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THE N-MEMBRANES PROBLEM FOR QUASILINEAR DEGENERATE SYSTEMS

ASSIS AZEVEDO, JOSÉ-FRANCISCO RODRIGUES AND LISA SANTOS

ABSTRACT: We study the regularity of the solution of the variational inequality for the problem of N-membranes in equilibrium with a degenerate operator of p-Laplacian type, $1 < p < \infty$, for which we obtain the corresponding Lewy-Stampacchia inequalities. By considering the problem as a system coupled through the characteristic functions of the sets where at least two membranes are in contact, we analyze the stability of the coincidence sets.

1. Introduction

In an open bounded subset Ω of $\mathbb{R}^d$, $d \geq 1$, we consider the quasi-linear operator

$$Av = -\nabla \cdot a(x, \nabla v) \qquad \text{in } \mathscr{D}'(\Omega),$$

where $a : \Omega \times \mathbb{R}^d \longrightarrow \mathbb{R}^d$ is a Carathéodory function, and the following variational inequality for the N-membranes problem

$$(u_1, \dots, u_N) \in \mathbb{K}_N : \sum_{i=1}^{N} \int_\Omega a(x, \nabla u_i) \cdot \nabla(v_i - u_i) \;\geq\; \sum_{i=1}^{N} \int_\Omega f_i(v_i - u_i), \quad \forall\, (v_1, \dots, v_N) \in \mathbb{K}_N. \tag{1}$$

Here we shall consider the convex subset $\mathbb{K}_N$ of the Sobolev space $\left[W^{1,p}(\Omega)\right]^N$, $1 < p < \infty$, defined by

$$\mathbb{K}_N = \Big\{(v_1, \dots, v_N) \in \left[W^{1,p}(\Omega)\right]^N : v_1 \geq \dots \geq v_N, \text{ a. e. in } \Omega, \quad v_i - \varphi_i \in W_0^{1,p}(\Omega),\, i = 1, \dots, N\Big\}, \tag{2}$$

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where we give $\varphi_1, \dots, \varphi_N \in W^{1,p}(\Omega)$, such that $\mathbb{K}_N \neq \emptyset$. For instance, if $\partial\Omega \in C^{0,1}$ is a Lipschitz boundary, it suffices to assume, in the trace sense, that

$$\varphi_1 \geq \cdots \geq \varphi_N \qquad \text{on } \partial\Omega.$$

In (1) we shall assume the N forces given by the functions

$$f_1, \dots, f_N \in L^q(\Omega) \subset W^{-1,p'}(\Omega) \tag{3}$$

where $W^{-1,p'}(\Omega)$ denotes the dual space of $W_0^{1,p}(\Omega)$, so that $p' = \frac{p}{p-1}$ is the conjugate exponent of p and, by Sobolev imbeddings, $q = 1$ if $p > d$, $q > 1$ if $p = d$ or $q = \frac{dp}{dp+p-d}$ if $1 < p < d$. Under the following assumptions for a.e. $x \in \Omega$ and ξ, $\eta \in \mathbb{R}^d$:

$$a(x,\xi) \cdot \xi \geq \alpha\, |\xi|^p, \qquad 1 < p < \infty, \tag{4}$$

$$|a(x,\xi)| \leq \beta\, |\xi|^{p-1}, \tag{5}$$

$$\Big[a(x,\xi) - a(x,\eta)\Big] \cdot (\xi - \eta) > 0, \qquad \text{if } \xi \neq \eta, \tag{6}$$

for given constants α, $\beta > 0$, the general theory of variational inequalities for strictly monotone operators (see [14], [9]) immediately yields the existence and uniqueness of solution to the N-membranes problem (1). If we choose as a model for the N-membranes in equilibrium, each one under the action of the forces f_i and attached to rigid supports at height φ_i, the minimization functional

$$E(u_1, \dots, u_N) = \sum_{i=1}^{N} \int_\Omega \left[\frac{1}{p}|\nabla u_i|^p - f_i u_i\right]$$

in the convex set of admissible displacements given by (2), we obtain the variational inequality (1) associated with the p-Laplacian

$$Av = -\Delta_p v = -\nabla \cdot \left(|\nabla v|^{p-2}\nabla v\right), \qquad 1 < p < \infty.$$

The N-membranes problem was considered in [4] for linear elliptic operators, where for differentiable coefficients the regularity of the solution in Sobolev spaces $W^{2,p}(\Omega)$ was shown for $p \geq 2$ (hence also in $C^{1,\lambda}(\Omega)$ for $0 < \lambda = 1 - \frac{d}{p} < 1$) extending earlier results of [23] for the two membranes problem. Noting the analogy (and relation) with the one obstacle problem, it was observed in those problems that the C^2-regularity of the solution cannot be expected in general, even with very smooth data. Considering the analogy of the two and three membranes problem, respectively with the one and the two obstacle problems, in [1] we have shown the Lewy-Stampacchia type inequalities

$$\bigwedge_{j=1}^{i} f_j \leq Au_i \leq \bigvee_{j=i}^{N} f_j \qquad \text{a.e. in } \Omega,\ i = 1, \dots, N, \tag{7}$$

for general second order linear elliptic operators with measurable coefficients and, in cases $N = 2$ and $N = 3$ we have established sufficient conditions on the external forces for the stability of the coincidence sets

$$\{x \in \Omega : u_j(x) = u_{j+1}(x)\}, \qquad j = 1, \dots, N-1, \tag{8}$$

where two consecutive membranes touch each other. In (7) we use the notation

$$\bigvee_{i=1}^{k} \xi_i = \xi_1 \vee \ldots \vee \xi_k = \sup\{\xi_1, \dots, \xi_k\}$$

and

$$\bigwedge_{i=1}^{k} \xi_i = \xi_1 \wedge \ldots \wedge \xi_k = \inf\{\xi_1, \dots, \xi_k\}$$

and we also denote $\xi^+ = \xi \vee 0$ and $\xi^- = -(\xi \wedge 0)$. In order to prove (7) we shall approximate, in Section 2, the solution $(u_1, \dots, u_N)$ of (1) by solutions $(u_1^\varepsilon, \dots, u_N^\varepsilon)$ of a suitable system of Dirichlet problems for the operator A associated to a particular new monotone perturbation that extends the bounded penalization, as $\varepsilon \to 0$, of obstacle problems (see [9] or [19] and their references). Under the further assumptions on the strong monotonicity of the vector field $a(x, \xi)$ with respect to ξ, i.e., if for some $\alpha > 0$,

$$\Big[a(x,\xi) - a(x,\eta)\Big] \cdot (\xi - \eta) \geq \begin{cases} \alpha|\xi - \eta|^p & \text{if } p \geq 2, \\ \\ \alpha\left(|\xi| + |\eta|\right)^{p-2}|\xi - \eta|^2 & \text{if } 1 < p < 2, \end{cases} \tag{9}$$

we are able to establish that the error of the approximating solutions in the $W^{1,p}(\Omega)$-norm is of order $\varepsilon^{1/p}$, if $p > 2$, and of the order $\varepsilon^{1/2}$, if $1 < p \leq 2$, with a constant that depends only on $\alpha > 0$ and on the L^q-norms of $f_1, \dots, f_N$. This type of estimate that appears in [20] for the obstacle problem in case $p \geq 2$ seems new for $1 < p < 2$. The inequalities (7) are a consequence of the fact that each Au_i is a L^q function and we can regard u_1 and u_N as solutions of one obstacle problems and all the other u_i, $1 < i < N$, as solutions of two obstacles problems, to which we can apply the well-known

Lewy-Stampacchia inequalities (see, for instance [19], [22], [20] or [17] and their references). Another important consequence of these properties is the reduction of the regularity of the solution of the N-membranes problem to the regularity of each equation

$$Au_i = h_i \qquad \text{a.e. in } \Omega,\ i = 1, \dots, N. \tag{10}$$

Therefore, in Section 3, we conclude from the well-known properties for weak solutions of quasilinear elliptic equations (see [11] and [16]) that the solutions u_i are in fact Hölder continuous, provided $q > \frac{d}{p}$ in (3), or have Hölder continuous gradient (see [5]) if $q > \frac{dp}{p-1}$ and the operator A has the stronger structural properties, for a.e. $x \in \Omega$,

$$\sum_{i,j=1}^{d} \frac{\partial a_i}{\partial \eta_j}(x,\eta)\,\xi_i\xi_j \geq \alpha_0 |\eta|^{p-2}|\xi|^2 \tag{11}$$

$$\left|\frac{\partial a_i}{\partial \eta_j}(x,\eta)\right| \leq \alpha_1 |\eta|^{p-2} \qquad \text{and} \qquad \left|\frac{\partial a_i}{\partial x_j}(x,\eta)\right| \leq \alpha_1 |\eta|^{p-1} \tag{12}$$

for some positive constants α_0, α_1 and all $\eta \in \mathbb{R}^d \setminus \{0\}$, $\xi \in \mathbb{R}^d$ and all $i, j = 1, \dots, d$. We may even conclude that each

$$u_i \in C^{0,\lambda}(\overline{\Omega}) \qquad \text{or} \qquad u_i \in C^{1,\lambda}(\overline{\Omega}), \qquad i = 1, \dots, N,$$

provided the Dirichlet data φ_i and $\partial\Omega$ have the corresponding required regularity (see Section 3). Finally in Section 4 we study the stability of the coincidence sets (8) in terms of the convergence of their characteristic functions. For this purpose, we define, for a.e. $x \in \Omega$ and for $1 \leq j < k \leq N$, the following $\frac{N(N-1)}{2}$ coincidence sets

$$I_{j,k} = \{x \in \Omega : u_j(x) = \dots = u_k(x)\} \tag{13}$$

and notice that the coincidence sets defined in (8) are simply $I_{j,j+1}$. Besides that, $I_{j,k} = I_{j,j+1} \cap \ldots \cap I_{k-1,k}$. Set

$$\chi_{j,k}(x) = \chi_{I_{j,k}}(x) = \begin{cases} 1 & \text{if } u_j(x) = \dots = u_k(x) \\ \\ 0 & \text{otherwise .} \end{cases} \tag{14}$$

In [1] we have shown that the solution (u_1, u_2, u_3) of (1) for $N = 3$ with a linear operator in fact satisfies a.e. in Ω

$$\begin{cases} Au_1 = f_1 + \frac{1}{2}(f_2 - f_1)\chi_{1,2} + \frac{1}{6}(2f_3 - f_2 - f_1)\chi_{1,3} \\ Au_2 = f_2 - \frac{1}{2}(f_2 - f_1)\chi_{1,2} + \frac{1}{2}(f_3 - f_2)\chi_{2,3} + \frac{1}{6}(2f_2 - f_1 - f_3)\chi_{1,3} \\ Au_3 = f_3 - \frac{1}{2}(f_3 - f_2)\chi_{2,3} + \frac{1}{6}(2f_1 - f_2 - f_3)\chi_{1,3} \end{cases} \tag{15}$$

which extends the remark of [24] for the case $N = 2$ that corresponds to the first two equations of (15) with $\chi_{2,3} \equiv 0$ (and consequently also $\chi_{1,3} \equiv 0$). As

$$f_1 \neq f_2 \qquad \text{a.e. in } \Omega$$

is a sufficient condition for the convergence of the unique coincidence set $I_{1,2}$ in case $N = 2$, additionally

$$f_2 \neq f_3, \qquad f_1 \neq \frac{f_2 + f_3}{2}, \qquad f_3 \neq \frac{f_1 + f_2}{2} \qquad \text{a.e. in } \Omega$$

in case $N = 3$ are sufficient conditions for the convergence of the three coincidence sets $I_{1,2}$, $I_{2,3}$ and $I_{1,3}$, with respect to the perturbation of the forces f_1, f_2, f_3 (see [1] for a direct proof). In section 4 we extend to arbitrary N the system (15) by showing that, for given forces $(f_1, \ldots, f_N)$ the solution $(u_1, \ldots, u_N)$ of (1) solves a system of the form

$$Au_i = f_i + \sum_{1 \leq j < k \leq N,\ j \leq i \leq k} b_i^{j,k}[f]\,\chi_{j,k} \qquad \text{a.e. in } \Omega, \qquad i = 1, \ldots, N, \tag{16}$$

where, in (16), each $b_i^{j,k}[f]$ represents a certain linear combination of the forces. We denote the average of $f_j, \ldots, f_k$ by

$$\langle f \rangle_{j,k} = \frac{f_j + \cdots + f_k}{k - j + 1}, \qquad 1 \leq j \leq k \leq N, \tag{17}$$

and we shall establish the following assumption on the averages of the forces

$$\langle f \rangle_{i,j} \neq \langle f \rangle_{j+1,k} \ \text{ a.e. in } \Omega, \ \text{ for all } i, j, k \in \{1, \ldots, N\} \text{ with } i \leq j < k, \tag{18}$$

as a sufficient condition for the stability of the coincidence sets $I_{j,k}$ in the N-membranes problem.

2. Approximation by bounded penalization

In this section we approximate the variational inequality using a bounded penalization. Defining

$$\begin{aligned} &\xi_0 = \max\left\{\tfrac{f_1+\cdots+f_i}{i} : i=1,\ldots,N\right\} \\ &\xi_i = i\,\xi_0 - (f_1+\cdots+f_i), \qquad \text{for } i=1,\ldots,N, \end{aligned} \tag{19}$$

we observe that

$$\begin{cases} \xi_i \geq 0 & \text{if } i \geq 1 \\ (\xi_{i-1}-\xi_{i-2}) - (\xi_i - \xi_{i-1}) = f_i - f_{i-1} & \text{if } i \geq 2. \end{cases} \tag{20}$$

For $\varepsilon > 0$, let θ_ε be defined as follows:

$$\begin{aligned} \theta_\varepsilon :\ \overline{\mathbb{R}} &\longrightarrow \overline{\mathbb{R}} \\ s &\mapsto \begin{cases} 0 & \text{if } s \geq 0 \\ \frac{s}{\varepsilon} & \text{if } -\varepsilon < s < 0 \\ -1 & \text{if } s \leq -\varepsilon. \end{cases} \end{aligned} \tag{21}$$

The approximated problem is given by the system

$$\begin{cases} Au_i^\varepsilon + \xi_i\theta_\varepsilon(u_i^\varepsilon - u_{i+1}^\varepsilon) - \xi_{i-1}\theta_\varepsilon(u_{i-1}^\varepsilon - u_i^\varepsilon) \ = \ f_i & \text{in } \Omega, \\ {u_i^\varepsilon}_{|\partial\Omega} = \varphi_i, & i=1,\cdots,N, \end{cases} \tag{22}$$

with the convention $u_0^\varepsilon = +\infty$, $u_{N+1}^\varepsilon = -\infty$.

Proposition 2.1. *If the operator A satisfies the assumptions* (4), (5) *and* (6), *problem* (22) *has a unique solution* $(u_1^\varepsilon,\ldots,u_N^\varepsilon) \in \left[W^{1,p}(\Omega)\right]^N$. *This solution satisfies*

$$u_i^\varepsilon \leq u_{i-1}^\varepsilon + \varepsilon, \qquad \textit{for } i=2,\ldots,N. \tag{23}$$

Proof: Existence and uniqueness of solution of the problem (22) is an immediate consequence of the theory of strictly monotone and coercive operators (see [14]). In fact, summing the N equations of the system, each one multiplied by a test function w_i, problem (22) implies that

$$\sum_{i=1}^N \int_\Omega \langle Au_i^\varepsilon, w_i\rangle + \langle Bv, w\rangle = \sum_{i=1}^N \int_\Omega f_i w_i, \quad \forall\, w=(w_1,\ldots,w_N) \in \left[W^{1,p}(\Omega)\right]^N,$$

where

$$\langle Bv, w\rangle = \sum_{i=1}^{N} \int_\Omega \Big(\xi_i\theta_\varepsilon(v_i - v_{i+1}) - \xi_{i-1}\theta_\varepsilon(v_{i-1} - v_i)\Big)\, w_i$$

with the same convention $v_0^\varepsilon = +\infty$, $v_{N+1}^\varepsilon = -\infty$, satisfies

$$\langle Bv - Bw, v - w\rangle =$$

$$\sum_{i=1}^{N-1} \int_\Omega \xi_i \Big(\theta_\varepsilon(v_i - v_{i+1}) - \theta_\varepsilon(w_i - w_{i+1})\Big)\,\Big((v_i - v_{i+1}) - (w_i - w_{i+1})\Big) \geq 0,$$

since $\xi_i \geq 0$, for $i = 1, \ldots, N$ and θ_ε is monotone nondecreasing.

To prove (23), multiplying the $i-$th equation of (22) by $(u_i^\varepsilon - u_{i-1}^\varepsilon - \varepsilon)^+$ and integrating on Ω, noticing that $(u_i^\varepsilon - u_{i-1}^\varepsilon - \varepsilon)^+_{|\partial\Omega} = 0$ we obtain,

$$\begin{aligned}
&\int_\Omega A\, u_i^\varepsilon (u_i^\varepsilon - u_{i-1}^\varepsilon - \varepsilon)^+ = \\
&\qquad\int_\Omega [f_i - \xi_i\theta_\varepsilon(u_i^\varepsilon - u_{i+1}^\varepsilon) + \xi_{i-1}\theta_\varepsilon(u_{i-1}^\varepsilon - u_i^\varepsilon)]\,(u_i^\varepsilon - u_{i-1}^\varepsilon - \varepsilon)^+ \\
&\qquad= \int_\Omega [f_i - \xi_i\theta_\varepsilon(u_i^\varepsilon - u_{i+1}^\varepsilon) - \xi_{i-1}]\,(u_i^\varepsilon - u_{i-1}^\varepsilon - \varepsilon)^+
\end{aligned}$$

since $\theta_\varepsilon(u_{i-1}^\varepsilon - u_i^\varepsilon)(u_i^\varepsilon - u_{i-1}^\varepsilon - \varepsilon)^+ = -(u_i^\varepsilon - u_{i-1}^\varepsilon - \varepsilon)^+$. In particular, because $\theta_\varepsilon(u_i^\varepsilon - u_{i+1}^\varepsilon) \geq -1$, we have

$$\int_\Omega A\, u_i^\varepsilon (u_i^\varepsilon - u_{i-1}^\varepsilon - \varepsilon)^+ \;\leq\; \int_\Omega [f_i + \xi_i - \xi_{i-1}]\,(u_i^\varepsilon - u_{i-1}^\varepsilon - \varepsilon)^+. \tag{24}$$

With similar arguments, if we multiply, for $i \geq 2$, the $(\,i-1)-$th equation of (22) by $(u_i^\varepsilon - u_{i-1}^\varepsilon - \varepsilon)^+$ and integrate on Ω we obtain,

$$\int_\Omega A\, u_{i-1}^\varepsilon (u_i^\varepsilon - u_{i-1}^\varepsilon - \varepsilon)^+ \geq \int_\Omega [f_{i-1} + \xi_{i-1} - \xi_{i-2}]\,(u_i^\varepsilon - u_{i-1}^\varepsilon - \varepsilon)^+. \tag{25}$$

From inequalities (24) and (25) we have, using (20),

$$\begin{aligned}
&\int_\Omega \Big(a(x, \nabla u_i^\varepsilon) - a(x, \nabla u_{i-1}^\varepsilon)\Big)\cdot \nabla(u_i^\varepsilon - u_{i-1}^\varepsilon - \varepsilon)^+ = \\
&\qquad\int_\Omega (A\, u_i^\varepsilon - A\, u_{i-1}^\varepsilon)\,(u_i^\varepsilon - u_{i-1}^\varepsilon - \varepsilon)^+ \\
&\qquad\leq \int_\Omega \Big[f_i - f_{i-1} + \big(\xi_i - \xi_{i-1}\big) - \big(\xi_{i-1} - \xi_{i-2}\big)\Big]\,(u_i^\varepsilon - u_{i-1}^\varepsilon - \varepsilon)^+ = 0.
\end{aligned}$$

From the strict monotony (6) of a, it follows that $u_i^\varepsilon \leq u_{i-1}^\varepsilon + \varepsilon$ a.e. in Ω. ∎

Proposition 2.2. *If* $(u_1^\varepsilon, \dots, u_N^\varepsilon)$ *and* $(u_1, \dots, u_N)$ *are respectively the solution of the problem* (22) *and the solution of the problem* (1) *then*

$$(u_1^\varepsilon, \dots, u_N^\varepsilon) \rightharpoonup (u_1, \dots, u_N) \quad \text{in} \quad \left[W^{1,p}(\Omega)\right]^N\text{-weak}, \quad \text{when } \varepsilon \to 0.$$

Proof: Multiplying the i-th equation of (1) by $v_i - u_i^\varepsilon$, where $(v_1, \dots, v_N) \in \mathbb{K}$ and $u^\varepsilon = (u_1^\varepsilon, \dots, u_N^\varepsilon)$, integrating over Ω and summing, we obtain

$$\sum_{i=1}^N \int_\Omega a(x, u_i^\varepsilon) \cdot \nabla(v_i - u_i^\varepsilon) + \langle Bu^\varepsilon, v - u^\varepsilon \rangle = \sum_{i=1}^N \int_\Omega f_i(v_i - u_i^\varepsilon).$$

Noticing that $\langle Bv, v - u^\varepsilon \rangle = 0$ and due to the monotonicity of the operator B proved in (24),

$$\sum_{i=1}^N \int_\Omega a(x, \nabla u_i^\varepsilon) \cdot \nabla(v_i - u_i^\varepsilon) \;\geq\; \sum_{i=1}^N \int_\Omega f_i(v_i - u_i^\varepsilon) \tag{26}$$

and using (6) we conclude that

$$\sum_{i=1}^N \int_\Omega a(x, \nabla v_i) \cdot \nabla(v_i - u_i^\varepsilon) \;\geq\; \sum_{i=1}^N \int_\Omega f_i(v_i - u_i^\varepsilon). \tag{27}$$

From (4) and (5) we easily deduce the uniform boundedeness of $\{(u_1^\varepsilon, \dots, u_N^\varepsilon)\}_\varepsilon$ in $\left[W^{1,p}(\Omega)\right]^N$. So, there exists $(u_1^*, \dots, u_N^*) \in \left[W^{1,p}(\Omega)\right]^N$ such that

$$(u_1^\varepsilon, \dots, u_N^\varepsilon) \rightharpoonup (u_1^*, \dots, u_N^*) \quad \text{in} \quad \left[W^{1,p}(\Omega)\right]^N\text{-weak}, \quad \text{when } \varepsilon \to 0$$

and, letting $\varepsilon \to 0$ in (27) we obtain

$$\sum_{i=1}^N \int_\Omega a(x, \nabla v_i) \cdot \nabla(v_i - u_i^*) \;\geq\; \sum_{i=1}^N \int_\Omega f_i(v_i - u_i^*) \qquad \forall\, (v_1, \dots, v_N) \in \mathbb{K}.$$

Besides that, using (23), $u_1^* \geq \cdots \geq u_n^*$. Since we also have $u^*_{i\,|\partial\Omega} = \varphi_i$, for $i = 1, \dots, N$, then $(u_1^*, \dots, u_N^*) \in \mathbb{K}$. The hemicontinuity of the operator A allows us to conclude that $(u_1^*, \dots, u_N^*)$ actually solves the variational inequality (1) and the uniqueness of solution of the variational inequality implies that $u_i^* = u_i$, $i = 1, \dots, N$. ∎

We present now two lemmas that will be used to prove the next theorem. The first lemma states that, under certain circunstancies, weak convergence implies strong convergence. The second lemma is a reverse type Hölder inequality.

Lemma 2.3. ([3], p. 190) *Under the assumptions* (4), (5) *and* (6), *when* $\varepsilon \to 0$, *if*

$$u^\varepsilon - u \longrightarrow 0 \qquad \text{in } W_0^{1,p}(\Omega) \tag{28}$$

and

$$\int_\Omega [a(x, \nabla u^\varepsilon) - a(x, \nabla u)] \cdot \nabla(u^\varepsilon - u) \longrightarrow 0 \tag{29}$$

then

$$u^\varepsilon - u \longrightarrow 0 \qquad \text{in } W_0^{1,p}(\Omega)\text{-strong.}$$

Lemma 2.4. ([21], p. 8) *Let* $0 < r < 1$ *and* $r' = \frac{r}{r-1}$. *If* $F \in L^r(\Omega)$, $FG \in L^1(\Omega)$ *and* $\int_\Omega |G(x)|^{r'} dx < \infty$ *in a bounded domain* Ω *of* $\mathbb{R}^d$, *then one has*

$$\left(\int_\Omega |F(x)|^r dx\right)^{\frac{1}{r}} \leq \left(\int_\Omega |F(x)G(x)| dx\right)\left(\int_\Omega |G(x)|^{r'} dx\right)^{-\frac{1}{r'}}. \tag{30}$$

Theorem 2.5. *Let* $(u_1^\varepsilon, \ldots, u_N^\varepsilon)$ *and* $(u_1, \ldots, u_N)$ *denote, respectively, the solutions of the problems* (22) *and* (1). *Under the assumptions* (4), (5) *and* (6),

i) $(u_1^\varepsilon, \ldots, u_N^\varepsilon) \underset{\varepsilon \to 0}{\longrightarrow} (u_1, \ldots, u_N)$ *in* $\left[W^{1,p}(\Omega)\right]^N$.

ii) If, in addition, a is strongly monotone, i.e., satisfies (9), *then there exists a positive constant* C, *independent of* ε, *such that, for all* $i = 1, \ldots, N$,

$$\|\nabla(u_i^\varepsilon - u_i)\|_{L^p(\Omega)} \leq \begin{cases} C\varepsilon^{\frac{1}{p}} & \text{if } p \geq 2, \\ C\varepsilon^{\frac{1}{2}} & \text{if } 1 < p \leq 2. \end{cases}$$

Proof: *i)* Choose, for $i = 1, \ldots, N$, $v_i = \bigvee_{k=i}^N u_k^\varepsilon$ in (1). Indeed, since $v_{i-1} \geq v_i$ a.e. in Ω and $v_i - \varphi_i \in W_0^{1,p}(\Omega)$, $(v_1, \ldots, v_N) \in \mathbb{K}$ and we have

$$\sum_{i=1}^N \int_\Omega a(x, \nabla u_i) \cdot \nabla\left(\bigvee_{k=i}^N u_k^\varepsilon - u_i\right) \geq \sum_{i=1}^N \int_\Omega f_i \left(\bigvee_{k=i}^N u_k^\varepsilon - u_i\right).$$

So,

$$\sum_{i=1}^{N}\int_\Omega a(x,\nabla u_i)\cdot\nabla\left(u_i^\varepsilon-u_i\right)\geq\sum_{i=1}^{N}\int_\Omega f_i\left(u_i^\varepsilon-u_i\right)$$

$$+\sum_{i=1}^{N}\int_\Omega a(x,\nabla u_i)\cdot\nabla\left(u_i^\varepsilon-\bigvee_{k=i}^{N}u_k^\varepsilon\right)+\sum_{i=1}^{N}\int_\Omega f_i\left(\bigvee_{k=i}^{N}u_k^\varepsilon-u_i^\varepsilon\right).$$

On the other hand, by (26),

$$\sum_{i=1}^{N}\int_\Omega a(x,\nabla u_i^\varepsilon)\cdot\nabla(u_i^\varepsilon-u_i)\leq\sum_{i=1}^{N}\int_\Omega f_i(u_i^\varepsilon-u_i),$$

and we conclude that

$$\begin{aligned}\sum_{i=1}^{N}\int_\Omega \Big[a(x,\nabla u_i^\varepsilon)-a(x,\nabla u_i)\Big]\cdot\nabla(u_i^\varepsilon-u_i)\leq\\ \sum_{i=1}^{N}\int_\Omega a(x,\nabla u_i)\cdot\nabla\left(\bigvee_{k=i}^{N}u_k^\varepsilon-u_i^\varepsilon\right)-\sum_{i=1}^{N}\int_\Omega f_i\left(\bigvee_{k=i}^{N}u_k^\varepsilon-u_i^\varepsilon\right)\\ =\sum_{i=1}^{N}\int_\Omega\left(Au_i-f_i\right)\left(\bigvee_{k=i}^{N}u_k^\varepsilon-u_i^\varepsilon\right).\end{aligned}\tag{31}$$

Here we have used the fact that $Au_i\in L^q(\Omega)$, for $i=1,\dots,N$, since we know that

$$f_i-\xi_{i-1}\leq Au_i^\varepsilon=-\xi_i\theta_\varepsilon(u_i^\varepsilon-u_{i+1}^\varepsilon)+\xi_{i-1}\theta_\varepsilon(u_{i-1}^\varepsilon-u_i^\varepsilon)+\ f_i\leq f_i+\xi_i,$$

by (22) and $-1\leq\theta_\varepsilon\leq 0$. Noticing that, from (23),

$$0\leq\bigvee_{k=i}^{N}u_k^\varepsilon-u_i^\varepsilon\leq u_i^\varepsilon+(N-i+1)\varepsilon-u_i^\varepsilon\leq(N-i+1)\,\varepsilon\tag{32}$$

it is immediate to conclude that

$$0\leq\sum_{i=1}^{N}\int_\Omega\Big[a(x,\nabla u_i^\varepsilon)-a(x,\nabla u_i)\Big]\cdot\nabla(u_i^\varepsilon-u_i)\leq C\varepsilon.\tag{33}$$

and, since (28) and (29) hold, then, by Lemma 2.3, for each $i=1,\dots,N$,

$$u_i^\varepsilon\longrightarrow u_i\qquad\text{when }\varepsilon\to 0\qquad\text{in }W^{1,p}(\Omega).$$

ii) From (33) and using the strong monotonicity of the operator a, we have

- if $p \geq 2$,

$$\alpha \sum_{i=1}^{N} \int_\Omega |\nabla(u_i^\varepsilon - u_i)|^p \leq$$
$$\sum_{i=1}^{N} \int_\Omega \Big[a(x, \nabla u_i^\varepsilon) - a(x, \nabla u_i)\Big] \cdot \nabla(u_i^\varepsilon - u_i) \leq C\varepsilon,$$

- if $1 < p < 2$, using also the strong monotonicity of the operator a and (33),

$$\begin{aligned} &\alpha \sum_{i=1}^{N} \int_\Omega \big(|\nabla u^\varepsilon| + |\nabla u_i|\big)^{p-2} |\nabla(u_i^\varepsilon - u_i)|^2 \leq \\ &\qquad \sum_{i=1}^{N} \int_\Omega \Big[a(x, \nabla u_i^\varepsilon) - a(x, \nabla u_i)\Big] \cdot \nabla(u_i^\varepsilon - u_i) \leq C\varepsilon, \end{aligned} \tag{34}$$

Let $\hat{\Omega}_i = \{x \in \Omega : |\nabla u_i^\varepsilon| + |\nabla u_i| \neq 0\}$. We may use the reverse inequality (30) with $r = \frac{p}{2}$, noticing that $0 < r < 1$ and $r' = \frac{p}{p-2}$, setting $F = |\nabla(u_i^\varepsilon - u_i)|^2$ and $G = \big(|\nabla u_i^\varepsilon| + |\nabla u_i|\big)^{p-2}$. Then we obtain, for $i = 1, \ldots, N$,

$$\left(\int_{\hat{\Omega}_i} |\nabla(u_i^\varepsilon - u_i)|^p\right)^{\frac{2}{p}} dx \leq$$
$$\left(\int_{\hat{\Omega}_i} |\nabla(u_i^\varepsilon - u_i)|^2 \big(|\nabla u_i^\varepsilon| + |\nabla u_i|\big)^{p-2} dx\right) \left(\int_{\hat{\Omega}_i} \big(|\nabla u_i^\varepsilon| + |\nabla u_i|\big)^p dx\right)^{\frac{2-p}{p}}.$$

Since by (34)

$$\int_{\hat{\Omega}_i} |\nabla(u_i^\varepsilon - u_i)|^2 \big(|\nabla u_i^\varepsilon| + |\nabla u_i|\big)^{p-2} dx \leq \frac{1}{\alpha} C\varepsilon$$

and

$$\exists\, M_p > 0 : \qquad \left(\int_{\hat{\Omega}_i} \big(|\nabla u_i^\varepsilon| + |\nabla u_i|\big)^p dx\right)^{\frac{2-p}{p}} \leq M_p,$$

the conclusion follows immediately summing the N inequalities above.

∎

3. Lewy-Stampacchia inequalities and regularity

As a consequence of the approximation by bounded penalization we know already that each

$$A\,u_i \in L^q(\Omega), \quad i = 1, \dots, N$$

and so, we can use the analogy with the obstacle problem to show further regularity of the solution u_i. In [12] Lewy and Stampacchia have shown that the solution of the obstacle problem for the Laplacian satisfies a dual inequality, which in fact holds in more general cases, as it was observed in [7] or [2] for nonlinear operators. Summarizing the known results for the one and the two obstacles problem that we shall apply to the N-membranes problem, the following theorem may be proved as in [19] or [17].

Theorem 3.1. *Given $\varphi, \psi_1, \psi_2 \in W^{1,p}(\Omega)$, $(1 < p < \infty)$, with f, $(A\,\psi_2 - f)^+$ and $(A\,\psi_1 - f)^-$ in $L^q(\Omega) \subset W^{-1,p'}(\Omega)$, ($q = 1$ if $p > d$, $q > 1$ if $p = d$ or $q = \frac{dp}{dp+p-d}$ if $1 < p < d$) such that*

$$\mathbb{K}_{\psi_2}^{\psi_1} = \Big\{ v \in W^{1,p}(\Omega) : \psi_1 \geq v \geq \psi_2 \text{ a.e. in } \Omega,\ v - \varphi \in W_0^{1,p}(\Omega) \Big\} \neq \emptyset, \quad (35)$$

the unique solution to the variational inequality

$$u \in \mathbb{K}_{\psi_2}^{\psi_1} : \int_\Omega a(x, \nabla u) \cdot \nabla(v - u) \geq \int_\Omega f(v - u),\ \forall v \in \mathbb{K}_{\psi_2}^{\psi_1} \qquad (36)$$

under the assumptions (4), (5) *and* (6) *satisfies the Lewy-Stampacchia inequality*

$$f \wedge A\,\psi_1 \ \leq A\,u \leq f \vee A\,\psi_2 \text{ a.e. in } \Omega. \qquad (37)$$

Remark 3.2. *Setting $\xi_1 = (A\,\psi_1 - f)^-$ and $\xi_2 = (A\,\psi_2 - f)^+$ and using the penalization function θ_ε of the previous section we may approach, as $\varepsilon \to 0$, the solution of* (36) *by the solutions u^ε of the equation*

$$A\,u^\varepsilon + \xi_2 \theta_\varepsilon\,(u^\varepsilon - \psi_2) - \xi_1 \theta_\varepsilon\,(\psi_1 - u^\varepsilon) = f \quad \text{in } \Omega \qquad (38)$$

with the Dirichlet boundary condition $u^\varepsilon = \varphi$ on $\partial\Omega$. Noting that

$$f \wedge A\,\psi_1 \ = f - (A\,\psi_1 - f)^- \quad \text{and} \quad f \vee A\,\psi_2 \ = f + (A\,\psi_2 - f)^+$$

we easily conclude (37) *from the analogous inequalities that are satisfied for each u^ε.*

Remark 3.3. *We may consider that Theorem* 3.1*, although stated for two obstacles problem, also contains the case of only one obstacle. Indeed, by taking* $\psi_1 \equiv +\infty$*,* (36) *is a lower obstacle problem and* (37) *reads*

$$f \le A\,u \le f \vee A\,\psi_2, \quad \textit{for } u \ge \psi_2, \textit{ a.e. in } \Omega \tag{39}$$

and by taking $\psi_2 \equiv -\infty$*,* (36) *is an upper obstacle problem for which* (37) *reads*

$$f \wedge A\,\psi_1 \le A\,u \le f, \quad \textit{for } u \le \psi_1, \textit{ a.e. in } \Omega. \tag{40}$$

Remark 3.4. *In* [17]*, for more general operators and under a strong monotonicity assumption of the type* (9)*, which however is not necessary in our Theorem* 3.1*, it was shown that each inequality of* (37) *still holds independently of each other in the duality sense, provided* $A\,\psi_1 - f$ *and/or* $A\,\psi_2 - f$ *are in* $V^*_{p'} = \left[W^{-1,p'}(\Omega)\right]^+ - \left[W^{-1,p'}(\Omega)\right]^+$*, i.e., in the ordered dual space of* $W^{1,p}_0(\Omega)$*.*

Theorem 3.5. *The solution* $(u_1, \ldots, u_N)$ *of the* N*-membranes problem, under the assumptions* (4)*,* (5) *and* (6)*, satisfies the following Lewy-Stampacchia type inequalities*

$$\left.\begin{array}{rcll} f_1 & \le & A\,u_1 & \le\ f_1 \vee \cdots \vee f_N \\ f_1 \wedge f_2 & \le & A\,u_2 & \le\ f_2 \vee \cdots \vee f_N \\ & & \vdots & \\ f_1 \wedge \cdots \wedge f_{N-1} & \le & A\,u_{N-1} & \le\ f_{N-1} \vee f_N \\ f_1 \wedge \cdots \wedge f_N & \le & A\,u_N & \le\ f_N \end{array}\right\} \textit{a.e. in } \Omega \tag{41}$$

Proof: Observe that choosing $(v, u_2, \ldots, u_N) \in \mathbb{K}_N$, with $v \in \mathbb{K}_{u_2}$, we see that $u_1 \in \mathbb{K}_{u_2}$ (as in (35) with $\psi_1 = +\infty$) solves the variational inequality (36) with $f = f_1$, and so by (39) we have

$$f_1 \le A\,u_1 \le f_1 \vee A\,u_2 \quad \text{a.e. in } \Omega.$$

Analogously, we see that $u_j \in \mathbb{K}^{u_{j-1}}_{u_{j+1}}$ solves the two obstacles problem (36) with $f = f_j$, $j = 2, \ldots, N-1$, and satisfies, by (37),

$$f_j \wedge A\,u_{j-1} \le A\,u_j \le f_j \vee A\,u_{j+1} \quad \text{a.e. in } \Omega.$$

Since $u_N \in \mathbb{K}^{u_{N-1}}$, by (40), also satisfies

$$f_N \wedge A\,u_{N-1} \le A\,u_N \le f_N \quad \text{a.e. in } \Omega,$$

(41) is easily obtained by simple iteration. ∎

Although for $p > d$, the Sobolev inclusion $W^{1,p}(\Omega) \subset C^{0,\lambda}(\Omega)$ for $0 < \lambda = 1 - \frac{d}{p} < 1$, immediately implies the Hölder continuity of the solutions u_i of the N-membranes problem, this property still holds for $1 < p \leq d$ by using the fact that each $A\,u_i$ is in the same $L^q(\Omega)$ as the forces f_i, $i = 1, \dots, N$. So under the classical assumptions of [11] (see also [16]) we may state for completeness the following regularity result.

Corollary 3.6. *Under the assumptions* (3)-(6) *for* $1 < p \leq d$ *with* $q > \frac{d}{p}$ *in* (3)*, the solution* $(u_1, \dots, u_N)$ *of* (1) *is such that,*

$$u_i \in C^{0,\lambda}(\Omega) \qquad \text{for some } 0 < \lambda < 1, \qquad i = 1, \dots, N$$

and is also in $C^{0,\lambda}(\overline{\Omega})$ *if, in addition, each* $\varphi_i \in C^{0,\lambda}(\partial\Omega)$ *and* $\partial\Omega$ *is smooth, for instance, of class* $C^{0,1}$*.* ■

Remark 3.7. *The above classical result for equations was also shown to hold for the one obstacle problem, for instance, in* [6] *or in* [15]*, or for the two obstacles problems in* [10]*, under more general assumptions on the data. It would be interesting to obtain the Hölder continuity of the solution of* (1) *directly under the classical and more general assumptions of each* $f_i \in W^{-1,s}(\Omega)$ *for* $s > \frac{d}{p-1}$*.*

A more interesting regularity is the Hölder continuity of the gradient of the solution, by analogy with the results for solutions of degenerate elliptic equations. For instance, as a consequence of the inequalities (41) and the results of [5] on the $C^{1,\lambda}$ local regularity of weak solutions, as well as in the regularity up to the boundary of [13], we may also state the following results.

Corollary 3.8. *Under the stronger differentiability properties* (11), (12)*, if* (3) *holds with* $q > \frac{dp}{p-1}$*, the solution* $(u_1, \dots, u_N)$ *of* (1) *is such that*

$$u_i \in C^{1,\lambda}(\Omega) \qquad \text{for some } 0 < \lambda < 1, \qquad i = 1, \dots, N$$

and is also in $C^{1,\lambda}(\overline{\Omega})$ *if, in addition, each* $\varphi_i \in C^{1,\gamma}(\partial\Omega)$*, for some* γ *(*$\lambda \leq \gamma < 1$*) and* $f_i \in L^\infty(\Omega)$ *for all* $i = 1, \dots, N$*.* ■

For differentiable strongly coercive vector fields satisfying the assumptions (11), (12), with $p = 2$, there is no degeneration of the operator A and stronger regularity in $W^{2,s}(\Omega)$ may be obtained also from the fact that (41) holds for the solution of the N-membranes problem. For instance, as in Theorem 3.3 of [9], page 114 (see also Remark 4.5 of [19], page 244), we may prove the following result.

Corollary 3.9. *Let* (11), (12) *hold for* $p = 2$*, suppose* $\partial\Omega \in C^{1,1}$ *and each* $f_i \in L^\infty(\Omega)$*,* $\varphi_i \in W^{2,\infty}(\Omega)$*,* $i = 1, \dots, N$*. Then the solution* $(u_1, \dots, u_N)$ *of* (1) *is such that*

$$u_i \in W^{2,s}(\Omega) \cap C^{1,\gamma}(\overline{\Omega}),\ i = 1, \dots, N,\ \textit{for all } 1 \le s < \infty \textit{ and } 0 \le \gamma < 1. \tag{42}$$

Remark 3.10. *For* N *linear operators of the form*

$$a_i^k(x, \xi) \;=\; \sum_{j=1}^{d} a_{ij}^k(x)\xi_j \qquad k = 1, \dots, N,$$

the regularity (42) *was shown in* [4] *for every* $s \ge 2$ *and, with the same operators with lower order terms in* [1] *for* $s > 1$ *if* $d = 2$ *and* $s \ge \frac{2d}{d+2}$ *if* $d \ge 3$*. For the case of two membranes with linear operators, earlier results were shown in* [23]*, by using similar regularity results for the one obstacle problem. In spite of this analogy, the optimal* $W^{2,\infty}$ *regularity of solutions to obstacle problems is an open problem for the* N*-membranes system.*

Remark 3.11. *In the case of two membranes with constant mean curvature, i.e., when* A *is the minimal surface operator and* f_1 *and* f_2 *are constants in a smooth domain with mean curvature* $H_{\partial\Omega}$ *of* $\partial\Omega$ *larger or equal to* $\frac{|f_1| \vee |f_2|}{d-1}$*, in* [24] *it was shown the existence of a unique solution with the regularity* (42)*. The* N*-membranes problem for the minimal surface operator, in general, is an open problem.*

4. The convergence of the coincidence sets

In this section we prove that, if $(u_1^n, \dots, u_N^n)$ is the solution of the N-membranes problem, under the assumptions (4), (5) and (6) with given data $(f_1^n, \dots, f_N^n)$, $n \in \mathbb{N}$, if $(f_1^n, \dots, f_N^n)$ converges in $\left[L^q(\Omega)\right]^N$ to $(f_1, \dots, f_N)$, we have the stability result in $L^s(\Omega)$, $1 \le s < \infty$, for the corresponding coincidence sets

$$\chi_{\{u_k^n = \cdots = u_l^n\}} \underset{n}{\longrightarrow} \chi_{\{u_k = \cdots = u_l\}}, \qquad \text{for } 1 \le k < l \le N.$$

We begin presenting a lemma that will be needed.

Lemma 4.1. [20] *Given functions* $u, v \in W^{1,p}(\Omega)$*,* $1 < p < \infty$*, such that* $Au, Av \in L^1(\Omega)$*, we have*

$$Au = Av \qquad \textit{a.e. in } \{x \in \Omega : u(x) = v(x)\}.$$

In what follows we continue using the convention $u_0 = +\infty$ and $u_{N+1} = -\infty$. Given $1 \le j \le k \le N$, we define the following sets

$$\Theta_{j,k} = \{x \in \Omega : u_{j-1}(x) > u_j(x) = \cdots = u_k(x) > u_{k+1}(x)\}. \tag{43}$$

The first part of the following proposition identifies the value of Au_i a.e. on each coincidence set $I_{j,k}$ defined in (13). The second part states a necessary condition satisfied by the forces in order to exist contact among consecutive membranes.

Proposition 4.2. *If $j, k \in \mathbb{N}$ are such that $1 \le j \le k \le N$, we have*

i) $$Au_i = \begin{cases} \langle f \rangle_{j,k} & \text{a.e. in } \Theta_{j,k} \quad \text{if } i \in \{j, \dots, k\} \\ \\ f_i & \text{a.e. in } \Theta_{j,k} \quad \text{if } i \notin \{j, \dots, k\} \end{cases}$$

ii) if $j < k$ then for all $i \in \{j, \dots, k\}$ $\langle f \rangle_{i+1,k} \ge \langle f \rangle_{j,i}$ a.e. in $\Theta_{j,k}$.

Proof: *i)* Suppose $i \in \{j, \dots, k\}$ (the other case has a similar and simpler proof). For a.e. $x \in \Theta_{j,k}$ we have $u_{j-1}(x) - u_j(x) = \alpha > 0$ and $u_k(x) - u_{k+1}(x) = \beta > 0$, for some $\alpha = \alpha(x)$, and $\beta = \beta(x)$. Since x belongs to the open set $\{y \in \Omega : u_{j-1}(y) - u_j(y) - \frac{\alpha}{2} > 0\} \cap \{y \in \Omega : u_k(y) - u_{k+1}(y) - \frac{\beta}{2} > 0\}$, there exists $\delta > 0$ such that, for all $\varphi \in \mathcal{D}(B(x,\delta))$, there exists $\varepsilon_0 > 0$ such that, if $0 < \varepsilon < \varepsilon_0$, then $u_{j-1} \ge u_j \pm \varepsilon\varphi$ and $u_k \ge u_{k+1} \pm \varepsilon\varphi$. Choose for test functions

$$v_r = \begin{cases} u_r & \text{if } r \notin \{j, \dots, k\} \\ \\ u_r \pm \varepsilon\varphi & \text{if } r \in \{j, \dots, k\}. \end{cases}$$

Then

$$\pm\varepsilon \sum_{r=j}^{k} \int_\Omega a(x, \nabla u_r) \cdot \nabla\varphi \ge \pm\varepsilon \sum_{r=j}^{k} \int_\Omega f_r\, \varphi, \quad \forall\, \varphi \in \mathcal{D}(B(x,\delta))$$

and

$$\sum_{r=j}^{k} \int_\Omega a(x, \nabla u_r) \cdot \nabla\varphi = \sum_{r=j}^{k} \int_\Omega f_r\, \varphi, \quad \forall\, \varphi \in \mathcal{D}(B(x,\delta)).$$

So we conclude that

$$\sum_{r=j}^{k} Au_r = \sum_{r=j}^{k} f_r, \quad \text{a.e. in } B(x,\delta).$$

We know that $Au_i \in L^1(\Omega)$, for all $i = 1, \dots, N$. So, using Lemma 4.1, we have

$$Au_j = \cdots = Au_i = \cdots = Au_k \quad \text{in } \Theta_{j,k}$$

and we conclude then that

$$(k - j + 1)Au_i = f_j + \cdots + f_k \text{ a.e. in } \Theta_{j,k}.$$

ii) The proof of this item is analogous to the previous one. We choose for test functions

$$v_r \;=\; \begin{cases} u_r & \text{if } r \notin \{j, \dots, i\} \\ \\ u_r + \varepsilon\varphi & \text{if } r \in \{j, \dots, i\} \end{cases}$$

with $\varphi \in \mathcal{D}(B(x, \delta))$, $\varphi \geq 0$, $\varepsilon > 0$ such that $(v_1, \dots, v_N) \in \mathbb{K}$. We conclude then that

$$\sum_{r=j}^{i} \int_\Omega a(x, \nabla u_r) \cdot \nabla\varphi \;\geq\; \sum_{j=r}^{i} \int_\Omega f_r\, \varphi, \quad \forall\, \varphi \in \mathcal{D}(B(x, \delta)),\ \varphi \geq 0,$$

and so, we have $Au_i \geq \langle f \rangle_{j,i}$ a.e. in $\Theta_{j,k}$. Then using the first part of this proposition we conclude that

$$\langle f \rangle_{j,k} \geq \langle f \rangle_{j,i} \qquad \text{a.e. in } \Theta_{j,k}$$

or equivalently, that

$$\langle f \rangle_{i+1,k} \geq \langle f \rangle_{j,i} \qquad \text{a.e. in } \Theta_{j,k}.$$

■

Our goal is to determine a system of N equations, coupled by the characteristic functions of the $\frac{N(N-1)}{2}$ coincidence sets, which is equivalent to the problem (1). This was done in [23] for the case $N = 2$ and in [1] for the case $N = 3$. The system for $N = 2$ is simply

$$\begin{cases} Au_1 &= f_1 + \frac{f_2 - f_1}{2} \chi_{\{u_1 = u_2\}} \\ \\ Au_2 &= f_2 - \frac{f_2 - f_1}{2} \chi_{\{u_1 = u_2\}} \end{cases}$$

and for $N = 3$ is the system (15). From these two examples we see that the determination of the coefficients of this system is not a very simple problem of combinatorics. We present the result for the case N in Theorem 4.5.

Definition 4.3. *Given $f_1, \dots, f_N \in L^q(\Omega)$ we define, for $j, k, i \in \{1, \dots, N\}$, with $j < k$ and $j \leq i \leq k$,*

$$b_i^{j,k}[f] \;=\; \begin{cases} \langle f\rangle_{j,k} - \langle f\rangle_{j,k-1} & \text{if } \; i = j \\ \\ \langle f\rangle_{j,k} - \langle f\rangle_{j+1,k} & \text{if } \; i = k \\ \\ \frac{2}{(k-j)(k-j+1)}\big(\langle f\rangle_{j+1,k-1} - \frac{1}{2}(f_j + f_k)\big) & \text{if } \; j < i < k. \end{cases}$$

Observe that, if $j < i < k$, then $b_i^{j,k}[f]$ does not depends on i. It is also not difficult to see that $\displaystyle\sum_{i=j}^{k} b_i^{j,k}[f] = 0$. We notice first some auxiliary results concerning the coefficients $b_i^{j,k}[f]$ that will be needed. From now on we drop the dependence of $b_i^{j,k}[f]$ on f.

Lemma 4.4.

i) If $j \leq l < r$ then

$$\sum_{k=l+1}^{r} b_j^{j,k} \;=\; \frac{r-l}{r-j+1}\Big(\langle f\rangle_{l+1,r} - \langle f\rangle_{j,l}\Big);$$

In particular $\displaystyle\sum_{k=l+1}^{r} b_j^{j,k}$ is positive if and only if the average of $f_{l+1}, \dots, f_r$ is greater or equal then the average of $f_j, \dots, f_l$.

ii) If $m < i$ then

$$\forall\, r \in \{i, \dots, N\} \qquad \sum_{k=i}^{r} b_i^{m,k} \;=\; b_r^{m,r}.$$

Proof: *i)* We have

$$\begin{aligned}
\sum_{k=l+1}^{r} b_j^{j,k} &= \sum_{k=l+1}^{r} \Big(\langle f\rangle_{j,k} - \langle f\rangle_{j,k-1}\Big) = \langle f\rangle_{j,r} - \langle f\rangle_{j,l} \\
&= \frac{f_j + \cdots + f_r}{r-j+1} - \frac{f_j + \cdots + f_l}{l-j+1} \\
&= \frac{f_j + \cdots + f_l}{r-j+1} + \frac{f_{l+1} + \cdots + f_r}{r-j+1} - \frac{f_j + \cdots + f_l}{l-j+1} \\
&= \frac{f_{l+1} + \cdots + f_r}{r-j+1} - \frac{(r-l)(f_j + \cdots + f_l)}{(r-j+1)(l-j+1)} \\
&= \frac{r-l}{r-j+1}\left(\frac{f_{l+1} + \cdots + f_r}{r-l} - \frac{f_j + \cdots + f_l}{l-j+1}\right) \\
&= \frac{r-l}{r-j+1}\Big(\langle f\rangle_{l+1,r} - \langle f\rangle_{j,l}\Big).
\end{aligned}$$

ii) We prove the equality by induction over r. If $r = i$, the equality is trivial. For $r > i$

$$\begin{aligned}
\sum_{k=i}^{r+1} b_i^{m,k} &= \sum_{k=i}^{r} b_i^{m,k} + b_i^{m,r+1} \\
&= b_r^{m,r} + b_i^{m,r+1}, \qquad \text{by induction hypothesis,} \\
&= \langle f\rangle_{m,r} - \langle f\rangle_{m+1,r} + \frac{2}{(r-m+1)(r-m+2)}\Big(\langle f\rangle_{m+1,r} - \frac{1}{2}\big(f_m + f_{r+1}\big)\Big) \\
&= \frac{f_m + \cdots + f_r}{r-m+1} - \frac{f_{m+1} + \cdots + f_r}{r-m} + \\
&\quad \frac{2(f_{m+1} + \cdots + f_r)}{(r-m)(r-m+1)(r-m+2)} - \frac{f_m + f_{r+1}}{(r-m+1)(r-m+2)}.
\end{aligned}$$

Then

$$\begin{aligned}
\sum_{k=i}^{r+1} b_i^{m,k} &= \left(\frac{1}{r-m+1} - \frac{1}{(r-m+1)(r-m+2)}\right) f_m - \frac{1}{(r-m+1)(r-m+2)} f_{r+1} + \\
&\quad \left(\frac{1}{r-m+1} - \frac{1}{r-m} + \frac{2}{(r-m)(r-m+1)(r-m+2)}\right)(f_{m+1} + \cdots + f_r) \\
&= \frac{f_m}{r-m+2} - \frac{f_{r+1}}{(r-m+1)(r-m+2)} - \frac{f_{m+1} + \cdots + f_r}{(r-m+1)(r-m+2)} \\
&= \frac{f_m + \cdots + f_{r+1}}{r-m+2} - \frac{f_{m+1} + \cdots + f_{r+1}}{r-m+1} \\
&= b_{r+1}^{m,r+1}.
\end{aligned}$$

■

We are now able to deduce the system of equations involving the characteristic functions of the coincidence sets which is equivalent to the problem (1).

Theorem 4.5.

$$Au_i = f_i + \sum_{1\le j<k\le N,\ j\le i\le k} b_i^{j,k}\, \chi_{j,k} \quad a.e.\ in\ \Omega. \tag{44}$$

Proof: We prove that the equality is valid a.e. in $\Theta_{m,r}$ for m, r such that $1 \le m \le r \le N$. This is enough because $\bigcup_{1\le m\le r\le N} \Theta_{m,r} = \Omega$. If $i \notin \{m, \dots, r\}$, then (44) results immediately from Proposition 4.2– i). Supposing then that $i \in \{m, \dots, r\}$, using Lemma 4.2, the equality (44), for $x \in \Theta_{m,r}$ becomes

$$f_i + \sum_{m\le j<k\le r,\ j\le i\le k} b_i^{j,k} = \langle f\rangle_{m,r}.$$

We prove now this equality by induction on $i - m$.
If $i - m = 0$, then

$$\begin{aligned}
f_i + \sum_{m\le j<k\le r,\ j\le i\le k} b_i^{j,k} &= f_m + \sum_{m<k\le r} b_m^{m,k} \\
&= f_m + \sum_{m<k\le r} \Big(\langle f\rangle_{m,k} - \langle f\rangle_{m,k-1}\Big) \\
&= \langle f\rangle_{m,r}.
\end{aligned}$$

By the induction step, if $i-m>0$, then

$$\begin{aligned} f_i + \sum_{m\le j<k\le r,\ j\le i\le k} b_i^{j,k} &= f_i + \sum_{m+1\le j<k\le r,\ j\le i\le r} b_i^{j,k} + \sum_{i\le k\le r} b_i^{m,k} \\ &= \langle f\rangle_{m+1,r} + \sum_{k=i}^{r} b_i^{m,k} \quad \text{by induction hypothesis} \\ &= \langle f\rangle_{m+1,r} + b_r^{m,r} \quad \text{by Lemma 4.4–ii)} \\ &= \langle f\rangle_{m,r}. \end{aligned}$$

■

We have now the main result of this section.

Theorem 4.6. *Given $n\in\mathbb{N}$, let $(u_1^n,\ldots,u_N^n)$ denote the solution of problem (1) with given data $(f_1^n,\ldots,f_N^n)\in[L^q(\Omega)]^N$, with q as in (3). Suppose that*

$$f_i^n \xrightarrow[n]{} f_i \quad \text{in} \quad L^q(\Omega), \qquad i=1,\ldots,N. \tag{45}$$

Then

$$u_i^n \xrightarrow[n]{} u_i \quad \text{in} \quad W^{1,p}(\Omega), \qquad i=1,\ldots,N. \tag{46}$$

If, in addition, the limit forces satisfy

$$\langle f\rangle_{i,j} \ne \langle f\rangle_{j+1,k}, \qquad \text{for all } i,j,k\in\{1,\ldots,N\} \text{ with } i\le j<k, \tag{47}$$

then, for any $1\le s<\infty$,

$$\forall\, j,k\in\{1,\ldots,N\},\ j<k \quad \chi_{\{u_j^n=\cdots=u_k^n\}} \xrightarrow[n]{} \chi_{\{u_j=\cdots=u_k\}} \quad \text{in} \quad L^s(\Omega). \tag{48}$$

Before proving the theorem we need another auxiliary lemma:

Lemma 4.7. *Let $n\in\mathbb{N}$ and $a_1,\ldots,a_n\in\mathbb{R}$ be such that $\sum_{r=j}^{n} a_r>0$ for all $j=1,\ldots,n$. Then the inequality*

$$a_1Y_1+\cdots+a_nY_n \ \le\ 0,$$

with the restrictions $0\le Y_1\le\cdots\le Y_n$, has only the trivial solution $Y_1=\cdots=Y_n=0$.

Proof: If $n=1$ the conclusion is immediate. Supposing the result proved for n, let us prove it for $n+1$:

$$\begin{aligned} 0 &\ge a_1Y_1+\cdots+a_nY_n+a_{n+1}Y_{n+1} \\ &\ge a_1Y_1+\cdots+a_nY_n+a_{n+1}Y_n \end{aligned}$$

since $Y_{n+1} \geq Y_n \geq 0$ and $a_{n+1} > 0$. Then

$$0 \geq a_1 Y_1 + \cdots + (a_n + a_{n+1)}\, Y_n$$

and, because the result is true for n, then $Y_1 = \cdots = Y_n = 0$ and, therefore, since $a_{n+1} > 0$, we also have $Y_{n+1} = 0$. ■

Proof of Theorem 4.6: The convergence (46) of the solutions is an immediate consequence of a theorem due to Mosco. For simplicity, we write $\chi_{\{u_j=\cdots=u_k\}} = \chi_{j,k}$ and we denote $\chi_{\{u_j^n=\cdots=u_k^n\}}$ by $\chi_{j,k}^n$ Let $j, k \in \{1, \ldots, N\}$ with $j < k$. Since $0 \leq \chi_{j,k} \leq 1$, there exists $\chi_{j,k}^* \in L^q(\Omega)$ such that $\left(\chi_{j,k}^n\right)_{n\in\mathbb{N}}$ converges to $\chi_{j,k}^*$ in $L^q(\Omega)$-weak. Of course we have

$$\begin{cases} 0 \leq \chi_{j,k}^* \leq 1, & \text{because } 0 \leq \chi_{j,k}^n \leq 1 \\ \\ \chi_{m,r}^* \leq \chi_{j,k}^* \text{ (if } m \leq j < k \leq r), & \text{because } \chi_{m,r}^n \leq \chi_{j,k}^n. \end{cases} \tag{49}$$

Besides that, letting $n \to \infty$ in the equality $\chi_{j,k}^n \left(u_j^n - u_k^n\right)^+ \equiv 0$, we conclude

$$\chi_{j,k}^* \,(u_j - u_k)^+ = 0 \qquad \text{a.e. in } \Omega. \tag{50}$$

Consider now the system (44), with the coefficients b substituted by b_n, for data $f_1^n, \ldots, f_N^n$, with $n \in \mathbb{N}$,

$$Au_i^n \;=\; f_i^n + \sum_{j<k\leq N,\ j\leq i\leq k} \left(b_n\right)_i^{j,k} \chi_{j,k}^n \qquad \text{a.e. in } \Omega, \qquad i = 1, \ldots, N.$$

Passing to the weak limit in $L^q(\Omega)$, when $n \to \infty$, we have

$$Au_i \;=\; f_i + \sum_{j<k\leq N,\ j\leq i\leq k} b_i^{j,k} \chi_{j,k}^* \qquad \text{a.e. in } \Omega, \qquad i = 1, \ldots, N.$$

Subtracting the equality (44) for the limit solution from this one, we obtain

$$\sum_{j<k\leq N,\ j\leq i\leq k} b_i^{j,k} \left(\chi_{j,k} - \chi_{j,k}^*\right) \;=\; 0 \qquad \text{a.e. in } \Omega, \qquad i = 1, \ldots, N. \tag{51}$$

For $k > j$, let $Y_{j,k}$ denote $\chi_{j,k} - \chi_{j,k}^*$. To complete the proof we only need to show that, for $j < k$, $Y_{j,k} \equiv 0$, i.e., $\left(\chi_{j,k}^n\right)_{n\in\mathbb{N}}$ converges to $\chi_{j,k}$ in $L^q(\Omega)$-weak. From equation (50) we know that

$$\forall\, j < k \quad Y_{j,k} \equiv 0 \quad \text{in } \{u_j \neq u_k\} = \{u_j > u_k\}. \tag{52}$$

Fix j_0 and k_0 such that $j_0 < k_0$. Using (52), we only need to see that $Y_{j_0,k_0} \equiv 0$ in $I_{j_0,k_0} = \{u_{j_0} = \cdots = u_{k_0}\}$. It is enough then to prove it in two cases: *i)* $\Theta_{j_0,r}$, for $r \geq j_0$; *ii)* $\Theta_{m,r}$ for $m < j_0$ and $r \geq k_0$.

In the first case, using (52), we have $Y_{j,k} \equiv 0$ in $\Theta_{j_0,r}$, if $j < j_0$ or $k > r$. So, letting $i = j_0$ on equation (51), we have, in $\Theta_{j_0,r}$,

$$\begin{aligned} 0 &= \sum_{j<k\leq N,\, j\leq j_0\leq k} b_{j_0}^{j,k}\, Y_{j,k} \\ &= \sum_{j_0\leq j<k\leq N,\, j\leq j_0\leq k\leq r} b_{j_0}^{j,k}\, Y_{j,k} \\ &= \sum_{k=j_0+1}^{r} b_{j_0}^{j_0,k}\, Y_{j_0,k}. \end{aligned}$$

We can apply now Lemma 4.7 to conclude that $Y_{j_0,k} = 0$ in $\Theta_{j_0,r}$ for $k \in \{j_0+1,\ldots,r\}$, since

- for $x \in \Theta_{j_0,r}$, $Y_{j_0,r}(x) = 1 - \chi^*_{j_0,k}(x)$ and, using (49), $Y_{j_0,j_0+1}(x) \leq \cdots \leq Y_{j_0,r}(x)$;
- for $l \geq j_0$, by Lemma 4.4–i),

$$\sum_{k=l+1}^{r} b_{j_0}^{j_0,k} = \frac{r-l}{r-j_0+1}\big(\langle f\rangle_{l+1,r} - \langle f\rangle_{j_0,l}\big),$$

  which is positive, by Lemma 4.2–ii), as $x \in \Theta_{j_0,r}$, and (47).

In the second case, in $\Theta_{m,r}$ ($m < j_0$ and $r \geq k_0$)

$$\begin{aligned} 0 \leq Y_{j_0,k_0} &= \chi_{j_0,k_0} - \chi^*_{j_0,k_0} \\ &= 1 - \chi^*_{j_0,k_0} && \text{since } m < j_0 < k_0 \leq r \\ &\leq 1 - \chi^*_{m,k_0} && \text{by (49)} \\ &= \chi_{m,k_0} - \chi^*_{m,k_0} \\ &= Y_{m,k_0} \\ &= 0 && \text{as in the previous case.} \end{aligned}$$

Notice that, since χ_{j_0,k_0} is a characteristic function, $\big(\chi^n_{j_0,k_0}\big)_{n\in\mathbb{N}}$ converges in fact to χ_{j_0,k_0} in $L^s(\Omega)$-strong, for all $1 \leq s < \infty$. ∎

Remark 4.8. *Notice that, arguing as in Theorem* 2.5*, under the strong monotonicity assumption* (9)*, it is easy to show the following continuous dependence result on the data,*

$$\sum_{i=1}^{N} \|u_i^n - u_i\|_{W_0^{1,p}(\Omega} \leq C_q \sum_{i=1}^{N} \|f_i^n - f_i\|_{L^q(\Omega)},$$

for q defined as in (3)*. However, a corresponding L^1 estimate for the characteristic functions of the coincidence sets, similar to the obstacle problem* ([19]*,* [20]) *seems more difficult to obtain.*

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Assis Azevedo
Department of Mathematics, University of Minho, Campus de Gualtar, 4710–057 Braga, Portugal

E-mail address: assis@math.uminho.pt

José-Francisco Rodrigues
CMUC/University of Coimbra & University of Lisbon/CMAF, Av. Prof. Gama Pinto, 2, 1649–003 Lisbon, Portugal

E-mail address: rodrigue@fc.ul.pt

Lisa Santos
CMAF/University of Lisbon & Department of Mathematics, University of Minho, Campus de Gualtar, 4710–057 Braga, Portugal

E-mail address: lisa@math.uminho.pt