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MORITA EQUIVALENCE OF MANY-SORTED ALGEBRAIC THEORIES

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ABSTRACT: Algebraic theories are called Morita equivalent provided that the corresponding categories of algebras are equivalent. Generalizing Dukarm's result from one-sorted theories to general algebraic theories, we prove that all theories Morita equivalent to a theory $\mathcal{T}$ are obtained as idempotent modifications of $\mathcal{T}$. This is analogous to the classical result of Morita: all rings Morita equivalent to a ring R are obtained as idempotent modifications of matrix rings of R.

1. Introduction

The classical results of Kiiti Morita characterizing equivalence of categories of modules, see [9], have been generalized to one-sorted algebraic theories in several articles. The aim of the present paper is to generalize one of the basic characterizations to many-sorted theories, and to illustrate the situation on concrete examples.

Let us first recall the classical results concerning

$$R\text{-}\mathbf{Mod}$$

the category of left R-modules for a given ring R. Two rings R and S are called *Morita equivalent* if the corresponding categories R-**Mod** and S-**Mod** are equivalent. (For distinction we speak about *categorical equivalence* whenever the equivalences of categories in the usual sense is discussed.) K. Morita provided two types of characterizations:

Type 1: Rings R and S are Morita equivalent iff there exist an R-S-bimodule M and an S-R-bimodule M' such that

$$M \otimes M' \cong S \text{ and } M' \otimes M \cong R\,.$$

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This result was fully generalized by F. Borceux and E. Vitale [4] to Lawvere's algebraic theories as follows: given algebraic theories $\mathcal{T}$ and $\mathcal{S}$, by a $\mathcal{T}$-$\mathcal{S}$-bimodel M is meant a model of $\mathcal{T}$ in the category of $\mathcal{S}$-algebras. Two algebraic theories $\mathcal{T}$ and $\mathcal{S}$ are *Morita equivalent*, i.e., their categories of algebras are (categorically) equivalent, iff there exist an $\mathcal{T}$-$\mathcal{S}$-bimodel M and an $\mathcal{S}$-$\mathcal{T}$-bimodel M' such that

$$M \otimes M' \cong \mathcal{S} \text{ and } M' \otimes M \cong \mathcal{T}$$

where $\cong$ means natural isomorphism and $\otimes$ is the tensor product corresponding to $\mathrm{Hom}(M, -)$ and $\mathrm{Hom}(M', -)$, respectively (i.e., the functors obtained by composing models with M (or M')).

Type 2: Two constructions on a ring R are specified yielding a Morita equivalent ring. Then it is proved that every Morita equivalent ring can be obtained from R by applying successively the two constructions.

(a) *Matrix Ring* $R^{[n]}$. This is the ring of all $n \times n$ matrices over R with the usual addition, multiplication, and unit matrix. This ring $R^{[n]}$ is always Morita equivalent to R.

(b) *Idempotent Modification* uRu. Let u be an idempotent element of R, $uu = u$, and let uRu be the ring of all elements of the form uxu (i.e., all elements $x \in R$ with $x = uxu$). The addition and multiplication of uRu is that of R, and u is the multiplicative unit. This ring uRu is Morita equivalent to R whenever u is pseudoinvertible, i.e., $eum = 1$ for some elements e and m of R.

K. Morita proved that two rings R and S are Morita equivalent iff S is isomorphic to the ring $uR^{[n]}u$ for some pseudoinvertible $n \times n$ matrix u over R.

This result was generalized to one-sorted algebraic theories $\mathcal{T}$ (i.e., categories having as objects natural numbers and such that every object n is a product $1 \times 1 \times \cdots \times 1$) by J. J. Dukarm [5] as follows: he again introduced two constructions yielding from a given one-sorted theory a Morita equivalent theory:

(a) *Matrix Theory* $\mathcal{T}^{[n]}$. This is the full subcategory of $\mathcal{T}$ on all objects kn $(k \in \mathbb{N})$.

(b) *Idempotent Modification* $u\mathcal{T}u$. Given an idempotent $u : 1 \to 1$, i.e., $u \cdot u = u$, we denote by

$$u^k = u \times u \times \cdots \times u : k \to k$$

the corresponding idempotents of $\mathcal{T}$, and we call u *pseudoinvertible* if there is $k \geq 1$ such that

$$eu^k m = \mathrm{id}$$

for some morphisms $1 \xrightarrow{m} k \xrightarrow{e} 1$ of $\mathcal{T}$.

We denote, for every pseudoinvertible idempotent u, by $u\mathcal{T}u$ the theory of all those morphisms $f : n \to m$ of $\mathcal{T}$ which fulfil $f = u^m f u^n$. The composition is as in $\mathcal{T}$ and the identity morphisms are u^n.

J. J. Dukarm proved, again, that whenever $\mathcal{T}$ and $\mathcal{S}$ are one-sorted algebraic theories then they are Morita equivalent iff $\mathcal{S}$ is categorically equivalent to the theory $u\mathcal{T}^{[n]}u$ for some n and some pseudoinvertible idempotent u of $\mathcal{T}^{[n]}$.

We are going to generalize this to algebraic theories (i.e., small categories with finite products) without the assumption that they are one-sorted. The two constructions (a) and (b) above are put together by considering idempotent modifications $u\mathcal{T}u$ where u is an S-tuple of idempotents which is, in a technical sense defined below, pseudoinvertible. Then all Morita equivalent theories are precisely those idempotent modifications.

2. Morita Equivalence of Algebraic Theories

Notation 2.1. For an *algebraic theory* $\mathcal{T}$, i.e., a small category with finite products, we denote by

$$Alg\mathcal{T}$$

the category of algebras, i.e., the full subcategory of $\mathbf{Set}^{\mathcal{T}}$ formed by all functors preserving finite products.

Two algebraic theories $\mathcal{T}$ and $\mathcal{S}$ are called *Morita equivalent* provided that the categories $Alg\mathcal{T}$ and $Alg\,\mathcal{S}$ are categorically equivalent.

Remark 2.2. (a) We call a category *idempotent-complete* provided that every idempotent in it splits. Recall that every category $\mathcal{K}$ has an *idempotent completion* $\mathcal{L}$ (called Cauchy completion in [3]), i.e., $\mathcal{L}$ is an idempotent-complete category containing $\mathcal{K}$ as a full subcategory such that every object of $\mathcal{L}$ is obtained as a splitting of an idempotent of $\mathcal{K}$.

(b) For two small categories $\mathcal{T}$ and $\mathcal{S}$ the presheaf categories $\mathbf{Set}^{\mathcal{T}}$ and $\mathbf{Set}^{\mathcal{S}}$ are categorically equivalent iff $\mathcal{T}$ and $\mathcal{S}$ have the same idempotent completion, see [3], 6.5.11. If follows that Morita equivalence of algebraic theories is nothing else than the categorical equivalence of their idempotent completions. We provide a more concrete characterization below.

(c) Recall from [1] the concept of a *sifted colimit*. For the proof below all the reader has to know about sifted colimits is the following:

(i) If a category $\mathcal{D}$ has finite coproducts then every diagram with domain $\mathcal{D}$ is sifted.

(ii) A *strongly finitely presentable* object is an object whose hom-functor preserves sifted colimits. In categories $Alg\mathcal{T}$ of algebras strongly finitely presentable objects are precisely the retracts of the "free algebras"

$$YB : \mathcal{T} \to \mathbf{Set} \qquad \text{for } B \in \mathcal{T}$$

where $Y : \mathcal{T}^{\mathrm{op}} \to Alg\mathcal{T}$ is the Yoneda embedding and B an arbitrary object of $\mathcal{T}$.

Definition 2.3. A collection of idempotent morphisms

$$u_s : B_s \to B_s \qquad (s \in S)$$

of an algebraic theory $\mathcal{T}$ is called *pseudoinvertible* provided that for every object $T \in \mathcal{T}$ there exists a finite family $s_1, \ldots, s_n \in S$ and morphisms

$$T \xrightarrow{m} B_{s_1} \times \cdots \times B_{s_n} \xrightarrow{e} T$$

such that the square

$$\begin{tikzcd}
B_{s_1} \times \cdots \times B_{s_n} \arrow[r, "u_{s_1} \times \cdots \times u_{s_n}"] & B_{s_1} \times \cdots \times B_{s_n} \arrow[d, "e"] \\
T \arrow[u, "m"] \arrow[r, equal] & T
\end{tikzcd}$$

commutes.

Remark 2.4. A theory $\mathcal{T}$ is called R-sorted provided that a collection $(T_r)_{r \in R}$ of objects of $\mathcal{T}$ is given such that every object of $\mathcal{T}$ is (isomorphic to) a product of objects of the collection. For verification of pseudoinvertibility of a collection $u = (u_s)_{s \in S}$ of idempotents it is sufficient to find m and e above for all the objects $T = T_r$, $r \in R$. In particular, in case of one-sorted theories Definition 2.3 coincides with the pseudoinvertibility in the Introduction.

Notation 2.5. Let $u = (u_s)_{s \in S}$ be a pseudoinvertible collection of idempotents $u_s : B_s \to B_s$ of $\mathcal{T}$. We denote by

$$u\mathcal{T}u$$

the algebraic theory whose objects are all finite words $s_1 \dots s_n$ over the alphabet S (including the empty word) and whose morphisms from $s_1 \dots s_n$ to $t_1 \dots t_k$ are precisely those morphisms $f : B_{s_1} \times \cdots \times B_{s_n} \to B_{t_1} \times \cdots \times B_{t_k}$ of $\mathcal{T}$ for which the following square

$$\begin{tikzcd} B_{s_1} \times \cdots \times B_{s_n} \arrow[r, "f"] \arrow[d, "u_{s_1} \times \cdots \times u_{s_n}"'] & B_{t_1} \times \cdots \times B_{t_k} \\ B_{s_1} \times \cdots \times B_{s_n} \arrow[r, "f"'] & B_{t_1} \times \cdots \times B_{t_k} \arrow[u, "u_{t_1} \times \cdots \times u_{t_k}"'] \end{tikzcd} \tag{2.1}$$

commutes. The composition in $u\mathcal{T}u$ is that of $\mathcal{T}$, and the identity morphism of $s_1 \dots s_n$ is $u_{s_1} \times \cdots \times u_{s_n}$.

Remark 2.6. (1) If $S = \{s\}$ has just one element, i.e., a single endomorphism $u : B \to B$ is given, then $u\mathcal{T}u$ of 2.5 differs from $u\mathcal{T}u$ of Introduction only in calling the objects words $s \dots s$ (of length k) rather than the corresponding natural numbers k.

(2) The matrix theory $\mathcal{T}^{[n]}$ of Introduction has the obvious S-sorted generalization: given a collection $D = \{B_s;\ s \in S\}$ of objects of $\mathcal{T}$, we consider the full subcategory $\mathcal{T}^{[D]}$ of $\mathcal{T}$ on all finite products of these objects. This is a special case of $u\mathcal{T}u$: choose $u_s = \mathrm{id}_{B_s}$, for $s \in S$. Pseudoinvertibility means here that all objects are retracts of products $B_{s_1} \times \cdots \times B_{s_n}$.

Theorem 2.7. *Let $\mathcal{T}$ be an algebraic theory. Then an S-sorted algebraic theory $\mathcal{S}$ is Morita equivalent to $\mathcal{T}$ iff it is categorically equivalent to $u\mathcal{T}u$ for some pseudoinvertible collection $u = (u_s)_{s \in S}$ of idempotents in $\mathcal{T}$.*

Proof.

(1) Sufficiency: let

$$u_s : B_s \to B_s \qquad (s \in S)$$

be a pseudoinvertible collection of idempotents. Denote by

$$Y : \mathcal{T}^{\mathrm{op}} \to \mathit{Alg}\,\mathcal{T}$$

the Yoneda embedding. Since $\mathit{Alg}\,\mathcal{T}$ is complete, the idempotent Yu_s has a splitting

$$\begin{tikzcd} YB_s \arrow[r, shift left, "\varepsilon_s"] & A_s \arrow[l, shift left, "\mu_s"] \end{tikzcd}$$

in $Alg\,\mathcal{T}$: let μ_s be an equalizer of Yu_s and id, and ε_s the unique morphism with

$$\mu_s\varepsilon_s = Yu_s \quad \text{and} \quad \varepsilon_s\mu_s = \text{id} \qquad \text{in} \quad Alg\,\mathcal{T}\,. \tag{2.2}$$

Denote by

$$\mathcal{T}^{\langle u\rangle} \subseteq (Alg\,\mathcal{T})^{\text{op}} \tag{2.3}$$

the full subcategory of the dual of $Alg\,\mathcal{T}$ on all objects which are, in $(Alg\,\mathcal{T})^{\text{op}}$, finite products of the algebras A_s ($s \in S$).

(1a) We prove that $\mathcal{T}$ and $\mathcal{T}^{\langle u\rangle}$ are Morita equivalent. The closure $\mathcal{C}$ of $\mathcal{T}^{\langle u\rangle}$ under retracts in the (idempotent-complete) category $(Alg\,\mathcal{T})^{\text{op}}$ is an idempotent completion of $\mathcal{T}^{\langle u\rangle}$. It is sufficient to prove that

$$YB_s \in \mathcal{C}$$

for every $s \in S$: in fact, we then have $YT \in \mathcal{C}$ for every $T \in \mathcal{T}$ because T is a retract of a finite product $B_{s_1} \times \cdots \times B_{s_n}$ (use m and $\overline{e} = e(u_{s_1} \times \cdots \times u_{s_n})$ in Definition 2.3). Therefore, $Y^{\text{op}}[\mathcal{T}]$ is contained in $\mathcal{C}$. Moreover, since A_s is a retract of YB_s (use (2.2) above), we conclude that $\mathcal{C}$ is an idempotent completion of $Y^{\text{op}}[\mathcal{T}] \cong \mathcal{T}$, thus, $\mathcal{T}$ and $\mathcal{T}^{\langle u\rangle}$ are Morita equivalent.

For the proof of $YB_s \in \mathcal{C}$ apply Definition 2.3 to $T = B_s$ and consider the following morphisms of $Alg\,\mathcal{T}$:

$$\widetilde{e} \equiv YB_s \xrightarrow{Ye} YB_{s_1} + \cdots + YB_{s_n} \xrightarrow{\varepsilon_{s_1}+\cdots+\varepsilon_{s_n}} A_{s_1} + \cdots + A_{s_n}$$

and

$$\widetilde{m} \equiv A_{s_1} + \cdots + A_{s_n} \xrightarrow{\mu_{s_1}+\cdots+\mu_{s_n}} YB_{s_1} + \cdots + YB_{s_n} \xrightarrow{Ym} YB_s\,.$$

Since (2.2) implies $\widetilde{m}\widetilde{e} = Ym{\cdot}Y(u_{s_1}\times\cdots\times u_{s_n}){\cdot}Ye = Y(e{\cdot}u_{s_1}\times\cdots\times u_{s_n}{\cdot}m) =$ id, we see that YB_s is a retract of $A_{s_1} \times \ldots A_{s_n}$ in $(Alg\,\mathcal{T})^{\text{op}}$, thus, it lies in $\mathcal{C}$.

(1b) We prove next that $\mathcal{T}^{\langle u\rangle}$ is categorically equivalent to $u\mathcal{T}u$ – thus, by (1a), $u\mathcal{T}u$ is Morita equivalent to $\mathcal{T}$.

Define a functor

$$E : u\mathcal{T}u \to \mathcal{T}^{\langle u\rangle}$$

on objects by

$$E(s_1 \ldots s_n) = A_{s_1} \times \cdots \times A_{s_n}$$

and on morphisms $f : s_1 \ldots s_n \to t_1 \ldots t_k$ (which, recall, are special morphisms $f : B_{s_1} \times \cdots \times B_{s_n} \to B_{t_1} \times \cdots \times B_{t_k}$ of $\mathcal{T}$) by the commutativity of the following square in $Alg\,\mathcal{T}$:

$$\begin{tikzcd}
 & A_{s_1}+\cdots+A_{s_n} & A_{t_1}+\cdots+A_{t_k} \arrow[l, "Ef"'] \arrow[d, "\mu_{t_1}+\cdots+\mu_{t_k}"] \\
YB_{s_1}+\cdots+YB_{s_n}=Y(B_{s_1}\times\cdots\times B_{s_n}) \arrow[ur, "\varepsilon_{s_1}+\cdots+\varepsilon_{s_n}"] & & Y(B_{t_1}\times\cdots\times B_{t_k})=YB_{t_1}+\cdots+YB_{t_k} \arrow[ll, "Yf"]
\end{tikzcd} \tag{2.4}$$

It is easy to verify that E is well-defined, let us prove that it is an equivalence functor.

E is faithful because Y is faithful, and we have

$$\begin{aligned}
Yf &= Y(u_{s_1}\times\cdots\times u_{s_n})\cdot Yf\cdot Y(u_{t_1}\times\cdots\times u_{t_k}) && \text{see (2.1)}\\
&= (\mu_{s_1}+\cdots+\mu_{s_n})(\varepsilon_{s_1}+\cdots+\varepsilon_{s_n})\\
&\quad \cdot Yf\cdot(\mu_{t_1}+\cdots+\mu_{t_k})(\varepsilon_{t_1}+\cdots+\varepsilon_{t_k}) && \text{see (2.2)}\\
&= (\mu_{s_1}+\cdots+\mu_{s_n})\cdot Ef\cdot(\varepsilon_{t_1}+\cdots+\varepsilon_{t_k}) && \text{see (2.4).}
\end{aligned}$$

Since in the last composite the first morphism is a split epimoprhism and the last one a split monomoprhism, the faithfulness of E implies that of Y.

E is full because Y is full: given $h: A_{t_1}+\cdots+A_{t_k}\to A_{s_1}+\cdots+A_{s_n}$ in $Alg\,\mathcal{T}$, we have $f: B_{s_1}\times\cdots\times B_{s_n}\to B_{t_1}\times\ldots B_{t_k}$ in $\mathcal{T}$ with

$$Yf=(\mu_{s_1}+\cdots+\mu_{s_n})\cdot h\cdot(\varepsilon_{t_1}+\cdots+\varepsilon_{t_k})\,. \tag{2.5}$$

From (2.2) we conclude that

$$Yf=Y[(u_{t_1}\times\cdots\times u_{t_k})f(u_{s_1}\times\cdots\times u_{s_n})]\,,$$

hence f is a morphism of $u\mathcal{T}u$ (recall that Y is faithful). From (2.2), (2.4) and (2.5) we conclude $Ef=h$.

Since E is surjective on objects it is an equivalence functor.

(2) Necessity: given an algebraic theory $\mathcal{S}$ whose objects are finite products of C_s ($s\in S$) and given an equivalence functor

$$F: Alg\,\mathcal{S}\to Alg\,\mathcal{T}$$

we find a pseudoinvertible collection $u=(u_s)_{s\in S}$ of idempotents with $\mathcal{S}$ categorically equivalent to $u\mathcal{T}u$. Denote the corresponding Yoneda embeddings by $Y_{\mathcal{T}}:\mathcal{T}^{\mathrm{op}}\to Alg\,\mathcal{T}$ and $Y_{\mathcal{S}}:\mathcal{S}^{\mathrm{op}}\to Alg\,\mathcal{S}$. The $\mathcal{T}$-algebras

$$A_s=F(Y_{\mathcal{S}}C_s)\qquad(s\in S)$$

are strongly finitely presentable (since $Y_{\mathcal{S}} C_s$ are, see 2.2(c)). Thus, each A_s is a retract of some $Y_{\mathcal{T}} B_s$ for $B_s \in \mathcal{T}$. Choose homomorphisms

$$Y_{\mathcal{T}} B_s \underset{\mu_s}{\overset{\varepsilon_s}{\rightleftarrows}} A_s \quad \text{with} \quad \varepsilon_s \mu_s = \mathrm{id} \qquad (\text{in } \mathit{Alg}\,\mathcal{T})\,.$$

Then the idempotent $\mu_s \varepsilon_s$ has the form $Y_{\mathcal{T}} u_s$ for a unique idempotent $u_s : B_s \to B_s$ of $\mathcal{T}^{\mathrm{op}}$. And the codomain restriction of $(F \cdot Y_{\mathcal{S}})^{\mathrm{op}} : \mathcal{S} \to (\mathit{Alg}\,\mathcal{T})^{op}$ yields an equivalence functor between $\mathcal{S}$ and $\mathcal{T}^{\langle u \rangle}$, see (2.3) above. As in (1b), we deduce that $u\mathcal{T}u$ is categorically equivalent to $\mathcal{T}^{\langle u \rangle}$. It remains to show that u is pseudoinvertible.

For every object $T \in \mathcal{T}$ we will prove that $Y_{\mathcal{T}} T$ is a retract of an object of $(\mathcal{T}^{\langle u \rangle})^{\mathrm{op}}$ in $\mathit{Alg}\,\mathcal{T}$, i.e., that there exist homomorphisms $\overline{e} : A_{s_1} + \cdots + A_{s_n} \to Y_{\mathcal{T}} T$ and $\overline{m} : Y_{\mathcal{T}} T \to A_{s_1} + \cdots + A_{s_n}$ with $\overline{e} \cdot \overline{m} = \mathrm{id}$ in $\mathit{Alg}\,\mathcal{T}$. This will prove the pseudoinvertibility: we have unique morphisms m and e in $\mathcal{T}$ with

$$Y_{\mathcal{T}} e = Y_{\mathcal{T}} T \xrightarrow{\overline{m}} A_{s_1} + \cdots + A_{s_n} \xrightarrow{\mu_{s_1} + \cdots + \mu_{s_n}} Y_{\mathcal{T}}(B_{s_1} \times \cdots \times B_{s_n})$$

and

$$Y_{\mathcal{T}} m = Y_{\mathcal{T}}(B_{s_1} \times \cdots \times B_{s_n}) \xrightarrow{\varepsilon_{s_1} + \cdots + \varepsilon_{s_n}} A_{s_1} + \cdots + A_{s_n} \xrightarrow{\overline{e}} Y_{\mathcal{T}} T\,.$$

The desired square in 2.3 follows from the fact that $Y_{\mathcal{T}}$ is faithful:

$$\begin{tikzcd}
Y_{\mathcal{T}} T \arrow[rrrr, "{Y_{\mathcal{T}}[e(u_{s_1} \times \cdots \times u_{s_n})m] = \mathrm{id}}"] \arrow[d, "\overline{m}"'] & & & & Y_{\mathcal{T}} T \\
A_{s_1} + \cdots + A_{s_n} \arrow[d, "\mu_{s_1} + \cdots + \mu_{s_n}"'] \arrow[drr, "\mathrm{id}"] & & & & A_{s_1} + \cdots + A_{s_n} \arrow[u, "\overline{e}"'] \\
Y_{\mathcal{T}} B_{s_1} + \cdots + Y_{\mathcal{T}} B_{s_n} \arrow[rr, "\varepsilon_{s_1} + \cdots + \varepsilon_{s_n}"'] & & A_{s_1} + \cdots + A_{s_n} \arrow[urr, "\mathrm{id}"] \arrow[rr, "\mu_{s_1} + \cdots + \mu_{s_n}"'] & & Y_{\mathcal{T}} B_{s_1} + \cdots + Y_{\mathcal{T}} B_{s_n} \arrow[u, "\varepsilon_{s_1} + \cdots + \varepsilon_{s_n}"']
\end{tikzcd}$$

To prove that $Y_{\mathcal{T}} T$ is a retract of an object of $(\mathcal{T}^{\langle u \rangle})^{\mathrm{op}}$, observe that since the algebras $Y_{\mathcal{S}} C_s$ ($s \in S$) are dense in $\mathit{Alg}\,\mathcal{S}$, it follows that A_s ($s \in S$) are dense in $\mathit{Alg}\,\mathcal{T}$. And so is their closure $(\mathcal{T}^{\langle u \rangle})^{\mathrm{op}}$ under finite coproducts. Therefore, $Y_{\mathcal{T}} T$ is a canonical colimit of the diagram D of all homomorphisms $A \to Y_{\mathcal{T}} T$ with $A \in (\mathcal{T}^{\langle u \rangle})^{\mathrm{op}}$. The domain of this diagram, i.e., the comma-category $(\mathcal{T}^{\langle u \rangle})^{\mathrm{op}} / Y_{\mathcal{T}} T$ has finite coproducts (being closed under them in $\mathit{Alg}\,\mathcal{T} / Y_{\mathcal{T}} T$), thus, the diagram is sifted, see Remark 2.2(c); since $Y_{\mathcal{T}} T$ is strongly finitely presentable, it follows that one of the colimit morphisms of D is a split epimorphism. ■

3. Examples

Example 3.1. Modules. For one-sorted theories K. Morita covered the whole spectrum: there exist no other one-sorted theories of R-**Mod** than those canonically derived from Morita equivalent rings.

More detailed:

(i) Each R^n ($n \in \mathbb{N}$) has a natural structure of left R-module. The full subcategory

$$\mathcal{T}_R = \{R^n\,;\, n \in \mathbb{N}\}$$

of $(R\text{-}\mathbf{Mod})^{\mathrm{op}}$ is a one-sorted algebraic theory of R-**Mod**.

(ii) Consequently, for every ring S Morita equivalent to R, we have an algebraic theory $\mathcal{T}_S$ of R-**Mod**.

(iii) The above are, up to categorical equivalence, all one-sorted algebraic theories of R-**Mod**. In fact, let $\mathcal{T}$ be a one-sorted algebraic theory with objects n ($n \in \mathbb{N}$) and with an equivalence functor

$$E : Alg\mathcal{T} \to R\text{-}\mathbf{Mod}\,.$$

Then $\mathcal{T}$ is equivalent to $\mathcal{T}_S$ for a ring S Morita equivalent to R: indeed, following [6], $Alg\mathcal{T}$ is equivalent to S-**Mod**, with $S = \mathcal{T}(1,1)$. Moreover, the composition of the Yoneda embedding $Y : \mathcal{T}^{\mathrm{op}} \to Alg\mathcal{T}$ with the equivalence $Alg\mathcal{T} \to S$-**Mod** sends an object n to $\mathcal{T}(n,1)$ which, by additivity, is isomorphic to $\mathcal{T}(1,1)^n = S^n$. This shows that $\mathcal{T}$ is equivalent to $\mathcal{T}_S$, with S Morita equivalent to R.

Remark 3.2. There are, of course, many more algebraic theories of R-**Mod** which are not one-sorted. For example, in $\mathbf{Ab} = \mathbb{Z}$-**Mod** the theory $\mathcal{T}'$ generated by $\mathbb{Z}$ and $\mathbb{Z}_2 = \mathbb{Z}/2\mathbb{Z}$ is certainly Morita equivalent to $\mathcal{T}_{\mathbb{Z}}$, but it is not categorically equivalent to $\mathcal{T}_S$ for any Morita equivalent ring S (e.g., $\mathcal{T}'$ contains an object with a finite hom).

Example 3.3. All algebraic theories of **Set**. The one-sorted theories are well-known to be just the theories

$$\mathcal{T}^{[n]} \qquad (n = 1,\, 2,\, 3,\, \dots)$$

where $\mathcal{T} \subseteq \mathbf{Set}^{\mathrm{op}}$ is the full subcategory on all natural numbers, and $\mathcal{T}^{[n]}$ is the matrix theory, i.e., the full subcategory of $\mathcal{T}$ on 0, n, $2n$, $\dots$. And they are, obviously, pairwise categorically non-equivalent.

We now describe all many-sorted theories: they are precisely the matrix theories $\mathcal{T}^{[D]}$, see 2.6(2), for finite sets

$$D \subseteq \mathbb{N}$$

which are *sum-irreducible*, i.e., no number of D is a sum of more than one member of D. Recall that

$$\mathcal{T}^{[D]}$$

is the dual of the full subcategory of **Set** on all finite sums of members of D. Then we know that $\mathcal{T}^{[D]}$ is an algebraic theory of **Set**. We are going to prove that these are precisely all of them:

(a) Every algebraic theory $\mathcal{T}'$ is categorically equivalent to $\mathcal{T}^{[D]}$ for some finite sum-irreducible $D \subseteq \mathbb{N}$. In fact, consider a pseudoinvertible collection $u_s : B_s \to B_s$ $(s \in S)$ of idempotents in $\mathcal{T}$ with $\mathcal{T}'$ categorically equivalent to $u\mathcal{T}u$, where u_s has precisely r_s fixed points. Without loss of generality we can assume $u_s \neq \mathrm{id}_\emptyset$ for every s, i.e., $r_s \geq 1$. Let K be the subsemigroup of the additive semigroup $\mathbb{N}$ generated by $\{r_s\}_{s\in S}$. (That is, K is the set of all numbers of fixed points of the morphisms $u_{s_1} \times \cdots \times u_{s_n}$ in $\mathbf{Set}^{\mathrm{op}}$.) Then $u\mathcal{T}u$ is categorically equivalent to K as a full subcategory of $\mathbf{Set}^{\mathrm{op}}$. Recall that every subsemigroup K of the additive semigroup of natural numbers is finitely generated (see [10]). Therefore, if D is a minimum set of generators of K, then D is finite, sum-irreducible and K is categorically equivalent to $\mathcal{T}^{[D]}$.

(b) The theories $\mathcal{T}^{[D]}$ are pairwise nonequivalent categories. In fact, every element $n \in D$ defines an object n of $\mathcal{T}^{[D]}$ which is product-indecomposable and has n^n endomorphisms – this determines D categorically.

Example 3.4. M-sets. For monoids M the question of Morita equivalence (that is, given a monoid M' when are M-**Set** and M'-**Set** equivalent categories) was studied by B. Banaschewski [2] and V. Knauer [7] . The main result is formally very similar to that of K. Morita: let us say that an idempotent $u \in M$ is *pseudoinvertible* if there exist e, $m \in M$ with $eum = 1$. It follows that the monoid

$$uMu = \{umu \, : \, m \in M\}$$

whose unit is u and multiplication is as in M is Morita equivalent to M. And these are all monoids Morita equivalent to M, up to isomorphism.

Unlike Example 3.1, this does not describe all one-sorted theories of M-**Set**. In fact, if $M = \{1\}$ is the trivial one-element monoid, then M-**Set** = **Set** has infinitely many pairwise non-equivalent theories, as we saw in Example 3.3, although there are no nontrivial monoids Morita equivalent to $\{1\}$.

Remark 3.5. We saw above that all algebraic theories of **Set** are finitely-sorted (i.e., have finitely many objects whose finite products form all objects). This is not true for M-sets, in general. In fact, whenever M is a commutative monoid with uncountably many idempotents, then the "standard" algebraic theory $\mathcal{T}$ (dual to the category of all free M-sets on finitely many generators) has an idempotent completion $\mathcal{T}'$ which has uncountably many pairwise non-isomorphic objects. (Obviously, every idempotent m of M yields an idempotent endomorphism $m \cdot - : M \to M$ in $\mathcal{T}$, and the splittings of these endomorphisms produce pairwise non-isomoprhic objects A_m of $\mathcal{T}'$: indeed, whenever A_m is isomorphisc to A_n, then for every element x of M we see that $m \cdot x = x$ iff $n \cdot x = x$. By choosing $x = n$ and $x = m$ we conclude $m = n$.) Consequently, $\mathcal{T}'$ is an algebraic theory of M-sets which is not finitely-sorted.

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