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NEW GLOBAL A PRIORI ESTIMATES FOR THE THIRD-GRADE FLUID EQUATIONS

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ABSTRACT: This note bridges the gap between the existence and regularity classes for the solutions of the third-grade Rivlin-Ericksen fluid equations. We obtain a new global *a priori* estimate which conveys the precise regularity conditions that lead to the existence of a global in time regular solution.

KEYWORDS: third-grade fluid equations, global *a priori* estimates, regularity class.
AMS SUBJECT CLASSIFICATION (2000): 76A10, 76D03, 35Q35.

1. Introduction

Many models governing the motion of incompressible viscoelastic fluids are best described as systems of nonlinear parabolic-hyperbolic PDE's. Typically, existence results for this type of systems can only be obtained locally in time, or else globally, while assuming that the given data are sufficiently small, see *e.g.* [1, 2, 4, 5, 7, 9–11, 14]. For some models, the situation is better in a two-dimensional setting and solvability can be proved globally in time for any sufficiently regular set of data, cf. [3, 5]. Quite recently it was shown, without any smallness assumptions on the data, that the equations of third-grade Rivlin-Ericksen fluids admit global solutions if the initial fluid velocity belongs to $H^2(\mathbb{R}^n)$, $n = 2, 3$, see [3]. In the two-dimensional case, this regularity is enough to show uniqueness but in the 3-D case there is a gap between the existence and the uniqueness classes.

In this article, we will study the regularity of a global in time solution of the third-grade fluid equations in 3-D. Our analysis is based on a new global *a priori* estimate which allows for the study of the precise regularity conditions that lead to the existence of a global regular solution. Consequently, we obtain a regularity class, different from the existence class, but within which the uniqueness is also valid. Let us stress that although the existence of more regular (even classical) solutions for these equations has been studied previously, cf. [2, 14], these results, which are all only true under restrictive

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smallness (and regularity) conditions on the data or on the material constants, were obtained directly within the regularity and uniqueness classes. Hence, they can only be results "in the small".

Finally, it is interesting to note that the term which makes it possible to show global existence in 3-D for third-grade fluids (and not for the second-grade fluid equations in which this term is missing) is the same which Ladyzhenskaya added to the Navier-Stokes equations in order to prove global in time existence, cf. [12].

The paper is organized as follows: in section 2 we present the model and introduce some notation; section 3 gathers the basic *a priori* estimates for the solution of the problem; section 4 contains a uniqueness result; the main section 5 bridges the gap between the existence and regularity classes through the establishment of a new *a priori* estimate.

2. The equations

In an incompressible Rivlin-Ericksen fluid of grade three the extra-stress tensor is given by (see [13])

$$\mathbf{T}_E = \eta \mathbf{A}_1(\mathbf{v}) + \alpha_1 \mathbf{A}_2(\mathbf{v}) + \alpha_2 \mathbf{A}_1^2(\mathbf{v}) + \beta(\mathrm{tr}\, \mathbf{A}_1^2(\mathbf{v}))\mathbf{A}_1(\mathbf{v}) \ , \tag{1}$$

where $\mathbf{v}$ is the fluid velocity, $\mathbf{A}_1(\mathbf{v})$ and $\mathbf{A}_2(\mathbf{v})$ denote the first two Rivlin-Ericksen tensors

$$\begin{aligned} \mathbf{A}_1(\mathbf{v}) &= \nabla \mathbf{v} + (\nabla \mathbf{v})^T \ , \\ \mathbf{A}_2(\mathbf{v}) &= \left(\frac{\partial}{\partial t} + \mathbf{v} \cdot \nabla \right) \mathbf{A}_1(\mathbf{v}) + \mathbf{A}_1(\mathbf{v}) \nabla \mathbf{v} + (\nabla \mathbf{v})^T \mathbf{A}_1(\mathbf{v}) \ , \end{aligned} \tag{2}$$

and η, α_1, α_2 and β stand for material constants. In fact, the constitutive relation (1) is a degenerate form of a more general Rivlin-Ericksen fluid of grade three defined by

$$\mathbf{T}_E = \eta \mathbf{A}_1 + \alpha_1 \mathbf{A}_2 + \alpha_2 \mathbf{A}_1^2 + \beta_1 \mathbf{A}_3 + \beta_2 (\mathbf{A}_1 \mathbf{A}_2 + \mathbf{A}_2 \mathbf{A}_1) + \beta_3 (\mathrm{tr}\, \mathbf{A}_1^2) \mathbf{A}_1$$

and obtained by assuming, in view of thermodynamics, that $\beta_1 = \beta_2 = 0$ (see [8]).

A third-grade fluid is compatible with thermodynamics if the material constants in (1) satisfy the conditions (cf. [8]):

$$\eta \geq 0 \ , \qquad \alpha_1 \geq 0 \ , \qquad \beta \geq 0 \ , \qquad |\alpha_1 + \alpha_2| \leq \sqrt{24 \eta \beta} \ . \tag{3}$$

The constitutive law (1) includes as special cases the fluids of second-grade ($\beta = 0$), and the Newtonian fluids ($\beta = \alpha_1 = \alpha_2 = 0$). In particular, the second-grade fluids are consistent with thermodynamics if the material constants satisfy (cf. [6]):

$$\eta \geq 0, \qquad \alpha_1 \geq 0, \qquad \alpha_1 + \alpha_2 = 0 \ . \tag{4}$$

The constitutive relation (1), together with the equations of motion, leads to the following system of equations that governs the motion of an incompressible viscoelastic Rivlin-Ericksen fluid of grade three:

$$\begin{cases} \dfrac{\partial}{\partial t}(\mathbf{v} - \alpha_1 \Delta \mathbf{v}) - \nu \Delta \mathbf{v} - \beta \nabla \cdot \big(|\mathbf{A}(\mathbf{v})|^2 \mathbf{A}(\mathbf{v}) \big) + \nabla p & \\ \qquad\qquad = -\mathbf{v} \cdot \nabla (\mathbf{v} - \alpha_1 \Delta \mathbf{v}) + \nabla \cdot \mathbf{N}(\mathbf{v}) + \mathbf{f} & \text{in } \ \mathbb{R}^3 \times (0,T) \\ \nabla \cdot \mathbf{v} = 0 & \text{in } \ \mathbb{R}^3 \times (0,T) \\ \mathbf{v}(x,0) = \mathbf{v}_0(x) & x \in \mathbb{R}^3 \ . \end{cases} \tag{5}$$

Here we have set $\mathbf{A}(\mathbf{v}) = \mathbf{A}_1(\mathbf{v})$. Moreover, all the material constants are divided by the constant density ρ ($\nu = \eta/\rho$ denotes the kinematical viscosity coefficient), and

$$\mathbf{N}(\mathbf{v}) = \alpha_1 (\nabla \mathbf{v})^T \mathbf{A}(\mathbf{v}) + (\alpha_1 + \alpha_2) \mathbf{A}^2(\mathbf{v}).$$

3. Basic *a priori* estimates

Our main result is based on a new, global *a priori* estimate for the third-order spatial derivatives of the solution. Hence, for the sake of completeness, we gather in this section a few basic *a priori* estimates for the lower order derivatives that will be useful in the sequel and briefly recall how they can be derived. Let us start by recalling the existence result proven in [3].

Theorem 3.1. *Assume that* $\mathbf{f} \in L^\infty_{\text{loc}}\big([0,\infty); L^2(\mathbb{R}^3)\big)$ *and that* $\mathbf{v}_0 \in H^2\left(\mathbb{R}^3\right)$, *with* $\nabla \cdot \mathbf{v}_0 = 0$. *There exists a solution* $\mathbf{v} \in C_{\mathrm{w}}\big([0,T); H^2(\mathbb{R}^3)\big)$, *which is global in time* (*i.e., the solution exists for all* $T > 0$), *satisfying equations* (5) *in the sense of distributions.*

The result follows from an *a priori* estimate for the $L^2(\mathbb{R}^3)$-norm of $\mathbf{v} - \alpha_1 \Delta \mathbf{v}$ and from a subsequent application of the Galerkin method.

We next describe how to obtain the basic *a priori* estimates. Multiplying $(5)_1$ with $\mathbf{v}$, integrating over $\mathbb{R}^3$, performing several integrations by parts, and using the fact that $\nabla\cdot\mathbf{v}=0$, one obtains

$$\frac{1}{2}\frac{d}{dt}\Big(\int_{\mathbb{R}^3}|\mathbf{v}|^2\,dx+\alpha_1\int_{\mathbb{R}^3}|\nabla\mathbf{v}|^2\,dx\Big)+\nu\int_{\mathbb{R}^3}|\nabla\mathbf{v}|^2\,dx+\frac{\beta}{2}\int_{\mathbb{R}^3}|\mathbf{A}(\mathbf{v})|^4\,dx$$

$$=-\frac{\alpha_1+\alpha_2}{2}\int_{\mathbb{R}^3}\mathbf{A}^2(\mathbf{v}):\mathbf{A}(\mathbf{v})\,dx+\int_{\mathbb{R}^3}\mathbf{f}\cdot\mathbf{v}\,dx\ ,$$

where $\mathbf{A}:\mathbf{B}=A_{ij}B_{ij}$ denotes the usual double scalar product between two second-order tensors. Using Hölder's and Young's inequalities, we get

$$\frac{d}{dt}\left(\int_{\mathbb{R}^3}|\mathbf{v}|^2\,dx+\alpha_1\int_{\mathbb{R}^3}|\nabla\mathbf{v}|^2\,dx\right)+2\nu\int_{\mathbb{R}^3}|\nabla\mathbf{v}|^2\,dx+\frac{\beta}{2}\int_{\mathbb{R}^3}|\mathbf{A}(\mathbf{v})|^4\,dx \tag{6}$$

$$\le\frac{|\alpha_1+\alpha_2|^2}{2\,\alpha_1\,\beta}\left(\alpha_1\int_{\mathbb{R}^3}|\nabla\mathbf{v}|^2\,dx+\int_{\mathbb{R}^3}|\mathbf{v}|^2\,dx\right)+\frac{2\,\alpha_1\,\beta}{|\alpha_1+\alpha_2|^2}\int_{\mathbb{R}^3}|\mathbf{f}|^2\,dx$$

and, hence, Gronwall's inequality yields the first *a priori* estimate

$$\operatorname*{ess\,sup}_{0\le t\le T}\left(\|\mathbf{v}\|_{0,2}^2+\alpha_1\|\nabla\mathbf{v}\|_{0,2}^2\right)+2\,\nu\int_0^T\|\nabla\mathbf{v}\|_{0,2}^2\,dt+\frac{\beta}{2}\int_0^T\|\mathbf{A}(\mathbf{v})\|_{0,4}^4\,dt \tag{7}$$

$$\le\ \exp\left\{\frac{|\alpha_1+\alpha_2|^2}{\beta\alpha_1}T\right\}\left\{\|\mathbf{v}_0\|_{0,2}^2+\alpha_1\|\nabla\mathbf{v}_0\|_{0,2}^2+\frac{2\,\alpha_1\,\beta}{|\alpha_1+\alpha_2|^2}\int_0^T\|\mathbf{f}\|_{0,2}^2\,dt\right\}\ .$$

Next, let us (formally) multiply equation $(5)_1$ by $-\Delta\mathbf{v}$, integrate over $\mathbb{R}^3$, and again integrate by parts. This results in

$$\frac{1}{2}\frac{d}{dt}\left(\int_{\mathbb{R}^3}|\nabla\mathbf{v}|^2\,dx+\alpha_1\int_{\mathbb{R}^3}|\nabla^2\mathbf{v}|^2\,dx\right)+\nu\int_{\mathbb{R}^3}|\nabla^2\mathbf{v}|^2\,dx \tag{8}$$

$$+\frac{\beta}{2}\int_{\mathbb{R}^3}|\mathbf{A}(\mathbf{v})|^2\,|\nabla\mathbf{A}(\mathbf{v})|^2\,dx+\beta\sum_j\int_{\mathbb{R}^3}(\mathbf{A}(\mathbf{v}):\partial_j\mathbf{A}(\mathbf{v}))^2\,dx$$

$$\le|\alpha_1+\alpha_2|\int_{\mathbb{R}^3}|\nabla\mathbf{A}(\mathbf{v})|^2\,|\mathbf{A}(\mathbf{v})|\,dx+\frac{\alpha_1}{2}\int_{\mathbb{R}^3}|\mathbf{A}(\mathbf{v})|\,|\nabla\mathbf{A}(\mathbf{v})|^2\,dx$$

$$+\int_{\mathbb{R}^3}|\mathbf{v}|\,|\mathbf{A}(\mathbf{v})|\,|\nabla\mathbf{A}(\mathbf{v})|\,dx+|(\mathbf{f},\Delta\mathbf{v})|\ .$$

Using Hölder's and Young's inequalities, one can absorb part of the terms on the right-hand side of (8) to the other side. This leads to

$$\frac{d}{dt}\left(\int_{\mathbb{R}^3}|\nabla\mathbf{v}|^2\,dx+\alpha_1\int_{\mathbb{R}^3}|\nabla^2\mathbf{v}|^2\,dx\right)+\nu\int_{\mathbb{R}^3}|\nabla^2\mathbf{v}|^2\,dx \tag{9}$$

$$
+\frac{\beta}{4}\int_{\mathbb{R}^3}|\mathbf{A}(\mathbf{v})|^2\,|\nabla\mathbf{A}(\mathbf{v})|^2\,dx+2\,\beta\sum_j\int_{\mathbb{R}^3}(\mathbf{A}(\mathbf{v}):\partial_j\mathbf{A}(\mathbf{v}))^2\,dx
$$

$$
\leq\frac{4}{\beta}\left(4\,|\alpha_1+\alpha_2|^2+\alpha_1^2\right)\int_{\mathbb{R}^3}|\nabla^2\mathbf{v}|^2\,dx+\frac{4}{\beta}\int_{\mathbb{R}^3}|\mathbf{v}|^2\,dx+\frac{1}{\nu}\int_{\mathbb{R}^3}|\mathbf{f}|^2\,dx
$$

and one obtains, from Gronwall's inequality, the second *a priori* estimate

$$
\begin{aligned}
\operatorname*{ess\,sup}_{0\leq t\leq T}\left(\|\nabla\mathbf{v}\|_{0,2}^2+\alpha_1\|\nabla^2\mathbf{v}\|_{0,2}^2\right)\ &+\ \nu\int_0^T\|\nabla^2\mathbf{v}\|_{0,2}^2\,dt \\
&+\ \frac{\beta}{4}\int_0^T\int_{\mathbb{R}^3}|\mathbf{A}(\mathbf{v})|^2|\nabla\mathbf{A}(\mathbf{v})|^2\,dx\,dt \\
\leq\ \exp\left\{\left(\frac{384\,\nu}{\alpha_1}+\frac{4\,\alpha_1}{\beta}\right)T\right\}\,\cdot& \\
\cdot\left(\|\nabla\mathbf{v}_0\|_{0,2}^2+\alpha_1\|\nabla^2\mathbf{v}_0\|_{0,2}^2+\frac{4}{\beta}\int_0^T\|\mathbf{v}\|_{0,2}^2\,dt+\frac{1}{\nu}\int_0^T\|\mathbf{f}\|_{0,2}^2\,dt\right)&\ ,
\end{aligned}
\tag{10}
$$

after using the first estimate to control the term involving $\|\mathbf{v}\|_{0,2}^2$.

4. A uniqueness result

Here we show that an additional regularity assumption is enough to obtain uniqueness.

Theorem 4.1. *Let $\mathbf{v}_1,\mathbf{v}_2\in C_{\mathrm{w}}\left([0,T);H^2(\mathbb{R}^3)\right)$, be two solutions of equations (5). Moreover, assume that $\mathbf{v}_1\in L^1\left(0,T;W^{2,3}(\mathbb{R}^3)\right)$. Then $\mathbf{v}_1(t)\equiv\mathbf{v}_2(t)$ a.e. in $\mathbb{R}^3$ for all $t\geq 0$.*

Proof. Subtracting equation $(5)_1$ written for the two solutions $\mathbf{v}_1$ and $\mathbf{v}_2$, multiplying the resulting equation by $\mathbf{w}=\mathbf{v}_1-\mathbf{v}_2$, and integrating over $\mathbb{R}^3$ provides the identity

$$
\begin{aligned}
&\frac{1}{2}\frac{d}{dt}\int_{\mathbb{R}^3}(|\mathbf{w}|^2+\alpha_1|\nabla\mathbf{w}|^2)\,dx+\nu\int_{\mathbb{R}^3}|\nabla\mathbf{w}|^2\,dx \\
&\qquad+\frac{\beta}{2}\int_{\mathbb{R}^3}\left(|\mathbf{A}(\mathbf{v}_1)|^2\mathbf{A}(\mathbf{v}_1)-|\mathbf{A}(\mathbf{v}_2)|^2\mathbf{A}(\mathbf{v}_2)\right):\mathbf{A}(\mathbf{w})\,dx \\
&\quad=\int_{\mathbb{R}^3}\mathbf{w}\cdot\nabla\mathbf{v}_1\cdot\mathbf{w}\,dx-\frac{\alpha_1}{2}\int_{\mathbb{R}^3}\mathbf{w}\cdot\nabla\mathbf{A}(\mathbf{v}_1):\mathbf{A}(\mathbf{w})\,dx \\
&\qquad-\alpha_1\int_{\mathbb{R}^3}\left(\mathbf{A}^2(\mathbf{w}):\mathbf{A}(\mathbf{v}_1)+\mathbf{A}(\mathbf{v}_2)\mathbf{A}(\mathbf{w}):\nabla\mathbf{w}\right)\,dx
\end{aligned}
\tag{11}
$$

$$-\frac{\alpha_2}{2}\int_{\mathbb{R}^3}\left(\mathbf{A}^2(\mathbf{v}_1)-\mathbf{A}^2(\mathbf{v}_2)\right):\mathbf{A}(\mathbf{w})\,dx\ .$$

It follows that

$$\frac{1}{2}\frac{d}{dt}\left(\|\mathbf{w}\|_{0,2}^2+\alpha_1\|\nabla\mathbf{w}\|_{0,2}^2\right)+\nu\,\|\nabla\mathbf{w}\|_{0,2}^2 \tag{12}$$

$$+\frac{\beta}{4}\int_{\mathbb{R}^3}\left(|\mathbf{A}(\mathbf{v}_1)|^2-|\mathbf{A}(\mathbf{v}_2)|^2\right)^2\,dx+\frac{\beta}{4}\int_{\mathbb{R}^3}|\mathbf{A}(\mathbf{w})|^2\left(|\mathbf{A}(\mathbf{v}_1)|^2+|\mathbf{A}(\mathbf{v}_2)|^2\right)\,dx$$

$$\leq c\,\|\nabla\mathbf{v}_1\|_{1,2}\,\|\mathbf{w}\|_{1,2}^2+c\,\alpha_1\,\|\mathbf{w}\|_{0,6}\,\|\nabla\mathbf{A}(\mathbf{v}_1)\|_{0,3}\,\|\mathbf{A}(\mathbf{w})\|_{0,2}$$

$$+\frac{\beta}{8}\int_{\mathbb{R}^3}|\mathbf{A}(\mathbf{w})|^2\left(|\mathbf{A}(\mathbf{v}_1)|^2+|\mathbf{A}(\mathbf{v}_2)|^2\right)\,dx+c\,\frac{(|\alpha_1|+|\alpha_2|)^2}{\beta}\,\|\nabla\mathbf{w}\|_{0,2}^2\ ,$$

where we have used Hölder's and Young's inequalities. Hence, one obtains

$$\frac{1}{2}\frac{d}{dt}\left(\|\mathbf{w}\|_{0,2}^2+\alpha_1\|\nabla\mathbf{w}\|_{0,2}^2\right)\leq c\,\Big(\|\mathbf{v}_1\|_{2,2}+\max\{1,\alpha_1\}\|\nabla^2\mathbf{v}_1\|_{0,3}$$

$$+\frac{(|\alpha_1|+|\alpha_2|)^2}{\alpha_1\beta}\Big)\left(\|\mathbf{w}\|_{0,2}^2+\alpha_1\|\nabla\mathbf{w}\|_{0,2}^2\right)\ ,$$

which, in view of Gronwall's inequality, yields the result.

∎

5. A bridge between existence and regularity

We are now ready to prove our main result providing a regularity class for the weak solution.

Theorem 5.1. *Let* $\mathbf{f}\in L^2\left(0,T;H^1(\mathbb{R}^3)\right)$ *and* $\mathbf{v}_0\in H^3\left(\mathbb{R}^3\right)$. *Moreover, assume that there exists a weak solution* $\mathbf{v}\in C_{\mathrm{w}}\left([0,T);H^2(\mathbb{R}^3)\right)$ *to problem* (5) *such that* $\mathbf{v}\in L^2\left(0,T;W^{2,3}(\mathbb{R}^3)\right)$. *Then* $\mathbf{v}\in L^\infty\left(0,T;H^3(\mathbb{R}^3)\right)$, *for all* $T>0$.

Proof. The following calculations are formal but can be easily justified by a density argument. The idea is to test the equation

$$\partial_t\left(\mathbf{v}-\alpha_1\Delta\mathbf{v}\right)+\left(\mathbf{v}\cdot\nabla\right)\mathbf{v}-\nu\Delta\mathbf{v}-\beta\,\nabla\cdot\Big(|\mathbf{A}(\mathbf{v})|^2\,\mathbf{A}(\mathbf{v})\Big)$$

$$=\mathbf{f}-\nabla p+\alpha_2\nabla\cdot\mathbf{A}^2(\mathbf{v})+\alpha_1\nabla\cdot\Big(\mathbf{v}\cdot\nabla\mathbf{A}(\mathbf{v})+(\nabla\mathbf{v})^{\mathrm{T}}\,\mathbf{A}(\mathbf{v})+\mathbf{A}(\mathbf{v})\nabla\mathbf{v}\Big)$$

with $\Delta^2\mathbf{v}$, perform integration by parts (typically twice), and estimate the resulting terms using the usual inequalities. First, one easily sees that

$$\int_{\mathbb{R}^3} \partial_t \mathbf{v} \cdot \Delta^2 \mathbf{v}\, dx = \frac{1}{2}\frac{d}{dt}\int_{\mathbb{R}^3} \left|\nabla^2 \mathbf{v}\right|^2\, dx\,;$$

$$-\alpha_1 \int_{\mathbb{R}^3} \partial_t \Delta\mathbf{v} \cdot \Delta^2 \mathbf{v}\, dx = \alpha_1 \int_{\mathbb{R}^3} \partial_t \nabla\mathbf{v} : \nabla\Delta^2 \mathbf{v}\, dx = \frac{\alpha_1}{2}\frac{d}{dt}\int_{\mathbb{R}^3} \left|\nabla^3 \mathbf{v}\right|^2\, dx\ ;$$

$$-\nu \int_{\mathbb{R}^3} \Delta\mathbf{v} \cdot \Delta^2 \mathbf{v}\, dx = \nu \int_{\mathbb{R}^3} \nabla\mathbf{v} : \nabla\Delta^2 \mathbf{v}\, dx = \nu \int_{\mathbb{R}^3} \left|\nabla^3 \mathbf{v}\right|^2\, dx\ ;$$

$$\int_{\mathbb{R}^3} \mathbf{f} \cdot \Delta^2 \mathbf{v}\, dx = -\int_{\mathbb{R}^3} \partial_k f_i\, \partial_k \Delta v_i\, dx\ ;$$

$$\int_{\mathbb{R}^3} (\mathbf{v}\cdot\nabla)\,\mathbf{v}\cdot\Delta^2\mathbf{v}\, dx = \int_{\mathbb{R}^3} \partial_l v_j \partial_j \partial_k v_i \partial_k \partial_l v_i\, dx - \int_{\mathbb{R}^3} \partial_k v_j \partial_j v_i \partial_k \partial_l \partial_l v_i\, dx := I_1\,.$$

The next term is

$$\begin{aligned}
& - \beta \int_{\mathbb{R}^3} \nabla\cdot\Big(\left|\mathbf{A}(\mathbf{v})\right|^2 \mathbf{A}(\mathbf{v})\Big)\cdot\Delta^2\mathbf{v}\, dx \\
= \ & \frac{\beta}{2}\int_{\mathbb{R}^3} \left|\mathbf{A}(\mathbf{v})\right|^2 \mathbf{A}(\mathbf{v})\cdot\mathbf{A}\left(\Delta^2\mathbf{v}\right)\, dx \\
= \ & \frac{\beta}{2}\int_{\mathbb{R}^3} \partial_k\partial_l\Big(\left|\mathbf{A}(\mathbf{v})\right|^2 A_{ij}(\mathbf{v})\Big) A_{ij}\left(\partial_k\partial_l\mathbf{v}\right)\, dx \\
= \ & \frac{\beta}{2}\Bigg\{\int_{\mathbb{R}^3} \left|\mathbf{A}(\mathbf{v})\right|^2 A_{ij}(\partial_k\partial_l\mathbf{v}) A_{ij}\left(\partial_k\partial_l\mathbf{v}\right)\, dx \\
& + \int_{\mathbb{R}^3} \Big(\partial_k\partial_l\left|\mathbf{A}(\mathbf{v})\right|^2\Big) A_{ij}(\mathbf{v}) A_{ij}\left(\partial_k\partial_l\mathbf{v}\right)\, dx \\
& +2\int_{\mathbb{R}^3} \Big(\partial_k\left|\mathbf{A}(\mathbf{v})\right|^2\Big) A_{ij}(\partial_l\mathbf{v}) A_{ij}\left(\partial_k\partial_l\mathbf{v}\right)\, dx\Bigg\}\ ,
\end{aligned}$$

which can be rewritten in the form

$$\frac{\beta}{2}\left\{\int_{\mathbb{R}^3} \left|\mathbf{A}(\mathbf{v})\right|^2 \left|\nabla^2\mathbf{A}(\mathbf{v})\right|^2\, dx + \frac{1}{2}\int_{\mathbb{R}^3}\left|\nabla^2\left|\mathbf{A}(\mathbf{v})\right|^2\right|^2\, dx\right\} - I_2$$

with

$$\begin{aligned}
I_2 \ := \ & -\beta\int_{\mathbb{R}^3}\Big(\partial_k\left|\mathbf{A}(\mathbf{v})\right|^2\Big)\left(\partial_k\frac{\left|\mathbf{A}\left(\nabla\mathbf{v}\right)\right|^2}{2}\right)\, dx \\
& +\frac{\beta}{2}\int_{\mathbb{R}^3}\Big(\partial_k\partial_l\left|\mathbf{A}(\mathbf{v})\right|^2\Big) A_{ij}(\partial_k\mathbf{v}) A_{ij}\left(\partial_l\mathbf{v}\right)\, dx\,.
\end{aligned}$$

The nonlinear term multiplied by α_2 takes the form

$$
\begin{aligned}
\alpha_2 \int_{\mathbb{R}^3} \nabla \cdot \mathbf{A}^2(\mathbf{v}) \cdot \Delta^2 \mathbf{v}\, dx &= -\frac{\alpha_2}{2} \int_{\mathbb{R}^3} \mathbf{A}^2(\mathbf{v}) \,:\, \mathbf{A}\left(\Delta^2 \mathbf{v}\right)\, dx \\
&= -\,\alpha_2 \left\{ \int_{\mathbb{R}^3} A_{ik}(\partial_l \partial_m \mathbf{v}) A_{kj}(\mathbf{v})\, A_{ij}\left(\partial_l \partial_m \mathbf{v}\right)\, dx \right. \\
&\qquad \left. + \int_{\mathbb{R}^3} A_{ik}(\partial_l \mathbf{v}) A_{kj}(\partial_m \mathbf{v})\, A_{ij}\left(\partial_l \partial_m \mathbf{v}\right)\, dx \right\} \\
&:= I_3 \ ,
\end{aligned}
$$

and a similar reasoning shows that

$$
\begin{aligned}
&\alpha_1 \int_{\mathbb{R}^3} \nabla \cdot \Big(\mathbf{v} \cdot \nabla \mathbf{A}(\mathbf{v}) + (\nabla \mathbf{v})^{\mathrm{T}}\, \mathbf{A}(\mathbf{v}) + \mathbf{A}(\mathbf{v}) \nabla \mathbf{v} \Big) \cdot \Delta^2 \mathbf{v}\, dx \\
&= -\frac{\alpha_1}{2} \left\{ \int_{\mathbb{R}^3} \mathbf{v} \cdot \nabla \mathbf{A}(\mathbf{v}) \,:\, \mathbf{A}\left(\Delta^2 \mathbf{v}\right)\, dx + \int_{\mathbb{R}^3} \mathbf{A}^2(\mathbf{v}) \,:\, \mathbf{A}\left(\Delta^2 \mathbf{v}\right)\, dx \right\} \\
&= -\,\frac{\alpha_1}{2} \left\{ \int_{\mathbb{R}^3} A_{ml}(\mathbf{v})\, \partial_l A_{ij}(\partial_k \mathbf{v})\, A_{ij}\left(\partial_k \partial_m \mathbf{v}\right)\, dx \right. \\
&\qquad \left. + \int_{\mathbb{R}^3} \partial_k \partial_m v_l\, \partial_l A_{ij}(\mathbf{v})\, A_{ij}\left(\partial_k \partial_m \mathbf{v}\right)\, dx \right\} \\
&\quad -\,\alpha_1 \left\{ \int_{\mathbb{R}^3} A_{ik}(\partial_l \partial_m \mathbf{v}) A_{kj}(\mathbf{v})\, A_{ij}\left(\partial_l \partial_m \mathbf{v}\right)\, dx \right. \\
&\qquad \left. + \int_{\mathbb{R}^3} A_{ik}(\partial_l \mathbf{v}) A_{kj}(\partial_m \mathbf{v})\, A_{ij}\left(\partial_l \partial_m \mathbf{v}\right)\, dx \right\} \ := I_4 \ ,
\end{aligned}
$$

again because $\operatorname{div}\, \mathbf{v} = 0$. This finally gives

$$
\begin{aligned}
&\frac{d}{dt} \left(\int_{\mathbb{R}^3} \left|\nabla^2 \mathbf{v}\right|^2\, dx + \alpha_1 \int_{\mathbb{R}^3} \left|\nabla^3 \mathbf{v}\right|^2\, dx \right) + 2\nu \int_{\mathbb{R}^3} \left|\nabla^3 \mathbf{v}\right|^2\, dx \\
&\qquad + \beta \left\{ \int_{\mathbb{R}^3} |\mathbf{A}(\mathbf{v})|^2 \left|\nabla^2 \mathbf{A}(\mathbf{v})\right|^2\, dx + \frac{1}{2} \int_{\mathbb{R}^3} \left|\nabla^2 |\mathbf{A}(\mathbf{v})|^2\right|^2\, dx \right\} \qquad (13) \\
&\qquad\qquad = -2 \int_{\mathbb{R}^3} \partial_k f_i\, \partial_k \Delta v_i\, dx - 2I_1 + 2I_2 + 2I_3 + 2I_4 \, .
\end{aligned}
$$

The right-hand side in (13) can be bounded from above by

$$
2\, \|\nabla \mathbf{f}\|_{0,2}\, \|\nabla \Delta \mathbf{v}\|_{0,2} + 2\,\beta\, \left\| \nabla\, |\mathbf{A}(\mathbf{v})|^2 \right\|_{0,6} \|\nabla \mathbf{A}(\mathbf{v})\|_{0,3}\, \|\nabla^2 \mathbf{A}(\mathbf{v})\|_{0,2}
$$

$$+\beta \left\| \nabla^2 |\mathbf{A}(\mathbf{v})|^2 \right\|_{0,2} \|\nabla \mathbf{A}(\mathbf{v})\|_{0,3} \|\nabla \mathbf{A}(\mathbf{v})\|_{0,6} + 2 \|\nabla \mathbf{v}\|_{0,2} \|\nabla^2 \mathbf{v}\|_{0,4}^2$$
$$+2 \|\nabla \mathbf{v}\|_{0,4}^2 \|\nabla^3 \mathbf{v}\|_{0,2} + \left(2 |\alpha_1 + \alpha_2| + \alpha_1 \right)$$
$$\Big(\| |\mathbf{A}(\mathbf{v})| \, |\nabla^2 \mathbf{A}(\mathbf{v})| \|_{0,2} + 2 \|\nabla^2 \mathbf{v}\|_{0,3} \|\nabla \mathbf{A}(\mathbf{v})\|_{0,6} \Big) \|\nabla^2 \mathbf{A}(\mathbf{v})\|_{0,2} \,,$$

where we have taken into account that $|\mathbf{A}(\mathbf{v})| \leq 2\,|\nabla \mathbf{v}|$. In view of Young's and Sobolev's inequalities, we get the estimates

$$2 \|\nabla \mathbf{f}\|_{0,2} \|\nabla \Delta \mathbf{v}\|_{0,2} \leq \frac{1}{\nu} \|\nabla \mathbf{f}\|_{0,2}^2 + \nu \left\|\nabla^3 \mathbf{v}\right\|_{0,2}^2 \,;$$

$$\begin{aligned}
\beta \|\nabla \mathbf{A}(\mathbf{v})\|_{0,3} &\Big(2 \left\| \nabla |\mathbf{A}(\mathbf{v})|^2 \right\|_{0,6} \|\nabla^2 \mathbf{A}(\mathbf{v})\|_{0,2} \\
&\qquad + \left\| \nabla^2 |\mathbf{A}(\mathbf{v})|^2 \right\|_{0,2} \|\nabla \mathbf{A}(\mathbf{v})\|_{0,6} \Big) \\
&\leq \frac{\beta}{4} \left\| \nabla^2 |\mathbf{A}(\mathbf{v})|^2 \right\|_{0,2}^2 + c\,\beta \|\nabla^2 \mathbf{v}\|_{0,3}^2 \|\nabla^3 \mathbf{v}\|_{0,2}^2 \,;
\end{aligned}$$

$$\begin{aligned}
&\|\nabla \mathbf{v}\|_{0,2} \|\nabla^2 \mathbf{v}\|_{0,4}^2 + \|\nabla \mathbf{v}\|_{0,4}^2 \|\nabla^3 \mathbf{v}\|_{0,2} \\
&\qquad \leq c \Big(1 + \|\nabla \mathbf{v}\|_{1,2}^2 \Big) \|\nabla^3 \mathbf{v}\|_{0,2}^2 + c \|\nabla \mathbf{v}\|_{1,2}^2 \,;
\end{aligned}$$

$$\begin{aligned}
&(2|\alpha_1 + \alpha_2| + \alpha_1) \Big(\| |\mathbf{A}(\mathbf{v})| \, |\nabla^2 \mathbf{A}(\mathbf{v})| \|_{0,2} \\
&\qquad\qquad + 2\|\nabla^2 \mathbf{v}\|_{0,3} \|\nabla \mathbf{A}(\mathbf{v})\|_{0,6} \Big) \|\nabla^2 \mathbf{A}(\mathbf{v})\|_{0,2} \\
&\qquad \leq \frac{\beta}{2} \| |\mathbf{A}(\mathbf{v})| \, |\nabla^2 \mathbf{A}(\mathbf{v})| \|_{0,2}^2 + c \Big(\nu + \frac{{\alpha_1}^2}{\beta} \\
&\qquad\qquad + (|\alpha_1 + \alpha_2| + \alpha_1) \|\nabla^2 \mathbf{v}\|_{0,3} \Big) \|\nabla^3 \mathbf{v}\|_{0,2}^2 \,,
\end{aligned}$$

where we have also recalled that $|\alpha_1 + \alpha_2| \leq 24\sqrt{\nu\beta}$ (cf. (3)), and used the interpolation inequality

$$\|\mathbf{v}\|_{0,4} \leq c \|\mathbf{v}\|_{0,2}^{1/4} \|\nabla \mathbf{v}\|_{0,2}^{3/4} \,.$$

In view of these estimates, we obtain, from (13), the inequality

$$
\begin{aligned}
\frac{d}{dt}\left(\|\nabla^2\mathbf{v}\|_{0,2}^2+\alpha_1\|\nabla^3\mathbf{v}\|_{0,2}^2\right)+\nu\,\|\nabla^3\mathbf{v}\|_{0,2}^2 \;&+\; \frac{\beta}{2}\left\|\,|\mathbf{A}(\mathbf{v})|\;\left|\nabla^2\mathbf{A}(\mathbf{v})\right|\,\right\|_{0,2}^2 \\
&+\; \frac{\beta}{4}\left\|\nabla^2\,|\mathbf{A}(\mathbf{v})|^2\right\|_{0,2}^2 \\
\le \frac{1}{\nu}\|\nabla\mathbf{f}\|_{0,2}^2+c\|\nabla\mathbf{v}\|_{1,2}^2+c\Big(C(\nu,\alpha_1,\alpha_2,\beta) \;&+\; \|\nabla\mathbf{v}\|_{1,2}^2 \\
&+\; (\beta+1)\|\nabla^2\mathbf{v}\|_{0,3}^2\Big)\|\nabla^3\mathbf{v}\|_{0,2}^2\,,
\end{aligned}
$$

where $C(\nu,\alpha_1,\alpha_2,\beta)=c(1+\nu+\frac{\alpha_1^2}{\beta}+(|\alpha_1+\alpha_2|+\alpha_1)^2)$. From Gronwall's inequality it then follows that

$$
\begin{aligned}
&\operatorname*{ess\,sup}_{0\le t\le T}\left(\|\nabla^2\mathbf{v}\|_{0,2}^2+\alpha_1\|\nabla^3\mathbf{v}\|_{0,2}^2\right)+\nu\int_0^T\|\nabla^3\mathbf{v}\|_{0,2}^2\,dt \\
&\le \exp\left\{C(\nu,\alpha_1,\alpha_2,\beta)\,T+\int_0^T\|\nabla\mathbf{v}\|_{1,2}^2\,dt+(\beta+1)\int_0^T\|\nabla^2\mathbf{v}\|_{0,3}^2\,dt\right\}\cdot \\
&\quad\cdot\left(\left\|\nabla^2\mathbf{v}_0\right\|_{0,2}^2+\alpha_1\left\|\nabla^3\mathbf{v}_0\right\|_{0,2}^2+\frac{1}{\nu}\int_0^T\|\nabla\mathbf{f}\|_{0,2}^2\,ds+c\int_0^T\|\nabla\mathbf{v}\|_{1,2}^2\,dt\right),
\end{aligned}
$$

which concludes the proof in view of the first two *a priori* estimates (7) and (10).

■

We can obtain further regularity by testing equation $(5)_1$ with $-\Delta^3\mathbf{v}$ and performing again some integrations by parts. We obtain the inequality

$$
\begin{aligned}
\frac{d}{dt}\left(\|\nabla^3\mathbf{v}\|_{0,2}^2+\alpha_1\|\nabla^4\mathbf{v}\|_{0,2}^2\right)+2\,\nu\|\nabla^4\mathbf{v}\|_{0,2}^2 \;&+\; \beta\int_{\mathbb{R}^3}|\mathbf{A}(\mathbf{v})|^2|\nabla^3\mathbf{A}(\mathbf{v})\,|^2\,dx \\
&+\; \frac{\beta}{2}\int_{\mathbb{R}^3}\left|\,\nabla^3|\mathbf{A}(\mathbf{v})|^2\,\right|^2\,dx \\
\le\;\; & c\int_{\mathbb{R}^3}\left(|\nabla\,\mathbf{v}|\,|\nabla^3\mathbf{v}|^2+|\nabla^2\mathbf{v}|^2\,|\nabla^3\mathbf{v}|\right)\,dx+2\,\left|(\nabla^2\mathbf{f},\nabla^4\mathbf{v})\right| \\
&+C(\alpha_1,\alpha_2)\int_{\mathbb{R}^3}\left(|\mathbf{A}(\mathbf{v})|\,|\nabla^3\mathbf{A}(\mathbf{v})|^2+|\nabla^2\mathbf{v}|\,|\nabla^3\mathbf{v}|\,|\nabla^4\mathbf{v}|\right)\,dx \\
&+c(\beta)\int_{\mathbb{R}^3}|\mathbf{A}(\mathbf{v})|\,|\nabla\mathbf{A}(\mathbf{v})|\,|\nabla^2\mathbf{A}(\mathbf{v})|\,|\nabla^3\mathbf{A}(\mathbf{v})|+|\nabla\mathbf{A}(\mathbf{v})|^2\,|\nabla^2\mathbf{A}(\mathbf{v})|^2\,dx,
\end{aligned}
$$

from which one easily concludes, using the previous estimates, that

$$\mathbf{v} \in L^\infty\left(0, T; H^4(\mathbb{R}^3)\right), \qquad \text{for all } T > 0.$$

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