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MATRIX REALIZATIONS OF PAIRS OF YOUNG TABLEAUX, KEYS AND SHUFFLES

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ABSTRACT: Let $\sigma \in \mathcal{S}_t$ and $\mathcal{H}_\sigma$ an associated *key* [6, 7, 12]. Given the pair of Young tableaux $(\mathcal{T}, \mathcal{H}_\sigma)$, where $\mathcal{T}$ is a skew-tableau with the same evaluation as $\mathcal{H}_\sigma$, we consider the problem of a matrix realization of the pair $(\mathcal{T}, \mathcal{H}_\sigma)$, over a local principal ideal domain [2, 3, 4]. It has been shown that, when σ is the identity [2], the reverse permutation in $\mathcal{S}_t$ [4], or any permutation in $\mathcal{S}_3$ [3], $(\mathcal{T}, \mathcal{H}_\sigma)$ has a matrix realization if and only if the word of $\mathcal{T}$ is in the plactic class of $\mathcal{H}_\sigma$. Here, we extend the only if condition of this result to any $\sigma \in \mathcal{S}_t$, $t \geq 1$. For a sequence of nonnegative integers $(m_1, \ldots, m_t)$ and $1 \leq i \leq t$, this amounts to give a criterion which extends a word congruent with the key of evaluation $(m_1, \ldots, m_{i-1}, m_{i+1} \ldots, m_t)$, to one congruent with the key of evaluation $(m_1, \ldots, m_t)$.

For the identity, the reverse permutation in $\mathcal{S}_t$, or an arbitrary permutation in $\mathcal{S}_3$, the plactic class of an associated key may be described as shuffles of their rows [3]. For $t \geq 4$, this is no longer true for an arbitrary permutation $\sigma \in \mathcal{S}_t$. Instead, we find that shuffling together the rows of a key always leads to a congruent word. The permutations $\sigma \in \mathcal{S}_t$ whose plactic classes of associated keys are described by shuffling together their rows, are identified. For $\sigma \in \mathcal{S}_4$, we show that we may describe the plactic class of any associated key, in terms of shuffling, by adding, in those cases where the rows of the key are not enough, just one single word.

1. Introduction

Given $\sigma \in \mathcal{S}_t$, let $\mathcal{H}_\sigma$ be an associated key [6, 7, 11, 12]. That is, a (strictly row) tableau with rows pairwise comparable for the inclusion order, by taking a sequence of left reordered factors of σ, considered as a word, by decreasing order of lengths. Given the pair of Young tableaux (strictly row) $(\mathcal{T}, \mathcal{H}_\sigma)$, where $\mathcal{T}$ is a skew-tableau with same evaluation as $\mathcal{H}_\sigma$, we consider the problem of a matrix realization, over a local principal ideal domain, for the pair $(\mathcal{T}, \mathcal{H}_\sigma)$ [Section 5, Definition 5.1].

When σ is the identity [1, 2], the reverse permutation in $\mathcal{S}_t$ [4], or any permutation in $\mathcal{S}_3$ [3], it has been shown that $(\mathcal{T}, \mathcal{H}_\sigma)$ has a matrix realization if and only if the word of $\mathcal{T}$ is in the plactic class of $\mathcal{H}_\sigma$. In the first two cases, this means that $\mathcal{T}$ is a Littlewood-Richardson and a dual Littlewood-Richardson skew-tableau, respectively. For these permutations, the elements

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of the plactic class of $\mathcal{H}_\sigma$ are shuffles of the rows of $\mathcal{H}_\sigma$ and this property has been used to exhibit a matrix realization $(\mathcal{T}, \mathcal{H}_\sigma)$.

Here, in Theorem 5.3, we show that, for any $\sigma \in \mathcal{S}_t$, $t \geq 1$, $(\mathcal{T}, \mathcal{H}_\sigma)$ has a matrix realization only if the word of $\mathcal{T}$ is in the plactic class of $\mathcal{H}_\sigma$. This amounts to give, in Theorem 3.10, a recursive criterion, which extends a word congruent with the key of evaluation $(m_1, \dots, m_{i-1}, m_{i+1}, \dots, m_t)$, to one congruent with the key of evaluation $(m_1, \dots, m_{i-1}, m_i, m_{i+1}, \dots, m_t)$. Moreover, our matrix problem verifies that criterion.

Due to the embedding of the symmetric group in the set of tableaux, originally defined Ehresmann in [6], the symmetric group acts on set of keys $\mathcal{H}_\sigma$, $\sigma \in \mathcal{S}_t$, in the obvious way. This action coincides with that one defined by operations on words described by A. Lascoux and M. P. Schutzenberger in the plactic monoid [10, 13]. On the other hand, these operations involve a special parentheses matching on words of a two-letters alphabet. In [3], we have generalized these operations considering more general parentheses matchings [Section 3]. Corollary 3.11 presents a recursive characterization of those which are compatible with the plactic classes of $\mathcal{H}_\sigma$, $\sigma \in \mathcal{S}_t$. Again our matrix problem verifies those generalized operations and their action on the plactic classes of $\mathcal{H}_\sigma$, $\sigma \in \mathcal{S}_t$.

Two rows commute (in the plactic sense) if and only if they are comparable for the inclusion order. In fact, when $t = 2, 3$, the words of the plactic class of $\mathcal{H}_\sigma$ are shuffles of their rows [3]. For $t \geq 4$, this property does not remain true, in general. Nevertheless, it is shown, in Theorem 4.1, that shuffling together the rows of $\mathcal{H}_\sigma$ always leads to a word in the plactic class of $\mathcal{H}_\sigma$. We characterize the permutations $\sigma \in \mathcal{S}_t$, $t \geq 1$, for which the plactic class of $\mathcal{H}_\sigma$ may be described in terms of shuffling together their rows. The identity and the reverse permutations in $\mathcal{S}_t$, $t \geq 1$, are simple examples of such permutations. Finally, for $\sigma \in \mathcal{S}_4$, we show that we may describe the plactic class of any associated key, in terms of shuffling, by adding, in those cases where the rows of the key are not enough, one single word 434121.

The paper is organized as follows. In the next section we collect some notation and background necessary in the sequel. The relationship between shuffling and Knuth operations on words is discussed. The following question is addressed: *if the rows* $u_1, \dots, u_k$ *are pairwise comparable for the inclusion order, under what conditions* $Sh(u_1, \dots, u_k)$ *is the plactic class of* $u_1 \dots u_k$*?* Some preliminary results are given.

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In Section 3, the set of keys, that is, the tableaux with pairwise comparable rows for the inclusion order, is considered. In fact, they are introduced as the image of the embedding of the symmetric group in the set of tableaux, originally defined by Erhesmann [6]. σ- Yamanouchi words are introduced as words congruent with a key of the permutation σ, and their relationship with the action of the symmetric group, defined by the operations on words described by A. Lascoux and M. P. Schutzenberger in the plactic monoid [10, 13], is shown. A recursive criterion to characterize σ- Yamanouchi words is presented to answer the question: *given* $I = [t] \setminus \{i\}$, *with* $i \in [t]$, *and* $u \in I^*$ *congruent with the tableau of evaluation* $(m_1, \ldots, m_{i-1}, m_{i+1}, \ldots, m_t)$ *and shape* $\sum_{j=1}^{i-1}(1^{m_j}) + \sum_{j=i+1}^{t}(1^{m_j})$, *under what conditions can* u *be a subword of a word* $w \in [t]^*$ *congruent with the tableau of evaluation* $(m_1, \ldots, m_{i-1}, m_i, m_{i+1}, \ldots, m_t)$ *and shape* $\sum_{j=1}^{t}(1^{m_j})$? Considering more general parentheses matchings in the operations on words, defined by A. Lascoux and M. P. Schutzenberger, that criterion also characterizes those which are compatible with the plactic classes of $\mathcal{H}_\sigma$, $\sigma \in \mathcal{S}_t$.

In Section 4, the answer to question addressed in Section 2 is given. In the case of $\mathcal{S}_4$, a full description of the plactic classes of the associated keys, in terms of shuffling, is shown. In Section 5, the matrix problem is considered, and the answer, using the recursive criterion for σ-Yamanouchi words, given in Section 3, is presented. Finally, in Appendix, the permutations in $\mathcal{S}_5$ and $\mathcal{S}_6$ giving a positive answer to question addressed in Section 2, are listed.

2. Words, shuffles and plactic congruence

2.1. Words and tableaux. Let $\mathbb{N}$ be the set of nonnegative integers with the usual order " $\leq$ ". Given $i \leq j \in \mathbb{N}$, $[i, j]$ is an interval in $\mathbb{N}$ with the usual order. If $t \in \mathbb{N}$, $[t]$ denotes the set $\{1, \ldots, t\}$.

Let A be a subset of $\mathbb{N}$. We denote by A^* the free monoid in the alphabet A. That is, the collection of all finite words over the alphabet A, with the concatenation operation. The neutral element is the empty word denoted by λ.

Given $w = x_1 \cdots x_k$ over the alphabet A, we denote by $|w|_j$ the multiplicity of the letter $j \in A$ in the word w. The sequence $(|w|_1, \ldots, |w|_t)$ is called the *evaluation* of w. Here k is the length of w, denoted by $|w|$. We have $|w| = |w|_1 + \cdots + |w|_t$ and the length of λ is zero. Let I be a subset of A. We denote by $w|I$ the word $x_{i_1} \cdots x_{i_k}$, if $I = \{i_1 < i_2 < \cdots < i_k\}$. Such a word $w|I$ is called a *subword* of w.

Let q words $u_1, \ldots, u_q \in A^*$ of lengths $k_1, \ldots, k_q$ respectively. Put $k = k_1 + \cdots + k_q$ and let $[k] = \cup_{j=1}^{q} I_j$, where $(I_1, \ldots, I_q)$ is a q-tuple of pairwise disjoint subsets of $[k]$ with $|I_j| = k_j$, $j = 1, \ldots, q$. Then the word $w|(I_1, \ldots, I_q)$ is defined by $w|I_j = u_j$, for $j = 1, \ldots, q$ [16].

If the letters in w are by strictly decreasing order, that is, $x_i > x_{i+1}$, $1 \leq i \leq k$, w is called a *row* in A^*. The underlying set of a row defines a bijection $w \to \{w\} = \{x_1, \ldots, x_k\}$ between the set of rows in A^* and the family 2^A of subsets of A [12]. Two rows $w = x_1 \cdots x_k$ and $v = y_1 \cdots y_s$ are said *comparable* with respect to the inclusion order if the sets $\{w\} = \{x_1, \ldots, x_k\}$ and $\{v\} = \{y_1, \ldots, y_s\}$ are comparable with respect to the inclusion order. That is either v is a subrow of w or *vice-versa*.

We define another order $\rhd$ on rows by putting $w \rhd v$ if $k \geq s$ and $x_{k-i} \leq y_{s-i}$, $0 \leq i \leq s-1$. That is, there is a decreasing injection of $\{v\}$ into $\{w\}$. For example, $5321 \rhd 41 \rhd 2 \rhd 4$.

A *standard* or *permutation* word is a word without repeated letters.

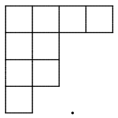
Given B a subalphabet of A, $w_{|B}$ (not to be confused with $w|B$) denotes the word obtained by erasing the letters not in B.

A *partition* is a sequence of nonnegative integers $a = (a_1, a_2, \ldots)$, all but a finite number of which are non zero, such that $a_1 \geq a_2 \geq \cdots$ The number $|a| := \sum_i a_i$ is called the *weight* of a and the maximum value of i for which $a_i > 0$ is called the *length* of a. The null partition $a = (0, 0, \ldots)$ is the partition of length zero. If $l(a) = k$, we shall often write $a = (a_1, \ldots, a_k)$. Sometimes it is convenient to use the notation

$$a = (a_1^{m_1}, a_2^{m_2}, ..., a_k^{m_k}),$$

where $a_1 > a_2 > ... > a_k$ and $a_i^{m_i}$, with $m_i \geq 0$, means that a_i appears m_i times as a part of a.

Suppose $a = (a_1, \ldots, a_k)$ is a partition of weight n. The *Young diagram* of a is an array of n boxes having k left-justified rows with row i containing a_i boxes for $1 \leq i \leq k$. We shall identify a partition with its Young diagram. For example, the Young diagram of $a = (4, 2, 2, 1)$ is:


.

$\mathcal{S}_6$			
$ww'w''$,	$w \in \mathcal{S}_{\{3,6\}}$,	$w' \in \mathcal{S}_{\{4,5\}}$,	$w'' \in \mathcal{S}_{\{1,2\}}$;
$w4w'$,	$w \in \mathcal{S}_{\{2,5,6\}}$,	$w' \in \mathcal{S}_{\{1,3\}}$;	
$w5w'$,	$w \in \mathcal{S}_{\{2,4,6\}}$,	$w' \in \mathcal{S}_{\{1,3\}}$;	
$w46w'$,	$w \in \mathcal{S}_{\{2,5\}}$,	$w' \in \mathcal{S}_{\{1,3\}}$;	
$w52w'$,	$w \in \mathcal{S}_{\{3,6\}}$,	$w' \in \mathcal{S}_{\{1,4\}}$;	
$w3w'$,	$w \in \mathcal{S}_{\{2,5,6\}}$,	$w' \in \mathcal{S}_{\{1,4\}}$;	
$w5w'$,	$w \in \mathcal{S}_{\{2,3,6\}}$,	$w' \in \mathcal{S}_{\{1,4\}}$;	
$w36w'$,	$w \in \mathcal{S}_{\{2,5\}}$,	$w' \in \mathcal{S}_{\{1,4\}}$;	
$w42w'$,	$w \in \mathcal{S}_{\{3,6\}}$,	$w' \in \mathcal{S}_{\{1,5\}}$;	
$w3w'$,	$w \in \mathcal{S}_{\{2,4,6\}}$,	$w' \in \mathcal{S}_{\{1,5\}}$;	
$w4w'$,	$w \in \mathcal{S}_{\{2,3,6\}}$,	$w' \in \mathcal{S}_{\{1,5\}}$;	
$ww'w''$,	$w \in \mathcal{S}_{\{2,5\}}$,	$w' \in \mathcal{S}_{\{3,4\}}$,	$w'' \in \mathcal{S}_{\{1,6\}}$;
ww',	$w \in \mathcal{S}_{\{1,4,5,6\}}$,	$w' \in \mathcal{S}_{\{2,3\}}$;	
ww',	$w \in \mathcal{S}_{\{1,3,5,6\}}$,	$w' \in \mathcal{S}_{\{2,4\}}$;	
ww',	$w \in \mathcal{S}_{\{1,3,4,6\}}$,	$w' \in \mathcal{S}_{\{2,5\}}$;	
$w3w'$,	$w \in \mathcal{S}_{\{1,4,5\}}$,	$w' \in \mathcal{S}_{\{2,6\}}$;	
$w4w'$,	$w \in \mathcal{S}_{\{1,3,5\}}$,	$w' \in \mathcal{S}_{\{2,6\}}$;	
$w35w'$,	$w \in \mathcal{S}_{\{1,4\}}$,	$w' \in \mathcal{S}_{\{2,6\}}$;	
ww',	$w \in \mathcal{S}_{\{1,2,5,6\}}$,	$w' \in \mathcal{S}_{\{3,4\}}$;	
ww',	$w \in \mathcal{S}_{\{1,2,4,6\}}$,	$w' \in \mathcal{S}_{\{3,5\}}$;	
$w41w'$,	$w \in \mathcal{S}_{\{2,5\}}$,	$w' \in \mathcal{S}_{\{3,6\}}$;	
$w2w'$,	$w \in \mathcal{S}_{\{1,4,5\}}$,	$w' \in \mathcal{S}_{\{3,6\}}$;	
$w4w'$,	$w \in \mathcal{S}_{\{1,2,5\}}$,	$w' \in \mathcal{S}_{\{3,6\}}$;	
$w25w'$,	$w \in \mathcal{S}_{\{1,4\}}$,	$w' \in \mathcal{S}_{\{3,6\}}$;	
ww',	$w \in \mathcal{S}_{\{1,2,3,6\}}$,	$w' \in \mathcal{S}_{\{4,5\}}$;	
$w31w'$,	$w \in \mathcal{S}_{\{2,5\}}$,	$w' \in \mathcal{S}_{\{4,6\}}$;	
$w2w'$,	$w \in \mathcal{S}_{\{1,3,5\}}$,	$w' \in \mathcal{S}_{\{4,6\}}$;	
$w3w'$,	$w \in \mathcal{S}_{\{1,2,5\}}$,	$w' \in \mathcal{S}_{\{4,6\}}$;	
$ww'w''$,	$w \in \mathcal{S}_{\{1,4\}}$,	$w' \in \mathcal{S}_{\{2,3\}}$,	$w'' \in \mathcal{S}_{\{5,6\}}$.

There are a total of 52 permutations in $\mathcal{S}_5$ and 488 permutations in $\mathcal{S}_6$ that fail to satisfy the conditions of theorem 3.19.

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We consider the case $m_{t-1} \geq m_t$, the other one is similar. There exists an unimodular matrix $U' \in \mathcal{U}_n$ such that

$$\Delta_a U D_{[m_1]} \cdots D_{[m_{t-1}]} D_{[m_t]} \sim_L \Delta' U' D_{[m_{t-1}]} D_{[m_t]},$$

where $\Delta' = diag_p(a + \chi^{J_1} + \cdots + \chi^{J_{t-2}})$. Since $m_{t-1} \geq m_t$, by the case $t = 2$, the sequence $\Delta' U', D_{[m_{t-1}]}, D_{[m_t]}$ realizes a tableau whose word $w_{|\{t-1,t\}}$ is a Yamanouchi word. Finally, consider the sequence

$$\Delta_a U, D_{[m_1]}, \cdots, D_{[m_{t-2}]}, D_{[m_t]},$$

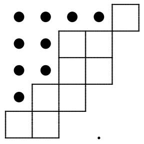
and let w' be the word of the corresponding tableau. Attending to previous proposition, we must have $w' = (\theta_{t-1}(w))_{|[t-1]}$, for some operation θ_{t-1}, and by the inductive step, $P(w') = \mathcal{H}_{s_{t-1}\sigma_{|[t-1]}}$. By theorem 3.10, we find that w is a σ-Yamanouchi word. □

6. Appendix

Below are listed the permutations σ in $\mathcal{S}_5$ and $\mathcal{S}_6$ for which the set $Sh((r_t)^{l_t}, (r_{\sigma,t-1})^{l_{t-1}}, \ldots, (r_{\sigma,2})^{l_2}, (r_{\sigma,1})^{l_1})$, with $l_i > 0$, $i = 1, \ldots, t$, $t = 5, 6$, is not the whole plactic class of $\mathcal{H}_\sigma$.

$\mathcal{S}_5$		
$w4w'$,	$w \in \mathcal{S}_{\{2,5\}}$,	$w' \in \mathcal{S}_{\{1,3\}}$;
$w3w'$,	$w \in \mathcal{S}_{\{2,5\}}$,	$w' \in \mathcal{S}_{\{1,4\}}$;
ww',	$w \in \mathcal{S}_{\{1,4,5\}}$,	$w' \in \mathcal{S}_{\{2,3\}}$;
ww',	$w \in \mathcal{S}_{\{1,3,5\}}$,	$w' \in \mathcal{S}_{\{2,4\}}$;
$w3w'$,	$w \in \mathcal{S}_{\{1,4\}}$,	$w' \in \mathcal{S}_{\{2,5\}}$;
$w2w'$,	$w \in \mathcal{S}_{\{1,4\}}$,	$w' \in \mathcal{S}_{\{3,5\}}$;
ww',	$w \in \mathcal{S}_{\{1,2,5\}}$,	$w' \in \mathcal{S}_{\{3,4\}}$.

Given two partitions a and c, we write $a \subseteq c$ to mean $a_i \leq c_i$, for all i. Graphically, this means that the Young diagram of a is contained in the Young diagram of c. A *skew diagram* c/a is the diagram obtained by removing the smaller diagram of a from the diagram of c. For example, if $a = (4, 2, 2, 1)$ and $c = (5, 4, 4, 3, 2)$, the following shows the skew diagram c/a:



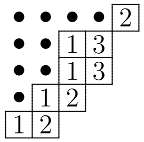
We write $|c/a| := |c| - |a|$. A skew-diagram is called a vertical *m-strip*, where $m > 0$, if it has m boxes and at most one box in each row.

For convenience we shall consider strictly row tableaux. This will be clear in the last section.

Let a and c be partitions such that $a \subseteq c$, and $(m_1, ..., m_t)$ a sequence of nonnegative integers. A *Young tableau* (strictly row) $\mathcal{T}$ of evaluation $(m_1, ..., m_t)$ and shape c/a is a sequence of partitions [14]

$$\mathcal{T} = (a^0, a^1, ..., a^t) \tag{1}$$

where $a = a^0 \subseteq a^1 \subseteq ... \subseteq a^t = c$, such that for each $k = 1, ..., t$, the skew-diagram a^k/a^{k-1} is a vertical strip labelled by k, with $m_k = |a^k/a^{k-1}|$. When $a^0 \neq 0$, $\mathcal{T}$ is often called a skew tableau. For example,



(2)

is a (skew) tableau of evaluation $(4, 3, 2)$, and shape $(5, 4, 4, 3, 2)/\ (4, 2, 2, 1)$.

The word $w(\mathcal{T})$ of the (skew) tableau $\mathcal{T}$ (1) is the word in $[t]^*$ obtained by listing the labels from right to left in each row, starting in the top and moving to bottom. The evaluation of $w(\mathcal{T})$ equals the evaluation of $\mathcal{T}$. In example (2), we have $w(\mathcal{T}) = 2\,31\,31\,21\,21$.

When $a = 0$, the Young tableau $\mathcal{T}$ is the planar representation of the word $w(\mathcal{T}) = w_1 \ldots w_k$, where w_i are rows satisfying $w_1 \rhd \ldots \rhd w_k$. Such a word is called a *tableau*, and is identified with its planar representation. Note that the shape of $\mathcal{T}$ is $(|w_1|, \ldots, |w_k|)$. For example, the planar representation of

the word 5421 521 2 is

$$\begin{array}{llll} 1 & 2 & 4 & 5 \\ 1 & 2 & 5 & \\ 2 & & & \end{array}.$$

(For simplicity, we have dropped the boxes' frame of the labelled diagram.)

Given the tableau $\mathcal{T}$, (1), let

$$\begin{pmatrix} \pi_1 & \cdots & \pi_k \\ u_1 & \cdots & u_k \end{pmatrix}, \tag{3}$$

be the biword where the bottom word is $w(\mathcal{T}) = u_1 \cdots u_k$, and the top word $\pi_1\pi_2\cdots\pi_k$ is such that $\pi_1 \leq \pi_2 \leq \cdots \leq \pi_k$ with π_j the row indices of the box in c/a labelled with u_j, $1 \leq j \leq k$. Let J_i be the set of the row indices of the boxes in c/a labelled with i, $1 \leq i \leq t$. Then, by sorting the biletters $\begin{pmatrix} \pi_j \\ u_j \end{pmatrix}$ in (3), we obtain the biword

$$\begin{pmatrix} J_1 & J_2 & \cdots & J_t \\ 1^{m_1} & 2^{m_2} & \cdots & t^{m_t} \end{pmatrix}, \tag{4}$$

where $\begin{pmatrix} J_i \\ 1^{m_i} \end{pmatrix}$ means the biword with bottom word 1^{m_i} and top word a standard word $y^i_1 \ldots y^i_{m_i}$ such that $J_i = \{y^i_1, \ldots, y^i_{m_i}\}$.

Identifying the biwords with the same billeters, a tableau determines a unique biword. For example, the biword of the tableau (2) is

$$\begin{pmatrix} 1 & 2 & 2 & 3 & 3 & 4 & 4 & 5 & 5 \\ 2 & 3 & 1 & 3 & 1 & 2 & 1 & 2 & 1 \end{pmatrix} = \begin{pmatrix} 2 & 3 & 4 & 5 & 1 & 4 & 5 & 2 & 3 \\ 1 & 1 & 1 & 1 & 2 & 2 & 2 & 3 & 3 \end{pmatrix}.$$

2.2. Shuffles and plactic congruence. Now suppose u, v are words in $[t]^*$ of lengths r and s, respectively. Take any set $P \subseteq \mathbb{N}$ with the following properties

$$P \subseteq [r+s], |P| = r. \tag{5}$$

Let $P' = [r+s] \setminus P$ the complement of P in $[r+s]$ [8, 16].

Definition 2.1. Let P satisfying (5). The word $w|(P,P')$ defined by $w|P = u$ and $w|P' = v$, is called a *shuffle of* u *and* v, denoted by $sh_P(u,v)$.

Clearly, $sh_P(u,v) = sh_{P'}(v,u)$.

For example, if $u = x_1x_2, v = y_1y_2y_3 \in [t]^*$, we may take for P any 2-element subset of [5], say $P = \{2,5\}$. Then, $sh_P(u,v)$ is the word of length

II. For each $r \in \{0,1,...,t\}$, the matrix $A_r := A_0B_1...B_r$ has invariant partition a^r.
III. For each $r \in \{1,...,t\}$, the matrix $B_1...B_r$ has invariant partition b^r.

$(\mathcal{T},\mathcal{F})$ is called an *admissible pair* of tableaux.

Conditions (I) and (II) alone are trivially feasible. But in conjunction with condition (III) they impose a non trivial argument on the concept of matrix realization. Here, we restrict ourselves to pairs $(\mathcal{T},\mathcal{H}_{\sigma(l_t,\ldots,l_1)})$. Next theorem, proved in [3, 4], shows that, without loss of generality, we may consider matrix realizations of $(\mathcal{T},\mathcal{H}_{\sigma(l_t,\ldots,l_1)})$ with a particular simple form.

Theorem 5.1. *The following conditions are equivalent.*

(a) $(\mathcal{T},\mathcal{H}_{\sigma(l_t,\ldots,l_1)})$ *is an admissible pair.*
(b) *There exists* $U \in \mathcal{U}_n$ *such that* $\Delta_aU, D_{[m_1]}, ..., D_{[m_t]}$ *realizes the pair* $(\mathcal{T},\mathcal{H}_{\sigma(l_t,\ldots,l_1)})$.

The characterization of σ-Yamanouchi words as shuffles of the rows $\mathcal{H}_\sigma$ has been used to determine necessary and sufficient conditions for the admissibility of a pair of Young tableaux $(\mathcal{T},\mathcal{H}_{\sigma(l_t,\ldots,l_1)})$, when $t \leq 3$ [3]. Next theorem extends the necessary condition of this result to any $t \geq 1$, and verifies the recursive criterion for σ-Yamanouchi words given in theorem 3.10. The proof of this theorem needs the following proposition, proved in [3].

Proposition 5.2. [3] *Let* (m_1,m_2) *be a partition. Let* w, w' *be the words of the tableaux realized by the sequences* $\Delta_aU, D_{[m_1]}, D_{[m_2]}$ *and* $\Delta_aU, D_{[m_2]}, D_{[m_1]}$, *respectively. Then, there exists an operation* θ_1 *such that* $w' = \theta_1 w$.

Theorem 5.3. *Let* $\sigma \in \mathcal{S}_t$. *The pair* $(\mathcal{T},\mathcal{H}_{\sigma(l_t,\ldots,l_1)})$ *is admissible only if* $w(\mathcal{T}) \equiv \mathcal{H}_{\sigma(l_t,\ldots,l_1)}$.

Proof: By induction on $t \geq 1$. When $t = 1$ there is nothing to prove, and the case $t = 2$ has been proved in [2, 4]. Assume the claim for $t-1 \geq 2$. Thanks to theorem 5.1 we may assume the existence of an unimodular matrix $U \in \mathcal{U}_n$ such that $\Delta_a, UD_{[m_1]}, \ldots, D_{[m_t]}$ realizes $(\mathcal{T},\mathcal{H}_{\sigma(l_t,\ldots,l_1)})$. Let $w := w(\mathcal{T})$ and (4) the biword of $\mathcal{T}$. By the inductive step, the word $w_{|[t-1]}$ of the tableau realized by the sequence

$$\Delta_a, UD_{[m_1]}, \ldots, D_{[m_{t-1}]}$$

satisfy $P(w_{|[t-1]}) = \mathcal{H}_{\sigma|[t-1]}$.

5. Matrix realizations of pairs of tableaux

Let $\mathcal{R}_p$ be a local principal ideal domain with maximal ideal (p). The matrices under consideration have entries in $\mathcal{R}_p$.

Let $\mathcal{U}_n$ be the group of $n \times n$ unimodular matrices over $\mathcal{R}_p$. Given $n \times n$ matrices A and B, we say that B is *left equivalent* to A (written $B \sim_L A$) if $B = UA$ for some unimodular matrix U; B is *right equivalent* to A (written $B \sim_R A$) if $B = AV$ for some unimodular matrix V; and B is *equivalent* to A (written $B \sim A$) if $B = UAV$ for some unimodular matrices U, V. The relations $\sim_L$, $\sim_R$ and $\sim$ are equivalence relations in the set of all $n \times n$ matrices over $\mathcal{R}_p$.

Let A be an $n \times n$ nonsingular matrix. By the Smith normal form theorem [5, 15], there exist nonnegative integers $a_1, ..., a_n$ with $a_1 \geq ... \geq a_n$ such that A is equivalent to the diagonal matrix

$$diag(p^{a_1}, ..., p^{a_n}).$$

The sequence $a = (a_1, ..., a_n)$ of exponents of the p-powers in the Smith normal form of A is a partition of length $\leq n$ uniquely determined by the matrix A. The partition a is the invariant of the equivalence class containing A and we call a the *invariant partition* of A. More generally, if we are given a sequence $(f_1, ..., f_n)$ of nonnegative integers, the following notation for p-powered diagonal matrices will be used:

$$diag_p(f_1, ..., f_n) := diag(p^{f_1}, ..., p^{f_n}).$$

Given $A \subseteq [n]$, we write $\chi^A = (f_1, ..., f_n)$ with $f_i = 1$ if $i \in A$ and 0 otherwise, thus, we put $D_A := diag_p(\chi^A)$. Given a partition a of length $\leq n$, we write $\Delta_a = diag_p(a)$. If $a = 0$, $\Delta_0 = I$.

Let $\sigma \in \mathcal{S}_t$, $t \geq 1$, and let m be a partition of length t such that $\sigma m = (m_1, \ldots, m_t)$. In what follows, $\mathcal{T}$ will denote a skew-tableau of evaluation $(m_1, ..., m_t)$ and shape c/a where the length of c is $\leq n$. Next we introduce the definition of matrix realization of a pair of Young tableaux $(\mathcal{T}, \mathcal{F})$, with $\mathcal{F}$ a tableau of evaluation $(m_1, ..., m_t)$ and shape b, following [2, 3, 4].

Definition 5.1. Let $\mathcal{T} = (a^0, a^1, ..., a^t)$ and $\mathcal{F} = (0, b^1, ..., b^t)$ be Young tableaux both of evaluation $(m_1, ..., m_t)$. We say that a sequence of $n \times n$ nonsingular matrices $A_0, B_1, ..., B_t$ is a *matrix realization* of the pair of Young tableaux $(\mathcal{T}, \mathcal{F})$ (or realizes $(\mathcal{T}, \mathcal{F})$) if:

I. For each $r \in \{1, ..., t\}$, the matrix B_r has invariant partition $(1^{m_r}, 0^{n-m_r})$.

5, having x_1, x_2 in places $2, 5$ respectively, and y_1, y_2, y_3 in the remaining places $1, 3, 4$. Thus,

$$sh_P(u, v) = y_1 x_1 y_2 y_3 x_2 = sh_{P'}(v, u).$$

If $P = \emptyset$, then $u = \lambda$ and $sh_\emptyset(u, v) = v$. The order of the letters in u is preserved in each shuffle $sh_P(u, v)$, and similarly for v. Note that distinct sets, say P and Q, may give the same shuffle $sh_P(u, v) = sh_Q(u, v)$. For example, if $u = xx$ and $v = xxx$ for some element $x \in [t]$, then all the 10 shuffles of u and v are equal to $xxxxx$ [8].

There are $\binom{r+s}{r}$ sets P which satisfy (5), and each of these gives a shuffle $sh_P(u, v)$ of u and v. For example, if $u = 1\,2$ and $v = 1\,3$ there are 6 sets P. These sets give rise to all shuffles of u and v, namely $\{1213; 1123; 1132; 1312\}$, where the words 1123 1132 have multiplicity 2.

We define $Sh(u, v) = \{sh_P(u, v) : P \subseteq [r + s], |P| = r\}$. When there is no danger of confusion, we drop the P in the notation $sh_P(u, v)$ and write $sh(u, v)$. We have the following properties:

$$\begin{aligned} Sh(u, v) &= Sh(v, u), && (6) \\ Sh(u_{|[t-1]}, v_{|[t-1]}) &= Sh(u, v) \cap [t-1]^*. && (7) \end{aligned}$$

Given q words $u_1, \ldots, u_q \in [t]^*$ of lengths $k_1, \cdots, k_q$ respectively, put $k = k_1 + \cdots + k_q$. We define the set of all words obtained by shuffling together these q words

$$Sh(u_1, \ldots, u_q) =$$
$$= \{w|(I_1, \cdots, I_q) : \cup_{j=1}^q I_j = [k],\ |I_j| = k_j,\ w|I_j = u_j,\ 1 \leq j \leq q\},$$

where the q-tuple $(I_1, \cdots, I_q)$ is pairwise disjoint. When $q = 2$, this is concordant with definition 2.1. In particular, $Sh(u_1) = \{u_1\}$.

The elements of $Sh(u_1, \ldots, u_q)$ are denoted by $sh(u_1, \ldots, u_q)$. The evaluation of a word in $Sh(u_1, \ldots, u_q)$ is the sum of the evaluations of $u_1, \ldots, u_q$ respectively.

Given a finite set $A = \{u_1, \ldots, u_q\}$, $q \geq 0$, over the alphabet $[t]$, we define $Sh(A) = Sh(u_1, \ldots, u_q)$, with $Sh(\emptyset) = \emptyset$, if $q = 0$. If C is another finite set, we have

$$Sh(A \cup C) = Sh(Sh(A), Sh(C)).$$

Knuth's congruence $\equiv$ [9] on words over the alphabet $[t]$ is the congruence generated by the so-called elementary transformations, where x, y, z are

letters and u, v are words in $[t]$:

$$uxzxv \equiv uzxxv, \quad uzzxv \equiv uzxzv, \quad x < z, \tag{8}$$

$$uxzyv \equiv uzxyv, \quad x < y < z, \tag{9}$$

$$uyzxv \equiv uyxzv, \quad x < y < z. \tag{10}$$

These relations (8),(9),(10), also called *plactic*, are the algebraic version of the plactic congruence [7, 9, 13].

C. Schensted [7, 17] has described an algorithm, known as *Schensted's insertion algorithm*, which associates to each word w a tableau $P(w)$. The elementary step, in our version, consists in the insertion of a letter x into a strictly row tableau $\mathcal{T}$, denoted $P(x.\mathcal{T})$. It takes a positive integer x and a tableau $\mathcal{T}$ and puts x in a new box at the end of the first row if possible, that is, if x is strictly larger than all the entries of the row. If not, it bumps the smallest entry of that row that is larger or equal to x. This bumped entry moves to the next row, going to the end if possible, and bumping an element to the next row, otherwise. The process continues until the bumped entry can go at the end of the next row, or until it becomes the only entry of a new row. Here is an example of the insertion of 3 in a tableau:

$$3 \to \begin{array}{cccc} 1 & 2 & \underline{4} & 5 \\ 1 & 2 & 5 & \\ 2 & & & \end{array} \qquad 4 \to \begin{array}{cccc} 1 & 2 & 3 & 5 \\ 1 & 2 & \underline{5} & \\ 2 & & & \end{array} \qquad 5 \to \begin{array}{cccc} 1 & 2 & 3 & 5 \\ 1 & 2 & 4 & \\ 2 & & & \end{array} \qquad \begin{array}{cccc} 1 & 2 & 3 & 5 \\ 1 & 2 & 4 & \\ 2 & 5 & & \end{array}.$$

For an arbitrary word $w = x_1 \cdots x_k$ in $[t]$, one defines $P(w)$ as the result of inserting x_{k-1} into the unitary tableau $x_k = P(x_k)$, then inserting x_{k-2} into the resulting tableau $P(x_{k-1}.P(x_k))$, and so on. As an example of the general case, the successive steps of the calculation of $P(434231)$ are:

$$1 \;\to\; 1\;3 \;\to\; \begin{array}{cc} 1 & 2 \\ 3 & \end{array} \;\to\; \begin{array}{ccc} 1 & 2 & 4 \\ 3 & & \end{array} \;\to\; \begin{array}{ccc} 1 & 2 & 3 \\ 3 & 4 & \end{array} \;\to\; \begin{array}{cccc} 1 & 2 & 3 & 4 \\ 3 & 4 & & \end{array}. \tag{11}$$

Two words w, w' are plactic equivalent if and only if $P(w) = P(w')$ [7, 9, 13]. For example, the word 434231 and the tableau 4321 43 in (11) are plactic equivalent. The set of all tableaux is a *section* of the plactic congruence. This means that every word can be obtained by a finite sequence of elementary Knuth transformations on a tableau.

Let $u_1, \ldots, u_k$ in $[t]^*$ be rows by decreasing order of length. If $\mathcal{T} = u_1 \ldots u_k$ is a tableau, then $\mathcal{T}$ is the unique tableau of $Sh(u_1, \ldots, u_k)$ only if the rows of the tableau $\mathcal{T}$ are pairwise comparable for the inclusion order. That is

theorem shows that it is necessary and sufficient to include the word 431421 in the set of the rows of $\mathcal{H}_{\sigma(l_4,\ldots,l_1)}$ to characterize by shuffle operations the σ-Yamanouchi words over the alphabet [4], for any $\sigma \in \mathcal{S}_4$.

Theorem 4.11. *Let $\sigma \in \mathcal{S}_4$ and $(l_4, \ldots, l_1)$ a sequence of positive integers. Then, the plactic class of $\mathcal{H}_{\sigma(l_4,\ldots,l_1)}$ is*

$$Sh((r_4)^{l_4}, (r_{\sigma,3})^{l_3}, (r_{\sigma,2})^{l_2}, (r_{\sigma,1})^{l_1}), \quad \textit{if } \sigma \in \mathcal{S}_4 \setminus \{1423, 1432, 4123, 4132\}, \tag{21}$$

and, otherwise,

$$\begin{aligned} & Sh((r_4)^{l_4}, (r_{\sigma,3})^{l_3}, (r_{\sigma,2})^{l_2}, (r_{\sigma,1})^{l_1}) \cup \\ \cup \;\; & Sh((431421)^{n_5}, (r_4)^{n_4}, (r_{\sigma,3})^{n_3}, (r_{\sigma,2})^{n_2}, (r_{\sigma,1})^{n_1}), \end{aligned} \tag{22}$$

where $\sum_{j=1}^{4} l_j |r_{\sigma,j}|_i = n_5 |431421|_i + \sum_{j=1}^{4} n_j |r_{\sigma,j}|_i$, for $i = 1, 2, 3, 4$.

Proof: When $\sigma \in \mathcal{S}_4 \setminus \{1423, 1432, 4123, 4132\}$, we have already proved in corollary 4.10 that the plactic class of $\mathcal{H}_{\sigma(l_4,\ldots,l_1)}$ if the set

$$Sh(\mathcal{H}_{\sigma(l_4,\ldots,l_1)}) = Sh((r_4)^{l_4}, (r_{\sigma,3})^{l_3}, (r_{\sigma,2})^{l_2}, (r_{\sigma,1})^{l_1}).$$

Assume now that $\sigma \in \{1423, 1432, 4123, 4132\}$. Note that $r_{\sigma,2} = 41$.

As discussed before, the only case to consider when analyzing the effect of a single Knuth transformation on a shuffle of the rows of $\mathcal{H}_{\sigma(l_t,\ldots,l_1)}$, is when the Knuth transformation $zxy \equiv xzy$ or $yzx \equiv yxz$ involves three distinct letters, $z > y > x$, with zx a factor of a row , and y a letter of another row of $\mathcal{H}_\sigma$. Thus, using lemma 4.5, we find that any word obtained by application of a single Knuth transformation on a shuffle of two rows of $\mathcal{H}_{\sigma(l_4,\ldots,l_1)}$, other than $sh(r_4, r_{\sigma,2}) = 43\,\underline{4}\,\underline{1}\,21$, is still a shuffle of the rows of $\mathcal{H}_{\sigma(l_4,\ldots,l_1)}$.

In the case of the shuffle $sh(r_4, r_{\sigma,2}) = 43\,\underline{4}\,\underline{1}\,21$, the application of the transformation $341 \equiv 314$ or $412 \equiv 142$ gives the word 431421, which is not a shuffle of the rows of $\mathcal{H}_{\sigma(l_4,\ldots,l_1)}$.

Now, an exhaustive analysis of the effect of a single Knuth's transformation on all possible shuffles between any two words from the set $\{431421, r_4, r_{\sigma,3}, r_{\sigma,2}, r_{\sigma,1}\}$, shows that the resulting word is still a shuffle of two, or more, words of this set.

Thus, if $w \equiv \mathcal{H}_{\sigma(l_4,\ldots,l_1)}$, w is obtained by a finite number of Knuth's transformations on $\mathcal{H}_{\sigma(l_4,\ldots,l_1)}$. Hence, it must be a shuffle of the words $431421, r_4, r_{\sigma,3}, r_{\sigma,2}$, and $r_{\sigma,1}$, with appropriate multiplicities. $\square$

Since the planar representation of $\sigma = 75416832 \in \mathcal{S}_8$, given in (19), has a gap of length 2 in row 3, the shuffling of the rows of r_8, $r_{\sigma,7}$, $r_{\sigma,6}$, $\ldots$, $r_{\sigma,2}$, $r_{\sigma,1}$ are not enough to give the entire plactic class of $\mathcal{H}_{\sigma(1,1,\ldots,1)}$. Nevertheless, the plactic class of $\mathcal{H}_{\sigma(1,\ldots,1,0)} = r_{\sigma,7}\ r_{\sigma,6}\ \ldots r_{\sigma,2}\ r_{\sigma,1}$ is $Sh(r_{\sigma,7},\ r_{\sigma,6},\ \ldots, r_{\sigma,2},\ r_{\sigma,1})$, and the plactic class of $\mathcal{H}_{\sigma(1,1,0,0,0,0,1,1)} = r_{\sigma,8}\ r_{\sigma,7}\ r_{\sigma,2}\ r_{\sigma,1}$ is $Sh(r_{\sigma,8},\ r_{\sigma,7},\ r_{\sigma,2},\ r_{\sigma,1})$ as well.

Consider now the permutation $\sigma = 35476182 \in \mathcal{S}_8$, whose planar representation is given below.

$$\begin{array}{c|cccccccc}
 & 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 \\
\hline
1 & \bullet & \bullet & \bullet & \bullet & \bullet & \bullet & \bullet & \bullet \\
2 & \bullet & & \bullet & \bullet & \bullet & \bullet & \bullet & \bullet \\
3 & \bullet & & \bullet & \bullet & \bullet & \bullet & \bullet & \\
4 & & & \bullet & \bullet & \bullet & \bullet & \bullet & \\
5 & & & \bullet & \bullet & \bullet & & \bullet & \\
6 & & & \bullet & \bullet & \bullet & & & \\
7 & & & \bullet & & \bullet & & & \\
8 & & & \bullet & & & & &
\end{array} \tag{20}$$

Notice that each row of the planar representation of $\sigma = 35476182$ has, at most, one gap of length 1: rows $1, 4, 6$, and 8 have no gaps, and the remaining rows have one and only one gap of length 1. Thus, by previous lemma, we find that the plactic class of $\mathcal{H}_{\sigma(1,\ldots,1)}$ is the set $Sh(\mathcal{H}_{\sigma(1,\ldots,1)})$.

As we have seen in the example above, in $\mathcal{S}_t$, $t > 3$, there are permutations that do not satisfy the conditions of theorem 4.7. For $s_3 s_2 = 1423 \in \mathcal{S}_4$ we have $\mathcal{H}_{s_3 s_2} = (4321)^{l_4}(421)^{l_3}(41)^{l_2}1^{l_1}$ and row 41 has a gap of length 2. The word $w = 431421$ is a $s_3 s_2$-Yamanouchi word and is not a shuffle of the rows of the tableau $\mathcal{H}_{s_3 s_2} = 432141$. It is easy to check that in $\mathcal{S}_4$, the only permutations that fail to satisfy the conditions of theorem 4.7 are 1423, 1432, 4123, and 4132.

Corollary 4.10. *Let $\sigma \in \mathcal{S}_4$ and (l_4, l_3, l_2, l_1) a sequence of positive integers. The plactic class of $\mathcal{H}_{\sigma(l_4,l_3,l_2,l_1)}$ is $Sh((r_4)^{l_4},\ (r_{\sigma,3})^{l_3},\ (r_{\sigma,2})^{l_2}), (r_{\sigma,1})^{l_1})$ if and only if σ is in $\mathcal{S}_4 \setminus \{1423,\ 1432,\ 4123, 4132\}$.*

In Appendix the permutations of $\mathcal{S}_5$ and $\mathcal{S}_6$ such that the set $Sh((r_t)^{l_t},$ $(r_{\sigma,t-1})^{l_{t-1}}, \ldots, (r_{\sigma,2})^{l_2},\ (r_{\sigma,1})^{l_1})$, with $l_i > 0$, $i = 1, \ldots, t$, $t = 5, 6$, is not the whole plactic class of $\mathcal{H}_\sigma$, are listed.

For $t > 3$ the rows of $\mathcal{H}_\sigma$ are not enough to characterize the σ-Yamanouchi words in terms of shuffling together those rows. In the case of $\mathcal{S}_4$, next

$\{u_k\} \subseteq \cdots \subseteq \{u_1\}$. Note, for instance, that $Sh(41, 3) = \{3\,41, 41\,3, 431\}$ has two tableaux $41\,3$ and 431.

An elementary Knuth transformation (8), (9) or (10) on a shuffle of rows, say $u_1, \cdots, u_k$, involves at least two of these rows. If an elementary Knuth transformation (9) or (10) involves three distinct letters $x < y < z$ of $sh(u_1, \cdots, u_k)$, each one belonging to a different row u_i, then the output word is still a shuffle of $u_1, \ldots, u_k$.

Proposition 2.1. *Let $u_1, \ldots, u_k$, $k \geq 3$, be rows in $[t]^*$, and x, z, u and v as in (9), (10) such that each letter x, y and z appears in a distinct row. Then*

$$uxzyv \in Sh(u_1, \ldots, u_k) \Leftrightarrow uzxyv \in Sh(u_1, \ldots, u_k);$$
$$uyzxv \in Sh(u_1, \ldots, u_k) \Leftrightarrow uyxzv \in Sh(u_1, \ldots, u_k).$$

For instance, in the alphabet [5], consider the word $5\,\overline{2}\,4\,\underline{4}\,\overline{1}\,2\,\underline{2}\,1\,\underline{1} \in Sh(5421, 421, 21)$, where the underlined letters define the word 421, the overlined letters define the word 21, and the remaining letters define the word 5421. The application of the elementary Knuth transformation $\underline{4}\,\overline{1}\,2 \equiv \overline{1}\,\underline{4}\,2$ to $5\,\overline{2}\,4\,\underline{4}\,\overline{1}\,2\,\underline{2}\,1\,\underline{1}$ gives the word $w = 5\,\overline{2}\,4\,\overline{1}\,\underline{4}\,2\,\underline{2}\,1\,\underline{1}$, which is still a shuffle of 5421, 421 and 21.

If the Knuth transformation involves only two distinct letters of $sh(u_1 \ldots, u_q)$, it is also clear that the output word is still a shuffle of $u_1 \ldots, u_q$.

Proposition 2.2. *Let $u_1 \ldots, u_k$, $k \geq 2$, be rows in $[t]^*$, and x, z, u and v as in (8). Then*

$$uzxxv \in Sh(u_1, \ldots, u_k) \Leftrightarrow uxzxv \in Sh(u_1, \ldots, u_k);$$
$$uzzxv \in Sh(u_1, \ldots, u_k) \Leftrightarrow uzxzv \in Sh(u_1, \ldots, u_k).$$

Corollary 2.3. *Let $u_1 \ldots, u_k$, $k \geq 2$, be rows in $[2]^*$, and $w \in Sh(u_1, \ldots, u_k)$ of evaluation (p, q). Then*
(a) $P(w) \in Sh(u_1, \ldots, u_k)$.
(b) $Sh(u_1, \ldots, u_k)$ is either
(i) the plactic class of $P(w) = (21)^q 1^p$ or $P(w) = (21)^p 2^q$, if the rows u_i are pairwise comparable for the inclusion order, for $i = 1, \ldots, k$; or
(ii) or the union of the plactic classes of $P(w) = (21)^r 1^s 2^v$, $r + s = p$, $r + v = q$, and $P(w) = (21)^{p+q}$, otherwise.

Let $x < y < z \in [t]$. The rows $\overline{z}\,\overline{y}$ and $\underline{y}\,\underline{x}$, in the elementary Knuth transformations $x\,\overline{z}\,\overline{y} \equiv \overline{z}\,x\,\overline{y}$ (9), and $\underline{y}\,z\,\underline{x} \equiv \underline{y}\,\underline{x}\,z$ (10), respectively,

are not broken by these transformations and, therefore, the shuffle of $\overline{z}\,\overline{y}$ and x, and $\underline{y}\,\underline{x}$ and z is preserved. The only row that is broken by these Knuth transformations is zx which is transformed into xz. Consider again the rows 41 and 3. We have $Sh(41,3)=\{3\,41, 41\,3, 431\}$ but $3\,41\equiv 31\,4\notin Sh(41,3)$ and $41\,3\equiv 143\notin Sh(3,41)$. Therefore, considering, for example, the tableau $4321\,41$, the Knuth transformation $341\equiv 314$ implies $43\underline{4}\ \underline{1}21 = sh(4321,41)\equiv 43\underline{1}\ \underline{4}21$, but $43\underline{1}\ \underline{4}21$ can not be obtained by a shuffle of the rows 4321 and 41.

Supposing that $u_1\rhd\ldots\rhd u_k$ are pairwise comparable for the inclusion order, we address the question: *Under what conditions $Sh(u_1,\ldots,u_k)$ equals the plactic class of the tableau $\mathcal{T}=u_1\ldots u_k$?*

Over a two or three letters alphabet we have equality [3]. This question will be full answered in section 4.

3. Keys and σ-Yamanouchi words

3.1. Parentheses matchings and actions of the symmetric group on words. Given a set I we let $\mathcal{S}_I$ be the set of all bijections on I, and $\mathcal{S}_t:=\mathcal{S}_{[t]}$ the symmetric group of order t.

The symmetric group $\mathcal{S}_t$, $t\geq 1$, is generated by the simple transpositions $s_i=(i\,i+1)$, $i=1,\ldots,t-1$, which satisfy the Moore-Coxeter relations:

$$(I)\ s_i^2=s_0,\quad (II)\ s_is_j=s_js_i,\ \text{ if } |i-j|\neq 1,\ \text{ and } (III)\ s_is_{i+1}s_i=s_{i+1}s_is_{i+1},$$

where s_0 denotes the identity.

Let w be a word over the alphabet $[t]$ and $i,i+1\in[t]$. An operation θ_i on w [3] consists of (a) a longest matching on $w_{|\{i,i+1\}}$ between letters $i+1$ and letters i to their right, by putting a left parenthesis on the left of each letter $i+1$, and a right parenthesis on the right of each letter i, such that the unmatched right and left parentheses indicate a subword of the form $i^s(i+1)^r$; (b) this subword will be replaced in $w_{|\{i,i+1\}}$ with $i^r(i+1)^s$.

We convention θ_0 to be the identity operator.

Let $w=31314221412$ be a word over the alphabet $[4]$. For example, inserting parentheses on the right of the letters 1 and on the left of the 2 of the word $w_{|\{1,2\}}=1122211$, we get $w_{|\{1,2\}}=1)1)(2(2(21)1)$. We may match the two left most letters 2 with the letters 1 to their right. The unmatched letters indicates the subword $w'=112=1^22^1$. Thus, $\theta_1(w_{|\{1,2\}})=\underline{1}\,\underline{2}\,2\,2\underline{2}\,1\,1$, where the underlined word is the subword w' replaced with 1^12^2. Finally, $\theta_1(w)=3\,1\,3\,2\,4\,2\,2\,4\,1\,1$.

Theorem 4.7. *Let $\sigma=a_1\ldots a_t\in\mathcal{S}_t$ such that the left reordered factor of length q, $\{a_1,\ \ldots,\ a_q\}=\cup_{j=1}^s X_j$, with X_j, $1\leq j\leq s$, pairwise disjoint nonempty subintervals of $[t]$, has $s-1$ gaps of length $\neq 0$. Let $\{q>p>\cdots>f\}\subseteq[t]$ and $\mathcal{H}_\sigma=(r_{\sigma,q})^{l_q}\ (r_{\sigma,p})^{l_p}\cdots(r_{\sigma,f})^{l_f}$, with $l_q,\ l_p,\ldots,\ l_f>0$. The plactic class of $\mathcal{H}_\sigma$ is $Sh((r_{\sigma,q})^{l_q},\ (r_{\sigma,p})^{l_p},\cdots,(r_{\sigma,f})^{l_f})$ if and only if, for $k=f,\ldots,p$, $j=1,\ldots,s$, $\{a_1,\ldots,\ a_k\}\cap X_j\neq\emptyset$ has at most a gap of length* 1.

In particular, the plactic class of $\mathcal{H}_{\sigma(1,\ldots,1)}$ is the set $Sh(r_t,r_{\sigma,t-1},\ \ldots,r_{\sigma,1})$ if and only if every left factor of σ has at most a gap of length 1.

Corollary 4.8. [3] *If $\sigma=s_0$ or the reverse permutation $t\,t-1\cdots 2\,1$, a word over the alphabet $[t]$ is a σ-Yamanouchi word if and only if it is a shuffle of the rows of $\mathcal{H}_\sigma$.*

Corollary 4.9. [3] *If $\sigma\in\mathcal{S}_t$, $t=2,3$, a word over the alphabet $[t]$ is a σ-Yamanouchi word if and only if it is a shuffle of the rows of $\mathcal{H}_\sigma$.*

Given $\sigma\in\mathcal{S}_t$, let $a_1\cdots a_t$ be the word of σ in the alphabet $[t]$. We may give a planar representation for σ by exhibiting the underlying sets of all its left factors in an array of $t+(t-1)+\cdots+1$ bullets having t rows with $t-i+1$ bullets in row i, for $1\leq i\leq t$, displayed as follows: put a bullet in the first t columns of row 1; and, for $i=2,\ldots,t$, row i is obtained by erasing in row $i-1$ the dot in column a_i.

For instance, the planar representation of $\sigma=75416832\in\mathcal{S}_8$ is

	1	2	3	4	5	6	7	8
1	$\bullet$	$\bullet$	$\bullet$	$\bullet$	$\bullet$	$\bullet$	$\bullet$	$\bullet$
2	$\bullet$		$\bullet$	$\bullet$	$\bullet$	$\bullet$	$\bullet$	$\bullet$
3	$\bullet$			$\bullet$	$\bullet$	$\bullet$	$\bullet$	$\bullet$
4	$\bullet$			$\bullet$	$\bullet$	$\bullet$	$\bullet$	
5	$\bullet$			$\bullet$	$\bullet$		$\bullet$	
6				$\bullet$	$\bullet$		$\bullet$	
7					$\bullet$		$\bullet$	
8							$\bullet$	

(19)

The planar representation of $\sigma=75416832\in\mathcal{S}_8$ (19) shows a gap of length 1 in row 2, 6 and 7, a gap of length 2 in row 3, 4, and two gaps in row 5, one of length 1 and the other one of length 2. There are no gaps in the remaining rows.

or $l = r$ and $l' = q$, where $t \geq s \geq r \geq z > y > x \geq q \geq b \geq 1$. By lemma 4.5, if we perform a Knuth transformation (i) or (ii) on $sh(u, v)$, we do still obtain a shuffle of u and v. Therefore, we find that $w \in Sh((r_{\sigma,t})^{l_t} \cdots (r_{\sigma,1})^{l_1})$.

The *only if part*. Suppose that there exists an integer $k \in \{2, \dots, t-1\}$ with $l_k > 0$ such that the row $r_{\sigma,k}$ fails (13), (14) and (15). Without loss of generality, assume that $r_{\sigma,k}$ has one of the following forms:

(i) $r_{\sigma,k} = u_2 d a u_3$, where $u_2 = s(s-1) \cdots d+1$, $u_3 = (a-1) \cdots (r+1)r$, with $s, r, d, a \in [t]$ such that $d - a = 3$;

(ii) $r_{\sigma_k} = u_2 f d u_3 c a u_4$, where $u_2 = s(s-1) \cdots (f+1)$, $u_3 = (d-1) \cdots (c+1)$, and $u_4 = (a-1) \cdots (r+1)r$, with $f - d, c - a = 2$.

In case (i), write $r_t = u_1 u_2 d c b a u_3 u_4$. Since $r_{\sigma,k}$ and r_t are rows of $\mathcal{H}_\sigma$, the word

$$sh(r_t, r_{\sigma,k}) = \overline{u}_2\, u_1\, u_2\, d\, c\, (\overline{d}\, \overline{a}\, b)\, a\, u_3\, u_4\, \overline{u}_3 \tag{16}$$

is a σ-Yamanouchi word, where the overlined letters define the word $r_{\sigma,k}$. Applying the elementary Knuth transformation $dab \equiv adb$ on (16), we obtain

$$(16) \equiv \overline{u}_2\, u_1\, u_2\, d\, c\, (\overline{a}\, \overline{d}\, b)\, a\, u_3\, u_4\, \overline{u}_3. \tag{17}$$

Clearly, the right member of (17) is a σ-Yamanouchi word, but not a shuffle of the rows r_t and $r_{\sigma,k}$.

In case (ii), write $r_t = u_1 u_2 f e d u_3 c b a u_4 u_5$, and consider the following σ-Yamanouchi word

$$sh(r_t, r_{\sigma,k}) = u_1\, u_2\, f\, e\, d\, u_3\, c\, \overline{u}_2\, \overline{f}\, \overline{d}\, \overline{u}_3 (\overline{c}\, \overline{a}\, b)\, \overline{u}_4\, a\, u_4\, u_5, \tag{18}$$

where the overlined letters define the word $r_{\sigma,k}$. Performing the elementary Knuth transformation $cab \equiv acb$ on (18), we obtain the word

$$(18) \equiv u_1\, u_2\, f\, e\, d\, u_3\, c\, \overline{u}_2\, \overline{f}\, \overline{d}\, \overline{u}_3 (\overline{a}\, \overline{c}\, b)\, \overline{u}_4\, a\, u_4\, u_5,$$

which is a σ-Yamanouchi word, but not a shuffle of the rows r_t and $r_{\sigma,k}$. $\square$

In order to state the general characterization, we need the following definition.

Definition 4.1. Let A be a nonempty subset of $\mathbb{N}$. If $j, j+k \in A$ with $k \geq 1$ are such that $[j+1, j+k-1] \cap A = \emptyset$, then we say that $[j+1, j+k-1]$ is a *gap* of A of length $k-1$. A word is said to have a *gap* if the underlying set has a gap.

A. Lascoux and M. P. Schutzenberger [10, 13] have introduced the following involutions θ_i^*, for $i = 1, \dots, t-1$, on words over the alphabet $[t]$, a special case of operations θ_i. Let w be a word over the alphabet $[t]$. To compute $\theta_i^*(w)$, first extract from w the subword v containing the letters i and $i+1$ only. Second, bracket every factor $i+1\ i$ of v. The letters which are not bracketed constitute a subword v_1 of v. Then bracket every factor $i+1\ i$ of v_1. There remains a subword v_2. Continue this procedure until it stops, giving a word v_k of type $i^r\ (i+1)^s$. Then, replace it with the word $i^s\ (i+1)^r$ and, after this, recover all the removed letters of w, including the ones different from i and $i+1$. Moreover, the operations θ_i^*, $1 \leq i \leq t-1$, satisfy the Moore-Coxeter relations [10, 13] and define an action of $\mathcal{S}_t$ over $[t]^*$.

Let $w = 31314221412$ as above. To compute $\theta_1^*(w)$, we get $v = 112(21)12$, $v_1 = 11(21)2$ and $v_2 = 112 = 1^2 2^1$. Thus,

$$\theta_1^*(w) = 3\, \underline{1}\, 3\, \underline{2}\, 4\, 2\, 2\, 1\, 4\, 1\, \underline{2},$$

where the underlined word is the subword v_2 replaced with $1^1 2^2$. To compute $\theta_2^*(w)$, we get $v = (3(32)2)2)$ and $v_1 = 2 = 2^1 3^0$. Thus,

$$\theta_2^*(w) = 3\, 1\, 3\, 1\, 4\, 2\, 2\, 1\, 4\, 1\, \underline{3},$$

where the underlined word is the subword v_1 replaced with $2^0 3^1$. Clearly, $\theta_3^*(w) = w$, since in this case $v = 3344 = 3^2 4^2$.

The operations θ_i^*, $1 \leq i \leq t-1$, are compatible with the plactic equivalence.

Proposition 3.1. [10, 13] *Let w, w' be words in $[t]^*$, and let $i \in [t]$. Then, $w \equiv w'$ if and only if $\theta_i^* w \equiv \theta_i^* w'$. In particular, $P(\theta_i^*(w)) = \theta_i^*(P(w))$.*

3.2. Keys and σ-Yamanouchi words. By definition, a *key* is a tableau such that its rows are pairwise comparable for the inclusion order [12]. For instance, 65431 641 41 is a key.

Let $(l_t, \dots, l_2, l_1)$ be a sequence of nonnegative integers. Then, $m = (l_1 + \cdots + l_t, \dots, l_{t-1} + l_t, l_t)$ is a partition and $(t^{l_t}, \dots, 2^{l_2}, 1^{l_1})$ its conjugate. For instance, $(1, 1, \dots, 1)$ defines the self-conjugate partition $(t, t-1, \dots, 1)$.

Let $\sigma \in \mathcal{S}_t$ written as a word $a_1 \cdots a_t$ in $[t]^*$. For $k = 1, \dots, t$, denote by $r_{\sigma,k}$ the row with underlying set $\{a_1, \dots, a_k\}$. In particular, when $\sigma = 12 \dots t$, we get $r_k = k \dots 21$. Clearly, $\{r_t\} \supseteq \{r_{\sigma,t-1}\} \supseteq \dots \supseteq \{r_{\sigma,1}\}$.

Definition 3.1. *Key of a permutation* [12]. To each pair consisting of a permutation $\sigma \in \mathcal{S}_t$ and a sequence of nonnegative integers $(l_t, \dots, l_1)$, Ehresmann [6] associated a key of shape $(t^{l_t} \dots, 2^{l_2}, 1^{l_1})$, here noted by $\mathcal{H}_{\sigma(l_t,\dots,l_1)}$, by taking the sequence $(r_t)^{l_t}, (r_{\sigma,t-1})^{l_{t-1}}, \dots, (r_{\sigma,1})^{l_1}$ of left reordered factors of σ. That is,

$$\mathcal{H}_{\sigma(l_t,\dots,l_1)} := (r_t)^{l_t}\,(r_{\sigma,t-1})^{l_{t-1}} \cdots (r_{\sigma,1})^{l_1},$$

is the key of σ of shape $(t^{l_t} \dots, 2^{l_2}, 1^{l_1})$. In particular,

$$\mathcal{H}_{(l_t,\dots,l_1)} = (t \cdots 21)^{l_t} \cdots (21)^{l_2}(1)^{l_1}.$$

For $i = 1, \dots, t$, the letter a_i appears only in the rows $r_t, \dots, r_{\sigma,i}$. Hence, the multiplicity of a_i in $\mathcal{H}_{\sigma(l_t,\dots,l_1)} = (r_t)^{l_t}\,(r_{\sigma,t-1})^{l_{t-1}} \dots (r_{\sigma,1})^{l_1}$ is $\sum_{k=i}^{t} l_k$, for $i = 1, \dots, t$. We put $\sigma m = (m_1, \dots, m_t)$, where $m_i = \sum_{k=\sigma^{-1}(i)}^{t} l_k$, $i = 1, \dots, t$.

Hence $\mathcal{H}_{\sigma(l_t,\dots,l_1)} := (r_t)^{l_t}\,(r_{\sigma,t-1})^{l_{t-1}} \dots (r_{\sigma,1})^{l_1}$ is also the key of σ with evaluation $(\sum_{k=\sigma^{-1}(1)}^{t} l_k, \dots, \sum_{k=\sigma^{-1}(t-1)}^{t} l_k, \sum_{k=\sigma^{-1}(t)}^{t} l_k)$; equivalently, the unique tableau of evaluation σm and shape $\sum_{i=1}^{t}(1^{m_i})$.

Note that $\mathcal{H}_{\sigma(l_j e_j)} = (r_{\sigma,j})^{l_j}$, and $\mathcal{H}_{\sigma(l_t,\dots,l_1)} = (\mathcal{H}_{\sigma(l_t e_t)})^{l_t} \dots (\mathcal{H}_{\sigma(l_1 e_1)})^{l_1}$, with $e_j = (\delta_{i,j})$, $j = 1, \dots, t$. Also notice that $\mathcal{H}_{op(l_t,\dots,l_1)}$, where op denotes the reverse permutation, is congruent with the dual of $\mathcal{H}_{(l_t,\dots,l_1)}$.

On the other hand, if $\mathcal{T} = q_t \rhd \dots \rhd q_2 \rhd q_1$ is a key, with $q_t = t \dots 2\,1$ and $|q_{t-1}| > \dots > |q_1|$, we get that row q_k is such that $\{q_k\} = \{q_t\} \setminus \{a_t, \dots, a_{k+1}\}$ with $\{a_t, \dots, a_{k+1}\} \subseteq \{1, \dots, t\}$, $1 \le k \le t-1$. Putting $\sigma := a_1 \dots a_t$ this shows $\mathcal{H}_{\sigma(1,\dots,1)} = \mathcal{T}$. Therefore, given a sequence $(l_t, \dots, l_1)$ of positive integers,

$$\sigma \longrightarrow \mathcal{H}_{\sigma(l_t,\dots,l_2,l_1)} = (r_t)^{l_t}\,(r_{\sigma,t-1})^{l_{t-1}} \dots (r_{\sigma,1})^{l_1},$$

defines an embedding of $\mathcal{S}_t$ into the set of tableaux of shape $(t^{l_t}, \dots, 2^{l_2}, 1^{l_1})$.

For example, with $\sigma = 3124 \in \mathcal{S}_4$, we have $r_4 = 4321$, $r_{s_2,3} = 321$, $r_{s_2,2} = 31$, $r_{s_2,1} = 3$, and

(*a*)

$$\mathcal{H}_{\sigma(1,1,1,1)} = \begin{array}{llll} 1 & 2 & 3 & 4 \\ 1 & 2 & 3 & \\ 1 & 3 & & \\ 3 & & & \end{array}.$$

where $w_1 = sh(u_2, u_1u_2z)$, $w_2 = sh(xu_3u_4, u_3)$. By lemma 4.4 (i), we must have $w_1 = sh_P(u_2z, u_1u_2)$, for some set P. Thus, we may write the right side of (12)

$$w_1xzyw_2 = sh(u_2z\underline{x}u_3, u_1u_2\underline{z}\,\overline{y}\,xu_3u_4) \in Sh(u,v).$$

The case of the elementary Knuth transformation $yzx \equiv yxz$ is analogous to the previous one. $\square$

As an illustration of the lemma above, consider, in the alphabet [6], the rows $u = 654321$ and $v = 542$, and let $sh(u,v) = \underline{5}\;6\;5\;4\;(\underline{4}\;\underline{2}\;3)\;2\;1$, where the underlined letters define the word v, and the remaining letters define u. Applying the Knuth transformation $423 \equiv 243$, to $sh(u,v)$, we get the word $\underline{5}\;6\;5\;\underline{4}\;(\underline{2}\;4\;3)\;2\;1$, which is also a shuffle of u and v.

Next, we identify the permutations in $\mathcal{S}_t$ for which Knuth transformations on the tableau $\mathcal{H}_{\sigma(l_t,\dots,l_1)}$ are equivalent to shuffling together the rows of $\mathcal{H}_{\sigma(l_t,\dots,l_1)}$. We start with the analysis of the case $l_t > 0$ and $l_i \ge 0$, $i = 1, \dots, t-1$.

Proposition 4.6. *Let $\sigma \in \mathcal{S}_t$ and $\mathcal{H}_\sigma = (r_{\sigma,t})^{l_t} \dots (r_{\sigma,1})^{l_1}$, with $l_t > 0$, and $l_i \ge 0$, $1 \le i \le t-1$. The plactic class of $\mathcal{H}_\sigma$ is $Sh((r_{\sigma,t})^{l_t}, \dots, (r_{\sigma,1})^{l_1})$ if and only if, for $k = 2, \dots, t-1$ with $l_k > 0$, either $r_{\sigma,k}$ is*

$$s(s+1)\cdots(z+1)(z-1)(z-2)\cdots b \tag{13}$$
$$\text{or} \qquad s(s+1)\cdots(z+1) \tag{14}$$
$$\text{or} \qquad (z-1)(z-2)\cdots b, \tag{15}$$

for some $t \ge s \ge z+1 > z-1 \ge b \ge 1$.

Proof: If $l_i = 0$ for $i = 2, \dots, t-1$, there is nothing to prove. Consider $l_i > 0$, for some $i \in \{2, \dots, t-1\}$.

The *if part.* Assume that for each $l_k > 0$, $\{r_{\sigma,k}\}$ is either (13) or (14) or (15), and let w be a σ-Yamanouchi word. By proposition 4.3, $w \equiv sh((r_{\sigma,t})^{l_t} \dots (r_{\sigma,1})^{l_1})$, and we may assume, without loss of generality, that w is obtained performing a single elementary Knuth transformation (i) $xzy \equiv zxy$, or (ii) $yzx \equiv yxz$, with $x < y < z$, on a shuffle of two rows of $\mathcal{H}_\sigma$, say u and v, such that zx is a factor of v and y is a letter of u.

Note that $\{v\} \subseteq \{u\}$, so there exist $k > k'$ such that $u = r_{\sigma,k} = u_1u_2zyxu_3u_4$, and $v = r_{\sigma,k'} = u_2zxu_3$, with $x < y < z$ consecutive letters in $[t]$, and $u_1 = s(s-1)\cdots(r+1)l$, $u_2 = (r-1)(r-2)\cdots(z-1)$, $u_3 = (x-1)(x-2)\cdots(q+1)l'$, and $u_4 = (q-1)(q-2)\cdots b$, with either $l = r$ and $l' = \lambda$, or $l = \lambda$ and $l' = q$,

Let w be a word obtained by applying an elementary Knuth transformation to $sh(u_1, \ldots, u_k)$, $k \geq 2$, where $u_1, \ldots, u_k$ are rows pairwise comparable for the inclusion order. From propositions 2.1, 2.2 and the discussion therein, we may assume w obtained by applying an elementary Knuth transformation, involving three distinct letters $x < y < z$, to a shuffle of two of these rows, say u and v, with zx a factor of v and y a letter of u. Since the letter y is in u, but not in v, and u and v are comparable for the inclusion order, we have $\{v\} \subseteq \{u\}$.

Lemma 4.4. *Let u_1, u_2 be rows in $[t]^*$, and $x, z \in [t]$ such that zu_1u_2x is a row. Then,*
(i) $sh(u_2, u_1u_2x) \in Sh(u_2x, u_1u_2)$.
(ii) $sh(u_1, zu_1u_2) \in Sh(zu_1, u_1u_2)$.

Proof: (*i*) Write $u_2 = a_1 \cdots a_r$, and $sh(u_2, u_1u_2z) = c_1 \cdots c_l$. For each $j = 1, \ldots, r$, let $p_j := min\{i : c_i = a_j\}$, and let $p' \in [l]$ such that $c_{p'} = z$. Then, it is clear that
$$sh(u_2, u_1u_2z) = sh_P(u_2z, u_1u_2),$$
where $P := \{p_1, \ldots, p_r, p'\}$.
(*ii*) Write $u_1 = a_1 \cdots a_r$, and $sh(u_1, xu_1u_2) = c_1 \cdots c_l$. For each $j = 1, \ldots, r$, let $p_j := max\{i : c_i = a_j\}$, and let $p' \in [l]$ such that $c_{p'} = x$. Then, we have
$$sh(u_1, xu_1u_2) = sh_P(xu_1, u_1u_2),$$
where $P := \{p_1, \ldots, p_r, p'\}$. □

Lemma 4.5. *Let u and v be rows in $[t]^*$ such that $u = u_1u_2zyxu_3u_4$ and $v = u_2zxu_3$, where $z > y > x$ are consecutive letters in $[t]$, and $u_1 = s(s-1)\cdots(r+1)l$, $u_2 = (r-1)(r-2)\cdots(z-1)$, $u_3 = (x-1)(x-2)\cdots(q+1)l'$, and $u_4 = (q-1)(q-2)\cdots b$, with either $l = r$ and $l' = \lambda$, or $l = \lambda$ and $l' = q$, or $l = r$ and $l' = q$, where $t \geq s \geq r \geq z > y > x \geq q \geq b \geq 1$. Then, the word obtained by a single elementary Knuth transformation on $sh(u, v)$ is still a word in $Sh(u, v)$.*

Proof: As before, the only cases to consider is the application of the elementary Knuth transformations $zxy \equiv xzy$ and $yzx \equiv yxz$ respectively to $sh(u, v) = w_1\, \underline{z}\, \underline{x}\, \overline{y}\, w_2$ and $sh(u, v) = w_1\, \overline{y}\, \underline{z}\, \underline{x}\, w_2$, where y is a letter of u, zx a factor of v, $w_1 = sh(u_2, u_1u_2z)$ and $w_2 = sh(xu_3u_4, u_3)$.
In the case of the elementary Knuth transformation $zxy \equiv xzy$, we have
$$sh(u, v) = w_1\underline{z}\;\underline{x}\;\overline{y}w_2 \equiv w_1\underline{x}\;\underline{z}\;\overline{y}w_2, \tag{12}$$

(*b*)
$$\mathcal{H}_{\sigma(1,1,2,0)} = (4321)^1(321)^1(31)^2(3)^0 = \begin{array}{llll} 1 & 2 & 3 & 4 \\ 1 & 2 & 3 & \\ 1 & 3 & & \\ 1 & 3 & & \end{array},$$
and with $s_2 = 1324$, we have $r_4 = 4321, r_{s_2,3} = 321, r_{s_2,2} = 31$, $r_{s_2,1} = 1$, and
$$\mathcal{H}_{s_2(1,1,2,0)} = (4321)^1(321)^1(31)^2(1)^0 = \begin{array}{llll} 1 & 2 & 3 & 4 \\ 1 & 2 & 3 & \\ 1 & 3 & & \\ 1 & 3 & & \end{array} = \mathcal{H}_{\sigma(1,1,2,0)}.$$

Let $I := [t] \setminus \{i\}$, with $i \in [t]$, and let $\sigma_{|I} := a_1 \ldots a_{t|I} \in \mathcal{S}_I$. If $\sigma^{-1}(i) = p$, then letter $a_p = i$ appears only in rows $r_t \ldots, r_{\sigma,p}$. Hence, when we erase letter i in row $r_{\sigma,p}$, we obtain row $r_{\sigma,p-1}$, and we have
$$\mathcal{H}_{\sigma(l_t,\ldots,l_1)_{|I}} = \mathcal{H}_{(\sigma_{|I})(l_t,\ldots,l_{p+1},l_p+l_{p-1},\ldots,l_1)}.$$
With $\sigma = s_4 = 12354$, we have $r_5 = 54321, r_{s_4,4} = 5321, r_{s_4,3} = 321$, $r_{s_4,2} = 21$, $r_{s_4,1} = 1$ and
$$\mathcal{H}_{s_4(0,1,1,2,1)} = (54321)^0(5321)^1(321)^1(21)^2(1)^1 = \begin{array}{llll} 1 & 2 & 3 & 5 \\ 1 & 2 & 3 & \\ 1 & 2 & & \\ 1 & 2 & & \\ 1 & & & \end{array};$$
and with $I = [5] \setminus \{2\}$,
$$(\mathcal{H}_{s_4(0,1,1,2,1)})_{|I} = \mathcal{H}_{(s_{4|I})(0,1,1,2+1)} = (5431)^0(531)^1(31)^1(1)^3 = \begin{array}{lll} 1 & 2 & 5 \\ 1 & 2 & \\ 1 & & \\ 1 & & \\ 1 & & \end{array}.$$

Proposition 3.2. *Let $\theta^* \in <\theta_1^* \ldots \theta_{t-1}^*>$ and $\sigma \in \mathcal{S}_t$ with same reduced decomposition. Then*
(*a*) $\theta^*(r_k) = r_{\sigma,k}$, $1 \leq k \leq t$.
(*b*) $\theta^*(\mathcal{H}_{(l_t,\ldots,l_1)}) = \mathcal{H}_{\sigma(l_t,\ldots,l_1)} = (\theta^*(r_t))^{l_t}\ (\theta^*(r_{t-1}))^{l_{t-1}} \cdots (\theta^*(r_1))^{l_1}$.

Proof: Follows from proposition 3.1. □

When there is no danger of confusion, we will drop the "$(l_t,\ldots,l_1)$" in the notation $\mathcal{H}_{\sigma(l_t,\ldots,l_1)}$, to denote a key of σ.

An operation θ_i may not act on the set $\{\mathcal{H}_\sigma : \sigma\in\mathcal{S}_t\}$. For example, consider $\mathcal{H}=4321\,21$, and $\theta_2(\mathcal{H})=\theta_2(4321\,21)=433121\notin\{\mathcal{H}_\sigma:\sigma\in\mathcal{S}_t\}$. Although $433121\equiv 4321\,31=\mathcal{H}_{s_2}$, we may have even worse $w=314321\equiv\mathcal{H}_{s_2s_1}$ and $\theta_2(w)=314221\equiv 321\rhd 14\rhd 2\notin\{\mathcal{H}_\sigma:\sigma\in\mathcal{S}_t\}$. We will comeback to this in the next section.

We define the union of keys, associated to a permutation σ, in the obvious way, $\mathcal{H}_{\sigma(l_t,\ldots,l_1)}\cup\mathcal{H}_{\sigma(l'_t,\ldots,l'_1)}:=\mathcal{H}_{\sigma(l_t+l'_t,\ldots,l_1+l'_1)}$.

A word w over the alphabet $[t]$ is said *Yamanouchi* [13] if any right factor v of w satisfies $|v|_1\geq|v|_2\geq\cdots\geq|v|_t$. This is equivalent to say that $w\in Sh((r_t)^{l_t},\cdots,(r_1)^{l_1})$, where the evaluation $(|v|_1,|v|_2,\cdots,|v|_t)=(l_1+\ldots+l_t,\ldots,l_{t-1}+l_t,l_t)$. Thus, for each $(l_t,\ldots,l_2,l_1)$, the tableau $\mathcal{H}_{(l_t,\ldots,l_2,l_1)}$ is a Yamanouchi word. Clearly, any shuffle of Yamanouchi words is still a Yamanouchi word.

Proposition 3.3. [13, Lemma 5.4.7] *The set of Yamanouchi words with evaluation m forms a single plactic class, whose representative word is the tableau $\mathcal{H}_{(l_t,\ldots,l_2,l_1)}$.*

Thus, the set of words obtained by shuffling together the rows of $\mathcal{H}_{(l_t,\ldots,l_2,l_1)}$ and the one obtained by applying a finite sequence of Knuth transformations on the tableau $\mathcal{H}_{(l_t,\ldots,l_2,l_1)}$ are the same. The characterization of Yamanouchi words given by proposition 3.3, leads to the following definition.

Definition 3.2. Let $t\geq 1$ and $\sigma\in\mathcal{S}_t$. A word w over the alphabet $[t]$ is said σ-Yamanouchi if $w\equiv\mathcal{H}_\sigma$. In particular, when σ is the identity, w is a Yamanouchi word, and when σ is the reverse permutation, w is a dual Yamanouchi word.

Since operations θ_i^* are compatible with the plactic equivalence, we may characterize σ-Yamanouchi words using operations θ_i^* [10, 13] as well.

Proposition 3.4. *Let $t\geq 1$ and $\sigma\in\mathcal{S}_t$. Let w be a word over the alphabet $[t]$. Then, w is a σ-Yamanouchi word iff $\theta_{i_r}^*\cdots\theta_{i_1}^*(w)$ is a Yamanouchi word, where $s_{i_1}\cdots s_{i_r}$ is a reduced decomposition of σ.*

Proof: We have $w\equiv\mathcal{H}_\sigma$ iff $\theta^*(w)\equiv\theta^*(\mathcal{H}_\sigma)=\mathcal{H}$, where $\theta^*=\theta_{i_r}^*\cdots\theta_{i_1}^*$. □

Proposition 3.5. [12, 13] *If B is an interval of $[t]$, then*

$$w\equiv w'\Rightarrow w_{|B}\equiv w'_{|B}.$$

alphabet $[t]$, with evaluations σm and $\sigma m'$, respectively, for some partitions m,m' with indices $(l_t,\ldots,l_1)$ and $(l'_t,\ldots,l'_1)$.

From corollary 3.6, we find that $w_{|[t-1]},w'_{|[t-1]}\equiv\mathcal{H}_{\sigma|[t-1]}$. Therefore, by the inductive step, $sh(w_{|[t-1]},w'_{|[t-1]})=sh(w,w')_{|[t-1]}\equiv\mathcal{H}_{\sigma|[t-1]}$. We consider the case, $m_{t-1}\geq m_t$ (and thus, $m'_{t-1}\geq m'_t$). The other case is similar.

The subwords $w_{|\{t-1,t\}},w'_{|\{t-1,t\}}$ are both Yamanouchi words and, thus, using step $t=2$, we find that $sh(w_{|\{t-1,t\}},w'_{|\{t-1,t\}})=sh(w,w')_{|\{t-1,t\}}$ is also a Yamanouchi word. Since w,w' are σ-Yamanouchi words, $\theta_{t-1}^*(w),\theta_{t-1}^*(w')$ must be $s_{t-1}\sigma$-Yamanouchi words. Thus, by corollary 3.6,

$$(\theta_{t-1}^*(w))_{|[t-1]},(\theta_{t-1}^*(w'))_{|[t-1]}\equiv\mathcal{H}_{s_{t-1}\sigma|[t-1]}$$

and, again by the inductive step, the word

$$sh((\theta_{t-1}^*(w))_{|[t-1]},(\theta_{t-1}^*(w'))_{|[t-1]})=sh(\theta_{t-1}^*(w),\theta_{t-1}^*(w'))_{|[t-1]}\equiv\mathcal{H}_{s_{t-1}\sigma|[t-1]}$$

Finally, note that there is a operation θ_{t-1} satisfying

$$\theta_{t-1}(sh(w,w'))=sh(\theta_{t-1}^*(w),\theta_{t-1}^*(w')),$$

and, therefore,

$$[\theta_{t-1}(sh(w,w'))]_{|[t-1]}=sh(\theta_{t-1}^*(w),\theta_{t-1}^*(w'))_{|[t-1]}\equiv\mathcal{H}_{s_{t-1}\sigma|[t-1]}.$$

By proposition 3.7 we may conclude that $sh(w,w')\equiv\mathcal{H}_{\sigma(l_t,\ldots,l_1)}\cup\mathcal{H}_{\sigma(l'_t,\ldots,l'_1)}$, that is, a σ-Yamanouchi word. □

Corollary 4.2. *Let $i\in\{1,\ldots,t-1\}$, $w=sh((r_t))^{l_t},\ldots,(r_1)^{l_1})$ and let $\theta_i(w)=sh(\,(\theta_i^*(r_t))^{l_t},\ldots,(\theta_i^*(r_1))^{l_1})$. Then $\theta_i(w)$ is a s_i-Yamanouchi word.*

By induction, we may easily extend proposition 4.1 to a shuffle of k σ-Yamanouchi words, for every $k\in\mathbb{N}$.

By proposition 3.2, the word $r_{\sigma,k}$ is a σ-Yamanouchi word for all $k\in[t]$. Thus we find that $sh((r_t)^{l_t},(r_{\sigma,t-1})^{l_{t-1}},\ldots,(r_{\sigma,1})^{l_1})$ is also a σ-Yamanouchi word, with $l_i\geq 0$. Therefore,

Theorem 4.3. $Sh((r_t)^{l_t},(r_{\sigma,t-1})^{l_{t-1}},\ldots,(r_{\sigma,1})^{l_1})\subseteq\{w\in[t]^*:w\equiv\mathcal{H}_{\sigma(l_t,\ldots,l_1)}\}$.

From this theorem, we find that shuffling together the rows of $\mathcal{H}_\sigma$ has the same effect as performing Knuth transformations on the tableau $\mathcal{H}_\sigma$. The reciprocal is not true in general. In what follows, we will determine the permutations $\sigma\in\mathcal{S}_t$ for which Knuth transformations on the tableau $\mathcal{H}_\sigma$ and shuffling together the rows of $\mathcal{H}_\sigma$ leads to the same words.

Now, since $(\theta_i w)_{|\{i+1,i+2\}}$ is either a Yamanouchi or a dual Yamanouchi word, and there is an operation θ_{i+1} such that $\theta_{i+1}(\theta_i(w))_{|[i+1]} \equiv \mathcal{H}_{s_{i+1}s_i\sigma_{|[i+1]}}$, again by the previous theorem, we find that

$$\theta_i(w)_{|[i+2]} \equiv \mathcal{H}_{s_i\sigma_{|[i+2]}}.$$

Finally, note that since $\theta_i(w)_{|\{j,j+1\}} = w_{|\{j,j+1\}}$, the word $\theta_i(w)_{|\{j,j+1\}}$ is either a Yamanouchi or a dual Yamanouchi word, for $j = i+2, \dots, t-1$. Therefore, repeating the process above for $j = i+2, \dots, t-1$, we obtain $\theta_i(w)_{|[t]} \equiv \mathcal{H}_{s_i\sigma_{|[t]}}$. □

Remark 3.1. In particular, it follows from previous theorem and the corresponding statement for dual words, that $\theta_1(w) \equiv \mathcal{H}_{s_1\sigma}$ if and only if $\theta_1(w)_{|[2,t]} \equiv \mathcal{H}_{s_1\sigma_{|[2,t]}}$.

Consider $w = 43143213321$ a σ-Yamanouchi word, where $\sigma = 3124$, and the keys $\mathcal{H}_{\sigma(2,0,1,1)} = 4321\,4321\,31\,3$ and $\mathcal{H}_{s_2\sigma(2,0,1,1)} = 4321\,4321\,21\,2$. We have $\theta_2^*(w) = 42143212321 \equiv \mathcal{H}_{s_2\sigma}$, but $\theta_2(w) = 43142213221$ is not a $s_2\sigma$-Yamanouchi word. Note that although $\theta_2(w)_{|\{3,4\}} = 4343$ is Yamanouchi, $\theta_2 w_{|[2]} = 1221221$ is not in the plactic class of $\mathcal{H}_{s_2\sigma_{|[2]}}$. The word $\theta_2' w = 42143213221$ is a $s_2\sigma$-Yamanouchi word, since $\theta_2' w_{|\{3,4\}} = 4433$ is Yamanouchi, $\theta_2' w_{|[2]} = 2121221 \equiv \mathcal{H}_{s_2\sigma_{|[2]}}$, and $\theta_3^*(\theta_2' w)_{|[3]} = \theta_2' w_{|[2]} = 213213221 \equiv \mathcal{H}_{s_3s_2\sigma_{|[3]}}$.

4. σ-Yamanouchi words as shuffles of rows of a key

It has been shown in [3] that, when $t \leq 3$, a word in $[t]^*$ of evaluation m is σ-Yamanouchi iff it is a shuffle of the rows of $\mathcal{H}_{\sigma(l_t,\dots,l_1)}$. For $t \geq 4$ this is no longer true, in general. For example, consider the Yamanouchi tableau $4321\,21$ and $421321 = sh(4321, 21)$. We have $\theta_3^*\theta_2^*(421321) = \theta_3^*(431321) = 431421 \equiv sh(4321, 41)$ but 431421 is not a shuffle of the rows of s_3s_2-Yamanouchi tableau $4321\,41$. For $t \geq 4$, we shall see that a word in $[t]^*$ of evaluation m is σ-Yamanouchi if and only if it is plactic equivalent to a shuffle of the rows of $\mathcal{H}_{\sigma(l_t,\dots,l_1)}$.

Next proposition shows that a shuffle of σ-Yamanouchi words is still a σ-Yamanouchi word, for $\sigma \in \mathcal{S}_t$.

Proposition 4.1. *Let $\sigma \in \mathcal{S}_t$. If w and w' are σ-Yamanouchi words over the alphabet $[t]$, then $sh(w, w')$ is also a σ-Yamanouchi word.*

Proof: By induction on t. When $t = 1$ there is nothing to prove, and the case $t = 2$ is trivial [3]. Let $t > 2$ and w, w' be σ-Yamanouchi words over the

Corollary 3.6. *Let w be a word over the alphabet $[t]$. Then, $P(w_{|[t-1]}) = P(w)_{|[t-1]}$.*

Proof: From the previous proposition, we have $w_{|[t-1]} \equiv P(w)_{|[t-1]}$. Thus $P(w_{|[t-1]})$ is obtained from $P(w)$ removing the letters t. □

If w is a σ-Yamanouchi word then the word $w_{|\{i,i+1\}} \equiv \mathcal{H}_{\sigma|\{i,i+1\}}$ and, thus, $w_{|\{i,i+1\}}$ is either a Yamanouchi or dual Yamanouchi word for $1 \leq i \leq t-1$. Moreover, if w has evaluation $(m_1, \dots, m_t)$, then $w = sh(u, t^{m_t})$ with $u \equiv \mathcal{H}_{\sigma(l_t,\dots,l_2,l_1)_{|[t-1]}}$ and $w_{|\{t-1,t\}}$ a Yamanouchi or dual Yamanouchi word.

The word $3121 \equiv 321 \rhd 1$ is a s_1-Yamanouchi word and 434 a dual Yamanouchi word. Nevertheless, considering the words in $\{w \in Sh(3121, 44) : w_{|\{3,4\}} = 434\}$ except $\underline{4}3\underline{4}121$, $\underline{4}31\underline{4}21 \equiv 4321 \rhd 41$, the words $\underline{4}312\underline{4}1$, $\underline{4}3121\underline{4} \equiv 4321 \rhd 1 \rhd 4$ are not σ-Yamanouchi words, whatever $\sigma \in \{4213; 2413; 2143; 2134\} \subseteq \mathcal{S}_4$.

This leads to the following question: *given $I = [t] \setminus \{i\}$, with $i \in [t]$, and $u \in I^*$ congruent with the tableau of evaluation $(m_1, \dots, m_{i-1}, m_{i+1}, \dots, m_t)$ and shape $\sum_{j=1}^{i-1}(1^{m_j}) + \sum_{j=i+1}^{t}(1^{m_j})$, under what conditions can u be a subword of a word $w \in [t]^*$ congruent with the tableau of evaluation $(m_1, \dots, m_t)$ and shape $\sum_{j=1}^{t}(1^{m_j})$?*

The answer is given by

Proposition 3.7. *Let $w \in [t]^*$ and $\sigma \in \mathcal{S}_t$. Given $i \in [t]$, let $I = [t] \setminus \{i\}$, and suppose $w_{|I} \equiv \mathcal{H}_{\sigma|I}$. Then, w is a σ-Yamanouchi word if and only if $w_{|\{j,j+1\}}$ is either a Yamanouchi or a dual Yamanouchi word, for $j = i-1, i$, and $\theta_j^*(w)_{|[j]} \equiv \mathcal{H}_{s_j\sigma_{|[j]}}$, for $j = i-1, \dots, t-1$.*

Proof: The conditions are clearly necessary. Suppose now that $w_{|\{j,j+1\}}$ is either a Yamanouchi or a dual Yamanouchi word, for $j = i-1, i$, and $\theta_j^*(w)_{|[j]} \equiv \mathcal{H}_{s_j\sigma_{|[j]}}$, for $j = i-1, \dots, t-1$.

We start with the case $i = t$, and thus, $I = [t-1]$. Consider $(m_1, \dots, m_t)$ the evaluation of w, and assume without loss of generality that $m_{t-1} \geq m_t$. Noticing that from the equality $P(w_{|[t-1]}) = P(w)_{|[t-1]} = \mathcal{H}_{\sigma|[t-1]}$, we find that the letters $t-1$ of w are inserted in the first m_{t-1} rows of $P(w)$.

Since $w_{|\{t-1,t\}} \equiv P(w)_{|\{t-1,t\}}$ is a Yamanouchi word, the m_t letters t of w are displayed in the first m_{t-1} rows of $P(w)$. On the other hand, the tableau $P(\theta_{t-1}^*(w)_{|[t-1]})$ is obtained by erasing the m_{t-1} letters t in the first m_{t-1} rows of the tableau $P(\theta_{t-1}^*(w)) = \theta_{t-1}^*(P(w))$. So if the letters t in tableau $P(w)$

are not in the first m_t rows, the letters $t-1$ are not in the first m_t rows of $P(\theta^*_{t-1}(w)_{|[t-1]}) = \mathcal{H}_{s_{t-1}\sigma|[t-1]}$. This is absurd.

Assume now that $i \neq t$. Then, $w_{|[i-1]} \equiv \mathcal{H}_{\sigma|[i-1]}$, and since $w_{|\{i-1,i\}}$ is either a Yamanouchi or a dual Yamanouchi word, and $\theta^*_{i-1}(w)_{|[i-1]} \equiv \mathcal{H}_{s_{i-1}\sigma|[i-1]}$, by the case $i=t$ we find that

$$w_{|[i]} \equiv \mathcal{H}_{\sigma|[i]}.$$

Next, note that $w_{|\{i,i+1\}}$ is either a Yamanouchi or a dual Yamanouchi word, and that $\theta^*_i(w)_{|[i]} \equiv \mathcal{H}_{s_i\sigma|[i]}$. Thus, again by the case $i=t$, we must have

$$w_{|[i+1]} \equiv \mathcal{H}_{\sigma|[i+1]}.$$

Noticing that since $w_{|I} \equiv \mathcal{H}_{\sigma|I}$, the word $w_{|\{j,j+1\}}$ is either Yamanouchi or dual Yamanouchi, for $j = i+1,\ldots,t-1$, we may repeat the process described above for $j = i+1,\ldots,t-1$, obtaining $w_{|[t]} \equiv \mathcal{H}_{\sigma|[t]}$. □

For instance, in the alphabet $[5]$, let $I := [5] \setminus \{3\}$, $\sigma = 51324 \in \mathcal{S}_5$, and consider the key $\mathcal{H}_{\sigma(1,0,2,0,1)} = 54321\,531\,531\,5$, associated with σ and $(1,0,2,0,1)$, and the word $w_{|I} := 555415211 \equiv \mathcal{H}_{\sigma(1,0,2,0,1)|I}$. Define $w = 553541352131$. Since $w_{|\{2,3\}} = 3323$ is dual Yamanouchi, $w_{|\{3,4\}} = 3433$ is Yamanouchi, $\theta^*_2 w_{|[2]} = 212121 \equiv \mathcal{H}_{s_2\sigma|[2]}$, $\theta^*_3 w_{|[3]} = 13211 \equiv \mathcal{H}_{s_3\sigma|[3]}$, and $\theta^*_4 w_{|[4]} = 44341342131 \equiv \mathcal{H}_{s_4\sigma|[4]}$, by the previous theorem the word w is σ-Yamanouchi, and has $w_{|I}$ as a subword. Consider now the word $w' = 533554135211$, which also has $w_{|I}$ has a subword. Although $w'_{|\{2,3\}} = 3332$ is dual Yamanouchi, $w'_{|\{3,4\}} = 3343$ is Yamanouchi, $\theta^*_2 w_{|[2]} = 221211 \equiv \mathcal{H}_{s_2\sigma|[2]}$, and $\theta^*_3 w_{|[3]} = 13211 \equiv \mathcal{H}_{s_3\sigma|[3]}$, w' is not a σ-Yamanouchi word since $\theta^*_4 w'_{|[4]} = 43344134211$ is not in the plactic class of $\mathcal{H}_{s_4\sigma|[4]}$.

Corollary 3.8. *Let $w = sh_P(t^r, u) \equiv \mathcal{H}_\sigma$ with $u \in [t-1]^*$. If $w' = sh_Q(t^r, u)$ with $Q \le P$, then $w' \equiv \mathcal{H}_\sigma$.*

Proof: By induction on t. If $t = 1, 2$ it is obvious. Let $t \ge 3$ and write

$$\begin{aligned}\theta^*_{t-1}(w)_{|[t-1]} &= sh_{\widetilde{P}}((t-1)^r, w_{|[t-2]}) \equiv \mathcal{H}_{s_{t-1}\sigma|[t-1]},\\ \theta^*_{t-1}(w')_{|[t-1]} &= sh_{\widetilde{Q}}((t-1)^r, w_{|[t-2]}).\end{aligned}$$

Clearly, we must have $\widetilde{Q} \le \widetilde{P}$. Then, by induction, it follows $\theta^*_{t-1}(w')_{|[t-1]} \equiv \mathcal{H}_{s_{t-1}\sigma|[t-1]}$, and by previous proposition, we find that $w' \equiv \mathcal{H}_\sigma$. □

The criterion given by previous proposition can be generalized to operations θ_i.

Lemma 3.9. *Let $w \in [t]^*$ and $w_{|\{t-1,t\}}$ a Yamanouchi or dual Yamanouchi word. Let $u = \theta_{t-1}(w)$, $z = \theta^*_{t-1}(w)$ and $v = w_{|[t-2]}$. Then*

$$u_{|[t-1]} = sh_P(u_{|\{t-1\}}, v) \quad and \quad z_{|[t-1]} = sh_Q(z_{|\{t-1\}}, v), \quad with\ Q \le P.$$

Proof: It is enough to consider the cases $w_{|\{t-1,t\}} = t\,t-1\,t-1$ and $w_{|\{t-1,t\}} = t\,t\,t-1$. Therefore, if $\theta_{t-1} \neq \theta^*_{t-1}$, we have

$$\begin{aligned}\theta_{t-1}(t\,t-1\,t-1) &= t\,t\,\underline{t-1},\\ \theta^*_{t-1}(t\,t-1\,t-1) &= t\,\underline{t-1}\,t,\end{aligned}$$

and

$$\begin{aligned}\theta_{t-1}(t\,t\,t-1) &= t\,\underline{t-1}\,\underline{t-1},\\ \theta^*_{t-1}(t\,t\,t-1) &= \underline{t-1}\,t\,\underline{t-1}.\end{aligned}$$

□

Theorem 3.10. *Let $w \in [t]^*$ and $\sigma \in \mathcal{S}_t$. Given $i \in [t]$, let $I = [t] \setminus \{i\}$, and suppose $w_{|I} \equiv \mathcal{H}_{\sigma|I}$. Then, w is a σ-Yamanouchi word if and only if $w_{|\{j,j+1\}}$ is either a Yamanouchi or a dual Yamanouchi word, for $j = i-1, i$, and $\theta_j(w)_{|[j]} \equiv \mathcal{H}_{s_j\sigma|[j]}$, for some operation θ_j, $j = i-1,\ldots,t-1$.*

Proof: Attending to previous lemma and corollary, if $\theta_j(w)_{|[j]} \equiv \mathcal{H}_{s_j\sigma|[j]}$, then $\theta^*_j(w)_{|[j]} \equiv \mathcal{H}_{s_j\sigma|[j]}$, for $j = i-1,\ldots,t-1$. By previous proposition, it follows $w \equiv \mathcal{H}_\sigma$. □

Corollary 3.11. *Let $w \in [t]^*$, $t \ge 2$, and $\sigma \in \mathcal{S}_t$. If $w \equiv \mathcal{H}_\sigma$, then $\theta_i(w) \equiv \mathcal{H}_{s_i\sigma}$ if and only if $(\theta_i w)_{|\{i+1,i+2\}}$ is either a Yamanouchi or a dual Yamanouchi word, $\theta_i(w)_{|[i]} \equiv \mathcal{H}_{s_i\sigma|[i]}$, and $\theta_j(\theta_i(w))_{|[j]} \equiv \mathcal{H}_{s_js_i\sigma|[j]}$, for some operation θ_j, $j = i+1,\ldots,t-1$.*

Proof: Taking $\theta_j = \theta^*_j$, $j = i+1,\ldots,t-1$, we find that the conditions are clearly necessary. Assume now that $w_{|\{i+1,i+2\}}$ is either a Yamanouchi or a dual Yamanouchi word, $\theta_i(w)_{|[i]} \equiv \mathcal{H}_{s_i\sigma|[i]}$, and $\theta_j(\theta_i(w))_{|[j]} \equiv \mathcal{H}_{s_js_i\sigma|[j]}$, for $j = i+1,\ldots,t-1$.

Since $\theta_i(w)_{|[i]} \equiv \mathcal{H}_{s_i\sigma|[i]}$, $\theta_i w_{|\{i,i+1\}}$ is either a Yamanouchi or a dual Yamanouchi word, and $\theta_i(\theta_i(w)_{|[i]}) = w_{|[i]} \equiv \mathcal{H}_{\sigma|[i]}$, by the previous theorem we must have

$$\theta_i(w)_{|[i+1]} \equiv \mathcal{H}_{s_i\sigma|[i+1]}.$$