_H(\omega), \quad \eta_3\in V_3(\omega),\\ &v_\alpha(x)=\eta_\alpha(x_1,x_2)-x_3\partial_\alpha\eta_3(x_1,x_2), \quad v_3(x)=\eta_3(x_1,x_2)\}, \end{aligned} \tag{30}$$

where

$$\begin{aligned} &V_H(\omega) = \{\eta=(\eta_1,\eta_2)\in[H^1(\omega)]^2: \quad \eta_{|\gamma_0}=0\},\\ &V_3(\omega) = \{\eta_3\in H^2(\omega): \quad {\eta_3}_{|\gamma_0}=0, \quad \partial_\nu{\eta_3}_{|\gamma_0}=0\}. \end{aligned} \tag{31}$$

Theorem 3.1. *There exist $u\in[H^1(\Omega)]^3$, $\kappa\in[L^2(\Omega)]^9$ and $\varphi\in L^2(\Omega)$ and subsequences $\{u(h)\}_{h>0}$, $\{\kappa(h)\}_{h>0}$ and $\{(h\partial_1\varphi(h),h\partial_2\varphi(h),\partial_3\varphi(h))\}_{h>0}$ (still indexed by h), such that the following weak convergences are satisfied, when $h\to 0^+$,*

$$\begin{aligned} &u(h)\rightharpoonup u \quad in \quad [H^1(\Omega)]^3,\\ &\kappa(h)\rightharpoonup \kappa \quad in \quad [L^2(\Omega)]^9,\\ &\varphi(h)\rightharpoonup \varphi \quad in \quad L^2(\Omega),\\ &(h\partial_1\varphi(h),h\partial_2\varphi(h),\partial_3\varphi(h))\rightharpoonup(0,0,\partial_3\varphi) \quad in \quad [L^2(\Omega)]^3. \end{aligned} \tag{32}$$

Moreover, the limits u, κ and φ belong to the spaces V_{KL}, $[L^2(\Omega)]^9$ and Ψ_l, respectively, $\varphi = \varphi_0$ on $\Gamma_\pm$ and

$$\begin{aligned}
&\kappa_{\alpha\beta} = e_{\alpha\beta}(u), \quad in \quad L^2(\Omega),\\
&\kappa_{33} = -\tfrac{1}{C_{3333}}(P_{333}\partial_3\varphi + C_{33\alpha\beta}e_{\alpha\beta}(u)), \quad in \quad L^2(\Omega),\\
&\kappa_{13} = -\frac{\frac{1}{2}det\begin{bmatrix} P_{313} & C_{1323}\\ P_{323} & C_{2323}\end{bmatrix}}{det\begin{bmatrix} C_{1313} & C_{1323}\\ C_{2313} & C_{2323}\end{bmatrix}}\partial_3\varphi, \quad in \quad L^2(\Omega),\\
&\kappa_{23} = -\frac{\frac{1}{2}det\begin{bmatrix} C_{1313} & P_{313}\\ C_{2313} & P_{323}\end{bmatrix}}{det\begin{bmatrix} C_{1313} & C_{1323}\\ C_{2313} & C_{2323}\end{bmatrix}}\partial_3\varphi, \quad in \quad L^2(\Omega).
\end{aligned} \tag{33}$$

Proof. Taking $(v, \psi) = (u(h), \bar{\varphi}(h))$ in (26) we obtain

$$\left\{\begin{array}{l} \|u(h)\|_V^2 + \int_\Omega \kappa(h) : \kappa(h)dx \\ \qquad +\|h\,\partial_1\varphi(h)\|^2_{L^2(\Omega)} + \|h\,\partial_2\varphi(h)\|^2_{L^2(\Omega)} + \|\partial_3\varphi(h)\|^2_{L^2(\Omega)} < c \end{array}\right. \tag{34}$$

where $c > 0$ is a constant independent of h. Arguing as in Sene ([13], proposition 3.1) the weak convergences in (32) are a direct consequence of this inequality.

The sequence $\kappa_{i3}(h)$ is bounded in $L^2(\Omega)$, because of (34), and consequently $e_{\alpha 3}(u(h)) = h\kappa_{\alpha 3}(h)$ and $e_{33}(u(h)) = h^2\kappa_{33}(h)$ strongly converge to zero in $L^2(\Omega)$. Thus $e_{i3}(u) = 0$, which means that $u \in V_{KL}$.

The first equation in (33) is a consequence of the first two convergences in (32). To obtain the remaining three equations of (33) we first multiply equation (26) by h^2 and consider $\psi = 0$, then we multiply (26) by h and take $(v_3, \psi) = (0, 0)$. In both cases we consider the limit when $h \to 0$. We obtain that

$$\begin{aligned}
&C_{33lm}\kappa_{lm} + P_{333}\partial_3\bar{\varphi} = -P_{333}\partial_3\varphi_0, \quad \text{in} \quad L^2(\Omega),\\
&C_{\alpha 3lm}\kappa_{lm} + P_{3\alpha 3}\partial_3\bar{\varphi} = -P_{3\alpha 3}\partial_3\varphi_0, \quad \text{in} \quad L^2(\Omega),
\end{aligned} \tag{35}$$

where $\bar{\varphi} = \varphi - \varphi_0$. Taking into account that $C_{\alpha\beta\gamma 3} = 0 = C_{\alpha 333}$, the system (35) reduces to

$$\begin{aligned} & C_{33\alpha\beta}\kappa_{\alpha\beta} + C_{3333}\kappa_{33} = -P_{333}\partial_3\varphi, \quad \text{in} \quad L^2(\Omega), \\ & 2C_{\alpha 3\beta 3}\kappa_{\beta 3} = -P_{3\alpha 3}\partial_3\varphi, \quad \text{in} \quad L^2(\Omega), \end{aligned} \tag{36}$$

and the solution of this system is precisely (33).■

As a consequence of the previous theorem we have the following limit result for the scaled stress tensor and the scaled electric displacement field.

Theorem 3.2. *There exist $\sigma \in [L^2(\Omega)]^9$, $D \in [L^2(\Omega)]^3$ and subsequences $\{\sigma_{ij}(h)(u(h), \varphi(h))\}_{h>0}$ and $\{D_i(h)(u(h), \varphi(h))\}_{h>0}$ (still indexed by h), such that the following weak convergences are satisfied, when $h \to 0^+$,*

$$\begin{aligned} & \sigma_{ij}(h)(u(h), \varphi(h)) \rightharpoonup \sigma_{ij} \quad \text{in} \quad [L^2(\Omega)]^9, \\ & D_i(h)(u(h), \varphi(h)) \rightharpoonup D_i \quad \text{in} \quad [L^2(\Omega)]^3, \end{aligned} \tag{37}$$

where

$$\begin{aligned} & \sigma_{i3} = 0, \\ & \sigma_{\alpha\beta} = \left[C_{\alpha\beta\gamma\rho} - \tfrac{C_{\alpha\beta 33}C_{33\gamma\rho}}{C_{3333}}\right] e_{\gamma\rho}(u) + \left[P_{3\alpha\beta} - \tfrac{C_{\alpha\beta 33}}{C_{3333}} P_{333}\right]\partial_3\varphi, \end{aligned} \tag{38}$$

and

$$\begin{aligned} D_i = {} & \left[P_{i\alpha\beta} - \tfrac{C_{\alpha\beta 33}}{C_{3333}} P_{i33}\right] e_{\alpha\beta}(u) - \Bigg\{ \varepsilon_{i3} + \frac{P_{i33}P_{333}}{C_{3333}} \\ & + \frac{1}{det \begin{bmatrix} C_{1313} & C_{1323} \\ C_{2313} & C_{2323} \end{bmatrix}} \begin{bmatrix} P_{323} \\ P_{313} \end{bmatrix}^T \begin{bmatrix} C_{1313} & -C_{1323} \\ -C_{2313} & C_{2323} \end{bmatrix} \begin{bmatrix} P_{i23} \\ P_{i13} \end{bmatrix} \Bigg\} \partial_3\varphi. \end{aligned} \tag{39}$$

Proof. Using the theorem 3.1 and taking the weak limit in (22) when $h \to 0^+$, we have that

$$\begin{aligned} & \sigma_{ij} = C_{ijlm}\kappa_{lm} + P_{3ij}\partial_3\varphi, \\ & D_i = P_{ilm}\kappa_{lm} - \varepsilon_{i3}\partial_3\varphi. \end{aligned} \tag{40}$$

But using (36) we immediately have $\sigma_{i3} = 0$. Then introducing the definition of κ_{lm} in (40) we obtain the definitions of $\sigma_{\alpha\beta}$ and D_i. ■

We observe that $\sigma_{i3} = 0$ means that at the limit the stresses are plane. This agrees with the a priori hypotheses on the stresses made by Bernadou and Hanel [1], p.4015, for piezoelectric thin shells.

The next theorem gives a first characterization of the weak limit (u, φ) defined in theorem 3.1. To that purpose we introduce the reduced elasticity coefficients $A_{\alpha\beta\gamma\rho}$ defined by

$$A_{\alpha\beta\gamma\rho} = C_{\alpha\beta\gamma\rho} - \frac{C_{\alpha\beta 33} C_{33\gamma\rho}}{C_{3333}}, \tag{41}$$

the coefficients $p_{3\alpha\beta}$

$$p_{3\alpha\beta} = P_{3\alpha\beta} - \frac{C_{\alpha\beta 33}}{C_{3333}} P_{333}, \tag{42}$$

and p_{33}

$$\left\{ \begin{array}{l} p_{33} = \varepsilon_{33} + \dfrac{P_{333} P_{333}}{C_{3333}} \\ + \dfrac{1}{det \begin{bmatrix} C_{1313} & C_{1323} \\ C_{2313} & C_{2323} \end{bmatrix}} \begin{bmatrix} P_{323} \\ -P_{313} \end{bmatrix}^T \begin{bmatrix} C_{1313} & C_{1323} \\ C_{2313} & C_{2323} \end{bmatrix} \begin{bmatrix} P_{323} \\ -P_{313} \end{bmatrix}. \end{array} \right. \tag{43}$$

We remark that the positivity hypothesis (10) guarantees that in $\bar{\omega} \times [-1, 1]$

$$C_{3333} > 0, \qquad \varepsilon_{33} > 0, \qquad det \begin{bmatrix} C_{1313} & C_{1323} \\ C_{2313} & C_{2323} \end{bmatrix} > 0, \tag{44}$$

and

$$\begin{bmatrix} C_{1313} & C_{1323} \\ C_{2313} & C_{2323} \end{bmatrix} \tag{45}$$

is a positive definite matrix, which implies that $p_{33} > c$ in $\bar{\Omega}$, with c a strictly positive constant. Also as a consequence of the hypothesis (10), $A = (A_{\alpha\beta\gamma\rho})$ is coercive (cf. Figueiredo and Leal [10]), that is, there exists a constant $c > 0$ independent of x

$$A_{\alpha\beta\gamma\rho}(x) M_{\gamma\rho} M_{\alpha\beta} \geq c \sum_{\alpha,\beta=1}^{2} (M_{\alpha\beta})^2 \tag{46}$$

for every symmetric 2×2 real matrix M and for every $x \in \bar{\omega} \times [-1, 1]$.

Theorem 3.3. *The weak limit (u,φ) is the unique solution of the variational problem*

$$\left\{\begin{array}{l} \textit{Find} \quad (u,\varphi)\in V_{KL}\times\Psi_l \quad \textit{such that:} \\ a\big((u,\varphi),(v,\psi)\big)=l(v,\psi), \quad \forall(v,\psi)\in V_{KL}\times\Psi_{l0}, \\ \varphi=\varphi_0, \quad \textit{on} \quad \Gamma_\pm, \end{array}\right. \tag{47}$$

where

$$\left[\begin{array}{l} a\big((u,\varphi),(v,\psi)\big)=\int_\Omega A_{\alpha\beta\gamma\rho}e_{\alpha\beta}(u)e_{\gamma\rho}(v)\,dx+\int_\Omega p_{33}\,\partial_3\varphi\,\partial_3\psi\,dx \\ \qquad\qquad -\int_\Omega p_{3\alpha\beta}\left[e_{\alpha\beta}(u)\partial_3\psi-e_{\alpha\beta}(v)\partial_3\varphi\right]dx, \end{array}\right. \tag{48}$$

and

$$l(v,\psi)=\int_\Omega f\cdot v\,dx+\int_{\Gamma_N} g\cdot v\,d\Gamma_N. \tag{49}$$

Proof. Considering $(v,\psi)\in V_{KL}\times\Psi$ in (27)-(28), taking the limit when $h\to 0^+$ and using theorem 3.1 we get

$$\left[\begin{array}{l} \lim\limits_{h\to0^+} a(h)\big((u(h),\bar\varphi(h)),(v,\psi)\big)= \\ \int_\Omega C_{\alpha\beta ij}\kappa_{ij}e_{\alpha\beta}(v)\,dx+\int_\Omega \varepsilon_{33}\,\partial_3\bar\varphi\,\partial_3\psi\,dx+ \\ \qquad\qquad +\int_\Omega P_{3\alpha\beta}\,\partial_3\bar\varphi\,e_{\alpha\beta}(v)\,dx-\int_\Omega P_{3lm}\,\partial_3\psi\,\kappa_{lm}\,dx, \end{array}\right. \tag{50}$$

and

$$\left[\begin{array}{l} \lim\limits_{h\to0^+} l(h)(v,\psi)=\int_\Omega f\cdot v\,dx+\int_{\Gamma_N} g\cdot v\,d\Gamma_N-\int_\Omega \varepsilon_{33}\,\partial_3\varphi_0\,\partial_3\psi\,dx \\ \qquad\qquad -\int_\Omega P_{3\alpha\beta}\,\partial_3\varphi_0\,e_{\alpha\beta}(v)\,dx. \end{array}\right. \tag{51}$$

Remarking that $\bar\varphi=\varphi-\varphi_0$ and introducing in (50)-(51) the definitions of κ_{ij}, given in theorem 3.1, we clearly have (47).

To prove the uniqueness of the solution we suppose that $(\hat u,\hat\varphi)$ is another solution of (47), and define $z=\hat u-u$ and $\chi=\hat\varphi-\varphi$. Then subtracting the equations

$$\begin{aligned} a((u,\varphi),(z,\chi))&=l(z,\chi) \\ a((\hat u,\hat\varphi),(z,\chi))&=l(z,\chi) \end{aligned} \tag{52}$$

we get

$$0=a((z,\chi),(z,\chi))=\int_\Omega A_{\alpha\beta\gamma\rho}e_{\alpha\beta}(z)e_{\gamma\rho}(z)\,dx+\int_\Omega p_{33}\,\partial_3\chi\,\partial_3\chi\,dx. \tag{53}$$

But because of the ellipticity of the coefficients $A_{\alpha\beta\gamma\rho}$, cf. (46), and the property of p_{33} we have

$$\begin{aligned} 0 &= \textstyle\int_\Omega A_{\alpha\beta\gamma\rho} e_{\alpha\beta}(z) e_{\gamma\rho}(z)\,dx + \int_\Omega p_{33}\,\partial_3\chi\,\partial_3\chi\,dx \\ &\geq c_1 \textstyle\sum_{\alpha,\beta=1}^2 \|e_{\alpha\beta}(z)\|_{L^2(\Omega)} + c_2\|\partial_3\chi\|^2_{L^2(\Omega)}, \end{aligned} \tag{54}$$

where c_1 and c_2 are strictly positive constants. Hence $z = 0$ in V_{KL}, $\partial_3\chi = 0$ in $L^2(\Omega)$ and $\chi = 0$ in Ψ_{l0}, and this finishes the proof. ■

Remark – Using arguments similar to those of Sene ([13], theorem 4.1) we can state that the weak convergences verified in theorem 3.1 are also strong.

We can also demonstrate that the limit electric potential φ has an explicit form as a function of the third component of the limit displacement u. This characterization of φ induces a simplification of the limit variational problem defined in (47); it can be reduced to a variational problem whose unknown is only the Kirchhoff-Love displacement u. These statements are summarized in the next theorem.

Theorem 3.4 (The reduced piezoelectric plate model)**.** *When $h \to 0$, the solution $(u(h), \varphi(h))$ of (2)-(4) converges strongly to (u, φ), in the functional space $H^1(\Omega) \times L^2(\Omega)$, and verifies the properties i) and ii) described below.*

i) The limit displacement u is a Kirchhoff-Love displacement vector field, that is

$$\begin{aligned} &u(x) = (u_1(x), u_2(x), u_3(x)), \quad x = (x_1, x_2, x_3) \in \bar{\Omega} \\ &u_1(x) = \zeta_1(x_1, x_2) - x_3\partial_1\zeta_3(x_1, x_2) \\ &u_2(x) = \zeta_2(x_1, x_2) - x_3\partial_2\zeta_3(x_1, x_2) \\ &u_3(x) = \zeta_3(x_1, x_2) \end{aligned} \tag{55}$$

where ζ_1, $\zeta_2 \in V_H(\omega)$, $\zeta_3 \in V_3(\omega)$, and u is the solution of the problem

$$\begin{cases} \textit{Find} \quad u \in V_{KL} \quad \textit{such that:} \\ \bar{a}(u, v) = \bar{l}(v), \quad \forall v \in V_{KL}, \end{cases} \tag{56}$$

where

$$\begin{aligned} &\bar{a}(u, v) = \textstyle\int_\Omega A_{\alpha\beta\gamma\rho} e_{\alpha\beta}(u) e_{\gamma\rho}(v)\,dx - \int_\Omega x_3\,\frac{p_{3\alpha\beta}p_{3\gamma\rho}}{p_{33}}\,\partial_{\gamma\rho}\zeta_3\,e_{\alpha\beta}(v)\,dx, \\ &\bar{l}(v) = \textstyle\int_\Omega f \cdot v\,dx + \int_{\Gamma_N} g \cdot v\,d\Gamma_N - \int_\Omega \frac{\varphi_0^+ - \varphi_0^-}{2}\,p_{3\alpha\beta}\,e_{\alpha\beta}(v)\,dx. \end{aligned} \tag{57}$$

ii) The limit electric potential φ is a second order polynomial in x_3, whose coefficients depend on ζ_3, and the exact analytic form of φ is the following

$$\begin{aligned}
&\varphi(x_1,x_2,x_3) = \textstyle\sum_{m=0}^{2} \psi^m(x_1,x_2)x_3^m,\\
&\psi^0 = \tfrac{\varphi_0^+ + \varphi_0^-}{2} + \tfrac{p_{3\alpha\beta}}{2p_{33}}\partial_{\alpha\beta}\zeta_3,\\
&\psi^1 = \tfrac{\varphi_0^+ - \varphi_0^-}{2},\\
&\psi^2 = -\tfrac{p_{3\alpha\beta}}{2p_{33}}\partial_{\alpha\beta}\zeta_3.
\end{aligned} \tag{58}$$

Proof. The proof of ii) is exactly the same as in Sene ([13], theorem 3.1). To demonstrate i) it is enough to take $\psi = 0$ in (47) and to replace φ by the expression (58). ■

4. The two-dimensional piezolectric plate model

In this section we give the formulation of the problem (56) as a boundary value problem, that we call two-dimensional piezoelectric plate model. The word two-dimensional means that the solution u is completely determined by the solution $\zeta = (\zeta_1, \zeta_2, \zeta_3)$ of a two-dimensional boundary value problem posed over the middle surface ω of the plate, and the word piezolectric signifies that this model depends on the elastic and electric data imposed on the three-dimensional problem. Besides, the theorem 4.2 gives the formulation of this two-dimensional piezolectric plate model and the expression of the limit electric potential, with respect to the original plate $\bar\Omega^h = \bar\omega^h \times [-h,h]$. This is achieved after de-scaling the limit displacement vector u, the limit electric potential φ and the variational equations (56) and (47).

To formulate the results of this section we must define the following coefficients

$$\begin{aligned}
&\mathcal{A}_{\alpha\beta\gamma\rho}(x_1,x_2) = \textstyle\int_{-1}^{+1} A_{\alpha\beta\gamma\rho}(x)dx_3\\
&\mathcal{B}_{\alpha\beta\gamma\rho}(x_1,x_2) = \textstyle\int_{-1}^{+1} x_3 A_{\alpha\beta\gamma\rho}(x)dx_3\\
&\mathcal{C}_{\alpha\beta\gamma\rho}(x_1,x_2) = \textstyle\int_{-1}^{+1} x_3^2 A_{\alpha\beta\gamma\rho}(x)dx_3
\end{aligned} \tag{59}$$

and

$$\begin{aligned}
&\mathcal{D}_{\alpha\beta\gamma\rho}(x_1,x_2) = \textstyle\int_{-1}^{+1} x_3 \frac{p_{3\alpha\beta}p_{3\gamma\rho}}{p_{33}}(x)dx_3\\
&\mathcal{E}_{\alpha\beta\gamma\rho}(x_1,x_2) = \textstyle\int_{-1}^{+1} x_3^2 \frac{p_{3\alpha\beta}p_{3\gamma\rho}}{p_{33}}(x)dx_3.
\end{aligned} \tag{60}$$

These new coefficients depend on $(x_1,x_2) \in \omega$ and are associated to the elasticity, piezoelectric and dielectric coefficients. Moreover we define also the tensors $N = (N_{\alpha\beta})$ and $M = (M_{\alpha\beta})$, associated to the Kirchhoff-Love displacement u, by

$$\begin{bmatrix} N_{\alpha\beta} \\ M_{\alpha\beta} \end{bmatrix} = \begin{bmatrix} \mathcal{A}_{\alpha\beta\gamma\rho} & -\mathcal{B}_{\alpha\beta\gamma\rho} - \mathcal{D}_{\alpha\beta\gamma\rho} \\ -\mathcal{B}_{\alpha\beta\gamma\rho} & \mathcal{C}_{\alpha\beta\gamma\rho} + \mathcal{E}_{\alpha\beta\gamma\rho} \end{bmatrix} \begin{bmatrix} e_{\gamma\rho}(\zeta) \\ \partial_{\gamma\rho}\zeta_3 \end{bmatrix} \tag{61}$$

that is

$$\begin{aligned} N_{\alpha\beta}(\zeta_1,\zeta_2,\zeta_3) &= \mathcal{A}_{\alpha\beta\gamma\rho} e_{\gamma\rho}(\zeta) - \Big(\mathcal{B}_{\alpha\beta\gamma\rho} + \mathcal{D}_{\alpha\beta\gamma\rho}\Big)\partial_{\gamma\rho}\zeta_3 \\ M_{\alpha\beta}(\zeta_1,\zeta_2,\zeta_3) &= -\mathcal{B}_{\alpha\beta\gamma\rho} e_{\gamma\rho}(\zeta) + \Big(\mathcal{C}_{\alpha\beta\gamma\rho} + \mathcal{E}_{\alpha\beta\gamma\rho}\Big)\partial_{\gamma\rho}\zeta_3, \end{aligned} \tag{62}$$

and we remark that both $N_{\alpha\beta}$ and $M_{\alpha\beta}$ depend on the three functions $(\zeta_1,\zeta_2,\zeta_3) = \zeta$ that define u. We also introduce the coefficients $\hat{N}_{\alpha\beta}$ and $\hat{M}_{\alpha\beta}$ by

$$\begin{aligned} \hat{N}_{\alpha\beta} &= N_{\alpha\beta} + \tfrac{\varphi_0^+ - \varphi_0^-}{2} \int_{-1}^{+1} p_{3\alpha\beta}(x)dx_3 \\ \hat{M}_{\alpha\beta} &= M_{\alpha\beta} - \tfrac{\varphi_0^+ - \varphi_0^-}{2} \int_{-1}^{+1} x_3 p_{3\alpha\beta}(x)dx_3, \end{aligned} \tag{63}$$

The following theorem formulates the variational problem (56) as a system of two coupled boundary value problems.

Theorem 4.1 (The two-dimensional piezoelectric plate model)**.** *The components ζ_i, for $i = 1,2,3$, of the Kirchhoff-Love displacement u, are the solution of the following system of dependent boundary value problems (64)-(65), defined on the middle surface of the plate, and depending on the piezoelectric, dielectric and elastic coefficients, the applied forces and the electric potential* *data:*

$$\left\{\begin{array}{l}\left[\begin{array}{l}\partial_{\alpha\beta}M_{\alpha\beta}=\int_{-1}^{1}(x_3\partial_\alpha f_\alpha+f_3)\,dx_3+g_3^++g_3^-+\partial_\alpha(g_\alpha^+-g_\alpha^-)\\ \qquad\qquad +\partial_{\alpha\beta}\Big(\frac{\varphi_0^+-\varphi_0^-}{2}\int_{-1}^{1}x_3\,p_{3\alpha\beta}\,dx_3\Big),\quad in\quad \omega,\end{array}\right.\\[2ex] \zeta_3=0=\frac{\partial\zeta_3}{\partial\nu},\quad on\quad \gamma_0,\\[1ex] M_{\alpha\beta}\nu_\alpha\nu_\beta=\frac{\varphi_0^+-\varphi_0^-}{2}\int_{-1}^{1}x_3\,p_{3\alpha\beta}\,\nu_\alpha\nu_\beta\,dx_3+\int_{-1}^{1}g_\alpha(-x_3)\,\nu_\alpha\,dx_3,\quad on\quad \gamma_1,\\[1ex] \left[\begin{array}{l}\partial_\alpha M_{\alpha\beta}\nu_\beta+\partial_\tau(M_{\alpha\beta}\nu_\alpha\tau_\beta)=\int_{-1}^{1}x_3f_\alpha\,\nu_\alpha\,dx_3+(g_\alpha^+-g_\alpha^-)\,\nu_\alpha\\ \qquad -\int_{-1}^{1}g_3\,dx_3-\partial_\tau\big(\int_{-1}^{1}g_\alpha\,x_3\,dx_3\,\tau_\alpha\big)\\ \qquad\qquad +\partial_\alpha\Big(\frac{\varphi_0^+-\varphi_0^-}{2}\int_{-1}^{1}x_3\,p_{3\alpha\beta}\,dx_3\Big)\nu_\beta\\ \qquad\qquad\quad +\partial_\tau\Big(\frac{\varphi_0^+-\varphi_0^-}{2}\int_{-1}^{1}x_3\,p_{3\alpha\beta}\,dx_3\,\nu_\alpha\tau_\beta\Big),\quad on\quad \gamma_1,\end{array}\right.\end{array}\right. \tag{64}$$

and

$$\left\{\begin{array}{l}\left[\begin{array}{l}-\partial_\alpha N_{\alpha\beta}=\int_{-1}^{1}f_\beta\,dx_3+(g_\beta^++g_\beta^-)\\ \qquad\qquad +\partial_\alpha\Big(\frac{\varphi_0^+-\varphi_0^-}{2}\int_{-1}^{1}p_{3\alpha\beta}\,dx_3\Big),\quad in\quad \omega,\end{array}\right.\\[2ex] (\zeta_1,\zeta_2)=(0,0),\quad on\quad \gamma_0,\\[1ex] N_{\alpha\beta}\nu_\alpha=\int_{-1}^{1}g_\beta\,dx_3-\frac{\varphi_0^+-\varphi_0^-}{2}\int_{-1}^{1}p_{3\alpha\beta}\,dx_3\,\nu_\alpha,\quad on\quad \gamma_1.\end{array}\right. \tag{65}$$

Moreover

$$\sigma_{\alpha\beta}=A_{\alpha\beta\gamma\rho}\big(e_{\gamma\rho}(\zeta)-x_3\partial_{\gamma\rho}\zeta_3\big)-x_3\frac{p_{3\alpha\beta}p_{3\gamma\rho}}{p_{33}}\partial_{\gamma\rho}\zeta_3+p_{3\alpha\beta}\frac{\varphi_0^+-\varphi_0^-}{2} \tag{66}$$

and

$$\hat{N}_{\alpha\beta}=\int_{-1}^{1}\sigma_{\alpha\beta}\,dx_3,\qquad -\hat{M}_{\alpha\beta}=\int_{-1}^{1}x_3\,\sigma_{\alpha\beta}\,dx_3 \tag{67}$$

which means that $\hat{N}_{\alpha\beta}$ are the stress resultants and $-\hat{M}_{\alpha\beta}$ are the stress couples (or bending moments), as in the pure elasticity case.

Proof. First we choose, in (56), $v\in V_{KL}$ with the components

$$v_\alpha(x)=-x_3\partial_\alpha\eta_3(x_1,x_2),\qquad v_3(x)=\eta_3(x_1,x_2), \tag{68}$$

and then

$$v_\alpha(x)=\eta_\alpha(x_1,x_2),\qquad v_3(x)=0. \tag{69}$$

The variational equation (56) may be written as

$$\begin{cases} \int_\omega M_{\alpha\beta}\,\partial_{\alpha\beta}\,\eta_3\,d\omega = -\int_\Omega \frac{\varphi_0^+-\varphi_0^-}{2} p_{3\alpha\beta}(-x_3\partial_\alpha\eta_3)\,dx \\ +\int_\Omega \big[f_\alpha(-x_3\partial_\alpha\eta_3)+f_3\eta_3\big]\,dx + \int_{\Gamma_N}\big[g_\alpha(-x_3\partial_\alpha\eta_3)+g_3\eta_3\big]\,d\Gamma_N, \end{cases} \tag{70}$$

for the first choice of v, and it may be written as

$$\begin{cases} \int_\omega N_{\alpha\beta}\,e_{\alpha\beta}(\eta)\,d\omega = -\int_\Omega \frac{\varphi_0^+-\varphi_0^-}{2} p_{3\alpha\beta}e_{\alpha\beta}(\eta)\,dx \\ \qquad +\int_\Omega f_\alpha\eta_\alpha\,dx + \int_{\Gamma_N} g_\alpha\eta_\alpha\,d\Gamma_N, \end{cases} \tag{71}$$

for the second choice of v, with $M_{\alpha\beta}$ and $N_{\alpha\beta}$ defined by (61)-(62). Applying the Green's formula to (70) and (71) we obtain (64) and (65), respectively. The definition of $\sigma_{\alpha\beta}$ in (66) is a consequence of the expression (38) and the following formula for $\partial_3\varphi$

$$\partial_3\varphi = -\frac{p_{3\alpha\beta}}{p_{33}}x_3\partial_{\alpha\beta}\zeta_3 + \frac{\varphi_0^+-\varphi_0^-}{2} \tag{72}$$

that is obtained directly from the definition (58) of φ. ■

We observe that the system (64)-(65) generalizes the usual pure elasticity plate model for anisotropic plates (cf. Green and Zerna [8] and Destuynder [7]). Moreover we have the following corollary of the previous theorem.

Corollary 4.1. *In particular, when C_{ijkl}, $p_{3\alpha\beta}$ and p_{33} are independent of x_3, then the problems (64)-(65) are also independent. More specifically, we have that:*

i) the component ζ_3 is the solution of the scalar boundary value equation

$$\begin{cases} \partial_{\alpha\beta}M_{\alpha\beta} = \int_{-1}^1 (x_3\partial_\alpha f_\alpha + f_3)\,dx_3 + g_3^+ + g_3^- + \partial_\alpha(g_\alpha^+ - g_\alpha^-), \quad in \quad \omega, \\ \zeta_3 = 0 = \frac{\partial\zeta_3}{\partial\nu}, \quad on \quad \gamma_0, \\ M_{\alpha\beta}\nu_\alpha\nu_\beta = \int_{-1}^1 g_\alpha(-x_3)\,\nu_\alpha\,dx_3, \quad on \quad \gamma_1, \\ \partial_\alpha M_{\alpha\beta}\nu_\beta + \partial_\tau(M_{\alpha\beta}\nu_\alpha\tau_\beta) = \int_{-1}^1 x_3 f_\alpha\,\nu_\alpha\,dx_3 + (g_\alpha^+ - g_\alpha^-)\,\nu_\alpha \\ \qquad -\int_{-1}^1 g_3\,dx_3 - \partial_\tau\big(\int_{-1}^1 g_\alpha\,x_3\,dx_3\,\tau_\alpha\big), \quad on \quad \gamma_1, \end{cases} \tag{73}$$

with

$$M_{\alpha\beta} = M_{\alpha\beta}(\zeta_3) = \frac{2}{3}\Big(A_{\alpha\beta\gamma\rho} + \frac{p_{3\alpha\beta}p_{3\gamma\rho}}{p_{33}}\Big)\partial_{\gamma\rho}\zeta_3, \tag{74}$$

ii) the pair (ζ_1, ζ_2) *is the solution of the boundary value system*

$$\begin{cases} -\partial_\alpha N_{\alpha\beta} = \int_{-1}^{1} f_\beta \, dx_3 + (g_\beta^+ + g_\beta^-) + \partial_\alpha \Big(p_{3\alpha\beta}(\varphi_0^+ - \varphi_0^-) \Big), \quad in \quad \omega, \\ (\zeta_1, \zeta_2) = (0,0), \quad on \quad \gamma_0, \\ N_{\alpha\beta}\nu_\alpha = \int_{-1}^{1} g_\beta \, dx_3 - p_{3\alpha\beta}(\varphi_0^+ - \varphi_0^-)\, \nu_\alpha, \quad on \quad \gamma_1. \end{cases} \tag{75}$$

with

$$N_{\alpha\beta} = N_{\alpha\beta}(\zeta_1, \zeta_2) = 2A_{\alpha\beta\gamma\rho}\, e_{\gamma\rho}(\zeta). \tag{76}$$

Proof. When C_{ijkl}, $p_{3\alpha\beta}$ and p_{33} are independent of x_3 the system (73)-(76) is obtained directly from (64)-(65). ■

We remark that the previous theorem 4.1 and corollary 4.1 generalize the results obtained by Sene [13]. In fact, for a homogeneous isotropic thin plate, that is, a plate where C_{ijkl} is defined by (11) and such that P_{ijk} and ε_{ij} are constants independent of x and h, then C_{ijkl}, $p_{3\alpha\beta}$ and p_{33} are independent of x_3, the system (73)-(76) is verified and coincides with the plate model deduced by Sene ([13], theorem 3.3).

To interpret the limit problem (64)-(65) and the expression of the limit electric potential (58), with respect to the original plate $\bar{\Omega}^h = \bar{\omega}^h \times [-h, h]$ it is convenient to formulate (64)-(65) and (58) in $\bar{\Omega}^h$. In order to do that we define the functions ζ_1^h, ζ_2^h and ζ_3^h by the de-scalings

$$\zeta_\alpha^h = h^2 \zeta_\alpha \quad \text{and} \quad \zeta_3^h = h\zeta_3, \quad \text{in} \quad \bar{\omega}, \tag{77}$$

and z_i^h by

$$z_\alpha^h(x^h) = h^2 u_\alpha(x) \quad \text{and} \quad z_3^h(x^h) = h u_3(x), \tag{78}$$

and ϕ^h by

$$\phi^h(x^h) = h^3 \varphi(x), \tag{79}$$

for all $x^h = (x_1, x_2, hx_3)$, with $x = (x_1, x_2, x_3) \in \bar{\Omega} = \bar{\omega} \times [-1, 1]$. The functions ζ_i^h are called the limit displacements of the middle surface ω of the plate $\bar{\Omega}^h = \bar{\omega} \times [-h, h]$. The functions ζ_α^h and ζ_3^h are respectively the in-plane, and transverse displacements. The functions z_i^h and ϕ^h are, respectively, the limit displacements and limit electric potential, inside the plate $\bar{\Omega}^h$.

We can now state the following immediate consequence of theorem 3.4, parts i) and ii).

Theorem 4.2. *The de-scaled functions* $(\zeta_1^h, \zeta_2^h, \zeta_3^h)$ *defined in* ω *are the solution of the system*

$$\begin{cases} \left[\begin{array}{l} \partial_{\alpha\beta} M^h_{\alpha\beta} = \int_{-h}^{+h} (x_3^h \partial_\alpha f_\alpha^h + f_3^h)\, dx_3^h + g_3^{h+} + g_3^{h-} + h\partial_\alpha (g_\alpha^{h+} - g_\alpha^{h-}) \\ \qquad\qquad + \partial_{\alpha\beta}\Big(\frac{\varphi_0^{h+} - \varphi_0^{h-}}{2h} \int_{-h}^{+h} x_3^h\, p_{3\alpha\beta}\, dx_3^h \Big), \quad in \quad \omega, \end{array} \right. \\ \zeta_3^h = 0 = \frac{\partial \zeta_3^h}{\partial \nu}, \quad on \quad \gamma_0, \\ M^h_{\alpha\beta} \nu_\alpha \nu_\beta = \frac{\varphi_0^{h+} - \varphi_0^{h-}}{2h} \int_{-h}^{+h} x_3^h\, p_{3\alpha\beta}\, dx_3^h \nu_\alpha \nu_\beta + \int_{-h}^{+h} g_\alpha^h (-x_3^h)\, \nu_\alpha\, dx_3^h, \ on\ \gamma_1, \\ \left[\begin{array}{l} \partial_\alpha M^h_{\alpha\beta} \nu_\beta + \partial_\tau (M^h_{\alpha\beta} \nu_\alpha \tau_\beta) = \int_{-h}^{+h} x_3^h f_\alpha^h\, \nu_\alpha\, dx_3^h + h(g_\alpha^{h+} - g_\alpha^{h-})\, \nu_\alpha \\ \quad - \int_{-h}^{+h} g_3^h\, dx_3^h - \partial_\tau \big(\int_{-h}^{+h} g_\alpha^h\, x_3^h\, dx_3^h\, \tau_\alpha \big) \\ \qquad + \partial_\alpha \Big(\frac{\varphi_0^{h+} - \varphi_0^{h-}}{2h} \int_{-h}^{+h} x_3^h\, p_{3\alpha\beta}\, dx_3^h \Big) \nu_\beta \\ \qquad\quad + \partial_\tau \Big(\frac{\varphi_0^{h+} - \varphi_0^{h-}}{2h} \int_{-h}^{+h} x_3^h\, p_{3\alpha\beta}\, dx_3^h\, \nu_\alpha \tau_\beta \Big), \quad on \quad \gamma_1, \end{array} \right. \end{cases} \tag{80}$$

and

$$\begin{cases} \left[\begin{array}{l} -\partial_\alpha N^h_{\alpha\beta} = \int_{-h}^{+h} f_\beta^h\, dx_3^h + (g_\beta^{h+} + g_\beta^{h-}) \\ \qquad\qquad + \partial_\alpha \Big(\frac{\varphi_0^{h+} - \varphi_0^{h-}}{2h} \int_{-h}^{+h} p_{3\alpha\beta}\, dx_3^h \Big), \quad in \quad \omega, \end{array} \right. \\ (\zeta_1^h, \zeta_2^h) = (0,0), \quad on \quad \gamma_0, \\ N^h_{\alpha\beta} \nu_\alpha = \int_{-h}^{+h} g_\beta^h\, dx_3^h - \frac{\varphi_0^{h+} - \varphi_0^{h-}}{2h} \int_{-h}^{+h} p_{3\alpha\beta}\, dx_3^h\, \nu_\alpha, \quad on \quad \gamma_1. \end{cases} \tag{81}$$

For any $(x_1, x_2) \in \bar{\omega}$,

$$g_i^{h\pm}(x_1, x_2) = g_i^h(x_1, x_2, \pm h), \qquad \varphi_0^{h\pm}(x_1, x_2) = \varphi_0^h(x_1, x_2, \pm h) \tag{82}$$

and $N^h_{\alpha\beta}$ *and* $M^h_{\alpha\beta}$ *are defined by*

$$\begin{cases} N^h_{\alpha\beta}(\zeta_1^h, \zeta_2^h, \zeta_3^h) = \\ \int_{-h}^{+h} A_{\alpha\beta\gamma\rho}\, dx_3^h\, e_{\gamma\rho}(\zeta) - \int_{-h}^{+h} x_3^h \Big(A_{\alpha\beta\gamma\rho} + \frac{p_{3\alpha\beta} p_{3\gamma\rho}}{p_{33}} \Big)\, dx_3^h\, \partial_{\gamma\rho} \zeta_3, \end{cases} \tag{83}$$

and

$$\left\{ \begin{array}{l} M^h_{\alpha\beta}(\zeta^h_1, \zeta^h_2, \zeta^h_3) = \\ - \int_{-h}^{+h} x^h_3 A_{\alpha\beta\gamma\rho}\, dx^h_3\, e_{\gamma\rho}(\zeta) + \int_{-h}^{+h} (x^h_3)^2 \Big(A_{\alpha\beta\gamma\rho} + \frac{p_{3\alpha\beta} p_{3\gamma\rho}}{p_{33}} \Big)\, dx^h_3\, \partial_{\gamma\rho} \zeta_3. \end{array} \right. \tag{84}$$

The vector field $z^h = (z^h_i)$ *defined in (78) is a Kirchhoff-Love displacement field, that is,*

$$e^h_{i3}(z^h) = \frac{1}{2}(\partial^h_i z_3 + \partial^h_3 z_i) = 0, \tag{85}$$

that verifies

$$z^h_\alpha = \zeta^h_\alpha - x^h_3 \partial_\alpha \zeta^h_3 \quad and \quad z^h_3 = \zeta^h_3 \quad in \quad \bar\Omega^h, \tag{86}$$

where ζ^h_i *are the solution of (80)-(81).*

The limit electric potential ϕ^h *in the plate* $\bar\Omega^h$ *is defined by*

$$\begin{array}{l} \phi^h(x_1, x_2, x^h_3) = \sum_{m=0}^2 \phi^m(x_1, x_2)(x^h_3)^m, \\ \phi^0(x_1, x_2) = \frac{\varphi^{h+}_0 + \varphi^{h-}_0}{2} + h^2 \frac{p_{3\alpha\beta}}{2p_{33}} \partial_{\alpha\beta} \zeta^h_3, \\ \phi^1(x_1, x_2) = \frac{\varphi^{h+}_0 - \varphi^{h-}_0}{2h}, \\ \phi^2(x_1, x_2) = -\frac{p_{3\alpha\beta}}{2p_{33}} \partial_{\alpha\beta} \zeta^h_3, \end{array} \tag{87}$$

or equivalently

$$\left\{ \begin{array}{l} \phi^h(x_1, x_2, x^h_3) = \\ \frac{\varphi^{h+}_0 + \varphi^{h-}_0}{2} + h^2 \frac{p_{3\alpha\beta}}{2p_{33}} \partial_{\alpha\beta} \zeta^h_3 + \frac{\varphi^{h+}_0 - \varphi^{h-}_0}{2h} x^h_3 - \frac{p_{3\alpha\beta}}{2p_{33}} \partial_{\alpha\beta} \zeta^h_3 (x^h_3)^2. \end{array} \right. \tag{88}$$

Proof. To obtain (80)-(84) we just consider the variational problem (56) formulated in Ω^h, the de-scalings (77)-(78) and argue as in the proof of theorem 4.1. The formulas (85)-(86) are a direct consequence of the de-scalings (77)-(78) and the properties of u. To establish (87) we consider $v = 0$ in the variational equation (47) and we follow the same reasoning of Sene ([13], theorem 3.1), but in the domain Ω^h, instead of Ω, and we use the de-scalings (77) and (79). ∎

5. Comments

We remark that if we do not assume the hypothesis $C_{\alpha\beta\gamma 3} = 0 = C_{\alpha 333}$, then we have the general case of anisotropy, with 21 independent elastic coefficients C_{ijkl}. Therefore the formulas (33) for κ, deduced in theorem 3.1, would be more complicated and, consequently, the expression of the reduced piezoelectric plate model, given in theorem 3.4, would be more complex. Nevertheless, we think that it is still possible to generalize the results of this paper to this case of a general anisotropic material.

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References

[1] M. Bernadou and C. Haenel, *Modelization and numerical approximation of piezoelectric thin shells. I. The continuous problems*, Computer Methods in Applied Mechanics and Engineering, 192, 37-38 (2003) pp. 4003-4030.
[2] M. Bernadou and C. Haenel, *Modelization and numerical approximation of piezoelectric thin shells. II. Approximation by finite element methods and numerical experiments*, Computer Methods in Applied Mechanics and Engineering, 192, 37-38 (2003) pp. 4045-4073.
[3] M. Bernadou and C. Haenel, *Modelization and numerical approximation of piezoelectric thin shells. III. From the patches to the active structures*, Computer Methods in Applied Mechanics and Engineering, 192, 37-38 (2003) pp. 4075-4107.
[4] P. Ciarlet, *Mathematical Elasticity, Volume II: Theory of Plates*, North-Holland, Amsterdam, 1997.
[5] P. Ciarlet, *Mathematical Elasticity, Volume II: Theory of Shells*, North-Holland, Amsterdam, 2000.
[6] C. Collard and B. Miara, *Two-dimensional models for geometrically nonlinear thin piezoelectric shells*, Asymptotic Analysis, 31, 2 (2003) pp. 113-151.
[7] P. Destuynder, *Une Théorie Asymptotique de Plaques Minces en Elasticité Linéaire*, Masson, Paris, 1986.
[8] A.E. Green and W. Zerna, *Theoretical Elasticity*, 2nd Edition, Oxford Press, 1968.
[9] T. Ikeda, *Fundamentals of Piezoelectricity*, Oxford University Press, 1990.
[10] I. Figueiredo and C. Leal, Ellipticity of the Koiter and Naghdi models for nonhomogeneous anisotropic shells, *Applicable Analysis*, 70, 1-2, (1998) pp. 75-84.
[11] A. Raoult and A. Sene, *Modelling of piezoelectric plates including magnetic effects*, Asymptotic Analysis, 34, 1 (2003) pp. 1-40.
[12] N. Sabu, *Vibrations of thin piezoelectric flexural shells: two dimensional approximation*, Journal of Elasticity, 68, (2002) pp. 145-165.
[13] A. Sene, *Modelling of piezoelectric static thin plates*, Asymptotic Analysis, 25, 1 (2001) pp. 1-20.

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