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FUNCTIONS q-ORTHOGONAL WITH RESPECT TO THEIR OWN ZEROS

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Abstract: In [4], G. H. Hardy proved that, under certain conditions, the only functions satisfying

$$\int_0^1 f(\lambda_m t) f(\lambda_n t) dt = 0 \tag{1}$$

where the λ_n´s are the zeros of f, are the Bessel functions. We replace the above integral by the Jackson q-integral and give the q-analogue of Hardy´s result.

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1. Introduction

The orthogonality relations

$$\int_0^1 \sin(m\pi t) \sin(n\pi t) dt = 0 \tag{2}$$

if $m \neq n$ and, for Bessel functions J_ν and their nth zero $j_{\nu n}$,

$$\int_0^1 t J_\nu(j_{\nu m} t) J_\nu(j_{\nu n} t) dt = 0 \tag{3}$$

lead J. M. Whittaker to call such functions *orthogonal with respect to their own zeros* [13]. It is known that, under some restrictions, the only such functions are the Bessel functions. This was shown by G. H. Hardy in [4]. For a remarkable big class of functions he proved that, denoting by λ_n the nth zero of f, if f satisfies

$$\int_0^1 f(\lambda_m t) f(\lambda_n t) dt = 0 \tag{4}$$

then f must be a Bessel function. The classes of functions considered by Hardy were defined in terms of the position of their zeros and their growth as entire functions, in the following terms:

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Definition 1. *The class A is constituted by all entire functions f of order less than two or of order two and minimal type of the form*

$$f(z) = z^{\nu} \prod_{n=1}^{\infty} \left(1 - \frac{z^2}{\lambda_n^2}\right) \tag{5}$$

where $\nu > -\frac{1}{2}$. The class B is constituted by all entire functions f of the form

$$f(z) = z^{\nu} F(z) \tag{6}$$

where $\nu > -\frac{1}{2}$ and $F(z)$ is an entire function with real but not necessarily positive zeros, and of order one, or of order one and minimal type, with $F(0) \neq 0$.

Hardy proved that, if they satisfy (4), the functions on the class A must be of the form $Kz^{\frac{1}{2}} J_{\nu-1/2}(cz)$ and the functions on the class B must be of the form $KJ_{2\nu}(cz^{1/2})$. We will replace (4) with the slightly more general orthogonality relation

$$\int_0^1 f(\lambda_m t) f(\lambda_n t) d\mu(t) = 0 \tag{7}$$

where $d\mu(t)$ is a positive defined measure in the real line and the λ_n´s are the zeros of f.

This paper is organized as follows. In the second section we derive a sampling theorem for this functions that was implicit in Hardy´s work. Then, specializing the measure in (7) in order to obtain the Jackson q-integral, we will formulate the q-version of the problem, and derive the q-difference equations satisfied by the functions f. For the class A we will recognize the resulting q-difference equation as being a parametrization of the second order q-difference equation derived by Meijer and Swarttouw [12] and thus prove that the only functions in class A that are q-orthogonal with respect to their own zeros are the third Jackson q-Bessel functions.

2. Kramer kernels and Lagrange type interpolation formulas

Suppose that f satisfies (7). If $f \in A$, it is possible to prove that the set $\{f(\lambda_n t)\}$ is complete in $L_q^2[\mu, (0,1)]$ and

$$\frac{\int_0^1 f(zt) f(\lambda_n t) d\mu(t)}{\int_0^1 |f(x\lambda_n)|^2 d\mu(x)} = \frac{2\lambda_n}{f'(\lambda_n)} \frac{f(z)}{z^2 - \lambda_n^2} \tag{8}$$

This was done in [4] for the case $d\mu(t) = dx$ and the proof remains the same if a general real positive measure $d\mu(t)$ is used. If $f \in B$, the set $\{f(\lambda_n t)\}$ is complete in $L_q^2[\mu,(0,1)]$ and

$$\frac{\int_0^1 f(zt)f(\lambda_n t)d\mu(t)}{\int_0^1 |f(t\lambda_n)|^2\, d\mu(t)} = \frac{f(z)}{f'(\lambda_n)(z-\lambda_n)} \tag{9}$$

In the next sections the measure $d\mu(t)$ will be specialized in order to obtain the q-integral.

The above formulas can be seen from the point of view of Kramer sampling Lema. Kramer sampling Lema [11] states that if $\{K(x,\lambda_n)\}$ is an orthogonal basis for $L^2(\mu, I)$ and, for some $u \in L^2(\mu, I)$ g can be written in the form

$$g(x) = \int_I u(t)K(t,x)d\mu(t) \tag{10}$$

then g admits the sampling expansion

$$g(x) = \sum_{n=1}^{\infty} g(\lambda_n)S_n(x) \tag{11}$$

where

$$S_n(x) = \frac{\int_I K(t,\lambda_n)K(t,x)d\mu(t)}{\int_I |K(t,\lambda_n)|^2\, d\mu(t)} \tag{12}$$

The kernel $K(x,t)$ is called a Kramer kernel. Sometimes the integral above can be evaluated explicitly. For instance, when $K(x,t)$ it is the solution of a regular Sturm Liounville eigenvalue problem, the Kramer type sampling expansion becomes a Lagrange type interpolation formula, with

$$S_n(x) = \frac{L(x)}{L\'(t)(x-\lambda_n)} \tag{13}$$

where

$$L(t) = \prod_{k=0}^{\infty}\left(1 - \frac{t}{\lambda_k}\right) \tag{14}$$

As remarked by Everitt, Nasri-Roudsari and Rehberg in [2], the question of whether there exists a Lagrange interpolation formula for every Kramer kernel is open. The identities (8) and (9) provide an answer to this question when $K(x,t) = f(xt)$ (these sort of kernels are usually said to be of the Watson type) and f in the classes A and B above. A simple application of Kramer´s Lema yields the following

Theorem 1. *Let f satisfy (7). If f is in the class A then, every function g of the form*

$$g(t)=\int_0^1 u(x)f(xt)d\mu(x) \tag{15}$$

has the sampling expansion

$$g(t)=2\sum_{n=1}^{\infty} g(\lambda_n)\frac{2\lambda_n}{f'(\lambda_n)}\frac{f(t)}{t^2-\lambda_n^2} \tag{16}$$

If f is in the class B then every function g of the form (15) has the sampling expansion

$$g(t)=\sum_{n=1}^{\infty} g(\lambda_n)\frac{f(t)}{f'(\lambda_n)(t-\lambda_n)} \tag{17}$$

Special cases of (16) are known when f is the Bessel [5] or the q-Bessel function [1]. These sampling theorems were originally obtained using special function formulae and the unitary property of the Hankel and the q-Hankel transform [10].

3. Functions $q-$orthogonal with respect to their own zeros

3.1. Basic definitions and facts. Following the standard notations in [3], consider $0<q<1$ and define the q-shifted factorial for n finite and different from zero as

$$(a;q)_n=(1-q)\,(1-aq)\ldots\left(1-aq^{n-1}\right) \tag{18}$$

and the zero and infinite cases as

$$(a;q)_0=1 \tag{19}$$

$$(a;q)_\infty=\lim_{n\to\infty}(a;q)_n \tag{20}$$

The $q-$difference operator D_q is

$$D_qf(x)=\frac{f(x)-f(qx)}{(1-q)x} \tag{21}$$

The q-analogue of the rule of the differentiation of a product is

$$D_q\left[f(x)g(x)\right]=f(qx)D_qg(x)+g(x)D_qf(x) \tag{22}$$

and the $q-$integral in the interval $(0,1)$ is

$$\int_0^z f(t)\, d_q t = (1-q)\sum_{k=0}^{\infty} f\left(zq^k\right) zq^k \tag{23}$$

It is possible to define an inner product by setting

$$< f, g >= \int_0^1 f(t)\, g(t) d_q t \tag{24}$$

The resulting Hilbert space is commonly denoted by $L_q^2(0,1)$. We will say that a function $f \in L_q^2(0,1)$ is q-orthogonal with respect to its own zeros in the interval $(0,1)$ if it satisfies the orthogonality relation

$$\int_0^1 f(\lambda_m t) f(\lambda_n t) d_q t = 0 \tag{25}$$

that is,

$$\sum_{k=0}^{\infty} f(\lambda_m q^k) f(\lambda_n q^k) q^k = 0 \tag{26}$$

if $n \neq m$. An example of a function satisfying such an orthogonality relation is the third Jackson q-Bessel function $J_\nu^{(3)}$ (also known in the literature as the Hahn-Exton q-Bessel function) defined by the power series

$$J_\nu^{(3)}(x;q) = \frac{(q^{\nu+1};q)_\infty}{(q;q)_\infty} \sum_{n=0}^{\infty} (-1)^n \frac{q^{n(n+1)/2}}{(q^{\nu+1};q)_n (q;q)_n} x^{2n+\nu} \tag{27}$$

Or equivalently, denoting by $j_{n\nu}(q)$ the nth zero of $J_\nu^{(3)}(x;q)$, by the infinite product representation

$$J_\nu^{(3)}(x;q) = \frac{(q^{\nu+1};q)_\infty}{(q;q)_\infty} x^\nu \prod_{n=1}^{\infty} \left(1 - \frac{x^2}{j_{n\nu}^2(q)}\right) \tag{28}$$

The equivalence of both definitions is an easy consequence of the Hadamard factorization theorem. It is well known [9] that, if $n \neq m$,

$$\int_0^1 x J_\nu^{(3)}(qxj_{n\nu}(q^2);q^2) J_\nu^{(3)}(qxj_{m\nu}(q^2);q^2) d_q x = 0 \tag{29}$$

This function was discussed in the context of quantum groups by Koelink [8] and the central concepts regarding its role in q-harmonic analysis were introduced by Koornwinder and Swarttouw [9]. The $J_\nu^{(k)}$, $k = 1, 2, 3$ notation for the Jackson analogues of the Bessel function is due to Ismail [6], [7].

3.2. $q-$**difference equations.** In [12], Meijer and Swarttouw proved that the general solution of the $q-$difference equation

$$D_q^2 y(z) + \frac{1}{z} D_q y(z) + \left[\frac{q^{2-\nu}}{(1-q)^2} - \frac{(1-q^\nu)(1-q^{-\nu})}{(1-q)^2 z^2} \right] y(qz) = 0 \tag{30}$$

is

$$H_\nu(x) = A J_\nu^{(3)}\left(x; q^2\right) - B Y_\nu\left(x; q^2\right) \tag{31}$$

where $Y_\nu\left(x; q^2\right)$ is a q-analogue of $Y_\nu(x)$, the classical second solution of the Bessel differential equation. The function $Y_\nu\left(x; q^2\right)$ is defined, if ν is not an integer, as

$$Y_\nu(x; q) = \frac{\Gamma_q(\nu)\Gamma_q(1-\nu)}{\pi} \{\cos(\pi\nu) q^{\nu/2} J_\nu^{(3)}(x; q) - J_{-\nu}^{(3)}\left(x q^{-\nu/2}; q\right)\} \tag{32}$$

and, for n an integer, as the limit

$$Y_n(x; q) = \lim_{\nu \to n} Y_\nu(x; q) \tag{33}$$

It is clear that, if $\nu > 0$ then Y_ν is unbounded near $x = 0$.

Lemma 1. *The general solution of the equation*

$$D_q^2 y(z) + \left[\frac{M^2 q^{\frac{3}{2}-\nu}}{(1-q^2)} - \frac{(1-q^{\nu-\frac{1}{2}})(1-q^{-\nu-\frac{1}{2}})}{(1-q^2) z^2} \right] y(qz) = 0 \tag{34}$$

is given by

$$f(z) = z^{\frac{1}{2}} \{A J_\nu^{(3)}\left(Mx; q^2\right) - B Y_\nu\left(Mx; q^2\right)\} \tag{35}$$

Proof: Set

$$y(x) = x^{\frac{1}{2}} H_\nu(Mx) \tag{36}$$

Apply the operator D_q to (36) and use (22) to obtain

$$M D_q H_\nu(Mx) = x^{-\frac{1}{2}} D_q y(x) + \frac{1-q^{-\frac{1}{2}}}{1-q} x^{-\frac{3}{2}} y(qx) \tag{37}$$

Now, to evaluate the second q-difference, apply again the operator D_q to (36), but switch the role of the functions f and g in the formula (22). The result is

$$M D_q H_\nu(Mx) = q^{-\frac{1}{2}} x^{-\frac{1}{2}} D_q y(x) + \frac{1-q^{-\frac{1}{2}}}{1-q} x^{-\frac{3}{2}} y(x) \tag{38}$$

Applying the operator D_q to both members gives

$$M^2 D_q^2 H_\nu(Mx) = \tag{39}$$

$$q^{-1}x^{-\frac{1}{2}}D_q^2 y(x) - q^{-1}x^{-\frac{3}{2}}D_q y(x) + \frac{(1-q^{-\frac{1}{2}})(1-q^{-\frac{3}{2}})}{(1-q)^2}x^{-\frac{5}{2}}y(qx) \tag{40}$$

Using these expressions it is not hard to see that the change of variable (36) transforms equation (30) in (34). This proves the lemma. ■

3.3. The main results. Observe that the $q-$integral (23) is a Riemann-Stieltjes integral with respect to a step function having infinitely many points of increase at the points q^k, with the jump at the point q^k being q^k. If we call this step function $\Psi_q(t)$ then $d\Psi_q(t) = d_q t$.

Theorem 2. *If f is in the class A and satisfies (25) then f must be of the form*

$$f(x) = z^{\frac{1}{2}} K J^{(3)}_{\nu-1/2}\left(Mx; q^2\right) \tag{41}$$

where

$$M^2 = -aq^{-3}(1-q^2)(1-q^{2\nu+1}) \tag{42}$$

$$a = -2\sum \frac{1}{\lambda_n^2} \tag{43}$$

and K is a real constant.

Proof: Take in (8) $\mu(t) = \Psi_q(t)$ to obtain

$$\int_0^1 f(zt)f(\lambda_n t)d_q t = \frac{2A_n\lambda_n}{f'(\lambda_n)}\frac{f(z)}{z^2-\lambda_n^2} \tag{44}$$

With minor adaptations, the argument used in [4, page 41] can be extended to the q-case to deduce, from the completeness of $\{f(\lambda_n t)\}$ and identity (44), the following q-integral equation for $f(z)$

$$a\int_0^z u^{\nu+2} f(u)\, d_q u = (az^2+2)\int_0^z u^\nu f(u) d_q u - 2\frac{1-q}{1-q^{2\nu+1}} z^{\nu+1} f(z) \tag{45}$$

where $a = -2\sum 1/\lambda_n^2$. Then, applying the operator D_q to both members of this equation and dividing by z produces

$$2q^{\nu+1}z^\nu \frac{1-q}{1-q^{2\nu+1}} D_q f(z) \tag{46}$$

$$-2q^{\nu+1}\frac{1-q^\nu}{1-q^{2\nu+1}} z^{\nu-1} f(z) - a(q+1)\int_0^{qz} u^\nu f(u) d_q u = 0 \tag{47}$$

Using the D_q operator again and multiplying the resulting equation by the factor

$$(1-q^{2\nu+1})(1-q)^{-1}q^{-2\nu-1}z^{-\nu}/2 \tag{48}$$

yields

$$D_q^2 f(z) - \left[\frac{(1-q^{\nu})(1-q^{\nu-1})q^{-\nu}}{(1-q)^2 z^2} + \frac{a(1+q)(1-q^{2\nu+1})q^{-\nu-1}}{1-q}\right] f(qz) = 0 \tag{49}$$

Observe that replacing ν by $\nu - \frac{1}{2}$ and M by the value given by (42) in (34) gives (49). Therefore, the general solution of (49) is

$$f(x) = z^{\frac{1}{2}}\{AJ^{(3)}_{\nu-1/2}\left(Mx;q^2\right) - BY_{\nu-1/2}\left(Mx;q^2\right)\} \tag{50}$$

with M as in (42). But as we have seen, Y_ν is unbounded near $x=0$ and f is analytic at $x=0$. This implies $B=0$. Therefore, (41) holds. ∎

Remark 1. *This agrees with orthogonality relation (29). To see this, just replace in (41) ν by $\nu + \frac{1}{2}$. The result is*

$$f(x) = Az^{\frac{1}{2}}J^{(3)}_{\nu}\left(Mx;q^2\right) \tag{51}$$

with

$$M^2 = -aq^{-3}(1-q^2)(1-q^{2\nu+2}) \tag{52}$$

To evaluate a, take the logarithmic derivative in (28) and set $x=0$. This yields

$$\sum_{k=0}^{\infty} \frac{1}{j^2_{nv}(q^2)} = \frac{q^2}{(1-q^2)(1-q^{2\nu+2})} \tag{53}$$

Therefore $M = q$.

If $f \in B$ it is also possible to find the q-difference equation satisfied by f.

Theorem 3. *If f is in the class B and f satisfies (25) then f must satisfy the following q-difference equation:*

$$D_q^2 f(z) + \frac{1}{qz}D_q f(z) - [\frac{(1-q^{\nu})(1-q^{\nu-1})}{(1-q^2)q^{\nu+1}z^2} - \frac{(1-q^{2\nu+1})(a+1)}{(1+q)q^{\nu+2}z}]f(qz) = 0 \tag{54}$$

where $a = F'(0)$.

Proof: Take in (9) $\mu(t) = \Psi_q(t)$ to obtain

$$\int_0^1 f(zt)f(\lambda_n t)d_qt = \frac{A_n}{f'(\lambda_n)}\frac{f(z)}{z-\lambda_n} \tag{55}$$

The integral equation obtained this time is

$$a\int_0^z u^{\nu+1}f(u)\,d_qu = (az+2)\int_0^z u^\nu f(u)d_qu - \frac{1-q}{1-q^{2\nu+1}}z^{\nu+1}f(z) \tag{56}$$

Use of the q-difference operator as in Theorem 2 establishes (54). ■

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