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THE HEIGHT CHARACTERISTIC AND THE GRAPH OF TRIANGULAR MATRICES

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ABSTRACT: In this paper, we discuss the relation between the graph and the spectral properties of a matrix, namely, algebraic multiplicity and degrees of the elementary divisors of an eigenvalue.

Hershkowitz and Schneider showed [5] that the *height characteristic* of a triangular matrix A majorizes the dual of the sequence of differences of maximal cardinalities of *singular k-paths* in the graph $G(A)$.

Other relations, not followed by that result, can also be obtained for these two sequences.

KEYWORDS: Graph, Height Characteristic, singular k-paths.
AMS SUBJECT CLASSIFICATION (1991): 15A21, 05C50, 15A29.

1. Introduction

The study of the relation between the graph and the spectral properties of a matrix has been of interest in the past ninety years. At first, and for about seventy years, this study was focused mostly on nonnegative matrices, since one of the major contributes was given by Frobenius in 1912, for those matrices (see [6, chapter 8]).

In the past fifteen years, this research was extended to general matrices. Several papers [1, 4, 5, 8] study the relation between the *height characteristic* (or equivalently, the sizes and number of the Jordan blocks associated to the eigenvalue 0) of a matrix and its graph (see the expository paper [3] and the references therein).

For nilpotent triangular matrices, a major result was given by Saks [8] and Gansner [1], who showed that the height characteristic of a matrix A *majorizes* the *dual* of the sequence of differences of maximal cardinalities of k-paths in the graph $G(A)$ of A, and that in the so called generic case the height characteristic is equal to the dual sequence.

For the triangular (not necessarily nilpotent) case, Hershkowitz and Schneider [5], extended those results showing that the height characteristic of A majorizes the dual of the sequence of differences of maximal cardinalities of

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singular k-paths in $G(A)$. They also proved that in the so called generic case the height characteristic is equal to the dual sequence.

A further generalization was obtained by Hershkowitz [4] for general matrices (where he considered *nonclosable* k-paths in $G(A)$). The fact that in the generic case the two above mentioned sequences are equal means that "almost" every real or complex matrix, verifies this equality, nevertheless has we will see, this equality is not true for all matrices.

Two natural questions arise. Given a graph G what are the possible height characteristic of the matrix A such that $G(A) = G$? And given two sequences η and π, is there a matrix A such that the height characteristic of A is η, and the sequence of differences of maximal cardinalities of nonclosable k-paths of $G(A)$ is π?

The answer of the first problem does not depend only on the sequence of differences of maximal cardinalities of nonclosable (or singular for the triangular case) k-paths in G.

In this paper we will discuss the second problem, but for triangular matrices, obtaining new relations between those two sequences (other than the one obtained by of Hershkowitz and Schneider). We have reasons to believe these are not the only ones.

For small order ($n \leq 7$) and nilpotent triangular matrices, the majorization theorem plus the results presented here completely describe the relations between the two sequences.

The first section contains notations and definitions, as well as some results considered important for the further section. In the following section we introduce the new relations between the sequences.

The matrices in this paper are considered to be over an arbitrary field F. Jordan blocks and Jordan canonical forms are assumed over the algebraic closure $\overline{F}$ of F. We deal with lower triangular matrices but, obviously we could also consider upper triangular and so our results (with the obvious modifications) are valid for those matrices.

Finally, we remark that even though we technically deal with the eigenvalue 0 of a matrix, these results may be applied to any eigenvalue λ of a matrix A, by studding the matrix $A - \lambda I$.

2. Notation and Definitions

In this section we recall definitions and notations we will use in this paper.

2.1. Sequences.

Notation 2.1. For a natural number n, we use the notation $\langle n\rangle$ for the set $\{1, 2, 3, \dots, n\}$, and $|C|$ for the cardinality of the finite set C.

Definition 2.2. Let $\alpha = (\alpha_1, \alpha_2, \dots, \alpha_t)$ be a monotone non-increasing sequence of positive integers. The sequence $\alpha^* = (\alpha_1^*, \alpha_2^*, \dots, \alpha_{\alpha_1}^*)$, is called *dual* to α, such that α_i^* is the number of elements of α greater than or equal to i.

Observation 2.3.

- The sequence α^* is monotone non-increasing;
- $(\alpha^*)^* = \alpha$;
- Constructing a diagram with t columns of *stars*, such that the j column (from the left) has α_j *stars*, then, α_i^* is equal to the number of *stars* of the i row (from the top).

Definition 2.4.

- Let α and β be sequences of nonnegative integers with the same number of elements:

$$\alpha = (\alpha_1, \alpha_2, \dots, \alpha_t) \quad \text{and} \quad \beta = (\beta_1, \beta_2, \dots, \beta_t).$$

  We say that β *majorizes* α, and denote it by $\beta \succeq \alpha$, if we arrange the entries of α and β in a decreasing order $\alpha_{i_1} > \alpha_{i_2} > \dots > \alpha_{i_t}$, $\beta_{j_1} > \beta_{j_2} > \dots > \beta_{j_t}$, and

$$\sum_{l=1}^{k} \beta_{j_l} \geq \sum_{l=1}^{k} \alpha_{i_l} \quad \text{for all} \quad k \in \langle t\rangle$$

  with equality for $k = t$;

- Let α and β be sequences of nonnegative integers:

$$\alpha = (\alpha_1, \alpha_2, \dots, \alpha_q) \quad \text{and} \quad \beta = (\beta_1, \beta_2, \dots, \beta_s).$$

  For $t = \max\{q, s\}$, let $\dot\alpha$ and $\dot\beta$ be the sequences:

$$\dot\alpha = (\dot\alpha_1, \dot\alpha_2, \dots, \dot\alpha_t) \quad \text{and} \quad \dot\beta = (\dot\beta_1, \dot\beta_2, \dots, \dot\beta_t),$$

such that $\dot{\alpha}_i = \alpha_i$ for $i \in \langle q \rangle$ and $\dot{\beta}_j = \beta_j$ for $j \in \langle s \rangle$. The remaining elements on both sequences being zero.

We say that β *majorizes* α, and denote it by $\beta \succeq \alpha$, if $\dot{\beta}$ majorizes $\dot{\alpha}$, according with the previous definition.

2.2. Matrices.

In the sequel, we assume that A is an $n \times n$ matrix over an arbitrary field F.

Notation 2.5. We denote,

- the *nullity of* A by $n(A)$, which is the dimension of the nullspace of A;
- the *rank of* A by $rank(A)$, which is equal to $n - n(A)$;
- the *algebraic multiplicity* of the eigenvalue zero of A, by $m(A)$.

Notice that the number of Jordan blocks associated with the eigenvalue zero is $n(A)$.

Definition 2.6. The monotone non-increasing sequence,

$$\xi(A) = (\xi_1(A), \xi_2(A), \dots, \xi_{n(A)}(A))$$

of sizes of the Jordan blocks associated with the eigenvalue zero (or equivalently, the non-increasing sequence of degrees of the elementary divisors associated to the eigenvalue 0) is called *Jordan* or *Segre characteristic of* A.

We call *index of* A to the size of the greatest Jordan block associated with the eigenvalue zero, $\xi_1(A)$.

Definition 2.7. Let i be a natural number, we denote by $\eta_i(A)$ to

$$\eta_i(A) = n(A^i) - n(A^{i-1}), \quad \text{with} \quad n(A^0) = 0\,.$$

The sequence $\eta(A) = (\eta_1(A), \eta_2(A), \dots, \eta_{\xi_1(A)}(A))$, is called *Weyr* or *height characteristic of* A.

Convention 2.8. We will use η_i, η, ξ_i and ξ for $\eta_i(A)$, $\eta(A)$, $\xi_i(A)$ and $\xi(A)$ respectively, if no confusion arise.

The next result, which can be found e.g. in [2, lema 2], establish a relation between η and ξ.

[2] Helene Shapiro, The Weyr Characteristic, *American Monthly Mathematics*, 106 (December 1999) 919-929.
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It remains, the last

$$(k+2)(\pi_2+2)+1-(\pi_2+2)=(k+1)(\pi_2+2)+1$$

singular vertices in P, that are smaller than l.

We need to compare the last $(k+1)(\pi_2+2)+1$ singular vertices of P, with the remaining k vertices with loop.

In this case the hypothesis of induction allows us to conclude that there is a set D of π_2+3 consecutive singular vertices in P, such that if v is a vertex with loop then he is either greater or smaller than every element of D.

We conclude by (3.8), that η has more than two elements. ∎

Example 3.13. Let A be an 17×17 lower triangular matrix such that $G(A)$ has one vertex with loop and the sequence $\pi=(9,2,2,2,1)$ then $\pi^*=(5,4,1,1,1,1,1,1,1)$.

By the previous result the sequence η cannot be $(8,8)$ since $(r+1)(\pi_2+2)+1=(1+1)(2+2)+1=9$.

Nevertheless π and $(8,8)$ are in condition of (2.17) and (3.4). This shows that (3.12) does not follow from those results.

We have checked that for nilpotent lower triangular matrices of order less than 8, the results (2.17), (3.4) and (3.10) completely describe all the relations between η and π. This means that, given two sequences η and π that verify those relations, then there is a nilpotent triangular matrix A such that $\eta(A)=\eta$ and $\pi(G(A))=\pi$.

However for order 8 and if $\eta=(4,4)$ and $\pi=(4,2,1,1)$ it can be shown that does not exist a matrix A with $\eta(A)=\eta$ and $\pi(G(A))=\pi$, although η and π verify (2.17), (3.4) and (3.10).

So these are not all the relations that can be obtained.

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References

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Theorem 2.9. Let A be an $n\times n$ matrix, then $\eta=\xi^*$.

From this result, and using the properties (2.3) of the *dual* sequence, η is a monotone non-increasing sequence and $\eta^*=\xi$.

We have also (see e.g. proof of [2, lema 2]):

- ξ_1 is the largest natural number, such that $\eta_{\xi_1}>0$;
- $n(A^i)=m(A)$ for every natural number $i\geq\xi_1$;
- $n(A^j)<m(A)$ for every natural number $j\in\langle\xi_1-1\rangle$.

2.3. Graphs.

Definition 2.10. A *direct graph* is a pair $G=(V,E)$, such that V is a nonempty finite set, and $E\subseteq V\times V$.

An element v_i of V is called *vertex* of G, and an element (v_i,v_j) of E is called *arc* in G from v_i to v_j.

The number of vertices of G is denoted by $|G|$.

Definition 2.11. Let $G=(V,E)$ be a graph:

- A *path* P in G is a sequence of pair wise distinct vertices $(v_1,v_2,\ldots,v_m)$, such that (v_i,v_{i+1}) is an arc in G for every $i\in\langle m-1\rangle$;
- Every sequence that consists of one vertex is a path;
- A *cycle* C in G is a sequence of pair wise distinct vertices $(v_1,v_2,\ldots,v_m)$ except for $v_1=v_m$, such that $m\geq 2$, and (v_i,v_{i+1}) is an arc in G, for every $i\in\langle m-1\rangle$;
- A cycle (v,v) in G is called a *loop*. If (v,v) is not a cycle in G then v is called a *loopless* or a *singular* vertex;
- Two paths in G are said to be *disjoint* if they have no common vertex.

For further information on this subject see [7].

Definition 2.12. The graph $G(A)=(V,E)$ of an $n\times n$ matrix $A=[a_{ij}]$ is, such that $V=\langle n\rangle$, and $(i,j)\in E$ if and only if $a_{ij}\neq 0$.

Hershkowitz and Schneider introduced the following sequence in [5] related to a graph.

Definition 2.13. Let G be a graph, S its set of singular vertices, and k a natural number. A *singular k-path* in G is a subset of S that can be covered by k or fewer pair wise disjoint paths in G.

Notation 2.14. Let G be a graph, S its set of singular vertices, and k a natural number:

- We denote by $p_k(G)$ the maximum cardinality of a singular k-path;
- Let t be the minimal number of pair wise disjoint paths needed to cover S. Note that $p_t(G) = |S|$, and $p_{t-1}(G) < |S|$. We put $\pi_k(G) = p_k(G) - p_{k-1}(G)$, where $k \in \langle t \rangle$, and $p_0(G) = 0$;
- We denote by $\pi(G)$, the sequence $(\pi_1(G), \pi_2(G), \ldots, \pi_t(G))$.

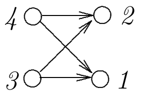
The sequence $\pi(G)$ is monotone non-increasing [5, Theorem(5.8)].

Convention 2.15. We will use p_i, π_i and π for $p_i(G)$, $\pi_i(G)$ and $\pi(G)$ respectively, if no confusion arise.

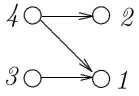
Example 2.16 Let G be a graph and

$$\eta(G) = \{\eta(A) : A \text{ is a matrix such that } G(A) = G\}.$$

Let G_1 and G_2 be two graphs with vertices $\langle 4 \rangle$:



G_1



G_2

Notice that $\pi(G_1) = \pi(G_2) = (2,2)$. As for the matrix:

$$A = \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 1 & 1 & 0 & 0 \\ 1 & 1 & 0 & 0 \end{bmatrix},$$

$$G(A) = G_1 \text{ and } \eta(A) = (3,1).$$

However, for every matrix B such that $G(B) = G_2$:

Corollary 3.10. Let A be an $n \times n$ nilpotent lower triangular matrix. If $\pi_1 \geq \pi_2 + 3$ then η has more than two elements.

Proof. Since $G(A)$ has no loops and $\pi_1 \geq \pi_2 + 3$ then there is a path P_1 with $\pi_2 + 3$ singular vertices in the path P that covers π_1 singular vertices. Like in the proof of (3.8), we can show that $G(A^2)$ has an arc (t_1, t_2) and so $A^2 \neq 0$.

We conclude that η has more than two elements. ■

Example 3.11. Let A be an 8×8 nilpotent lower triangular matrix such that the sequence π related to $G(A)$ is $(5,2,1)$ then $\pi^* = (3,2,1,1,1)$.

By the previous result the sequence η cannot be $(4,4)$.

Nevertheless π and $(4,4)$ are in condition of (2.17) and (3.4). This shows that (3.10) does not follow from those results.

Theorem 3.12. Let A be an $n \times n$ lower triangular matrix and $G(A)$ its graph with r vertices with loop such that $r \geq 1$. If $\pi_1 \geq (r+1)(\pi_2 + 2) + 1$, then η has more than two elements.

Proof. Let P be a path in $G(A)$ that covers π_1 singular vertices.

Let us show that there is a set D of $\pi_2 + 3$ consecutive singular vertices in P, such that if v is a vertex with loop then he is either greater or smaller than every element of D.

It is sufficient to prove the above statement for $\pi_1 = (r+1)(\pi_2 + 2) + 1$. We use induction on r.

- Case $r = 1$

  In this case, $\pi_1 = 2(\pi_2 + 2) + 1$. Suppose that the first $\pi_2 + 3$ singular vertices in P are not greater than the only vertex with loop, say vertex l. Then, at most, the first $\pi_2 + 2$ singular vertices in P verify that property.

  So the last $2(\pi_2 + 2) + 1 - (\pi_2 + 2) = \pi_2 + 3$ singular vertices in P, are smaller than l;
- Suppose the above statement true for $r = k$ let us prove it for $r = k+1$.

  If the greatest vertex with loop, say vertex l, is smaller than the first $\pi_2 + 3$ vertices of P then the remaining vertices with loop would also verify that property.

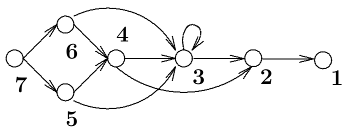
  Therefore, at most, the first $\pi_2 + 2$ singular vertices in P are greater than l.

(1) Suppose that the vertex with loop is 2. Notice that $G(A)$ has a path that covers 5 singular vertices, such that, at least, the first four of them are greater than the only vertex with loop. Since $\pi_2 = 1$ then, by the previous result, η has more than two elements.

(2) Suppose that the vertex with loop is 3. Let us consider A to be the matrix:

$$A = \begin{bmatrix} 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 1 & 0 & 0 & 0 & 0 \\ 0 & -1 & -1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & -1 & 0 \end{bmatrix}.$$

Since $G(A)$ is



we have in fact $\pi = (5, 1)$.

Also

$$A^2 = \begin{bmatrix} 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 1 & 1 & 1 & 0 & 0 & 0 & 0 \\ -1 & -1 & -1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}.$$

Notice that $n(A) = 3$ and so $n(A^2) = 6$.

In this case, $\eta = (3, 3)$ has only two elements.

If A is an $n \times n$ nilpotent lower triangular matrix, then $n(A^i) = m(A) = n$ for $i \geq \xi_1$. So $A^i = 0$ for $i \geq \xi_1$.

$$B = \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ * & 0 & 0 & 0 \\ * & * & 0 & 0 \end{bmatrix},$$

(where $*$ represent the nonzero entries) $\eta(B) = (2, 2)$. We conclude that $\pi(G_1) = \pi(G_2)$ does not mean $\eta(G_1) = \eta(G_2)$.

The following theorem was obtained by Hershkowitz and Schneider [5, Theorem(5.11)].

Theorem 2.17 Let A be a triangular $n \times n$ matrix over a field F, and $G(A)$ its graph. Then, $\pi^*(G(A)) \preceq \eta(A)$, and the index of A is less then or equal to p_1.

3. Other Relations Between η and π

In this section we study other relations between the sequences η and π, which does not follow from (2.17).

Definition 3.1. Let $A = [a_{ij}]$ be an $n \times n$ matrix. We define the *balance* of the position (i, j) of A, to be $j - i$.

Remark 3.2. Let k be a natural number, and A an $n \times n$ matrix. The positions that have balance equal to:

- zero are the ones in the main diagonal;
- k are the ones in the k sub-diagonal above the main diagonal;
- $-k$ are the ones in the k sub-diagonal below the main diagonal.

Moreover, the sum of the balance of all the positions in a generalized diagonal of A is equal to zero, since every row and every column appear just once.

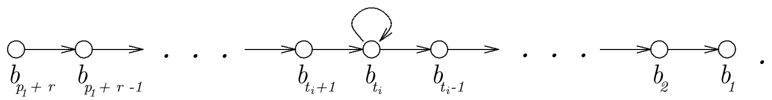
Reordering the elements in that sum we obtain:

$$(1 + 2 + 3 + \cdots + n) - (1 + 2 + \cdots + n) = 0\,.$$

Let $A = [a_{ij}]$ be an $n \times n$ lower triangular matrix, if (i, j) is an arc in $G(A)$, then $i \geq j$.

Theorem 3.3. Let A be an $n \times n$ lower triangular matrix. We have $rank(A) \geq \pi_1 - 1$.

Proof. Let $P = (b_{p_1+r}, b_{p_1+r-1}, \ldots, b_2, b_1)$ be a path in $G(A)$ that covers $\pi_1 = p_1$ singular vertices and the smallest number of vertices with loop:



We have $b_{p_1+r} > b_{p_1+r-1} > \cdots > b_2 > b_1$.

Call $\{b_{t_1}, b_{t_2}, \ldots, b_{t_r}\}$ the vertices of P that have a loop in $G(A)$.

Let us study the submatrix of A with rows $b_2, b_3, \ldots, b_{p_1+r}$ and columns $b_1, b_2, \ldots, b_{p_1+r-1}$.

All the elements in its main diagonal are nonzero, since they correspond to the arcs in P. The elements above the main diagonal, are equal to zero, except to the ones in the positions $(t_j - 1, t_j)$, that correspond to the loops (b_{t_j}, b_{t_j}) in $G(A)$, for every $j \in \langle r \rangle$. Notice that the balance of the positions of those elements in the submatrix is equal to one.

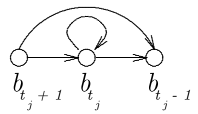
Let us proof that for this submatrix, the main diagonal is the only generalized diagonal without zero elements. Suppose there is another generalized diagonal without zero elements. Such diagonal must have an element above the main diagonal. Let t_j be the smallest natural number such that $(t_j - 1, t_j)$ is a position in the diagonal.

Let us see what are the possible values of w_1 of the (t_j, w_1) entry of the diagonal.

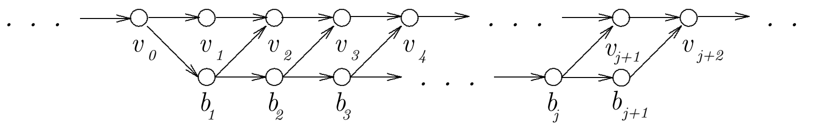
Such w_1 can not be greater than $t_j + 1$, because the balance of that position is greater than one.

Since t_j is already a position in the columns of the diagonal, then $w_1 \neq t_j$.

Also, $w_1 \neq t_j - 1$, otherwise (b_{t_j+1}, b_{t_j-1}) is an arc in $G(A)$:



This way, there is a path in $G(A)$ that covers p_1 singular vertices and $r - 1$ vertices with loop. Absurd.



This way, $G(A)$ has two disjoint paths (P and $(b_1, b_2, \ldots, b_j, b_{j+1})$), that covers $(\pi_1) + (j+1)$ singular vertices.

By definition p_2 is the maximum number of singular vertices that can be covered by two disjoint paths, so

$$\pi_2 = p_2 - p_1 \geq [(\pi_1) + (j+1)] - \pi_1 = j + 1 = \pi_2 + 1 \ .$$

Absurd, so there is an arc (t_1, t_2) in $G(A^2)$ such that if v is a vertex with loop then he is either greater or smaller than both t_1 and t_2.

Let $\{l_1, l_2, \ldots, l_r\}$ be the vertices with loop in $G(A)$ and suppose that: $l_1 < l_2 < \cdots < l_i < t_1, t_2 < l_{i+1} < \cdots < l_r$.

The matrix A^2 has, at least, r nonzero elements in the main diagonal, and one nonzero entry in the position (t_1, t_2).

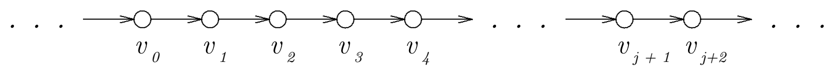
$$\begin{array}{ll}
\textbf{Columns:} & \quad 1 \ \ldots \ l_1 \ \ldots \ l_i \ \ldots \ t_2 \quad \ldots \quad l_{i+1} \ \ldots \ l_r \quad \ldots \quad n \\
\textbf{Row: } l_1 & (\ ? \ \ldots \ * \ \ 0 \qquad\qquad\qquad \ldots \qquad\qquad\qquad\qquad 0 \ ) \\
\quad \vdots & \quad \vdots \qquad\quad \ddots \qquad\qquad\qquad\qquad\qquad\qquad\qquad\quad \vdots \\
\textbf{Row: } l_i & (\ ? \ \ldots \ ? \ \ldots \ * \ \ 0 \qquad\qquad\quad \ldots \qquad\qquad\quad 0 \ ) \\
\textbf{Row: } t_1 & (\ ? \ \ldots \ ? \ \ldots \ ? \ \ldots \ * \ \ ?\ldots ? \ \ 0 \qquad \ldots \qquad 0 \ ) \\
\textbf{Row: } l_{i+1} & (\ ? \ \ldots \ ? \ \ldots \ ? \ \ldots \ ? \quad \ldots \quad * \ \ 0 \ \ \ldots \qquad 0 \ ) \\
\quad \vdots & \quad \vdots \qquad\qquad\qquad\qquad\qquad\qquad\quad \ddots \qquad\qquad \vdots \\
\textbf{Row: } l_r & (\ ? \ \ldots \ ? \ \ldots \ ? \ \ldots \ ? \quad \ldots \quad ? \ \ldots \ * \ \ 0 \ \ldots \ 0 \ )
\end{array}$$

where $(*)$ represent the nonzero entrances. In this way A^2 has $r + 1$ linear independent rows.

We conclude that $rank(A^2) > r$, then $n(A^2) < n - r$. Which means that $\xi_1 > 2$, so η has more than two elements. ■

The next example, will show the influence of the position of the vertices with loop relatively to the number of elements of η.

Example 3.9. Let A be an 7×7 lower triangular matrix, such that $G(A)$ has one loop, and $\pi = (5, 1)$:

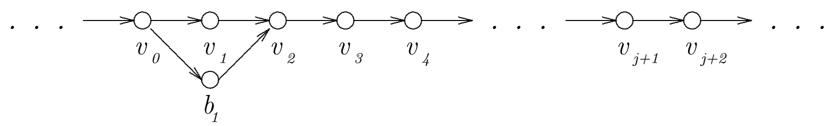


Let us show that there is an arc (t_1, t_2) in $G(A^2)$ such that if v is a vertex with loop then he is either greater or smaller than both t_1 and t_2.

Suppose that such arc does not exist.

By (3.6) there is an *alternative* path (v_0, b_1, v_2) for (v_0, v_1, v_2), otherwise (v_0, v_2) will be an arc in $G(A^2)$.

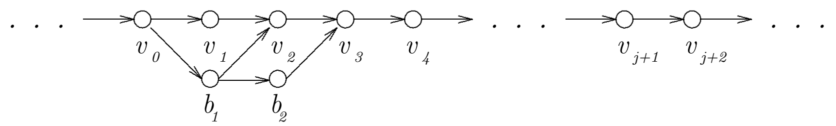
Since $b_1 \neq v_1$ and $v_0 > b_1 > v_2$, then b_1 is singular and not a vertex in P.



Notice that if v is a vertex with loop then he is either greater or smaller than every element of $D \cup \{b_1\}$.

Again by (3.6) there is an *alternative* path (b_1, b_2, v_3) for (b_1, v_2, v_3), otherwise (b_1, v_3) will be an arc in $G(A^2)$.

By (3.7) (b_1, v_1) is not an arc in $G(A)$, so $b_2 \neq v_1$. Since $b_2 \neq v_2$ and $v_0 > b_2 > v_3$, then b_2 is singular and not a vertex in P.



Once more, if v is a vertex with loop then he is either greater or smaller than every element of $D \cup \{b_1, b_2\}$.

Consecutively, by (3.6) there is an *alternative* path (b_k, b_{k+1}, v_{k+2}) for (b_k, v_{k+1}, v_{k+2}) where $k \in \{2, 3, \ldots, j\}$, otherwise (b_k, v_{k+2}) will be an arc in $G(A^2)$.

Since $b_{k+1} \neq v_{k+1}$, $v_0 > b_{k+1} > v_{k+2}$ and by (3.7), (b_k, v_i) is not an arc in $G(A)$ for $i \leq k$, then b_{k+1} is singular and not a vertex in P.

Each new vertex b_{k+1} is between v_0 and v_{k+2} so if v is a vertex with loop then he is either greater or smaller than every element of $D \cup \{b_1, b_2, \ldots, b_{k+1}\}$.

So we have the following possibilities:

- $w_1 \leq t_j - 2$, then the sum of the balance of the two positions is negative:

$$(w_1 - t_j) + (t_j - (t_j - 1)) \leq -2 + 1 = -1\,;$$

- $w_1 = t_j + 1$, then b_{t_j+1} is also a vertex with loop in $G(A)$.

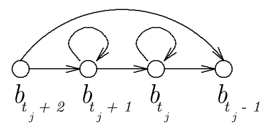
Let us show that the second case will end up with a negative balance. Note that the element in the position $(t_j, t_j + 1)$ is nonzero.

Let us see what are the possible values of w_2 of the $(t_j + 1, w_2)$ entry of the diagonal.

Like in the previous case, w_2 can not be greater than $t_j + 2$, because the balance of that position is greater than one.

Also, $w_2 \neq t_j$ and $w_2 \neq t_j + 1$.

Furthermore, $w_2 \neq t_j - 1$, otherwise (b_{t_j+2}, b_{t_j-1}) is an arc in $G(A)$:



This way, there is a path in $G(A)$ that covers p_1 singular vertices and $r - 2$ vertices with loop. Absurd.

So we have the following possibilities:

- $w_2 \leq t_j - 2$, then the sum of the balance of the three positions is negative:

$$(w_2 - (t_j + 1)) + (t_j + 1 - t_j) + (t_j - (t_j - 1)) \leq ((t_j - 2) - (t_j + 1)) + 1 + 1 = -1\,;$$

- $w_2 = t_j + 2$, then b_{t_j+2} is also a vertex with loop in $G(A)$.

Let us show again that the second possibility leads us to a negative balance.

As there is a finite number of vertices with loop in $G(A)$, after repeating this process a certain number of times, we are forced to choose a position which is not above the main diagonal.

So, the sum of the balance of the positions, obtained in this procedure, is negative.

The only way to *"stabilize"* the negative balance, is by assuming there is another nonzero element above the main diagonal. Let t_j^* be the smallest natural number such that $(t_j^* - 1, t_j^*)$ is one of those entries.

But, like on the previous case, this leads us to a negative balance, and once more, we have to assume there is another element above the main diagonal, and so successively.

We get to a point, in which there are no more nonzero entries above the main diagonal, so, is not possible to *"stabilize"* the balance.

In conclusion, there is no other generalized diagonal with all elements nonzero. So, the $(p_1 + r - 1) \times (p_1 + r - 1)$ submatrix is nonsingular. Then $rank(A) \geq p_1 + r - 1 \geq \pi_1 - 1$. ■

Note that the above proof show that if k is the number of vertices in the path that covers, p_1 singular vertices and the smallest number of vertices with loop, then $rank(A) \geq k - 1$.

Corollary 3.4. Let A be an $n \times n$ lower triangular matrix, then $\eta_1 \leq n - \pi_1 + 1$.

Proof. By definition, $\eta_1 = n(A) = n - rank(A)$. ■

Example 3.5. Let A be an 8×8 nilpotent lower triangular matrix such that the sequence π related to $G(A)$ is $(5, 2, 1)$, then $\pi^* = (3, 2, 1, 1, 1)$. By the previous result the sequence η cannot be $(5, 3)$ because $\eta_1 \leq 8 - 5 + 1 = 4$.

Nevertheless π^* and $(5, 3)$ are in condition of (2.17). This shows that (3.4) does not follow from (2.17).

Lemma 3.6. Let A be an $n \times n$ lower triangular matrix and v_1, v_2 singular vertices in $G(A)$. If $G(A)$ has only one path (v_1, v, v_2) for a vertex $v \in \langle n \rangle$ then (v_1, v_2) is an arc in $G(A^2)$.

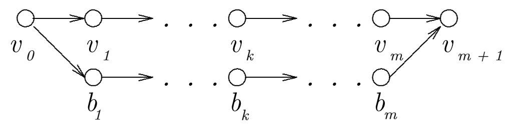
Proof. Let $A^2 = [c_{ij}]$. Since v_1, v_2 are singular vertices and $A = [a_{ij}]$ is a lower triangular matrix then

$$c_{v_1,v_2} = \sum_{\substack{v=1 \\ v_1 > v > v_2}}^{n} a_{v_1,v}\, a_{v,v_2}\,.$$

Notice that the element $a_{v_1,v}\ a_{v,v_2}$ in that sum is nonzero if and only if (v_1, v, v_2) is a path in $G(A)$. Since there is only one path in those conditions, then c_{v_1,v_2} is nonzero. ■

Lemma 3.7. Let A be an $n \times n$ lower triangular matrix, m a natural number and P a path in $G(A)$ that covers π_1 singular vertices. Suppose there are paths $P_1 = (v_0, v_1, v_2, \ldots, v_m, v_{m+1})$ and $P_2 = (v_0, b_1, b_2, \ldots, b_m, v_{m+1})$ of singular vertices in $G(A)$, such that P_1 is a path in P and $b_i \notin P$ for every $i \in \langle m \rangle$. For every natural numbers j and k such that $j \leq k$, (b_k, v_j) is not an arc in $G(A)$.

Proof. The graph $G(A)$ has two paths of singular vertices:



Suppose that (b_k, v_j) is an arc for a natural number $j \leq k$.

The path $P_1 = (v_0, v_1, v_2, \ldots, v_k, \ldots, v_m, v_{m+1})$ in P, can be replaced by $(v_0, b_1, b_2, \ldots, b_k, v_j, v_{j+1}, \ldots, v_k, \ldots, v_m, v_{m+1})$.

This way, we obtain a path in $G(A)$ that covers more than π_1 singular vertices. Absurd. ■

Notice that for an $n \times n$ lower triangular matrix A, such that $G(A)$ has r vertices with loop (which means that the main diagonal of A has r nonzero entries), $n(A^i) = m(A) = n - r$ for every $i \geq \xi_1$.

Theorem 3.8. Let A be an $n \times n$ lower triangular matrix. Suppose there is a set D, of $\pi_2 + 3$ consecutive singular vertices, in a path P that covers π_1 singular vertices, such that if v is a vertex with loop then he is either greater or smaller than every element of D. In this conditions η has more than two elements.

Proof. Let $j = \pi_2$ and $D = \{v_0, v_1, v_2, \ldots, v_{j+1}, v_{j+2}\}$ such that $v_0 > v_1 > v_2 > \cdots > v_{j+1} > v_{j+2}$.

Since the singular vertices in D are consecutive in P and without any vertex with loop between them, then $P_1 = (v_0, v_1, v_2, \ldots, v_{j+1}, v_{j+2})$ is a path in P.